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Weak solutions to the time-fractional g-Bénard equations

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Abstract

The Bénard problem consists in a system that couples the well-known Navier–Stokes equations and an advection-diffusion equation. In thin varying domains this leads to the g-Bénard problem, which turns out to be the classical Bénard problem when g is constant. The main goal of this paper is to, first of all, introduce the g-Bénard problem with time-fractional derivative of order $\alpha \in (0, 1)$. This formulation is new even in the classical Bénard problem, that is with constant g. The second goal of this paper is to prove the existence and uniqueness of a weak solution by means of the Faedo–Galerkin approximation method. Some recent works on time-fractional Navier–Stokes equations have opened new perspectives in studying variational aspects in problems involving time-fractional derivatives.

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1 Introduction

Fractional derivatives play a major role in modelling nonlocality, anomalous behaviour and memory effects, which are common characteristics of natural phenomena [15, 22] arising from complex systems. For instance, the memory effect results from the fact that fractional derivatives involve a convolution integral with a power-law memory kernel. This appears naturally when studying, for instance, viscoelastic materials and viscous fluid dynamics [22]. For more applications of fractional calculus, see, e.g. [14, 23–26, 30].

The introduction of time-fractional derivative in fluid dynamics goes back to Lions in [17], but for order less than 1/4 provided the space dimension is not further than 4. In recent works of Zhou and Peng [35], the question of weak solutions and an optimal control problem of time-fractional Navier–Stokes equations in fractal media were considered. Numerical results regarding such problems was treated firstly in [16] and constitute an emerging field of research.

More recently, a time-fractional g-Navier–Stokes problem has been introduced and results regarding the existence, uniqueness of solutions and optimal control have been proved [6]. This suggests, to the authors of the current paper, to consider various variations of g-Navier–Stokes equations that can be modelled by time-fractional derivatives instead of integer ones. It is, indeed, a general trend among researchers to try to find more applications where one can replace an integer derivative with various non-integer derivatives. In fluid dynamics the fact that fractal media exhibit memory-dependent behaviour justifies the use of time-fractional derivatives as suggested by Zhou and Peng [35].

It is worth mentioning that the theory of g-Navier–Stokes equations started with the works of Hale and Raugel [11, 12], Raugel and Sell [27] who studied 3D nonlinear equations and Navier–Stokes equations in thin domains. J. Roh [28], a student of Sell, generalised the previous works to thin domains of the form $\Omega_g = \Omega \times (0, g)$, where g is some smooth scalar function. The derived equations are called the g-Navier–Stokes equations. This theory has interested many researchers in recent years, see [4, 5, 13] and the references therein.

On the other hand, heat conduction based on the classical Fourier law, which relates the heat flux vector and the temperature gradient, has shown its limits. The time-fractional heat conduction model can be seen as a good alternative (see [9, 18, 29] and the references therein). Boussinesq (or Bénard) model is a combination of the heat conduction model and Navier–Stokes equations and is a well-developed subject in modelling heat conducting fluids [7, 10, 33, 34]. The aim of this paper is to generalise the setting in [20, 21], where g-Bénard equations were considered, to time-fractional g-Bénard equations.

The novelty of this paper is, first of all, to introduce a new fractional model in fluid dynamics and then to prove the existence and uniqueness of its solutions. This is a starting point for more questions to answer, particularly related to numerical analysis, stability and long-term behaviour. More precisely, let $\Omega_g = \Omega_2 \times (0, g)$, where Ω_2 is a bounded domain in $\mathbb{R}^2$ and g is some scalar nonnegative function. We introduce time-fractional g-Bénard equations of the following form:

$$\begin{aligned}
&\partial_t^\alpha u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = \xi \theta + f_1(t), \\
&\nabla \cdot gu = 0, \\
&\partial_t^\alpha \theta + (u \cdot \nabla)\theta - \kappa \Delta \theta = f_2(t),
\end{aligned}$$

where u is the fluid velocity, p is the pressure, θ is the temperature, f_1 is the external force function, f_2 is the heat source function, $\xi \in \mathbb{R}^3$ is a constant vector, ν is the kinematic viscosity and κ is the thermal diffusivity(ν and κ are positive constants). The derivative of order α is considered in the Caputo sense. The time-fractional g-Bénard problem consists in a system that couples time-fractional Navier–Stokes equations and time-fractional advection-diffusion heat equation in order to model a memory-dependent convection in a fluid considered in a fractal medium.

This paper is organised as follows: In Sect. 2, we recall some concepts and notations related to fractional calculus. Section 3 is devoted to the problem statement, and Sect. 4 is dedicated to the proof of the existence and uniqueness of weak solutions to time-fractional g-Bénard equations. In Sect. 5 we provide a conclusion.

2 Preliminaries on fractional calculus

In this section, we provide some notations and preliminary results concerning fractional calculus. For this purpose, assume X to be a Banach space. Let $\alpha \in (0, 1]$ and let k_α denote the Riemann–Liouville kernel

$$k_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

For a function $v : [0,T] \to X$, we give the following definitions of derivatives and integrals:

(1) The left Riemann–Liouville integral of v is defined by

$$I_t^\alpha v(t) = \int_0^t k_\alpha(t-s)v(s)\,ds, \quad t > 0,$$

provided the integral is point-wise defined on $[0,+\infty[$.

(2) The right Riemann–Liouville integral of v is defined by

$$I_{t,T}^\alpha v(t) = \int_t^T k_\alpha(t-s)v(s)\,ds, \quad t > 0,$$

provided the integral is point-wise defined on $[0,+\infty[$.

(3) The left Caputo fractional derivative of order α of v is defined by

$$D_t^\alpha v(t) = \int_0^t k_{1-\alpha}(t-s)\frac{d}{ds}v(s)\,ds.$$

(4) The right Riemann–Liouville fractional derivative of order α of v is defined by

$$D_{t,T}^\alpha v(t) = -\frac{d}{dt}\int_t^T k_{1-\alpha}(t-s)v(s)\,ds.$$

(5) The Liouville–Weyl fractional integral on the real axis for functions $v : \mathbb{R} \to X$ is defined as follows:

$$I_{-,t}^\alpha v(t) = \int_{-\infty}^t k_\alpha(t-s)v(s)\,ds.$$

(6) The Caputo fractional derivative on the real axis for functions $v : \mathbb{R} \to X$ is defined as follows:

$$D_{-,t}^\alpha v(t) = I_{-,t}^{1-\alpha}\frac{d}{dt}v(t).$$

Note that the notation ∂_t^α stands for Caputo fractional partial derivative, i.e. when functions have another argument than time. We have the following fractional integration by parts formula (see, e.g. [2]):

$$\begin{aligned}\int_0^T \big(\partial_t^\alpha u(t), \psi(t)\big)\,dt &= \int_0^T \big(u(t), D_{t,T}^\alpha \psi(t)\big)\,dt + \big(u(t), I_{t,T}^{1-\alpha}\psi(t)\big)\big|_0^T \\ &= \int_0^T \big(u(t), D_{t,T}^\alpha \psi(t)\big)\,dt - \big(u(0), I_T^{1-\alpha}\psi(t)\big),\end{aligned} \tag{2.1}$$

since for $\psi \in C_0^\infty([0,T],X)$ one has $\lim_{t\to T} I_{t,T}^{1-\alpha}\psi(t) = 0$.

