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Analysis of a hybrid integro-differential inclusion

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Abstract

Our main objectives in this paper are to investigate the existence of the solutions for an integro-differential inclusion of second order with hybrid nonlocal boundary value conditions. The sufficient condition for the uniqueness of the solution will be given and the continuous dependence of the solution on the set of selections and on other functions will be proved. As an application, the nonlocal problem of the Chandrasekhar hybrid second-order functional integrodifferential inclusion and some particular cases will be presented. Also, we provide some examples to illustrate our results.

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Keywords: Chandrasekhar integro-differential inclusion; Hybrid differential equation; Nonlocal hybrid boundary value problem; Continuous dependency of the solution

1 Introduction

Investigation of the fractional boundary value problems has received a great deal of attention due to the various applications and real-world problems [1–10].

The existence of the solutions for high-order fractional integrodifferential equations involving CFD and DCF have been studied in [11]. For the fractional differential inclusions and some existence results see [2] and [12].

Dhage and Lakshmikantham [13] introduced and initiated studying a new category of nonlinear differential equation called an ordinary hybrid differential equation.

Baleanu et al. [1] applied a generalization of the hybrid Dhage's fixed-point result for the sum of three fractional operators, with the aim of proving the existence of solutions for a fractional hybrid integrodifferential equation with mixed hybrid integral boundary value conditions [1].

An extension for the second-order differential equation of a thermostat model to the fractional hybrid equation and inclusion versions has been provided [14]. Also, hybrid boundary value conditions of this problem have been considered [14]. The complication of mumps-induced hearing loss in children has been modeled and studied in [15] by using the Caputo–Fabrizio fractional-order derivative that preserves the system's historical memory.

A new version for the mathematical model of HIV by using the fractional Caputo–Fabrizio derivative has been given [2]. The existence and uniqueness of the solution for that model by using fixed-point theory and by a combination of the Laplace transform and homotopy analysis method have been considered [2].

In 1997, [16] two new models involving delay-differential equations with hysteresis were developed to describe the dynamic behavior of an automotive thermostat and the solvability of those two models was obtained. A new mathematical model again for the dynamic behavior of a thermostat located in an engine's cooling system was published, along with an algorithm for numerical solutions [17].

In 2005, Webb [18] created the first mathematical model for thermostat control, which had the following structure.

$$\begin{cases} \mu''(t) + f(t)\mathcal{H}(t,\mu(t)) = 0, \\ \mu'(0) = 0, b\mu'(t) + \mu(\tau) = 0, \end{cases}$$

for $t \in [0,1]$ and $b > 0$. Shen, Zhou, and Yang analyzed the thermostat differential equation in noninteger format and with the identical boundary conditions as in [19].

$$\begin{cases} D^{\alpha}\mu(t) + \lambda\mathcal{H}(t,\mu(t)) = 0, \\ \mu'(0) = 0, bD^{\alpha-1}\mu(t) + \mu(\tau) = 0, \end{cases}$$

for $t \in [0,1]$, $b, \lambda > 0$, and $\alpha \in (1,2]$, $\tau \in (0,1)$, and $\mathcal{H} : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Many researchers looked at other structures of the fractional model of a thermostat [10] and [14]. In 2010, Dhage and Lakshmikantham [13] proposed hybrid differential equations.

Baleanu et al. established the hybrid fractional model of thermostat control for the first time, in [14], using Dhage's approach, which accepts such a structure

$${}^{c}D^{\alpha}\left(\frac{\mu(t)}{h(t,\mu(t))}\right) + \mathcal{H}\big(t,\mu(t)\big) = 0,$$

by means of hybrid boundary conditions

$$\begin{cases} D(\frac{\mu(t)}{h(t,\mu(t))})|_{t=0} = 0, \\ b\,{}^{c}D^{\alpha-1}(\frac{\mu(t)}{h(t,\mu(t))})|_{t=1} + (\frac{\mu(t)}{h(t,\mu(t))})|_{t=\tau} = 0 \end{cases}$$

in which $\alpha \in (1,2]$, $\tau \in (0,1)$, $b > 0$, $D = \frac{d}{dt}$, ${}^{c}D^{q}$ represent the Caputo derivative for given order $q \in \{\alpha, \alpha-1\}$ and $\mathcal{H}, h \in C([0,1] \times \mathbb{R}, \mathbb{R})$ with $h \neq 0$. Various thermostat models have been studied by a number of researchers. They have given some thermostat system models (see, for example, [7, 15, 19–24]).

Motivated by these results, we investigate some existence results for the nonlocal problem of the Chandrasekhar hybrid second-order functional integrodifferential inclusion

$$-\frac{d^2}{dt^2}\left(\frac{x(t)}{g(t,x(t))}\right) \in \int_0^1 \frac{t}{t+s}\Phi\left(s, \int_0^1 \frac{s}{s+\tau}\psi\big(\tau,x(\tau)\big)\,d\tau\right)ds, \quad t \in [0,1] = I \tag{1.1}$$

with the nonlocal hybrid boundary value conditions

$$\begin{cases} \mathcal{D}(\frac{x(t)}{g(t,x(t))})|_{t=0} = 0, \\ \lambda\, {}^{c}\mathcal{D}^{\varrho}(\frac{x(t)}{g(t,x(t))})|_{t=\sigma} + (\frac{x(t)}{g(t,x(t))})|_{t=\eta} = 0, \quad \varrho \in (0,1], \sigma \in (0,1], \eta \in (0,1], \end{cases} \tag{1.2}$$

where $\mathcal{D} = \frac{d}{dt}$, λ is a positive real parameter, ${}^{c}\mathcal{D}^{\varrho}$ is the Caputo derivative of order ϱ, $\Phi : I \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map, $\psi : I \times \mathbb{R} \to \mathbb{R}$ is continuous and $g \in C(I \times \mathbb{R}, \mathbb{R}\backslash\{0\})$.

The integral equations of Chandrasekhar's type have been studied in some papers and monographs (see [25, 26] for instance). It has received a lot of attention in recent years, because of its applicability in several different fields of science and engineering, such as radiative-transfer theory, kinetic theory of gases, neutron-transport theory, and traffic theory. Some authors have studied different kinds of quadratic Chandrasekhar integral equations in different classes (see [27–29]).

Here, we prove the existence of at least one solution $x \in C(I)$ of the problem (1.1)–(1.2).

As an application, the nonlocal problem of the Chandrasekhar hybrid second-order functional integrodifferential equation

$$-\frac{d^2}{dt^2}\left(\frac{x(t)}{g(t,x(t))}\right) = \int_0^1 \frac{t}{t+s} k_1(s)\phi_1\left(s, \int_0^1 \frac{s}{s+\tau} k_2(\tau)x(\tau)\,d\tau\right) ds, \quad t \in I \tag{1.3}$$

with the nonlocal hybrid boundary condition (1.2) will be considered.

The uniqueness of the solution $x \in C(I)$ of (1.3) and (1.2) and the continuous dependence of this solution on the two functions k_i, $(i = 1, 2)$ and the set of selections S_Φ, $\phi \in \Phi$ will be proved.

The remaining part of the paper is set as follows: In Sect. 2 some concepts are presented and we demonstrate the corresponding integral equation for the Thermostat Model (1.1)–(1.2). Section 3 establishes the main results, including the existence and continuous dependence of the solution. Finally, in Sect. 4 two examples are provided to highlight that our results are actually valid. The conclusions are given in Sect. 5.

2 Main result

Consider the nonlocal problem (1.1)–(1.2) with the following assumptions:

($\mathcal{H}_1$) Let $\Phi : I \times \mathbb{R} \to 2^{\mathbb{R}}$ be a nonempty, closed, and convex subset for all $(t, u) \in I \times \mathbb{R}$ such that

- (i) $\Phi(t, \cdot)$ is upper semicontinuous in $u \in \mathbb{R}$ for each $t \in I$.
- (ii) $\Phi(\cdot, u)$ is measurable in $t \in I$ for each $u \in \times\mathbb{R}$.
- (iii) There are two integrable functions $m, k_1 : I \to I$ such that

$$\left|\Phi(t,u)\right| = \sup\left\{|\phi| : \phi \in \Phi(t,u)\right\} \le m(t) + k_1(t)|u|, \quad t \in I$$

with

$$\int_0^1 \left|m(\tau)\right| d\tau = m \quad \text{and} \quad \int_0^1 \left|k_1(\tau)\right| d\tau = k_1.$$

Remark 1 We may derive from assumption ($\mathcal{H}_1$) that the set of selections S_Φ of the set valued function Φ is nonempty and that there exists a Carathéodory function $\phi \in \Phi$ (see

[30] and [31]) that is measurable in $t \in I$, $\forall x \in \mathbb{R}$ and continuous in $x \in \mathbb{R}$, $\forall t \in I$,

$$\left|\phi(t,u)\right| \leq m(t) + k_1(t)|u|, \quad t \in I$$

and satisfies the nonlocal problem of the Chandrasekhar hybrid second-order functional integrodifferential equation

$$-\frac{d^2}{dt^2}\left(\frac{x(t)}{g(t,x(t))}\right) = \int_0^1 \frac{s}{t+s}\phi\left(s, \int_0^1 \frac{s}{s+\tau}\psi\big(\tau,x(\tau)\big)\,d\tau\right)ds, \quad t \in I \tag{2.1}$$

with the conditions (1.2).

