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Asymptotic behavior of solution curves of nonlocal one-dimensional elliptic equations

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Abstract

We study the one-dimensional nonlocal elliptic equation

$$-A(\|u'\|_p^p)u''(x) = \lambda B(\|u'\|_q^q)u(x)^r, \quad x \in I := (0,1), u(x) > 0, x \in I,$$

$$u(0) = u(1) = 0,$$

where $A = A(y)$ and $B = B(y)$ are continuous functions, satisfying $A(y) > 0$, $B(y) > 0$ for $y > 0$, $p \geq 1$, $q \geq 1$, and $r > 1$ are given constants, and $\lambda > 0$ is a bifurcation parameter. We establish the global behavior of solution curves and precise asymptotic formulas for $u_\lambda(x)$ as $\lambda \to \infty$.

MSC: 34C23; 34F10

Keywords: Nonlocal elliptic equations; Bifurcation curves; Asymptotic formulas

1 Introduction

We consider the following one-dimensional nonlocal elliptic equation

$$\begin{cases} -A(\|u'\|_p^p)u''(x) = \lambda B(\|u'\|_q^q)u(x)^r, & x \in I := (0,1), \\ u(x) > 0, & x \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $A = A(y)$ and $B = B(y)$ are continuous functions with $A(y) > 0$, $B(y) > 0$ for $y > 0$, while $p \geq 1$, $q \geq 1$, $r > 1$ are given constants, $\|\cdot\|_m$ $(m \geq 1)$ denotes the usual L^m-norm of the real-valued functions on I, and $\lambda > 0$ is a bifurcation parameter. In this paper, we consider the following typical three cases.

(i) $A(y) = y$, $B(y) = y$,

(ii) $A(y) = e^y$, $B(y) = 1$, $p = 2$,

(iii) $A(y) = e^y$, $B(y) = y$.

The case (i) is a modification of a nonlocal problem of Kirchhoff type, which is motivated by the following problem (1.2) in [13]

$$\begin{cases} -A(\int_0^1 |u'(x)|^p\, dx)u''(x) = \lambda f(x, u(x)), & x \in I, \\ u(x) > 0, & x \in I, \\ u(0) = u'(1) = 0. \end{cases} \tag{1.2}$$

Besides, the cases (ii) and (iii) are motivated by the mean field equation and nonlocal Liouville-type equations.

Nonlocal problems have been of interest to many researchers from mathematical point of view, since many problems are derived from the phenomena of relevant physical, biological, and engineering problems. Therefore, nonlocal problems have been widely studied by many authors. We refer the reader to Goodrich [7–9], Lacey [11, 12], Stańczy [14], [2–4, 10], and the references therein. One of the main interest in this area is the existence of positive solutions. On the other hand, there seems to be a few studies on bifurcation problems. We refer the reader to [17] and the references therein. Roughly speaking, in [17], the case $A(y) = y^j + b$ and $B(y) = y^k$, where $j \ge 0$, $k > 0$ are constants, has been considered, and the existence of a branch of positive solutions bifurcating from infinity at $\lambda = 0$ has been discussed. Recently, the case, where $A(y) = y + b$, $B(y) \equiv 1$ and $p = 2$, has been studied in [15], where $b > 0$ is a constant, and the precise global behavior of solution curve has been obtained. For a standard bifurcation problems, we refer to [5].

The purpose of this paper is to consider more general nonlocal terms motivated by equations having background in physics and obtain the precise asymptotic behavior of bifurcation curves $\lambda = \lambda(\alpha)$ and u_λ as $\lambda \to \infty$. Here, $\alpha := \alpha_\lambda = \|u_\lambda\|_\infty$ for given $\lambda > 0$. The main tool here is time map method, also known as quadrature technique (cf. [12]). One can see the simple example of time map method in the Appendix.

Now, we state our main Theorems 1.1, 1.2, and 1.3. To do this, we prepare the following notation. For $r > 1$, let

$$\begin{cases} -W''(x) = W(x)^r, & x \in I, \\ W(x) > 0, & x \in I, \\ W(0) = W(1) = 0. \end{cases} \tag{1.3}$$

We know from [6] that there exists a unique solution $W_r(x)$ of (1.3). For $m \ge 1$, we put

$$L_m := \int_0^1 \frac{1}{\sqrt{1 - s^{m+1}}}\, ds, \qquad M_{r,m} := \int_0^1 \left(1 - s^{r+1}\right)^{(m-1)/2} ds. \tag{1.4}$$

We note that L_m is finite since $\sqrt{1 - s^2} \le \sqrt{1 - s^{m+1}}$ for $0 \le s \le 1$. Then, we have

$$\|W_r'\|_m^m = 2^{mr/(r-1)}(r+1)^{m/(r-1)} L_r^{(mr+m-r+1)/(r-1)} M_{r,m}, \tag{1.5}$$

$$\|W_r\|_\infty = \left(2(r+1)\right)^{1/(r-1)} L_r^{2/(r-1)}. \tag{1.6}$$

Equation (1.6) has been obtained in [16]. For completeness, the proof of (1.5) and (1.6) will be given in the Appendix.