To pass from weak convergence to strong convergence, we will need a compactness result. Let X_0, X, X_1 be Hilbert spaces with $X_0 \hookrightarrow X \hookrightarrow X_1$ being continuous and $X_0 \hookrightarrow X$ being compact. Assume that $v : \mathbb{R} \to X_1$ and denote by $\widehat{v}$ its Fourier transform:

$$\widehat{v}(\tau) = \int_{-\infty}^{+\infty} e^{-2i\pi t\tau} v(t)\,dt.$$

We have for $\gamma > 0$

$$\widehat{D_t^\gamma v}(\tau) = (2i\pi\tau)^\gamma \widehat{v}(\tau).$$

For given $0 < \gamma < 1$, we introduce the following space:

$$W^\gamma(\mathbb{R}, X_0, X_1) = \{v \in L^2(\mathbb{R}, X_0) : D_t^\gamma v \in L^2(\mathbb{R}, X_1)\}.$$

Clearly, it is a Hilbert space for the norm

$$\|v\|_\gamma = \left(\|v\|^2_{L^2(\mathbb{R},X_0)} + \big\||\tau|^\gamma \widehat{v}\big\|^2_{L^2(\mathbb{R},X_1)}\right)^{1/2}.$$

For any set $K \subset \mathbb{R}$, we associate with it the subspace $W_K^\gamma \subset W^\gamma$ defined as

$$W_K^\gamma(\mathbb{R}, X_0, X_1) = \{v \in W^\gamma(\mathbb{R}, X_0, X_1) : \text{support } u \subset K\}.$$

By similar discussion as in the proof of Theorem 2.2 in Temam [31], it is clear that $W_K^\gamma(\mathbb{R}, X_0, X_1) \hookrightarrow L^2(\mathbb{R}, X)$ is compact for any bounded set K and any $\gamma > 0$.

As a particular situation of the compactness result discussed above, let H, V be two Hilbert spaces endowed with the scalar product $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$ and the norms $|\cdot|_H$ and $\|\cdot\|_V$, respectively. Denote by $\langle\cdot,\cdot\rangle$ the dual pairing between V and V', the dual of V. Moreover assume that $V \hookrightarrow H \hookrightarrow V'$ continuously and compactly and note that the space

$$W^\gamma\left(0, T; V, V'\right) = \left\{v \in L^2(0, T; V) : \partial_t^\gamma v \in L^2\left(0, T; V'\right)\right\}$$

is compactly embedded in $L^2(0, T; H)$. It is then well known that

$$\partial_t^\gamma \left(u(t), v\right)_V = \left\langle \partial_t^\gamma u(t), v\right\rangle$$

for $u \in W^\gamma(0, T; V, V')$ and $v \in H$. Moreover, for a derivable function $v : [0, T] \to V$, we have from [3] that

$$\left(v(t), D_t^\gamma v(t)\right)_H \geq \frac{1}{2} D_t^\gamma \left|v(t)\right|^2.$$

We end this section by the following important result.

Lemma 2.1 *Suppose that a nonnegative function satisfies*

$${}_0^C D_t^\gamma v(t) + c_1 v(t) \leq c_2(t)$$

for $c_1 > 0$ and c_2 is a nonnegative integrable function for $t \in [0, T]$. Then

$$v(t) \leq v(0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} c_2(s)\, ds.$$

For more details about fractional calculus, we refer to the monograph [14].

3 Problem statement

We introduce the usual notation used in the context of the mathematical theory of Navier–Stokes equations [31]. Let $\Omega_g = \Omega_2 \times (0,g) = (0,1) \times (0,1) \times (0,g)$, where $g = g(y_1,y_2)$ is a smooth function defined on Ω_2. In addition, we assume that

$$\begin{aligned} &0 < m_0 < g(y_1,y_2) \le M_0 \quad \text{for all } (y_1,y_2) \in \Omega_2, \\ &|\nabla g|_\infty = \sup_{\Omega_2} |\nabla g| < \infty, \quad g \in C^\infty_{\text{per}}(\Omega_2). \end{aligned} \tag{3.1}$$

Let $L^2(\Omega,g)$ denote the Hilbert space, of weighted Sobolev spaces type, with the inner product

$$\langle u,v\rangle_g = \int_\Omega (u \cdot v) g\,dx$$

and the induced norm $|u|_g^2 = \langle u,u\rangle_g$. Similarly, we can define the weighted Sobolev space $H^1(\Omega,g)$ equipped with the norm

$$|u|^2_{H^1(\Omega,g)} = \langle u,u\rangle_g + \sum_{i=1}^n \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i}\right\rangle_g.$$

Moreover, we will need the following spaces:

$$\begin{aligned} \mathcal{V}_1 &= \left\{ u \in \left(C^\infty_{\text{per}}(\Omega)\right)^n : \nabla \cdot (gu) = 0, \int_\Omega u\,dx = 0 \text{ on } \Omega \right\}, \\ H_g &= \text{the closure of } \mathcal{V}_1 \text{ on } L^2(\Omega,g), \\ V_g &= \text{the closure of } \mathcal{V}_1 \text{ on } H^1(\Omega,g), \\ V'_g &= \text{the dual space of } V_g, \\ \mathcal{V}_2 &= \left\{ \varphi \in C^\infty_{\text{per}}(\Omega) : \int_\Omega \varphi\,dx = 0 \right\}, \\ W_g &= \text{the closure of } \mathcal{V}_2 \text{ on } H^1(\Omega,g), \\ W'_g &= \text{the dual space of } W_g, \\ Q &= \text{the closure of} \left\{ \nabla\varphi : \varphi \in C^1_{\text{per}}(\overline{\Omega},R)\right\} \text{ in } L^2(\Omega), \end{aligned}$$

where H_g is endowed with the inner product and the norm in $L^2(\Omega g)$. In addition, the spaces V_g and Wg are endowed with the inner product and the norm in $H^1(\Omega,g)$. Let us also remark that the inclusions

$$\begin{aligned} &V_g \subset H_g = H'_g \subset V'_g, \\ &W_g \subset L^2(\Omega,g) \subset W'_g \end{aligned}$$

are dense and continuous [19, 28]. By the Riesz representation theorem, it is possible to write

$$\langle f,u\rangle_g = (f,u)_g, \quad \forall f \in H_g, \forall u \in V_g.$$

Let us now define the orthogonal projection P_g as $P_g : L^2_{\text{per}}(\Omega, g) \to H_g$. It is clear that $Q \subseteq H_g^{\perp}$. Similarly, we define $\widetilde{P}_g$ as $\widetilde{P}_g : L^2_{\text{per}}(\Omega, g) \to W_g$. By taking into account the following equality [28]:

$$-\frac{1}{g}(\nabla \cdot g\nabla u) = -\Delta u - \frac{1}{g}(\nabla g \cdot \nabla)u,$$

we define the g-Laplace operator and g-Stokes operator as follows:

$$-\Delta_g u = -\frac{1}{g}(\nabla \cdot g\nabla u)$$

and

$$A_g u = P_g[-\Delta_g u],$$

respectively. We have the following result [28].