Hence, any solution of the problem (1.2) and (2.1) is a solution of the problem (1.1)–(1.2).

($\mathcal{H}_2$) $\psi \in C(I \times \mathbb{R}, \mathbb{R})$ and there exists a continuous function $k_2 : I \times I \to \mathbb{R}$ and a continuous nondecreasing map $\chi : [0,\infty) \to (0,\infty)$, such that

$$\left|\psi(t,\mu)\right| \leq k_2(t)\chi\left(\|\mu\|\right),$$

for all $t \in I$ and for all $\tau \in \mathbb{R}$ and

$$\int_0^1 \left|k_2(\tau)\right|d\tau = k_2.$$

($\mathcal{H}_3$) $g \in C(I \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and there is a positive constant ω, such that

$$\left|g(t,\mu_1) - g(t,\mu_2)\right| \leq \omega\left|\mu_1(t) - \mu_2(t)\right|,$$

for all $\mu_1, \mu_2 \in \mathbb{R}$ and $t \in I$.

($\mathcal{H}_4$) There is a positive root r of the equation

$$\big(m + k_1 k_2 \chi(r)\big)(r\omega + G)\Lambda = r,$$

where $G = \sup_{t\in I}|g(t,0)|$, $\Lambda = \lambda + 2$.

Remark 2 From assumptions ($\mathcal{H}_3$), we have

$$\left|g(t,\mu)\right| - \left|g(t,0)\right| \leq \left|g(t,\mu) - g(t,0)\right| \leq \omega|\mu - 0|,$$

then,

$$\left|g(t,\mu)\right| \leq \omega\left|\mu(t)\right| + G, \quad \text{with } G = \sup_{t\in I}\left|g(t,0)\right|.$$

Here, the existence of the solution $x \in C(I)$ for the nonlocal problem (1.2) and (2.1) is discussed. We begin by presenting a key lemma.

Lemma 1 *A function $x \in C[0,1]$ is a solution for the hybrid differential equation*

$$\frac{d^2}{dt^2}\left(\frac{x(t)}{g(t,x(t))}\right) + \varphi\big(t,x(t)\big) = 0, \quad t \in I \tag{2.2}$$

with the nonlocal hybrid condition (1.2) *if and only if $x \in C(I)$ is a solution for the integral equation*

$$\begin{aligned} x(t) = g\big(t,x(t)\big)\bigg[&-\int_0^t (t-s)\varphi\big(s,x(s)\big)\,ds \\ &+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\varphi\big(s,x(s)\big)\,ds + \int_0^\eta (\eta-s)\varphi\big(s,x(s)\big)\,ds\bigg]. \end{aligned} \tag{2.3}$$

Proof Let x be a solution for the hybrid fractional equation (2.2), then,

$$\frac{d}{dt}\left(\frac{x(t)}{g(t,x(t))}\right) = \frac{d}{dt}\left(\frac{x(t)}{g(t,x(t))}\right)\bigg|_{t=0} - \int_0^t \varphi\big(s,y(s)\big)\,ds = -\int_0^t \varphi\big(s,y(s)\big)\,ds. \tag{2.4}$$

Integrating both sides of (2.4), we obtain

$$\frac{x(t)}{g(t,x(t))} = c_\circ - I^2\varphi(t,y), \tag{2.5}$$

here, $c_\circ$ is a random constant. Then, at $t = \eta$,

$$\frac{x(t)}{g(t,x(t))}\bigg|_{t=\eta} = c_\circ - I^2\varphi(t,y)|_{t=\eta},$$

$$\lambda\, {}^c\mathcal{D}^\varrho\left(\frac{x(t)}{g(t,x(t))}\right)\bigg|_{t=\sigma} = \lambda I^{1-\varrho}\frac{d}{dt}\left(\frac{x(t)}{g(t,x(t))}\right)\bigg|_{t=\sigma} \tag{2.6}$$

and

$$\lambda\, {}^c\mathcal{D}^\varrho\left(\frac{x(t)}{g(t,x(t))}\right)\bigg|_{t=\sigma} = \lambda I^{2-\varrho}\varphi\big(t,y(t)\big)|_{t=\sigma}. \tag{2.7}$$

Using (2.5) and (2.7) in condition (1.2), we can obtain

$$\lambda I^{2-\varrho}\varphi\big(t,y(t)\big)|_{t=\sigma} + \big(c_\circ - I^2\varphi(t,y)|_{t=\eta}\big) = 0,$$

then,

$$c_\circ = -\lambda I^{2-\varrho}\varphi\big(t,y(t)\big)|_{t=\sigma} + I^2\varphi(t,y)|_{t=\eta}.$$

Substituting the value $c_\circ$ in (2.5), we obtain

$$x(t) = g\big(t,x(t)\big)\big[-\lambda I^{2-\varrho}\varphi\big(t,y(t)\big)|_{t=\sigma} + I^2\varphi(t,y)|_{t=\eta} - I^2\varphi\big(t,y(t)\big)\big].$$

Hence,

$$x(t) = g\big(t,x(t)\big)\bigg[-\int_0^t (t-s)\varphi\big(s,x(s)\big)\,ds + \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\varphi\big(s,x(s)\big)\,ds + \int_0^\eta (\eta-s)\varphi\big(s,x(s)\big)\,ds\bigg].$$

This proves that x is a solution of (2.3).

Conversely, from (2.3) we have

$$\frac{x(t)}{g(t,x(t))} = -I^2\varphi\big(t,x(t)\big) - \lambda I^\varrho\varphi\big(\sigma,x(\sigma)\big) + I^2\varphi\big(\eta,x(\eta)\big), \tag{2.8}$$

$$\frac{d}{dt}\left(\frac{x(t)}{g(t,x(t))}\right) = -I\varphi\big(t,x(t)\big)$$

and

$$\frac{d}{dt}\left(\frac{x(t)}{g(t,x(t))}\right)\bigg|_{t=0} = 0. \tag{2.9}$$

Also, for $t = \eta$ in (2.8), we have

$$\frac{x(t)}{g(t,x(t))}\bigg|_{t=\eta} = \lambda I^\varrho\varphi\big(\sigma,x(\sigma)\big). \tag{2.10}$$

Operating by $\lambda\,{}^c\mathcal{D}^\varrho$ to (2.8) with $t = \sigma$ and to (2.10), we obtain (1.2). □

Corollary 1 *If the solution $x \in C(I)$ of the nonlocal problem (1.2) and (2.1) exists then it is given by the integral equation*

$$\begin{aligned} x(t) = g\big(t,x(t)\big)\bigg[&-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\right)d\tau\,ds \\ &+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\right)d\tau\,ds \\ &+ \int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\right)d\tau\,ds\bigg]. \end{aligned} \tag{2.11}$$

Proof From Lemma 1, with

$$-\varphi\big(t,x(t)\big) = \int_0^1 \frac{t}{t+s}\phi\left(s,\int_0^1 \frac{s}{s+\tau}\psi\big(\tau,x(\tau)\big)\,d\tau\right)ds, \quad t \in I,$$

we obtain the result. □

For the existence of solutions $x \in C(I)$ of (1.2) and (2.1), we have the following theorem.

Theorem 1 *Assume that the assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ are satisfied, if $\omega(m + k_1k_2\chi(r))\Lambda < 1$. Then, there is at least one solution to the nonlocal problems (1.2), (2.1).*

Proof Allow the operator $\mathcal{A}$ to be defined as follows:

$$\begin{aligned}(\mathcal{A}x)(t) = g\big(t,x(t)\big)\Big[&-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\phi\Big(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\Big)\,d\tau\,ds \\ &+\lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\Big(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\Big)\,d\tau\,ds \\ &+\int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\phi\Big(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\Big)\,d\tau\,ds\Big],\end{aligned}$$

and consider the ball $\mathcal{V}_r = \{x \in C(I) : \|x\| = \|x\|_{C(I)} \le r\}$.

Clearly $\mathcal{V}_r$ is a closed, convex, and bounded subset of the Banach space $C(I) = C[0,1]$.