We begin with the first main Theorem 1.1, which will be proved in Sect. 2.

Theorem 1.1 *Let $A(y) = y$, $B(y) = y$ in (1.1). Assume that $p - q - r + 1 \neq 0$. Then, there exists a unique solution $u_\lambda(x)$ of (1.1) for any $\lambda > 0$, and it is represented as*

$$u_\lambda(x) = \lambda^{1/(p-q-r+1)} \frac{\| W_r' \|_q^{q/(p-q-r+1)}}{\| W_r' \|_p^{p/(p-q-r+1)}} W_r(x). \tag{1.7}$$

Further, λ is represented as the function of $\alpha := \|u_\lambda\|_\infty$ as

$$\lambda = \lambda(\alpha) = \frac{\| W_r' \|_p^p}{\| W_r' \|_q^q \| W_r \|_\infty^{p-q-r+1}} \alpha^{p-q-r+1}. \tag{1.8}$$

Next, we consider the case $A(y) = e^y$, $B(y) = 1$ and $p = 2$ and state Theorem 1.2. The proof will be given in Sect. 3. For $r > 1$, we put

$$R_r := \frac{r-1}{2} \left\{ 1 - \log(r-1) + \log 2 + 2 \log \| W_r' \|_2 \right\}. \tag{1.9}$$

Theorem 1.2 *Let $A(y) = e^y$, $B(y) = 1$ and $p = 2$. For $r > 1$, put $\lambda_r := e^{R_r}$.*

(i) *If $0 < \lambda < \lambda_r$, then there are no solutions of (1.1).*

(ii) *If $\lambda = \lambda_r$, then (1.1) has a unique solution $u_{1,\lambda}$.*

(iii) *If $\lambda > \lambda_r$, then there are exactly two solutions $u_{1,\lambda}$, $u_{2,\lambda}$ with $u_{1,\lambda}(x) < u_{2,\lambda}(x)$ for $x \in I$.*

(iv) *Let $\lambda > \lambda_r$ be fixed. Then, there are two numbers $\alpha_{1,\lambda} := \|u_{1,\lambda}\|_\infty$ and $\alpha_{2,\lambda} := \|u_{2,\lambda}\|_\infty$ satisfying:*

$$\lambda = 2(r+1) L_r^2 A\left(4 M_{r,2} L_r \alpha_{j,\lambda}^2\right) \alpha_{j,\lambda}^{1-r} \quad (j = 1, 2). \tag{1.10}$$

If $\lambda = \lambda_r$, then $\alpha_0 := \alpha_{1,\lambda} = \alpha_{2,\lambda}$ in (1.10).

(v) *As $\lambda \to \infty$,*

$$u_{1,\lambda}(x) = \lambda^{-1/(r-1)} \left\{ 1 + \frac{1}{r-1} \| W_r' \|_2^2 \lambda^{-2/(r-1)} \left(1 + o(1)\right) \right\} \| W_r' \|_2^{-1} W_r(x), \tag{1.11}$$

$$u_{2,\lambda}(x) = \left\{ \log \lambda + \frac{r-1}{2} \log(\log \lambda) \left(1 + o(1)\right) \right\}^{1/2} \| W_r' \|_2^{-1} W_r(x). \tag{1.12}$$

We finally state Theorem 1.3, which will be proved in Sect. 4.

Theorem 1.3 *Let $A(y) = e^y$, $B(y) = y$. Let*

$$C_0 = \frac{q+r-1}{p} \left(1 - \log \frac{q+r-1}{p} + \log \| W_r' \|_p^p - \frac{p}{q+r-1} \log \| W_r' \|_q^q \right) \tag{1.13}$$

and $\lambda_0 := e^{C_0}$.

(i) *If $0 < \lambda < \lambda_0$, then there are no solutions of (1.1).*

(ii) *If $\lambda = \lambda_0$, then (1.1) has a unique solution $u_{1,\lambda}$.*

(iii) *If $\lambda > \lambda_0$, then there are exactly two solutions $u_{1,\lambda}$, $u_{2,\lambda}$ with $u_{1,\lambda}(x) < u_{2,\lambda}(x)$ for $x \in I$.*