Proposition 3.1 *For the g-Stokes operator A_g, the following hold*:

(1) *The g-Stokes operator A_g is a positive, self-adjoint operator with compact inverse, where the domain of A_g is $D(A_g) = V_g \cap H^2(\Omega, g)$.*

(2) *There exist countable eigenvalues of A_g satisfying*

$$0 < \frac{4\pi^2 m_0}{M_0} \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots,$$

where λ_1 is the smallest eigenvalue of A_g. In addition, there exists the corresponding collection of eigenfunctions $\{w_i\}_{i\in\mathbb{N}}$ that forms an orthonormal basis for H_g.

The operators A_g and P_g are clearly self-adjoint, then by using integration by parts we have

$$\begin{aligned}\langle A_g u, u\rangle_g &= \left\langle P_g\left[-\frac{1}{g}(\nabla \cdot g\nabla)u\right], u\right\rangle_g \\ &= \int_\Omega (\nabla u \cdot \nabla u)_g \, dx \\ &= \langle \nabla u \cdot \nabla u\rangle_g.\end{aligned}$$

It then follows that for $u \in V_g$ we can write $|A^{1/2}u|_g = |\nabla u|_g = \|u\|_g$. On the other hand, since the functional

$$\tau \in W_g \quad \to \quad (\nabla\theta, \nabla\tau)_g \in \mathbb{R}$$

is a continuous linear mapping on W_g, we can define a continuous linear mapping $\widetilde{A}_g$ on W_g' such that

$$\forall \tau \in W_g, \quad \langle \widetilde{A}_g, \tau\rangle_g = (\nabla\theta, \nabla\tau)_g$$

for all $\theta \in W_g$. For u, v and w laying in an appropriate subspaces of $L^2_{\text{per}}(\Omega, g)$, we can define the bilinear operator

$$B_g(u,v) = P_g\big[(u\cdot\nabla)v\big]$$

and the trilinear form

$$b_g(u,v,w) = \sum_{i,j=1}^{n}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j g\,dx.$$

As a consequence, one obtains $b_g(u,v,w) = -b_g(u,w,v)$, which implies that $b_g(u,v,v) = 0$. Moreover, we have the following inequality on b_g(see, e.g. [31, 32]):

$$\big|b_g(u,v,w)\big|_g \le c|u|_g^{1/2}\|u\|_g^{1/2}|v|_g|w|_g^{1/2}\|w\|_g^{1/2}, \quad \forall u,v,w\in V_g. \tag{3.2}$$

Similarly, for $u\in V_g$ and $\theta,\tau\in W_g$, we define $\widetilde{B}_g(u,\theta) = \widetilde{P}_g[(u\cdot\nabla)\theta]$ and

$$\widetilde{b}_g(u,\theta,\tau) = \sum_{i,j=1}^{n}\int_\Omega u_i(x)\frac{\partial\theta(x)}{\partial x_j}\tau(x)g\,dx.$$

We denote the operators $C_g u = P_g[\frac{1}{g}(\nabla g\cdot\nabla)u]$ and $\widetilde{C}_g\theta = \widetilde{P}_g[\frac{1}{g}(\nabla g\cdot\nabla)\theta]$ such that

$$\langle C_g u, v\rangle_g = b_g\left(\frac{\nabla g}{g}, u, v\right),$$
$$\langle \widetilde{C}_g\theta, \tau\rangle_g = \widetilde{b}_g\left(\frac{\nabla g}{g}, \theta, \tau\right).$$

Finally, let $\widetilde{D}_g\theta = \widetilde{P}_g[\frac{\nabla g}{g}\theta]$ such that

$$\langle \widetilde{D}_g\theta, \tau\rangle_g = -\widetilde{b}_g\left(\frac{\nabla g}{g}, \theta, \tau\right) - \widetilde{b}_g\left(\frac{\nabla g}{g}, \theta, \tau\right).$$

We can now rewrite the system of g-Bénard equations in the following abstract time-fractional evolutionary equations:

$$\begin{aligned}
&\partial_t^\alpha u + B_g(u,u) + \nu A_g u + \nu C_g u = \xi\theta + f_1,\\
&\partial_t^\alpha \theta + \widetilde{B}_g(u,\theta) + \kappa\widetilde{A}_g\theta - \kappa\widetilde{C}_g\theta - \kappa\widetilde{D}_g\theta = f_2,\\
&u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x).
\end{aligned} \tag{3.3}$$

The proof of the following two lemmas can be found in [5].

Lemma 3.2 *For $n = 2$, there exists a positive constant c such that*

$$|u|_{L^4(\Omega,g)} \le c|u|_g^{1/2}|\nabla u|_g^{1/2}, \quad \forall u\in H^1(\Omega,g).$$

Lemma 3.3 *For $u\in L^2(0,T,V_g)$, we have*

$$B_g(u,u)(t)\in L^1\big(0,T,V_g'\big) \quad \textit{and} \quad C_g u(t)\in L^2(0,T,H_g).$$

4 Existence of weak solutions

In this section we prove the existence and uniqueness of the weak solution. The main technique is the Faedo–Galerkin approximation method, which allows to exhibit an approximating sequence that converges to the desired solution. The following gives the definition of weak solutions, that is, solutions in a variational sense.

Definition 4.1 A pair of functions $\{u,\theta\}$ is called a weak solution of system (3.3) if $u \in L^2(0,T;V_g)$ and $\theta \in L^2(0,T;W_g)$ satisfy the following equations:

$$\begin{aligned} &\partial_t^\alpha(u,v)_g + b_g(u,u,v) + \nu(\nabla u,\nabla v)_g + \nu(C_g u,v)_g = (\xi\theta,v)_g + (f_1,v)_g, \\ &\partial_t^\alpha(\theta,\tau)_g + \widetilde{b}_g(u,\theta,\tau) + \kappa(\nabla\theta,\nabla\tau)_g + \kappa\widetilde{b}_g\left(\frac{\nabla g}{g},\tau,\theta\right) = (f_2,v)_g \end{aligned} \tag{4.1}$$

for all $v_2 \in V_g$ and $\tau \in W_g$.

The following theorem contains the main result of this paper.

Theorem 4.2 *If* $f_1 \in L^{\frac{2}{\alpha_1}}(0,T;L^2(\Omega,g))$ *and* $f_2 \in L^{\frac{2}{\alpha_2}}(0,T;L^2(\Omega,g))$ $(\alpha_1,\alpha_2 < \alpha)$, $u_0 \in H_g$, $\theta_0 \in L^2(\Omega,g)$ *and* g *is a smooth function satisfying the conditions given in* (3.1) *defined on* Ω_2, *then there exists a unique weak solution* $\{u,\theta\}$ *of system* (3.3) *satisfying the periodic boundary conditions.*