Let $x \in \mathcal{V}_r$ and $t \in I$, hence,

$$\begin{aligned}&\big|\mathcal{A}x(t)\big| \\ &= \big|g\big(t,x(t)\big)\big|\Big|-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\phi\Big(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\Big)\,d\tau\,ds \\ &\quad+\lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\Big(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\Big)\,d\tau\,ds \\ &\quad+\int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\phi\Big(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\Big)\,d\tau\,ds\Big| \\ &\le \big|g\big(t,x(t)\big)\big|\Big(-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\Big|\phi\Big(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\Big)\Big|\,d\tau\,ds \\ &\quad+\lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\Big|\phi\Big(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\Big)\Big|\,d\tau\,ds \\ &\quad+\int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\Big|\phi\Big(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\Big)\Big|\,d\tau\,ds\Big) \\ &\le \big|g\big(t,x(t)\big)\big|\Big(\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\Big[m(\tau)+k_1(\tau)\int_0^1 \frac{\tau}{\tau+\varsigma}\big|\psi\big(\varsigma,x(\varsigma)\big)\big|\,d\varsigma\Big]\,d\tau\,ds \\ &\quad+\lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\Big[m(\tau)+k_1(\tau)\int_0^1 \frac{\tau}{\tau+\varsigma}\big|\psi\big(\varsigma,x(\varsigma)\big)\big|\,d\varsigma\Big]\,d\tau\,ds \\ &\quad+\int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\Big[m(\tau)+k_1(\tau)\int_0^1 \frac{\tau}{\tau+\varsigma}\big|\psi\big(\varsigma,x(\varsigma)\big)\big|\,d\varsigma\Big]\,d\tau\,ds\Big) \\ &\le \big[\omega\big|x(t)\big|+G\big] \\ &\quad\times\Big(\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\Big[\big|m(\tau)\big|+\big|k_1(\tau)\big|\int_0^1 \frac{\tau}{\tau+\varsigma}\big|k_2(\varsigma)\big|\chi\big(\|x\|\big)\,d\varsigma\Big]\,d\tau\,ds \\ &\quad+\lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\Big[\big|m(\tau)\big|+\big|k_1(\tau)\big|\int_0^1 \frac{\tau}{\tau+\varsigma}\big|k_2(\varsigma)\big|\chi\big(\|x\|\big)\,d\varsigma\Big]\,d\tau\,ds \\ &\quad+\int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\Big[\big|m(\tau)\big|+\big|k_1(\tau)\big|\int_0^1 \frac{\tau}{\tau+\varsigma}\big|k_2(\varsigma)\big|\chi\big(\|x\|\big)\,d\varsigma\Big]\,d\tau\,ds\Big) \\ &\le \big[\omega\big|x(t)\big|+G\big]\Big(\int_0^t\int_0^1\Big[\big|m(\tau)\big|+\big|k_1(\tau)\big|\int_0^1\big|k_2(\varsigma)\big|\chi\big(\|x\|\big)\,d\varsigma\Big]\,d\tau\,ds \\ &\quad+\lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1\Big[\big|m(\tau)\big|+\big|k_1(\tau)\big|\int_0^1\big|k_2(\varsigma)\big|\chi\big(\|x\|\big)\,d\varsigma\Big]\,d\tau\,ds\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\eta}\int_0^1\Bigl[\bigl|m(\tau)\bigr|+\bigl|k_1(\tau)\bigr|\int_0^1\bigl|k_2(\varsigma)\bigr|\chi\bigl(\|x\|\bigr)\,d\varsigma\Bigr]\,d\tau\,ds\Bigr)\\
&\leq\bigl[\omega\bigl|x(t)\bigr|+G\bigr]\biggl(\int_0^t\int_0^1\bigl[\bigl|m(\tau)\bigr|+\bigl|k_1(\tau)\bigr|k_2\chi\bigl(\|x\|\bigr)\bigr]\,d\tau\,ds\\
&+\lambda\int_0^{\sigma}\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1\bigl[\bigl|m(\tau)\bigr|+\bigl|k_1(\tau)\bigr|\bigl|k_2\chi\bigl(\|x\|\bigr)\bigr]\,d\tau\,ds\\
&+\int_0^{\eta}\int_0^1\bigl[\bigl|m(\tau)\bigr|+\bigl|k_1(\tau)\bigr|k_2\chi\bigl(\|x\|\bigr)\bigr]\,d\tau\,ds\biggr)\\
&\leq\bigl[\omega\bigl|x(t)\bigr|+G\bigr]\\
&\times\biggl(m+k_1k_2\chi\bigl(\|x\|\bigr)+\lambda\int_0^{\sigma}\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\bigl(m+k_1|k_2\chi\bigl(\|x\|\bigr)\bigr)\,ds+m+k_1k_2\chi\bigl(\|x\|\bigr)\biggr).
\end{aligned}
$$

Now, taking the supremum over $t\in I$, we have

$$
\begin{aligned}
\|\mathcal{A}x\|&\leq[r\omega+G]\biggl(\bigl(m+k_1k_2\chi(r)\bigr)+\frac{\lambda}{\Gamma(3-\varrho)}\bigl(m+k_1k_2\chi(r)\bigr)+\bigl(m+k_1k_2\chi(r)\bigr)\biggr)\\
&\leq\bigl(m+k_1k_2\chi(r)\bigr)(r\omega+G)\Lambda=r.
\end{aligned}
\tag{2.12}
$$

Then, $\|\mathcal{A}x\|\leq r$.

Hence, $\mathcal{A}:\mathcal{V}_r\to\mathcal{V}_r$, and the class $\{\mathcal{A}x\}$ is uniformly bounded on $\mathcal{V}_r$.

Let $\{x_n\}$ be a sequence that converges to a point $x\in\mathcal{V}_r$, then from our assumptions and the Lebesgue Dominated Convergence Theorem [32], we can obtain

$$
\begin{aligned}
&\lim_{n\to\infty}(\mathcal{A}x_n)(t)\\
&=\lim_{n\to\infty}g\bigl(t,x_n(t)\bigr)\biggl[-\lim_{n\to\infty}\int_0^t(t-s)\int_0^1\frac{s}{s+\tau}\phi\biggl(\tau,\int_0^1\frac{\tau}{\tau+\varsigma}\psi\bigl(\varsigma,x_n(\varsigma)\bigr)\,d\varsigma\biggr)\,d\tau\,ds\\
&+\lim_{n\to\infty}\lambda\int_0^{\sigma}\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1\frac{s}{s+\tau}\phi\biggl(\tau,\int_0^1\frac{\tau}{\tau+\varsigma}\psi\bigl(\varsigma,x_n(\varsigma)\bigr)\,d\varsigma\biggr)\,d\tau\,ds\\
&+\lim_{n\to\infty}\int_0^{\eta}\frac{(\eta-s)^{\varrho-1}}{\Gamma(\varrho)}\int_0^1\frac{s}{s+\tau}\phi\biggl(\tau,\int_0^1\frac{\tau}{\tau+\varsigma}\psi\bigl(\varsigma,x_n(\varsigma)\bigr)\,d\varsigma\biggr)\,d\tau\,ds\biggr]\\
&=g\bigl(t,x(t)\bigr)\biggl[-\int_0^t\frac{(t-s)^{\varrho-1}}{\Gamma(\varrho)}\int_0^1\frac{s}{s+\tau}\lim_{n\to\infty}\phi\biggl(\tau,\int_0^1\frac{\tau}{\tau+\varsigma}\psi\bigl(\varsigma,x_n(\varsigma)\bigr)\,d\varsigma\biggr)\,d\tau\,ds\\
&+\lambda\int_0^{\sigma}\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1\frac{s}{s+\tau}\lim_{n\to\infty}\phi\biggl(\tau,\int_0^1\frac{\tau}{\tau+\varsigma}\psi\bigl(\varsigma,x_n(\varsigma)\bigr)\,d\varsigma\biggr)\,d\tau\,ds\\
&+\int_0^{\eta}\frac{(\eta-s)^{\varrho-1}}{\Gamma(\varrho)}\int_0^1\frac{s}{s+\tau}\lim_{n\to\infty}\phi\biggl(\tau,\int_0^1\frac{\tau}{\tau+\varsigma}\psi\bigl(\varsigma,x_n(\varsigma)\bigr)\,d\varsigma\biggr)\,d\tau\,ds\biggr]\\
&=g\bigl(t,x(t)\bigr)\biggl[-\int_0^t\frac{(t-s)^{\varrho-1}}{\Gamma(\varrho)}\int_0^1\frac{s}{s+\tau}\phi\biggl(\tau,\int_0^1\frac{\tau}{\tau+\varsigma}\lim_{n\to\infty}\psi\bigl(\varsigma,x_n(\varsigma)\bigr)\,d\varsigma\biggr)\,d\tau\,ds\\
&+\lambda\int_0^{\sigma}\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1\frac{s}{s+\tau}\phi\biggl(\tau,\int_0^1\frac{\tau}{\tau+\varsigma}\lim_{n\to\infty}\psi\bigl(\varsigma,x_n(\varsigma)\bigr)\,d\varsigma\biggr)\,d\tau\,ds\\
&+\int_0^{\eta}\frac{(\eta-s)^{\varrho-1}}{\Gamma(\varrho)}\int_0^1\frac{s}{s+\tau}\phi\biggl(\tau,\int_0^1\frac{\tau}{\tau+\varsigma}\lim_{n\to\infty}\psi\bigl(\varsigma,x_n(\varsigma)\bigr)\,d\varsigma\biggr)\,d\tau\,ds\biggr]
\end{aligned}
$$

$$
\begin{aligned}
&= g\big(t,x(t)\big)\bigg[-\int_0^t \frac{(t-s)^{\varrho-1}}{\Gamma(\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \int_0^\eta \frac{(\eta-s)^{\varrho-1}}{\Gamma(\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds\bigg] \\
&= (\mathcal{A}x)(t).
\end{aligned}
$$

Thus, $\mathcal{A}x_n \to \mathcal{A}x$ and $\mathcal{A}$ is continuous. Now, for $x \in \mathcal{V}_r$, define the set

$$
\theta_g(\delta) = \sup\big\{\big|g(t_2,x) - g(t_1,x)\big| : t_1,t_2 \in I, t_1 < t_2, |t_2-t_1| < \delta, |x| \le \epsilon\big\},
$$

therefore, based on the uniform continuity of the function $\phi : I \times \mathcal{V}_r \to \mathbb{R}$ using the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$, we can conclude that $\theta_g(\delta) \to 0$, as $\delta \to 0$ independent of $x \in \mathcal{V}_r$,

Let $t_1, t_2 \in I$, $|t_2 - t_1| < \delta$. Then,

$$
\begin{aligned}
&\big|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\big| \\
&= \bigg|g\big(t_2,x(t_2)\big)\int_0^{t_2}(t_2-s)\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad - g\big(t_1,x(t_1)\big)\int_0^{t_1}(t_1-s)\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds\bigg| \\
&= \bigg|g\big(t_2,x(t_2)\big)\int_0^{t_2}(t_2-s)\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad - g\big(t_2,x(t_2)\big)\int_0^{t_2}(t_1-s)\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + g\big(t_2,x(t_2)\big)\int_0^{t_2}(t_1-s)\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad - g\big(t_2,x(t_2)\big)\int_0^{t_1}(t_1-s)\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + g\big(t_2,x(t_2)\big)\int_0^{t_1}(t_1-s)\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad - g\big(t_1,x(t_1)\big)\int_0^{t_1}(t_1-s)\int_0^1 \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds\bigg| \\
&\le \big|g\big(t_2,x(t_2)\big)\big| \\
&\quad \times \int_0^{t_2}\big((t_2-s)-(t_1-s)\big)\int_0^1 \frac{s}{s+\tau}\bigg|\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\bigg|\,d\tau\,ds \\
&\quad + \big|g\big(t_2,x(t_2)\big)\big|\int_{t_1}^{t_2}(t_1-s)\int_0^1 \frac{s}{s+\tau}\bigg|\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\bigg|\,d\tau\,ds \\
&\quad + \big|g\big(t_2,x(t_2)\big) - g\big(t_1,x(t_1)\big)\big| \\
&\quad \times \int_0^{t_1}(t_1-s)\int_0^1 \frac{s}{s+\tau}\bigg|\phi\bigg(\tau,\int_0^1 \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x(\varsigma)\big)\,d\varsigma\bigg)\bigg|\,d\tau\,ds \\
&\le \big[\big|x(t)\big|\omega + G\big]
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^{t_2} \big((t_2-s)-(t_1-s)\big) \int_0^1 \frac{s}{s+\tau}\left[m(\tau)+k_1(\tau)\int_0^1 \frac{\tau}{\tau+\varsigma}\big|\psi\big(\varsigma,x(\varsigma)\big)\big|\,d\varsigma\right] d\tau\,ds \\
&+ \big[\big|x(t)\big|\omega+G\big] \\
&\times \int_{t_1}^{t_2} (t_1-s) \int_0^1 \frac{s}{s+\tau}\left[m(\tau)+k_1(\tau)\int_0^1 \frac{\tau}{\tau+\varsigma}\big|\psi\big(\varsigma,x(\varsigma)\big)\big|\,d\varsigma\right] d\tau\,ds \\
&+ \theta_g(\delta)\int_0^{t_1} (t_1-s) \int_0^1 \frac{s}{s+\tau}\left[m(\tau)+k_1(\tau)\int_0^1 \frac{\tau}{\tau+\varsigma}\big|\psi\big(\varsigma,x(\varsigma)\big)\big|\,d\varsigma\right] d\tau\,ds \\
\le{}& \big[\|x\|\omega+G\big] \\
&\times \int_0^{t_2} \big((t_2-s)-(t_1-s)\big) \int_0^1 \left[\big|m(\tau)\big|+\big|k_1(\tau)\big|\int_0^1 \big|k_2(\varsigma)\big|\chi\big(\|x\|\big)\,d\varsigma\right] d\tau\,ds \\
&+ \big[\|x\|\omega+G\big]\int_{t_1}^{t_2} (t_1-s) \int_0^1 \left[\big|m(\tau)\big|+\big|k_1(\tau)\big|\int_0^1 \big|k_2(\varsigma)\big|\chi\big(\|x\|\big)\,d\varsigma\right] d\tau\,ds \\
&+ \theta_g(\delta)\int_{t_1}^{t_2} (t_1-s) \int_0^1 \left[\big|m(\tau)\big|+\big|k_1(\tau)\big|\int_0^1 \big|k_2(\varsigma)\big|\chi\big(\|x\|\big)\,d\varsigma\right] d\tau\,ds \\
\le{}& \big[\|x\|\omega+G\big] \\
&\times \int_0^{t_2} \big((t_2-s)-(t_1-s)\big) \int_0^1 \big[\big|m(\tau)\big|+\big|k_1(\tau)\big|k_2\chi\big(\|x\|\big)\big]\,d\tau\,ds \\
&+ \big[\|x\|\omega+G\big]\int_{t_1}^{t_2} (t_1-s) \int_0^1 \big[\big|m(\tau)\big|+\big|k_1(\tau)\big|k_2\chi\big(\|x\|\big)\big]\,d\tau\,ds \\
&+ \theta_g(\delta)\int_{t_1}^{t_2} (t_1-s) \int_0^1 \big[\big|m(\tau)\big|+\big|k_1(\tau)\big|k_2\chi\big(\|x\|\big)\big]\,d\tau\,ds \\
\le{}& [r\omega+G]\big[m+k_1k_2\chi(r)\big]\left[\int_0^{t_2}\big((t_2-s)-(t_1-s)\big)\,ds+\int_{t_1}^{t_2}(t_1-s)\,ds\right] \\
&+ \theta_g(\delta)\big[m+k_1k_2\chi(r)\big]\int_{t_1}^{0}(t_1-s)\,ds.
\end{aligned}
$$

Hence, the class $\{\mathcal{A}x\}$ is equicontinuous. Then, from the Arzela–Ascoli Theorem [32], the operator $\mathcal{A}$ is compact.

As a result, (see [33]), $\mathcal{A}$ has at least one fixed point $x \in \mathcal{V}_r$, then the problem (1.1)–(1.2) has a solution $x \in C(I)$. □

3 Continuous dependency

3.1 Uniqueness of the solution

To prove the uniqueness of the solution of (1.1)–(1.2) consider the following assumptions

$(\mathcal{H}_1)^*$ Let $\Phi : I \times \mathbb{R} \to 2^{\mathbb{R}}$ be a Lipschitzian set-valued map with a nonempty compact convex subset of 2^R such that

$$\big\|\Phi(t,\mu)-\Phi(t,\nu)\big\| \le k_1(t)|\mu-\nu|.$$

From this assumption we see that the assumption $(\mathcal{H}_1)$ is valid. Moreover, the set of Lipschitzian selections S_Φ is nonempty ([30]) and $\phi \in S_\Phi$ satisfies

$$\big|\phi(t,\mu)-\phi(t,\nu)\big| \le k_1(t)|\mu-\nu|,$$

from which we have

$$\left|\phi(t,\mu)\right| \le k_1(t)|\mu| + m, \qquad m = \sup_{t\in I}\left|\phi(t,0)\right|.$$

$(\mathcal{H}_2)^*$ $\psi(t,\mu(t)) = k_2(t)\mu(t)$.