Proof Since V_g is separable and $\mathcal{V}_1$ is dense in V_g, there exists a sequence $\{u_i\}_{i\in\mathbb{N}}$ forming a complete orthonormal system in H_g and a basis in V_g. Similarly, there exists a sequence $\{\theta_i\}_{i\in\mathbb{N}}$ forming a complete orthonormal system in $L^2(\Omega,g)$ and a basis in W_g. Let m be an arbitrary but fixed nonnegative integer. For each m, we define the following approximate solution $\{u^{(m)}(t),\theta^{(m)}(t)\}$ of (3.3):

$$u^{(m)}(t) = \sum_{j=1}^m f_j^{(m)}(t)u_j, \qquad \theta^{(m)}(t) = \sum_{j=1}^m g_j^{(m)}(t)\theta_j, \tag{4.2}$$

and we consider the following approximate problem (4.3)–(4.5):

$$\begin{aligned} &\partial_t^\alpha\left(u^{(m)},u_k\right)_g + b_g\left(u^{(m)},u^{(m)},u_k\right) + \nu\left(\left(u^{(m)},u_k\right)\right)_g + \nu b_g\left(\frac{\nabla g}{g},u^{(m)},u_k\right) \\ &\quad = \left(\xi\theta^{(m)},u_k\right)_g + (f_1,u_k)_g, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\partial_t^\alpha\left(\theta^{(m)},\theta_k\right)_g + \widetilde{b}_g\left(u^{(m)},\theta^{(m)},\theta_k\right) + \kappa\left(\left(\theta^{(m)},\theta_k\right)\right)_g + \kappa\widetilde{b}_g\left(\frac{\nabla g}{g},\theta_k,\theta^{(m)}\right) \\ &\quad = (f_2,\theta_k)_g \end{aligned} \tag{4.4}$$

and

$$u^{(m)}(0) = u_{m_0} = \sum_{j=1}^m (a_0,u_j)u_j, \qquad \theta^{(m)}(0) = \theta_{m_0} = \sum_{j=1}^m (\tau_0,\theta_j)\theta_j. \tag{4.5}$$

This system forms a nonlinear fractional order system of ordinary differential equations for the functions $f_j^{(m)}(t)$ and $g_j^{(m)}(t)$ and has a maximal solution on some interval $[0,T]$ (cf.

[6]). We multiply (4.3) and (4.4) by $f_j^{(m)}(t)$ and $g_j^{(m)}(t)$, respectively, and add these equations for $k = 1,\dots,m$. Taking into account $b_g(u^{(m)},u^{(m)},u^{(m)}) = 0$ and $\widetilde{b}g(u^{(m)},\theta^{(m)},\theta^{(m)}) = 0$, we get

$$\begin{aligned}&\left(D_t^\alpha u^{(m)},u^{(m)}\right)_g + \nu\left\|u^{(m)}(t)\right\|_g^2 + \nu b_g\left(\frac{\nabla g}{g},u^{(m)}(t),u^{(m)}(t)\right)\\&\quad= \left(\xi\theta^{(m)},u^{(m)}(t)\right)_g + \left(f_1,u^{(m)}(t)\right)\end{aligned}\tag{4.6}$$

and

$$\left(D_t^\alpha\theta^{(m)}(t),\theta^{(m)}(t)\right)_g + \kappa\left\|\theta^{(m)}(t)\right\|_g^2 + \kappa\widetilde{b}_g\left(\frac{\nabla g}{g},\theta^{(m)}(t),\theta^{(m)}(t)\right) = \left(f_2,\theta^{(m)}(t)\right)_g.\tag{4.7}$$

Using Schwarz and Young inequalities in (4.6) and (4.7),

$$\begin{aligned}&D_t^\alpha\left|u^{(m)}(t)\right|_g^2 + \nu\left\|u^{(m)}(t)\right\|_g^2 \le \frac{M_0|\xi|_\infty^2}{\pi^2 m_0\nu}\left|\theta^{(m)}(t)\right|_g^2 + \frac{4}{\nu}\left\|f_1(t)\right\|_{V_g'}^2 + \frac{2\nu|\nabla g|_\infty^2}{m_0^2}\left|u^{(m)}(t)\right|_g^2,\\&D_t^\alpha\left|\theta^{(m)}(t)\right|_g^2 + \kappa\left\|\theta^{(m)}(t)\right\|_g^2 \le \frac{2}{\kappa}\left\|f_2(t)\right\|_{W_g'}^2 + \frac{2\kappa|\nabla g|_\infty^2}{m_0^2}\left|\theta^{(m)}(t)\right|_g^2.\end{aligned}$$

By using the fact that $|\nabla g|_\infty^2 < \frac{\pi^2 m_0^3}{M_0}$ and noting $\nu' = \nu(1-\frac{M_0|\nabla g|_\infty^2}{2\pi^2 m_0^3})$, $\kappa' = \kappa(1-\frac{M_0|\nabla g|_\infty^2}{2\pi^2 m_0^3})$ and $c' = \frac{M_0^2\|\xi\|_\infty^2}{4\pi^4 m_0^2}$, we get the inequalities

$$D_t^\alpha\left|u^{(m)}(t)\right|_g^2 + \nu'\left\|u^{(m)}(t)\right\|_g^2 \le \frac{c'}{\nu}\left\|\theta^m(t)\right\|_g^2 + \frac{4}{\nu}\left\|f_1(t)\right\|_{V_g'}^2\tag{4.8}$$

and

$$D_t^\alpha\left|\theta^{(m)}(t)\right|_g^2 + \kappa'\left\|\theta^{(m)}(t)\right\|_g^2 \le \frac{2}{\kappa}\left\|f_2(t)\right\|_{W_g'}^2.\tag{4.9}$$

Integrating (4.9) from 0 to T, in the fractional sense, we obtain

$$\begin{aligned}&\left|\theta^{(m)}(t)\right|_g^2 + \frac{\kappa'}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|\theta^{(m)}(s)\right\|_g^2\,ds\\&\quad\le |\theta_{0m}|_g^2 + \frac{2}{\kappa\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|f_2(s)\right\|_{W_g'}^2\,ds\\&\quad\le |\theta_{0m}|_g^2 + \frac{2}{\kappa\Gamma(\alpha)}\int_0^t \left\|f_2(s)\right\|_{W_g'}^{2/\alpha_2}\,ds + \frac{2}{\kappa\Gamma(\alpha)}\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\\&\quad\le |\theta_{0m}|_g^2 + \frac{2}{\kappa\Gamma(\alpha)}\int_0^T \left\|f_2(s)\right\|_{W_g'}^{2/\alpha_2}\,ds + C_2,\end{aligned}$$

where $b_2 = \frac{\alpha-1}{1-\alpha_2}$ and $C_2 = \frac{2T^{1+b_2}}{\kappa(1+b_2)\Gamma(\alpha)}$. It follows that

$$\int_0^t (t-s)^{\alpha-1}\left\|\theta^{(m)}(s)\right\|_g^2\,ds \le \frac{\Gamma(\alpha)}{\kappa'}|\theta_{0m}|_g^2 + \frac{2}{\kappa\kappa'}\int_0^T \left\|f_2(s)\right\|_{W_g'}^{2/\alpha_2}\,ds + \frac{\Gamma(\alpha)}{\kappa'}C_2.\tag{4.10}$$