Theorem 2 *Assume that the assumptions of Theorem 1 are satisfied by replacing assumption $(\mathcal{H}_2)$ by $(\mathcal{H}_2)^*$ with $(\Lambda[\omega(m + k_1k_2r) + (\omega r + G)k_1k_2]) < 1$. Then, the hybrid problem (1.1)–(1.2) has a unique solution.*

Proof From our assumptions and Theorem 1, the solution of (2.11) exists. If x_1, x_2 are two solutions of the integral equation (2.11), then

$$\begin{aligned}
&\left|x_1(t) - x_2(t)\right| \\
&= g\big(t,x_1(t)\big)\bigg[-\int_0^t (t-s)\int_0^s \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_1(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^s \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_1(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \int_0^\eta (\eta-s)\int_0^s \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_1(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds\bigg] \\
&\quad - g\big(t,x_2(t)\big)\bigg[-\int_0^t (t-s)\int_0^s \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_2(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^s \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_2(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \int_0^\eta (\eta-s)\int_0^s \frac{s}{s+\tau}\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_2(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds\bigg] \\
&\le \left|g\big(t,x_1(t)\big) - g\big(t,x_2(t)\big)\right| \\
&\quad \times \int_0^t (t-s)\int_0^s \frac{s}{s+\tau}\left|\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_1(\varsigma)\big)\,d\varsigma\bigg)\right|\,d\tau\,ds \\
&\quad + \left|g\big(t,x_2(t)\big)\right|\int_0^t (t-s)\int_0^s \frac{s}{s+\tau}\bigg[\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_1(\varsigma)\big)\,d\varsigma\bigg) \\
&\quad - \phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_2(\varsigma)\big)\,d\varsigma\bigg)\bigg]\,d\tau\,ds \\
&\quad + \lambda\left|g\big(t,x_1(t)\big) - g\big(t,x_2(t)\big)\right| \\
&\quad \times \int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^s \frac{s}{s+\tau}\left|\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_1(\varsigma)\big)\,d\varsigma\bigg)\right|\,d\tau\,ds \\
&\quad + \lambda\left|g\big(t,x_2(t)\big)\right| \\
&\quad \times \int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^s \frac{s}{s+\tau}\bigg[\phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_1(\varsigma)\big)\,d\varsigma\bigg) \\
&\quad - \phi\bigg(\tau,\int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma,x_2(\varsigma)\big)\,d\varsigma\bigg)\bigg]\,d\tau\,ds \\
&\quad + \left|g\big(t,x_1(t)\big) - g\big(t,x_2(t)\big)\right|
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^\eta (\eta - s) \int_0^s \frac{s}{s+\tau} \left| \phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x_1(\varsigma)\big)\, d\varsigma\right) \right| d\tau\, ds \\
&+ \big| g\big(t, x_2(t)\big) \big| \\
&\times \int_0^\eta (\eta - s) \int_0^s \frac{s}{s+\tau} \left[\phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x_1(\varsigma)\big)\, d\varsigma\right) \right. \\
&\left. - \phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x_2(\varsigma)\big)\, d\varsigma\right) \right] d\tau\, ds \\
\leq{}& \big| g\big(t, x_1(t)\big) - g\big(t, x_2(t)\big) \big| \\
&\times \int_0^t (t - s) \int_0^s \frac{s}{s+\tau} \left[m(\tau) + k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} \big| \psi\big(\varsigma, x_1(\varsigma)\big) \big|\, d\varsigma) \right] d\tau\, ds \\
&+ \big| g\big(t, x_2(t)\big) \big| \\
&\times \int_0^t (t - s) \int_0^s \frac{s}{s+\tau} k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} \big| \psi\big(\varsigma, x_1(\varsigma)\big) - \psi\big(\varsigma, x_2(\varsigma)\big) \big|\, d\varsigma\, d\tau\, ds \\
&+ \lambda \big| g\big(t, x_1(t)\big) - g\big(t, x_2(t)\big) \big| \\
&\times \int_0^\sigma \frac{(\sigma - s)^{1-\varrho}}{\Gamma(2-\varrho)} \int_0^s \frac{s}{s+\tau} \left[m(\tau) + k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} k_1(\varsigma) \big| \psi\big(\varsigma, x_1(\varsigma)\big) \big|\, d\varsigma) \right] d\tau\, ds \\
&+ \lambda \big| g\big(t, x_2(t)\big) \big| \\
&\times \int_0^\sigma \frac{(\sigma - s)^{1-\varrho}}{\Gamma(2-\varrho)} \int_0^s \frac{s}{s+\tau} k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} \big| \psi\big(\varsigma, x_1(\varsigma)\big) - \psi\big(\varsigma, x_2(\varsigma)\big) \big|\, d\varsigma\, d\tau\, ds \\
&+ \big| g\big(t, x_1(t)\big) - g\big(t, x_2(t)\big) \big| \\
&\times \int_0^\eta (\eta - s) \int_0^s \frac{s}{s+\tau} \left[m(\tau) + k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} \big| \psi\big(\varsigma, x_1(\varsigma)\big) \big|\, d\varsigma) \right] d\tau\, ds \\
&+ \big| g\big(t, x_2(t)\big) \big| \\
&\times \int_0^\eta (\eta - s) \int_0^s \frac{s}{s+\tau} k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} \big| \psi\big(\varsigma, x_1(\varsigma)\big) - \psi\big(\varsigma, x_2(\varsigma)\big) \big|\, d\varsigma\, d\tau\, ds \\
\leq{}& \omega \big| x_1(t) - x_2(t) \big| \\
&\times \int_0^t (t - s) \int_0^1 \frac{s}{s+\tau} \left[\big| m(\tau) \big| + \big| k_1(\tau) \big| \int_0^1 \frac{\tau}{\tau+\varsigma} \big| k_2(\varsigma) \big| \big| x_1(\varsigma) \big|\, d\varsigma \right] d\tau\, ds \\
&+ \big[\omega \big| x_2(t) \big| + G \big] \int_0^t (t - s) \int_0^s \frac{s}{s+\tau} k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} k_2(\varsigma) \big| x_1(\varsigma) - x_2(\varsigma) \big|\, d\varsigma\, d\tau\, ds \\
&+ \lambda \omega \big| x_1(t) - x_2(t) \big| \\
&\times \int_0^\sigma \frac{(\sigma - s)^{1-\varrho}}{\Gamma(2-\varrho)} \int_0^1 \frac{s}{s+\tau} \left[\big| m(\tau) \big| + \big| k_1(\tau) \big| \int_0^1 \frac{\tau}{\tau+\varsigma} \big| k_2(\varsigma) \big| \big| x_1(\varsigma) \big|\, d\varsigma \right] d\tau\, ds \\
&+ \lambda \big[\omega \big| x_2(t) \big| + G \big] \\
&\times \int_0^\sigma \frac{(\sigma - s)^{1-\varrho}}{\Gamma(2-\varrho)} \int_0^s \frac{s}{s+\tau} \big| k_1(\tau) \big| \int_0^\tau \frac{\tau}{\tau+\varsigma} \big| k_2(\varsigma) \big| \big| x_1(\varsigma) - x_2(\varsigma) \big|\, d\varsigma\, d\tau\, ds \\
&+ \omega \big| x_1(t) - x_2(t) \big| \\
&\times \int_0^\eta (\eta - s) \int_0^1 \frac{s}{s+\tau} \left[\big| m(\tau) \big| + \big| k_1(\tau) \big| \int_0^1 \frac{\tau}{\tau+\varsigma} \big| k_2(\varsigma) \big| \big| x_1(\varsigma) \big|\, d\varsigma \right] d\tau\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \big[\omega|x_2(t)| + G\big]\int_0^\eta (\eta - s)\int_0^1 \frac{s}{s+\tau}k_1(\tau)\int_0^\tau \frac{\tau}{\tau+\varsigma}k_2(\varsigma)\big|x_1(\varsigma) - x_2(\varsigma)\big|\,d\varsigma\,d\tau\,ds \\
\le{}& \omega\big|x_1(t) - x_2(t)\big|\int_0^t\int_0^1 \big[|m(\tau)| + |k_1(\tau)|k_2\|x_1\|\big]\,d\tau\,ds \\
&+ \big[\omega|x_2(t)| + G\big]\int_0^t\int_0^s |k_1(\tau)|k_2\|x_1 - x_2\|\,d\tau\,ds \\
&+ \lambda\omega\big|x_1(t) - x_2(t)\big|\int_0^\sigma \frac{(\sigma - s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \big[|m(\tau)| + |k_1(\tau)|k_2\|x_1\|\big]\,d\tau\,ds \\
&+ \lambda\big[\omega|x_2(t)| + G\big]\int_0^\sigma \frac{(\sigma - s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^s |k_1(\tau)|k_2\|x_1 - x_2\|\,d\tau\,ds \\
&+ \omega\big|x_1(t) - x_2(t)\big|\int_0^\eta\int_0^1 \big[|m(\tau)| + |k_1(\tau)|k_2\|x_1\|\big]\,d\tau\,ds \\
&+ \big[\omega|x_2(t)| + G\big]\int_0^\eta\int_0^1 |k_1(\tau)|k_2\|x_1 - x_2\|\,d\tau\,ds \\
\le{}& \omega\|x_1 - x_2\|\big[m + k_1k_2\|x_1\|\big] + \big[\omega\|x_2\| + G\big]k_1k_2\|x_1 - x_2\| \\
&+ \frac{\lambda}{\Gamma(3-\varrho)}\omega\|x_1 - x_2\|\big[m + k_1k_2\|x_1\|\big] + \frac{\lambda}{\Gamma(3-\varrho)}\big[\omega|x_2(t)| + G\big]k_1k_2\|x_1 - x_2\| \\
&+ \omega\|x_1 - x_2\|\big[m + k_1k_2\|x_1\|\big] + \big[\omega\|x_2\| + G\big]k_1k_2\|x_1 - x_2\|.
\end{aligned}
$$

Taking the supremum over $t \in I$, we have

$$
\begin{aligned}
\|x_1 - x_2\| &\le [2+\lambda]\omega\|x_1 - x_2\|[m + k_1k_2r] + [2+\lambda][\omega r + G]k_1k_2\|x_1 - x_2\| \\
&\le \Lambda\|x_1 - x_2\|\big[\omega(m + k_1k_2r) + (\omega r + G)k_1k_2\big],
\end{aligned}
$$

and

$$
\big[1 - \big(\Lambda\big[\omega(m + k_1k_2r) + (\omega r + G)k_1k_2\big]\big)\big]\|x_1 - x_2\| \le 0,
$$

which implies

$$
x_1(t) = x_2(t). \qquad \square
$$

3.1.1 Continuous dependence on the set of selection S_Φ

Definition 1 The solutions of the hybrid problem (1.1)–(1.2) are continuously dependent on the set S_Φ, if $\forall\epsilon > 0$, $\exists\delta > 0$, such that

$$
\big|\phi(t,\mu) - \phi^*(t,\mu)\big| < \delta, \quad \text{implies} \quad \big\|\mu - \mu^*\big\| < \epsilon, \quad t \in I,
$$

with two solutions μ and μ^* of (1.1)–(1.2), which corresponds to the two selections $\phi, \phi^* \in S_\Phi$.

Theorem 3 *Assume that the conditions of Theorems 2 hold. Then, the solutions of the problem (1.1)–(1.2) depend continuously on the set S_Φ of all Lipschitzian selections of Φ.*

Proof For the inclusion problem (1.1)–(1.2) we have two solutions $x(t)$ and $x^*(t)$ related to the two selections $\phi, \phi^* \in S_\Phi$, and we obtain

$$
\begin{aligned}
&\big|x(t) - x^*(t)\big| \\
&= g\big(t,x(t)\big)\bigg[-\int_0^t (t-s)\int_0^s \frac{s}{s+\tau}\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^s \frac{s}{s+\tau}\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \int_0^\eta (\eta-s)\int_0^s \frac{s}{s+\tau}\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds\bigg] \\
&\quad - g\big(t,x^*(t)\big)\bigg[-\int_0^t (t-s)\int_0^s \frac{s}{s+\tau}\phi^*\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x^*(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^s \frac{s}{s+\tau}\phi^*\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x^*(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds \\
&\quad + \int_0^\eta (\eta-s)\int_0^s \frac{s}{s+\tau}\phi^*\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x^*(\varsigma)\big)\,d\varsigma\bigg)\,d\tau\,ds\bigg] \\
&\leq \big|g\big(t,x(t)\big) - g\big(t,x^*(t)\big)\big|\int_0^t (t-s)\int_0^s \frac{s}{s+\tau}\bigg|\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg)\bigg|\,d\tau\,ds \\
&\quad + \big|g\big(t,x^*(t)\big)\big| \\
&\quad \times \int_0^t (t-s)\int_0^s \frac{s}{s+\tau}\bigg[\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg) \\
&\quad - \phi^*\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x^*(\varsigma)\big)\,d\varsigma\bigg)\bigg]\,d\tau\,ds \\
&\quad + \lambda\big|g\big(t,x(t)\big) - g\big(t,x^*(t)\big)\big| \\
&\quad \times \int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^s \frac{s}{s+\tau}\bigg|\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg)\bigg|\,d\tau\,ds \\
&\quad + \lambda\big|g\big(t,x^*(t)\big)\big| \\
&\quad \times \int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^s \frac{s}{s+\tau}\bigg[\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg) \\
&\quad - \phi^*\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x^*(\varsigma)\big)\,d\varsigma\bigg)\bigg]\,d\tau\,ds \\
&\quad + \big|g\big(t,x(t)\big) - g\big(t,x^*(t)\big)\big| \\
&\quad \times \int_0^\eta (\eta-s)\int_0^s \frac{s}{s+\tau}\bigg|\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg)\bigg|\,d\tau\,ds \\
&\quad + \big|g\big(t,x^*(t)\big)\big| \\
&\quad \times \int_0^\eta (\eta-s)\int_0^s \frac{s}{s+\tau}\bigg[\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg) \\
&\quad - \phi^*\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x^*(\varsigma)\big)\,d\varsigma\bigg)\bigg]\,d\tau\,ds \\
&\leq \big|g\big(t,x(t)\big) - g\big(t,x^*(t)\big)\big|\int_0^t (t-s)\int_0^s \frac{s}{s+\tau}\bigg|\phi\bigg(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma}\psi\big(\varsigma, x(\varsigma)\big)\,d\varsigma\bigg)\bigg|\,d\tau\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \left|g\big(t,x^*(t)\big)\right| \int_0^t (t-s) \int_0^s \frac{s}{s+\tau} \left[\left|\phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x(\varsigma)\big)\, d\varsigma\right)\right.\right. \\
&\left. - \phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x^*(\varsigma)\big)\, d\varsigma\right)\right| \\
&\left.+ \left|\phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x^*(\varsigma)\big)\, d\varsigma\right) - \phi^*\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x^*(\varsigma)\big)\, d\varsigma\right)\right|\right] d\tau\, ds \\
&+ \lambda \left|g\big(t,x(t)\big) - g\big(t,x^*(t)\big)\right| \\
&\times \int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)} \int_0^s \frac{s}{s+\tau} \left|\phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x(\varsigma)\big)\, d\varsigma\right)\right| d\tau\, ds \\
&+ \lambda \left|g\big(t,x^*(t)\big)\right| \\
&\times \int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)} \int_0^s \frac{s}{s+\tau} \left[\left|\phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x(\varsigma)\big)\, d\varsigma\right)\right.\right. \\
&\left. - \phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x^*(\varsigma)\big)\, d\varsigma\right)\right| \\
&\left.+ \left|\phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x^*(\varsigma)\big)\, d\varsigma\right) - \phi^*\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x^*(\varsigma)\big)\, d\varsigma\right)\right|\right] d\tau\, ds \\
&+ \left|g\big(t,x(t)\big) - g\big(t,x^*(t)\big)\right| \\
&\times \int_0^\eta (\eta-s) \int_0^s \frac{s}{s+\tau} \left|\phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x(\varsigma)\big)\, d\varsigma\right)\right| d\tau\, ds \\
&+ \left|g\big(t,x^*(t)\big)\right| \\
&\times \left[\int_0^\eta (\eta-s) \int_0^s \frac{s}{s+\tau} \left|\phi\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x(\varsigma)\big)\, d\varsigma\right)\right.\right. \\
&\left.\left. - \phi^*\left(\tau, \int_0^\tau \frac{\tau}{\tau+\varsigma} \psi\big(\varsigma, x^*(\varsigma)\big)\, d\varsigma\right)\right| d\tau\, ds\right] \\
\le{}& \left|g\big(t,x(t)\big) - g\big(t,x^*(t)\big)\right| \\
&\times \int_0^t (t-s) \int_0^s \frac{s}{s+\tau} \left[m(\tau) + k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} \left|\psi\big(\varsigma, x(\varsigma)\big)\right| d\varsigma\right] d\tau\, ds \\
&+ \left|g\big(t,x^*(t)\big)\right| \\
&\times \int_0^t (t-s) \int_0^s \frac{s}{s+\tau} k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} \left[\left|\psi\big(\varsigma, x(\varsigma)\big) - \psi\big(\varsigma, x^*(\varsigma)\big)\right| + \delta\right] d\varsigma\, d\tau\, ds \\
&+ \lambda \left|g\big(t,x(t)\big) - g\big(t,x^*(t)\big)\right| \\
&\times \int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)} \int_0^s \frac{s}{s+\tau} \left[m(\tau) + k_1(\tau) \int_0^\tau \frac{\tau}{\tau+\varsigma} \left|\psi\big(\varsigma, x(\varsigma)\big)\right| d\varsigma\right] d\tau\, ds \\
&+ \lambda \left|g\big(t,x^*(t)\big)\right| \\
&\times \int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)} \int_0^s \frac{s}{s+\tau} k_1(\tau) \\
&\times \int_0^\tau \frac{\tau}{\tau+\varsigma} \left[\left|\psi\big(\varsigma, x(\varsigma)\big) - \psi\big(\varsigma, x^*(\varsigma)\big)\right| + \delta\right] d\varsigma\, d\tau\, ds \\
&+ \left|g\big(t,x(t)\big) - g\big(t,x^*(t)\big)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^{\eta}(\eta-s)\int_0^{s}\frac{s}{s+\tau}\left[m(\tau)+k_1(\tau)\int_0^{\tau}\frac{\tau}{\tau+\varsigma}\left|\psi\left(\varsigma,x(\varsigma)\right)\right|d\varsigma\right]d\tau\,ds \\
&+\left|g\left(t,x^*(t)\right)\right| \\
&\times \int_0^{\eta}(\eta-s)\int_0^{s}\frac{s}{s+\tau}k_1(\tau)\int_0^{\tau}\frac{\tau}{\tau+\varsigma}\left[\left|\psi\left(\varsigma,x(\varsigma)\right)-\psi\left(\varsigma,x^*(\varsigma)\right)\right|+\delta\right]d\varsigma\,d\tau\,ds \\
\leq{}& \omega\left|x(t)-x^*(t)\right|\int_0^{t}\int_0^{1}\left[\left|m(\tau)\right|+\left|k_1(\tau)\right|\int_0^{1}\left(\left|k_2(\varsigma)\right|\left|x_1(\varsigma)\right|d\varsigma\right]d\tau\,ds \\
&+\left[\omega\left|x^*(t)\right|+G\right]\int_0^{t}\int_0^{s}\left|k_1(\tau)\right|\int_0^{\tau}\left[\left|k_2(\varsigma)\right|\left|x(\varsigma)-x^*(\varsigma)\right|+\delta\right]d\varsigma\,d\tau\,ds \\
&+\lambda\omega\left|x_1(t)-x_2(t)\right| \\
&\times \int_0^{\sigma}\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^{1}\left[\left|m(\tau)\right|+\left|k_1(\tau)\right|\int_0^{1}\left(\left|k_2(\varsigma)\right|\left|x_1(\varsigma)\right|d\varsigma\right]d\tau\,ds \\
&+\lambda\left[\omega\left|x(t)\right|+G\right]\int_0^{\sigma}\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^{s}\left|k_1(\tau)\right|\int_0^{\tau}\left[\left|k_2(\varsigma)\right|\left|x(\varsigma)-x^*(\varsigma)\right|+\delta\right]d\varsigma\,d\tau\,ds \\
&+\omega\left|x(t)-x^*(t)\right|\int_0^{\eta}\int_0^{1}\left[\left|m(\tau)\right|+\left|k_1(\tau)\right|\int_0^{1}\left|k_2(\varsigma)\right|\left|x(\varsigma)\right|d\varsigma\right]d\tau\,ds \\
&+\left[\omega\left|x^*(t)\right|+G\right]\int_0^{\eta}\int_0^{1}\left|k_1(\tau)\right|\int_0^{\tau}\left[\left|k_2(\varsigma)\right|\left|x(\varsigma)-x^*(\varsigma)\right|+\delta\right]d\varsigma\,d\tau\,ds \\
\leq{}& \omega\left|x_1(t)-x_2(t)\right|\int_0^{t}\int_0^{1}\left[\left|m(\tau)\right|+\left|k_1(\tau)\right|k_2\|x_1\|\right]d\tau\,ds \\
&+\left[\omega\left|x_2(t)\right|+G\right]\int_0^{t}\int_0^{s}\left|k_1(\tau)\right|\left[k_2\|x_1-x_2\|+\delta\right]d\tau\,ds \\
&+\lambda\omega\left|x_1(t)-x_2(t)\right|\int_0^{\sigma}\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^{1}\left[\left|m(\tau)\right|+\left|k_1(\tau)\right|k_2\|x_1\|\right]d\tau\,ds \\
&+\lambda\left[\omega\left|x_2(t)\right|+G\right]\int_0^{\sigma}\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^{s}\left|k_1(\tau)\right|\left[k_2\|x_1-x_2\|+\delta\right]d\tau\,ds \\
&+\omega\left|x_1(t)-x_2(t)\right|\int_0^{\eta}\int_0^{1}\left[\left|m(\tau)\right|+\left|k_1(\tau)\right|k_2\|x_1\|\right]d\tau\,ds \\
&+\left[\omega\left|x_2(t)\right|+G\right]\int_0^{\eta}\int_0^{1}\left|k_1(\tau)\right|k_2\|x_1-x_2\|+\delta\right]d\tau\,ds \\
\leq{}& \omega\|x_1-x_2\|\left[m+k_1k_2\|x_1\|\right]+\left[\omega\|x_2\|+G\right]k_1\left[k_2\|x_1-x_2\|+\delta\right] \\
&+\frac{\lambda}{\Gamma(3-\varrho)}\omega\|x_1-x_2\|\left[m+k_1k_2\|x_1\|\right] \\
&+\frac{\lambda}{\Gamma(3-\varrho)}\left[\omega\left|x_2(t)\right|+G\right]k_1\left[k_2\|x_1-x_2\|+\delta\right] \\
&+\omega\|x_1-x_2\|\left[m+k_1k_2\|x_1\|\right]+\left[\omega\|x_2\|+G\right]k_1\left[k_2\|x_1-x_2\|+\delta\right].
\end{aligned}
$$