On the other hand, integrating (4.8) from 0 to T, in the fractional sense, we obtain

$$\begin{aligned}
&\left|u^{(m)}(t)\right|_g^2+\frac{\nu'}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|u^{(m)}(s)\right\|_g^2\,ds\\
&\quad\le |u_{0m}|_g^2+\frac{c'}{\nu\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|\theta^{(m)}(s)\right\|_g^2\,ds+\frac{4}{\nu\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|f_1(s)\right\|_{V_g'}^2\,ds\\
&\quad\le |u_{0m}|_g^2+\frac{c'}{\nu\kappa'}|\theta_{0m}|_g^2+\frac{2c'}{\nu\kappa\kappa'\Gamma(\alpha)}\int_0^t \left\|f_2(s)\right\|_{W_g'}^{2/\alpha_2}\,ds+\frac{c'}{\nu\kappa'}C_2\\
&\qquad+\frac{4}{\nu\Gamma(\alpha)}\int_0^t \left\|f_1(s)\right\|_{V_g'}^{2/\alpha_1}\,ds+\frac{4}{\nu\Gamma(\alpha)}\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}\,ds\\
&\quad\le |u_{0m}|_g^2+\frac{c'}{\nu\kappa'}|\theta_{0m}|_g^2+\frac{2c'}{\nu\kappa\kappa'\Gamma(\alpha)}\int_0^t \left\|f_2(s)\right\|_{W_g'}^{2/\alpha_2}\,ds+\frac{4}{\nu\Gamma(\alpha)}\int_0^t \left\|f_1(s)\right\|_{V_g'}^{2/\alpha_1}\,ds\\
&\qquad+C_1,
\end{aligned}$$

where $b_1=\frac{\alpha-1}{1-\alpha_1}$ and $C_1=\frac{c'}{\nu\kappa'}C_2+\frac{4T^{1+b_1}}{\nu(1+b_1)\Gamma(\alpha)}$. By using the fact that

$$\int_0^t (t-s)^{\alpha-1}\left\|u^{(m)}(s)\right\|_g^2\,ds\ge T^{\alpha-1}\int_0^t \left\|u^{(m)}(s)\right\|_g^2\,ds \tag{4.11}$$

and similarly

$$\int_0^t (t-s)^{\alpha-1}\left\|\theta^{(m)}(s)\right\|_g^2\,ds\ge T^{\alpha-1}\int_0^t \left\|\theta^{(m)}(s)\right\|_g^2\,ds, \tag{4.12}$$

it follows that

$$\begin{aligned}
&\left|u^{(m)}(t)\right|_g^2+\frac{\nu' T^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \left\|u^{(m)}(s)\right\|_g^2\,ds\\
&\quad\le |u_{0m}|_g^2+\frac{c'}{\nu\kappa'}|\theta_{0m}|_g^2 \qquad\qquad (4.13)\\
&\qquad+\frac{2c'}{\nu\kappa\kappa'\Gamma(\alpha)}\int_0^T \left\|f_2(s)\right\|_{W_g'}^{2/\alpha_2}\,ds+\frac{4}{\nu\Gamma(\alpha)}\int_0^T \left\|f_1(s)\right\|_{V_g'}^{2/\alpha_1}\,ds+C_1,
\end{aligned}$$

$$\left|\theta^{(m)}(t)\right|_g^2+\frac{\kappa' T^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \left\|\theta^{(m)}(s)\right\|_g^2\,ds\le |\theta_{0m}|_g^2+\frac{2}{\kappa\Gamma(\alpha)}\int_0^T \left\|f_2(s)\right\|_{V_g'}^{2/\alpha_2}\,ds+C_2. \tag{4.14}$$

Consequently,

$$\begin{aligned}
\sup_{t\in[0,T]}\left|u^{(m)}(t)\right|_g^2&\le |u_{0m}|_g^2+\frac{c'}{\nu\kappa'}|\theta_{0m}|_g^2+\frac{2c'}{\nu\kappa\kappa'\Gamma(\alpha)}\int_0^T \left\|f_2(s)\right\|_{W_g'}^{2/\alpha_2}\,ds \qquad (4.15)\\
&\quad+\frac{4}{\nu\Gamma(\alpha)}\int_0^T \left\|f_1(s)\right\|_{V_g'}^{2/\alpha_1}\,ds+C_1,
\end{aligned}$$

$$\sup_{t\in[0,T]}\left|\theta^{(m)}(t)\right|_g^2\le |\theta_{0m}|_g^2+\frac{2}{\kappa\Gamma(\alpha)}\int_0^T \left\|f_2(s)\right\|_{V_g'}^{2/\alpha_2}\,ds+C_2, \tag{4.16}$$

which implies that the sequences $\{u^{(m)}\}_m$ and $\{\theta^{(m)}\}_m$ remain in a bounded set of $L^\infty(0,T;H_g)$ and $L^\infty(0,T;L^2(\Omega,g))$, respectively. Moreover, for $t = T$, one obtains

$$\begin{aligned}
&\int_0^T \|u^{(m)}(s)\|_g^2\,ds \\
&\quad\le \frac{\Gamma(\alpha)}{\nu' T^{\alpha-1}}|u_{0m}|_g^2 + \frac{c'}{\nu\nu'\kappa' T^{\alpha-1}}|\theta_{0m}|_g^2 + \frac{2c'}{\nu\nu'\kappa\kappa' T^{\alpha-1}}\int_0^T \|f_2(s)\|_{W_g'}^{2/\alpha_2}\,ds \\
&\qquad + \frac{4}{\nu\nu' T^{\alpha-1}}\int_0^T \|f_1(s)\|_{V_g'}^{2/\alpha_1}\,ds + \frac{\Gamma(\alpha)}{\nu' T^{\alpha-1}}C_1,
\end{aligned} \tag{4.17}$$

$$\int_0^T \|\theta^{(m)}(s)\|_g^2\,ds \le \frac{\Gamma(\alpha)}{\kappa' T^{\alpha-1}}|\theta_{0m}|_g^2 + \frac{2}{\kappa\kappa' T^{\alpha-1}}\int_0^T \|f_2(s)\|_{V_g'}^{2/\alpha_2}\,ds + \frac{\Gamma(\alpha)}{\kappa' T^{\alpha-1}}C_2, \tag{4.18}$$

which implies that the sequences $\{u^{(m)}\}_m$ and $\{\theta^{(m)}\}_m$ remain in a bounded set of $L^2(0,T;V_g)$ and $L^2(0,T;W_g)$, respectively. Consequently, we can assert the existence of elements $u \in L^2(0,T;V_g) \cap L^\infty(0,T;H_g)$ and $\theta \in L^2(0,T;W_g) \cap L^\infty(0,T;L^2(\Omega,g))$ and the subsequences $\{u^{(m)}\}_m$ and $\{\theta^{(m)}\}_m$ such that $u^{(m)} \to u \in L^2(0,T;V_g)$ and $\theta^{(m)} \to \theta \in L^2(0,T;W_g)$ weakly and $u^{(m)} \to u \in L^\infty(0,T;H_g)$ and $\theta^{(m)} \to \theta \in L^\infty(0,T;L^2(\Omega,g))$ weakly-star as $m \to \infty$.

Let $\widetilde{u}^{(m)} : \mathbb{R} \to V_g$ and $\widetilde{\theta}^{(m)} : \mathbb{R} \to W_g$ be defined as

$$\widetilde{u}^{(m)}(t) = \begin{cases} u^{(m)}(t), & 0 \le t \le T, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{\theta}^{(m)}(t) = \begin{cases} \theta^{(m)}(t), & 0 \le t \le T, \\ 0, & \text{otherwise,} \end{cases}$$

and their Fourier transforms be denoted by $\widehat{u}^{(m)}$ and $\widehat{\theta}^{(m)}$, respectively. We show that the sequence $\{\tilde{u}^{(m)}\}_m$ remains bounded in $W^\gamma(\mathbb{R},V_g,H_g)$ and the sequence $\{\tilde{\theta}^{(m)}\}_m$ remains bounded in $W^\gamma(\mathbb{R},W_g,L^2(\Omega,g))$. To do so, we need to verify that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\big|\widehat{u}^{(m)}(\tau)\big|^2\,d\tau \le \text{const.} \quad \text{for some } \gamma > 0 \tag{4.19}$$

and

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\big|\widehat{\theta}^{(m)}(\tau)\big|^2\,d\tau \le \text{const.} \quad \text{for some } \gamma > 0. \tag{4.20}$$

In order to prove (4.19) and (4.20), we observe that

$$\big(D_t^\alpha \tilde{u}^{(m)}, u_k\big)_g = \big(\widetilde{F}_m^u, u_k\big)_g + (u_{m0}, u_k)_g I_{-,t}^{1-\alpha}\delta_0 - \big(u^{(m)}(T), u_k\big)_g I_{-,t}^{1-\alpha}\delta_T, \tag{4.21}$$

$$\big(D_t^\alpha \widetilde{\theta}^{(m)}, \theta_k\big)_g = \big(\widetilde{F}_m^\theta, \theta_k\big)_g + (\theta_{m0}, \theta_k)_g I_{-,t}^{1-\alpha}\delta_0 - \big(\theta^{(m)}(T), \theta_k\big)_g I_{-,t}^{1-\alpha}\delta_T, \tag{4.22}$$

where δ_0, δ_T are Dirac distributions at 0 and T and F_m^u and F_m^θ are defined by

$$\begin{aligned}
F_m^u &= \xi\theta^{(m)} + f_1 - B_g\big(u^{(m)}, u^{(m)}\big) - \nu A_g u^{(m)} - \nu C_g u^{(m)}, \\
F_m^\theta &= f_2 - \widetilde{B}_g\big(u^{(m)}, \theta^{(m)}\big) - \kappa\widetilde{A}_g\theta^{(m)} + \kappa\widetilde{C}_g\theta^{(m)} + \kappa\widetilde{D}_g\theta^{(m)}
\end{aligned}$$

for $k = 1,\dots,m$. Here $\widetilde{F}_m$ is defined as usual by

$$\widetilde{F}_m(t) = \begin{cases} F_m(t), & 0 \le t \le T, \\ 0, & \text{otherwise.} \end{cases} \tag{4.23}$$

Indeed, it is classical that since $\widetilde{u}^{(m)}$ and $\widetilde{\theta}^{(m)}$ have two discontinuities at 0 and T, the Caputo derivative of $\widetilde{u}^{(m)}$ is given by

$$D^{\alpha}_{-,t}\tilde{u}^{(m)} = I^{1-\alpha}_{-,t}\left(\frac{d}{dt}\tilde{u}^{(m)}\right) \tag{4.24}$$

$$= I^{1-\alpha}_{-,t}\left(\frac{d}{dt}u^{(m)} + u^{(m)}(0)\delta_0 - u^{(m)}(T)\delta_T\right) \tag{4.25}$$

$$= D^{\alpha}_{t}u^{(m)} + I^{1-\alpha}_{-,t}\left(u^{(m)}(0)\delta_0 - u^{(m)}(T)\delta_T\right) \tag{4.26}$$

and the one of $\widetilde{\theta}^{(m)}$ is given by

$$D^{\alpha}_{-,t}\widetilde{\theta}^{(m)} = D^{\alpha}_{t}\theta^{(m)} + I^{1-\alpha}_{-,t}\left(\theta^{(m)}(0)\delta_0 - \theta^{(m)}(T)\delta_T\right). \tag{4.27}$$

By the Fourier transform, (4.21) and (4.22) yield

$$(2i\pi\tau)^{\alpha}\left(\widehat{u}^{(m)}, u_k\right)_g = \left(\widehat{F}^{u}_m, u_k\right)_g + (u_{m0}, u_k)_g(2i\pi\tau)^{\alpha-1} \tag{4.28}$$

$$- \left(u^{(m)}(T), u_k\right)_g(2i\pi\tau)^{\alpha-1}e^{-2i\pi T\tau}, \tag{4.29}$$

$$(2i\pi\tau)^{\alpha}\left(\widehat{\theta}^{(m)}, \theta_k\right)_g = \left(\widehat{F}^{\theta}_m, \theta_k\right)_g + (\theta_{m0}, \theta_k)_g(2i\pi\tau)^{\alpha-1} \tag{4.30}$$

$$- \left(\theta^{(m)}(T), \theta_k\right)_g(2i\pi\tau)^{\alpha-1}e^{-2i\pi T\tau}. \tag{4.31}$$

Here $\widehat{u}^{(m)}$ and $\widehat{F}_m$ denote the Fourier transforms of $\tilde{u}^{(m)}$ and $\widetilde{F}_m$, respectively. We multiply (4.28) and (4.30) by $\widehat{f}^{(m)}_j$ and $\widehat{g}^{(m)}_j$, respectively, and add these equations for $k = 1,\dots,m$ to get

$$(2i\pi\tau)^{\alpha}\left|\widehat{u}^{(m)}(\tau)\right|^2_g = \left(\widehat{F}^{u}_m(\tau), \widehat{u}^{(m)}(\tau)\right)_g + \left(u_{m0}, \widehat{u}^{(m)}(\tau)\right)_g(2i\pi\tau)^{\alpha-1} \tag{4.32}$$

$$- \left(u^{(m)}(T), \widehat{u}^{(m)}(\tau)\right)_g(2i\pi\tau)^{\alpha-1}e^{-2i\pi T\tau}, \tag{4.33}$$

$$(2i\pi\tau)^{\alpha}\left|\widehat{\theta}^{(m)}(\tau)\right|^2_g = \left(\widehat{F}^{\theta}_m(\tau), \widehat{\theta}^{(m)}(\tau)\right)_g + \left(\theta_{m0}, \widehat{\theta}^{(m)}(\tau)\right)_g(2i\pi\tau)^{\alpha-1} \tag{4.34}$$

$$- \left(\theta^{(m)}(T), \widehat{\theta}^{(m)}(\tau)\right)_g(2i\pi\tau)^{\alpha-1}e^{-2i\pi T\tau}. \tag{4.35}$$

Since the integrals on the right-hand side of the inequalities

$$\int_0^T \left\|F^{u}_m(t)\right\|_{V'_g} dt \le \int_0^T c\Big(|\xi|_\infty\left\|\theta^{(m)}(t)\right\|_g + \left\|f_1(t)\right\|_{V'_g} + \left|u^{(m)}(t)\right|_g\left\|u^{(m)}\right\|_g + \left\|u^{(m)}(t)\right\|_g + |\nabla g|_\infty\left\|u^{(m)}(t)\right\|_g\Big)\,dt, \tag{4.36}$$

$$\int_0^T \left\|F^{\theta}_m(t)\right\|_{W'_g} dt \le \int_0^T c'\Big(\left\|f_2(t)\right\|_{W'_g} + \left|u^{(m)}(T)\right|_g\left\|\theta^{(m)}(t)\right\|_g + \left\|\theta^{(m)}(t)\right\|_g \tag{4.37}$$

$$+ |\nabla g|_\infty \left\|\theta^{(m)}(t)\right\|_g + |\Delta g|_\infty \left\|\theta^{(m)}(t)\right\|_g\Big)\, dt$$

remain bounded, $\|F_1(t)\|_{V'_g}$ and $\|F_2(t)\|_{W'_g}$ are bounded in $L^1(0,T;V'_g)$ and $L^1(0,T;W'_g)$, respectively. Therefore, for all m,

$$\sup_{\tau\in\mathbb{R}} \left\|\widehat{F}^u_m(\tau)\right\|_{V'_g} \le c_1 \quad \text{and} \quad \sup_{\tau\in\mathbb{R}} \left\|\widehat{F}^\theta_m(\tau)\right\|_{W'_g} \le c_2.$$