Taking the supremum over $t \in I$, we have

$$
\begin{aligned}
\|x_1-x_2\| &\leq [2+\lambda]\omega\|x_1-x_2\|[m+k_1k_2r]+[2+\lambda][\omega r+G]k_1\left[k_2\|x_1-x_2\|+\delta\right] \\
&\leq \Lambda\|x_1-x_2\|\left[\omega(m+k_1k_2r)+(\omega r+G)k_1k_2\right]+\Lambda(\omega r+G)k_1\delta.
\end{aligned}
$$

Hence,

$$\|x - x^*\| \leq \frac{\Lambda(\omega r + G)k_1\delta}{[1 - \Lambda(\omega(m + k_1k_2r) + (\omega r + G)k_1k_2)]} = \epsilon.$$

As a result of the previous inequality, we obtain

$$\|x - x^*\| \leq \epsilon.$$

This proves the continuous dependence of the solution on the set S_Φ. □

We can establish the following theorem in the same way.

Theorem 4 *Let the assumptions of Theorems 2 be satisfied. Then, the solutions for the problem (1.1)–(1.2) depend continuously on the function $\psi(t,x(t))$.*

4 Discussions and examples

- As an application, we consider the nonlocal problem of the Chandrasekhar hybrid second-order functional integrodifferential equation (1.3)

$$-\frac{d^2}{dt^2}\left(\frac{x(t)}{g(t,x(t))}\right) = \int_0^1 \frac{s}{t+s}k_1(s)\phi_1\left(s, \int_0^1 \frac{s}{s+\tau}k_2(\tau)x(\tau)\,d\tau\right)ds, \quad t \in [0,1]$$

  with the nonlocal hybrid boundary condition (1.2).

Theorem 5 *Let the hypotheses of Theorem 2 hold. Then, the problem (1.3) and (1.2) has a unique solution, which is given by*

$$\begin{aligned} x(t) = g\big(t,x(t)\big)\bigg[&-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}k_1(\tau)\phi_1\left(\tau, \int_0^1 \frac{\tau}{\tau+\varsigma}k_2(\varsigma)x(\varsigma)\,d\varsigma\right)d\tau\,ds \\ &+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}k_1(\tau)\phi_1\left(\tau, \int_0^1 \frac{\tau}{\tau+\varsigma}k_2(\varsigma)x(\varsigma)\,d\varsigma\right)d\tau\,ds \\ &+ \int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}k_1(\tau)\phi_1\left(\tau, \int_0^1 \frac{\tau}{\tau+\varsigma}k_2(\varsigma)x(\varsigma)\,d\varsigma\right)d\tau\,ds\bigg]. \end{aligned} \tag{4.1}$$

Proof Set

$$\phi\big(t,x(t)\big) = k_1(t)\cdot\phi_1\big(t,x(t)\big) \quad \text{and} \quad \psi\big(t,x(t)\big) = k_2(t)\cdot x(t),$$

in (2.1), then we see that all the assumptions of Theorems 1 and 2 are satisfied. Consequently, there exists a unique solution $x \in C[0,1]$ of the problem (1.3) and (1.2) and by using Lemma 1, this solution is given by (4.1). □

Remark 3 Also, from Theorems 3 and 4, the continuous dependence on the two functions k_1 and k_2 can be proved.

- As a particular case, letting $\varrho \to 1$, then we have the nonlocal problem of the Chandrasekhar hybrid second-order functional integrodifferential inclusion

$$-\frac{d^2}{dt^2}\left(\frac{x(t)}{g(t,x(t))}\right) \in \int_0^1 \frac{t}{t+s}\Phi\left(s, \int_0^1 \frac{s}{s+\tau}\psi\big(\tau,x(\tau)\big)\,d\tau\right)ds, \quad t \in [0,1]$$

with the nonlocal hybrid boundary value conditions

$$\begin{cases} \mathcal{D}(\frac{x(t)}{g(t,x(t))})|_{t=0} = 0, \\ \lambda\mathcal{D}(\frac{x(t)}{g(t,x(t))})|_{t=\sigma} + (\frac{x(t)}{g(t,x(t))})|_{t=\eta} = 0, \quad \sigma \in (0,1], \eta \in (0,1]. \end{cases}$$

- Letting $\varrho \to 1$ and $g(t,x) = 1$, then we have the nonlocal problem of the Chandrasekhar hybrid second-order functional integrodifferential inclusion

$$-x''(t) \in \int_0^1 \frac{t}{t+s}\Phi\left(s, \int_0^1 \frac{s}{s+\tau}\psi\big(\tau, x(\tau)\big)\,d\tau\right)ds, \quad t \in [0,1]$$

with the nonlocal hybrid boundary value conditions

$$\begin{cases} x'(0) = 0, \\ \lambda x'(\sigma) + x(\eta) = 0, \quad \sigma \in (0,1], \eta \in (0,1]. \end{cases}$$

- Letting $\varrho \to 1$, for all $\phi \in \Phi(t,x(t))$ with $\phi(t,x) = \psi(t,x) = x$ and $g(t,x) = 1$, then we have the nonlocal problem of the Chandrasekhar hybrid second-order functional integrodifferential inclusion

$$-x''(t) \in \int_0^1 K(t,\tau)x(\tau)\,d\tau, \quad t \in [0,1]$$

with the nonlocal hybrid boundary value conditions

$$\begin{cases} x'(0) = 0, \\ \lambda x'(\sigma) + x(\eta) = 0, \quad \sigma \in (0,1], \eta \in (0,1], \end{cases}$$

where $K(t,\tau) = \frac{-t^2}{\tau-t}\ln(\frac{1+t}{t}) + \frac{t\tau}{\tau-t}\ln(\frac{1+\tau}{\tau})$.

Now, we provide the following examples to illustrate our results.

Example 1 In the first example, we proceed to investigate the existence of a solution for the Chandrasekhar hybrid second-order integrodifferential inclusion

$$\begin{aligned} &-\frac{d^2}{dt^2}\left(\frac{x(t)}{\frac{t|x(t)|^2}{1+|x(t)|^2}+4}\right) \\ &\quad\in \left[\int_0^1 \frac{t}{t+s}\left(\frac{s}{100} + \frac{1}{10}\int_0^1 \frac{s}{s+\tau}\frac{\tau\cos^2(2\pi\tau)\cos(x(\tau))}{200}\,d\tau\right)ds, 0\right], \quad t \in [0,1] \end{aligned} \tag{4.2}$$

with the hybrid boundary value conditions

$$\begin{cases} \mathcal{D}(\frac{x(t)}{\frac{t|x(t)|^2}{1+|x(t)|^2}+4})|_{t=0} = 0, \\ \lambda\, {}^c\mathcal{D}^{\frac{4}{3}}(\frac{x(t)}{\frac{t|x(t)|^2}{1+|x(t)|^2}+4})|_{t=1} + (\frac{x(t)}{\frac{t|x(t)|^2}{1+|x(t)|^2}+4})|_{t=0.76} = 0. \end{cases} \tag{4.3}$$

Put $\varrho = \frac{4}{3}$, $\sigma = 1$, $\eta = 0.76$, and $\lambda = \frac{7}{5}$. Consider the continuous map $g : [0,1] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ by $g(t,x(t)) = \frac{t|x(t)|^2}{1+|x(t)|^2} + 4$, and the set-valued map $\Phi : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ by

$$\Phi\big(t,x(t)\big) = \left[\frac{t}{100} + \frac{1}{10}\int_0^1 \frac{t}{t+s}\frac{s\cos^2(2\pi s)\cos(x(s))}{200}\,ds, 0\right],$$

for all $\varphi \in \Phi(t,x(t))$, set $\phi : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ by

$$\phi\big(t,x(t)\big) = \frac{t}{100} + \frac{1}{10}x(t),$$

and

$$\psi\big(t,x(t)\big) = \int_0^1 \frac{t}{t+s}\frac{s\cos^2(2\pi s)\cos(x(s))}{200}\,ds.$$

It is evident that $\omega = 1$, $m(t) = \frac{t}{100}$, $k_1(t) = \frac{1}{10}$. Also, we have $k_2(t) = \frac{1}{200}$ and $\chi(\|x\|) = 1$. In this case, we obtain $\Lambda = 3.4$, we can choose $\epsilon > 0.061081$, and consequently, we have $\omega[m + kk^*\chi(\|x\|)]\Lambda = 0.0357 < 1$.

Now, by using Theorem 1, the fractional hybrid equation (4.2) with the three-point hybrid conditions (4.3) has at least one solution.

Example 2 Our second example specifies the Chandrasekhar hybrid second-order integrodifferential equation for the model

$$\begin{aligned} &-\frac{d^2}{dt^2}\left(\frac{x(t)}{\frac{\arctan(t)}{1+\frac{1}{5}|x|)}}\right) \\ &\quad= \int_0^1 \frac{t}{t+s}\left(\frac{s}{20} + \cos\left(\int_0^1 \frac{s}{s+\tau}e^{-\tau}\sin\left(\frac{|x(\tau)|}{1+|x(\tau)|}\right)d\tau\right)\right)ds, \quad t \in [0,1], \end{aligned} \tag{4.4}$$

with the three-point hybrid boundary value conditions

$$\begin{cases} \mathcal{D}\big(\frac{x(t)}{\frac{\arctan(t)}{1+\frac{1}{5}|x|}}\big)|_{t=0} = 0, \\ \lambda\, {}^c\mathcal{D}^{\frac{5}{4}}\big(\frac{x(t)}{\frac{\arctan(t)}{1+\frac{1}{5}|x|}}\big)|_{t=1} + \big(\frac{x(t)}{\frac{\arctan(t)}{1+\frac{1}{5}|x|}}\big)|_{t=0.89} = 0. \end{cases} \tag{4.5}$$

Put $\varrho = \frac{5}{4}$, $\sigma = 1$, $\eta = 0.89$, and $\lambda = \frac{9}{5}$. Consider the continuous map $g : [0,1] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ by $g(t,x(t)) = \frac{\arctan(t)}{1+\frac{1}{5}|x|}$, and we have

$$\begin{aligned} \big|g\big(t,x_1(t)\big) - g\big(t,x_2(t)\big)\big| &\le \left|\frac{\arctan(t)}{1+\frac{1}{5}|x_1|} - \frac{\arctan(t)}{1-\frac{1}{5}|x_2|}\right| \\ &\le \frac{\pi}{20}|x_1 - x_2|. \end{aligned}$$

Hence, assumption $(\mathcal{H}_3)$ holds with $\omega = \frac{\pi}{20}$, we also have $G = \sup_{t\in I}|g(t,0)| = \frac{\pi}{4}$. On the other hand, we formulate two continuous functions $\phi, \psi : [0,1] \times \mathbb{R} \to \mathbb{R}^+$, from which follows

$$\phi\big(t,x(t)\big) = \frac{t}{20} + \cos x(t), \quad \text{and} \quad \psi\big(t,x(t)\big) = \int_0^1 \frac{t}{t+s}e^{-s}\sin\frac{|x(s)|}{1+|x(s)|}\,ds.$$

In this case, we have $k_1 = \frac{1}{20}$, $k_2 = 0.02$, and $m = 0.05$. In this instance, the provided data yields $\Lambda = 3.8$. Hence, we can find $\epsilon > 22.595746$, and consequently, we have $(\Lambda[\omega(m + k_1k_2r) + (\omega r + G)k_1k_2]) \simeq 0.0341 < 1$.

Now, by using Theorem 2, the fractional hybrid equation (4.4) with the three-point hybrid conditions (4.5) has a unique solution.

5 Conclusions

Most natural phenomena are modeled by different kinds of differential equations that have been established by many authors from different viewpoints, for example [1, 2, 11, 12, 14, 16, 20].

Various kinds of fractional differential equations are used to model the majority of natural occurrences. This variety in approaches to studying difficult fractional differential equations improves the capacity for precise modeling of different phenomena.

In particular, our theory includes a discussion of a second-order functional integrodifferential inclusion with nonlocal boundary conditions of fractional order.

In this work, we investigate a hybrid integrodifferential inclusion via nonlocal three-point boundary value conditions. In this way, we use some fixed-point theorems to prove the existence and uniqueness of the solution for the nonlocal problem (1.1)–(1.2). Also, the continuous dependency of the solution of (1.1)–(1.2) on the set of selection S_Φ and on the function Ψ. Finally, some applications and examples are presented to illustrate our main result. The results described in the present paper are innovative, and they will mainly contribute to the literature already existing on boundary value problems.

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