Moreover, since $u^{(m)}(0)$, $u^{(m)}(T)$, $\theta^{(m)}(0)$ and $\theta^{(m)}(T)$ are bounded, we get

$$\begin{aligned} |\tau|^\alpha \left|\widetilde{u}^{(m)}(\tau)\right|_g^2 &\le c_1 \left\|u^{(m)}\right\|_{V_g} + c_2|\tau|^{\alpha-1}\left|u^{(m)}\right|_g \\ &\le c_3 \left\|u^{(m)}\right\|_{V_g}, \\ |\tau|^\alpha \left|\widetilde{\theta}^{(m)}(\tau)\right|_g^2 &\le c'_1 \left\|\theta^{(m)}\right\|_{W_g} + c'_2|\tau|^{\alpha-1}\left|\theta^{(m)}\right|_g \quad (4.38)\\ &\le c_3 \left\|\theta^{(m)}\right\|_{W_g}. \quad (4.39) \end{aligned}$$

For γ fixed, $\gamma < \alpha/4$, we observe that

$$|\tau|^{2\gamma} \le c(\gamma)\frac{1+|\tau|^\alpha}{1+|\tau|^{\alpha-2\gamma}}.$$

Then we can write

$$\begin{aligned} \int_{-\infty}^{+\infty} |\tau|^{2\gamma}\left|\widehat{u}^{(m)}(\tau)\right|_g^2 &\le c_5(\gamma)\int_{-\infty}^{+\infty} \frac{1+|\tau|^\alpha}{1+|\tau|^{\alpha-2\gamma}}\left|\widehat{u}^{(m)}(\tau)\right|_g^2 d\tau \\ &\le c_6(\gamma)\int_{-\infty}^{+\infty} \frac{1}{1+|\tau|^{\alpha-2\gamma}}\left\|\widehat{u}^{(m)}(\tau)\right\|_{V_g}^2 d\tau \\ &\quad + c_7(\gamma)\int_{-\infty}^{+\infty} \frac{|\tau|^{\alpha-1}}{1+|\tau|^{\alpha-2\gamma}}\left\|\widehat{u}^{(m)}(\tau)\right\|_{V_g}^2 d\tau, \\ \int_{-\infty}^{+\infty} |\tau|^{2\gamma}\left|\widehat{\theta}^{(m)}(\tau)\right|_g^2 &\le c'_6(\gamma)\int_{-\infty}^{+\infty} \frac{1}{1+|\tau|^{\alpha-2\gamma}}\left\|\widehat{\theta}^{(m)}(\tau)\right\|_{W_g}^2 d\tau \\ &\quad + c'_7(\gamma)\int_{-\infty}^{+\infty} \frac{|\tau|^{\alpha-1}}{1+|\tau|^{\alpha-2\gamma}}\left\|\widehat{\theta}^{(m)}(\tau)\right\|_{W_g}^2 d\tau. \end{aligned}$$

By the Parseval inequality, the first integral is bounded as $m \to \infty$. Applying the Schwarz inequality, the second integrals yield

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{|\tau|^{\alpha-1}}{1+|\tau|^{\alpha-2\gamma}}\left\|\hat{u}^{(m)}(\tau)\right\|_g^2 d\tau &\le \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{\alpha-2\gamma})^2}\right)^{1/2} \quad (4.40)\\ &\quad \times \left(\int_{-\infty}^{+\infty} |\tau|^{2\alpha-2}\left\|\hat{u}^{(m)}(\tau)\right\|_g^2 d\tau\right)^{1/2}, \quad (4.41)\\ \int_{-\infty}^{+\infty} \frac{|\tau|^{\alpha-1}}{1+|\tau|^{\alpha-2\gamma}}\left\|\widehat{\theta}^{(m)}(\tau)\right\|_g^2 d\tau &\le \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{\alpha-2\gamma})^2}\right)^{1/2} \quad (4.42)\\ &\quad \times \left(\int_{-\infty}^{+\infty} |\tau|^{2\alpha-2}\left\|\widehat{\theta}^{(m)}(\tau)\right\|_g^2 d\tau\right)^{1/2}. \quad (4.43) \end{aligned}$$

The first integrals are finite due to $\gamma < \alpha/4$. On the other hand, it follows from the Parseval equality that

$$\begin{aligned}\int_{-\infty}^{+\infty} |\tau|^{2\alpha-2}\big\|\hat{u}^{(m)}(\tau)\big\|_g^2\,d\tau &= \int_{-\infty}^{+\infty} \|{}_{-\infty}\mathrm{I}_t^{1-\alpha}\tilde{u}^{(m)}(t)\|_g^2\,dt \\ &= \int_0^T \|{}_0\mathrm{I}_t^{1-\alpha}u^{(m)}(t)\|_g^2\,dt \\ &\le \left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 \int_0^T \big\|u^{(m)}(t)\big\|_{V_g}^2\,dt, \\ \int_{-\infty}^{+\infty} |\tau|^{2\alpha-2}\big\|\widehat{\theta}^{(m)}(\tau)\big\|_g^2\,d\tau &\le \left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 \int_0^T \big\|\theta^{(m)}(t)\big\|_{W_g}^2\,dt,\end{aligned}$$

which implies that (4.19) and (4.20) hold. We know that there exists a subsequence of $\{u^{(m)}\}_m$ (which we will denote with the same symbols) that converges to some u weakly in $L^2(0,T;V_g)$ and weakly-star in $L^\infty(0,T;H_g)$ with $u \in L^2(0,T;V_g)\cap L^\infty(0,T;H_g)$. Similarly, there exists a subsequence of $\{\theta^{(m)}\}_m$ (which we will denote with the same symbol) that converges to some θ weakly in $L^2(0,T;W_g)$ and weakly-star in $L^\infty(0,T;L^2(\Omega,g))$ with $\theta \in L^2(0,T;W_g)\cap L^\infty(0,T;L^2(\Omega,g))$. As $W^\gamma(0,T,V_g;H_g)$ is compactly embedded in $L^2(0,T;H_g)$ and $W^\gamma(\mathbb{R},W_g,L^2(\Omega,g))$ in $L^2(0,T;L^2(\Omega,g))$, then $\{u^{(m)}\}_m$ strongly converges in $L^2(0,T;H_g)$ and $\{\theta^{(m)}\}_m$ in $L^2(0,T;L^2(\Omega,g))$, respectively.

In order to pass to the limit, we consider scalar functions $\Psi_1(t)$ and $\Psi_2(t)$ that are continuously differentiable on $[0,T]$ and such that $\Psi_1(T)=\Psi_2(T)=0$. We multiply (4.3) and (4.4) by $\Psi_1(t)$ and $\Psi_2(t)$, respectively, and then integrate by parts. This leads to the equations

$$\begin{aligned}&\int_0^T \big(u^{(m)}(t),\mathrm{D}_{t,T}^\alpha\Psi_1(t)u_k\big)_g\,dt + \int_0^T b_g\big(u^{(m)}(t),u^{(m)}(t),\Psi_1u_k\big)\,dt \\ &\quad + \nu\int_0^T \big(\big(u^{(m)}(t),\Psi_1u_k\big)\big)_g + \nu\int_0^T b_g\left(\frac{\nabla g}{g},u^{(m)}(t),\Psi_1u_k\right)dt \\ &= \big(u_{0m},\mathrm{I}_{0,T}^{1-\alpha}\Psi_2(t)u_k\big)_g \\ &\quad + \int_0^T \big(\xi\theta^{(m)}(t),\Psi_1u_k\big)_g\,dt + \int_0^T \big(f_1(t),u_k\big)_g\,dt, \\ &\int_0^T \big(\theta^{(m)}(t),\mathrm{D}_{t,T}^\alpha\Psi_2(t)\theta_k\big)_g\,dt + \int_0^T \widetilde{b}_g\big(u^{(m)}(t),\theta^{(m)}(t),\Psi_2\theta_k\big)\,dt \\ &\quad + \kappa\int_0^T \big(\big(\theta^{(m)}(t),\Psi_2\theta_k\big)\big)_g\,dt + \kappa\int_0^T \widetilde{b}_g\left(\frac{\nabla g}{g},\theta_k,\Psi_2\theta^{(m)}(t)\right)dt \\ &= \big(\theta_{0m},\mathrm{I}_{0,T}^{1-\alpha}\Psi_2(t)\theta_k\big)_g \\ &\quad + \int_0^T \big(f_2(t),\Psi_2\theta_k\big)_g\,dt.\end{aligned}$$

Following the same lines as in [8, 31], we obtain, as $m\to\infty$,

$$\begin{aligned}&\int_0^T \big(u(t),\mathrm{D}_{t,T}^\alpha\Psi_1(t)u_k\big)_g\,dt + \int_0^T b_g\big(u(t),u(t),\Psi_1u_k\big)\,dt + \nu\int_0^T \big(\big(u(t),\Psi_1u_k\big)\big)_g \\ &\quad + \nu\int_0^T b_g\left(\frac{\nabla g}{g},u(t),\Psi_1u_k\right)dt\end{aligned} \tag{4.44}$$

$$= \left(u_0, \mathrm{I}_{0,T}^{1-\alpha}\Psi_1 u_k\right)_g + \int_0^T \left(\xi\theta(t), \Psi_1 v\right)_g dt + \int_0^T \left(f_1(t), u_k\right)_g dt, \tag{4.45}$$

$$\begin{aligned} &\int_0^T \left(\theta(t), \mathrm{D}_{t,T}^{\alpha}\Psi_2(t)\theta_k\right)_g dt + \int_0^T \widetilde{b}_g\left(u(t), \theta(t), \Psi_2\theta_k\right) dt \\ &\quad + \kappa \int_0^T \left(\left(\theta(t), \Psi_2\theta_k\right)\right)_g dt + \kappa \int_0^T \widetilde{b}_g\left(\frac{\nabla g}{g}, \theta_k, \Psi_2\theta(t)\right) dt \\ &= \left(\theta_0, \mathrm{I}_{0,T}^{1-\alpha}\Psi_2(t)\theta_k\right)_g \\ &\quad + \int_0^T \left(f_2(t), \Psi_2\theta_k\right)_g dt. \end{aligned} \tag{4.46}$$

These equations hold for v and τ that are finite linear combination of u_k and θ_k, respectively ($k = 1, \ldots, m$), and by continuity the equations hold for any v in V_g and $\tau \in H_g$. It then follows that $\{u, \theta\}$ satisfies the two first equations of (3.3). To end the proof, we still need to check that $\{u, \theta\}$ satisfies the initial conditions $u(0) = u_0$ and $\theta(0) = \theta_0$. To do so, it suffices to multiply the two first equations in (3.3) by Ψ_1 and Ψ_2, respectively, and then to integrate. By making use of the integration by part and comparing with (4.44) and (4.46), one can find that

$$\left(u_0 - u(0), v\right)_g \mathrm{I}_{0,T}^{1-\alpha}\Psi_2(t) = 0, \quad \text{and} \quad \left(\theta_0 - \theta(0), \tau\right)_g \mathrm{I}_{0,T}^{1-\alpha}\Psi_2(t) = 0,$$

which leads to the desired result by taking a particular choice of Ψ_1 and Ψ_2.

For the uniqueness of the weak solutions, let (u_1, θ_1) and (u_2, θ_2) be two weak solutions with the same initial condition. Let $w = u_1 - u_2$ and $\widetilde{w} = \theta_1 - \theta_2$. Then we have

$$\begin{aligned} &D_t^{\alpha}(w, v)_g + b_g(u_1, u_1, v) - b_g(u_2, u_2, v) + \nu(\nabla w, \nabla v)_g + \nu(C_g w, v)_g = (\xi\widetilde{w}, v)_g, \\ &D_t^{\alpha}(\widetilde{w}, \tau)_g + \widetilde{b}_g(u_1, \theta_1, \tau) - \widetilde{b}_g(u_2, \theta_2, \tau) + \kappa(\nabla\widetilde{w}, \nabla\tau)_g + \kappa\widetilde{b}_g\left(\frac{\nabla g}{g}, \tau, \widetilde{w}\right) = 0. \end{aligned}$$

Taking $v = w(t)$ and $\tau = \widetilde{w}(t)$, one obtains

$$\begin{aligned} &D_t^{\alpha}(w, w)_g + b_g(w, u_2, w) + \nu\left|A_g^{1/2}w\right|_g^2 + \nu(C_g w, w)_g = (\xi\widetilde{w}, w)g, \\ &D_t^{\alpha}(\widetilde{w}, \widetilde{w})_g + \widetilde{b}_g(u_1, \theta_1, \widetilde{w}) - \widetilde{b}_g(u_2, \theta_2, \widetilde{w}) + \kappa\left|\widetilde{A}_g^{1/2}\widetilde{w}\right|_g^2 + \kappa\widetilde{b}_g\left(\frac{\nabla g}{g}, \widetilde{w}, \widetilde{w}\right) = 0. \end{aligned}$$

By applying the bounds on the terms b_g, $\widetilde{b}_g$, it then follows by the Cauchy–Schwarz inequality and Gronwall-like inequality that $w(t) = 0$ and $\widetilde{w}(t) = 0$ for all $t \geq 0$, since we have $w(0) = 0$ and $\widetilde{w}(0) = 0$. Thus the theorem is proved. □

5 Conclusion

In this paper, we have introduced a new variation of Navier–Stokes equations. It consists in time-fractional Bénard equations in fractal thin media. The main technique to prove the existence of solutions to this problem is the Faedo–Galerkin approximation method. The deduced estimates allow us to get a (sub)sequence that converges to a solution. The uniqueness follows immediately from a Gronwall type inequality.

There is still a lot to do with this subject, namely a numerical study should be conducted in future works. Moreover, the analysis of the stochastic version, the attractors and the long-term behaviour should be of great interest.

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Declarations

Competing interests
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