(iv) *As $\lambda \to \infty$*

$$u_{1,\lambda}(x) = \lambda^{-1/(q+r-1))} \| W_r' \|_q^{-q/(q+r-1)} \tag{1.14}$$

$$\times \left\{ 1 + \frac{1}{q+r-1} \| W_\lambda' \|_p^p \| W_\lambda' \|_q^{pq/(q+r-1)} \lambda^{-p/(q+r-1)} \left(1 + o(1)\right) \right\} W_r(x),$$

$$\alpha_{1,\lambda} = \lambda^{-1/(q+r-1))} \| W_r' \|_q^{-q/(q+r-1)} \times \left\{ 1 + \frac{1}{q+r-1} \| W_\lambda' \|_p^p \| W_\lambda' \|_q^{pq/(q+r-1)} \lambda^{-p/(q+r-1)} \bigl(1+o(1)\bigr) \right\} \| W_\lambda \|_\infty, \tag{1.15}$$

$$u_{2,\lambda}(x) = \| W_r' \|_q^{-1} (\log\lambda)^{1/p} \left\{ 1 + \frac{p^2}{(q+r-1)^3} \frac{\log(\log\lambda)}{\log\lambda} \bigl(1+o(1)\bigr) \right\} W_r(x), \tag{1.16}$$

$$\alpha_{2,\lambda} = \| W_r' \|_q^{-1} (\log\lambda)^{1/p} \left\{ 1 + \frac{p^2}{(q+r-1)^3} \frac{\log(\log\lambda)}{\log\lambda} \bigl(1+o(1)\bigr) \right\} \| W_\lambda \|_\infty. \tag{1.17}$$

The rest of this paper is organized as follows. We prove Theorems 1.1, 1.2, and 1.3 in Sects. 2, 3, and 4, respectively. In the proofs, the time map method and the argument from [1] play important roles. Finally, we prove (1.5) and (1.6) in the Appendix.

2 Proof of Theorem 1.1

Let $\lambda > 0$ be fixed. We write $w_r(x)$ as a unique solution of

$$\begin{cases} -w''(x) = \lambda w(x)^r, & x \in I, \\ w(x) > 0, & x \in I, \\ w(0) = w(1) = 0. \end{cases} \tag{2.1}$$

We look for the solution $u_\lambda(x)$ of the form

$$u_\lambda(x) = t_\lambda w_r(x), \tag{2.2}$$

where $t_\lambda > 0$ is a constant. By (1.1) and (2.1), we have

$$-t_\lambda^p \| w_r' \|_p^p t_\lambda w_r''(x) = \lambda t_\lambda^q \| w_r' \|_q^q t_\lambda^r w_r(x)^r. \tag{2.3}$$

Since $w_r(x) = \lambda^{-1/(r-1)} W_r(x)$, we find from (2.3) that if

$$t_\lambda^{p-q-r+1} = \lambda^{(p-q)/(r-1)} \frac{\| W_r' \|_q^q}{\| W_r' \|_p^p}, \tag{2.4}$$

then (2.2) satisfies (1.1). By this, we have

$$t_\lambda = \lambda^{(p-q)/((p-q-r+1)(r-1))} \frac{\| W_r' \|_q^{q/(p-q-r+1)}}{\| W_r' \|_p^{p/(p-q-r+1)}}. \tag{2.5}$$

By this and (2.2), we have

$$\begin{aligned} u_\lambda(x) &= \lambda^{(p-q)/((p-q-r+1)(r-1))} \frac{\| W_r' \|_q^{q/(p-q-r+1)}}{\| W_r' \|_p^{p/(p-q-r+1)}} \lambda^{-1/(r-1)} W_r(x) \\ &= \lambda^{1/(p-q-r+1)} \frac{\| W_r' \|_q^{q/(p-q-r+1)}}{\| W_r' \|_p^{p/(p-q-r+1)}} W_r(x). \end{aligned} \tag{2.6}$$

This implies (1.7). Since $\alpha = \|u_\lambda\|_\infty = u_\lambda(1/2)$, we put $x = 1/2$ in (2.6). Then, we obtain

$$\alpha = \lambda^{(1/(p-q-r+1))} \frac{\|W_r'\|_q^{q/(p-q-r+1)}}{\|W_r'\|_p^{p/(p-q-r+1)}} \|W_r\|_\infty. \tag{2.7}$$

By this, we obtain (1.8). Thus, the proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

We first show the existence of u_λ. We follow the argument in [1]. Let $t > 0$ and $A(t) = e^t$. We consider the following equation for $t > 0$.

$$A(t) = \|w_r'\|_2^{1-r} t^{(r-1)/2}. \tag{3.1}$$

Assume that $t_\lambda > 0$ satisfies (3.1). We put $\gamma := t_\lambda^{1/2}\|w_r'\|_2^{-1}$ and $u_\lambda := \gamma w_r$. Then, we have

$$A\left(\|\gamma w_r'\|_2^2\right) = A(t_\lambda) = \gamma^{r-1}.$$

By this, we have

$$\begin{aligned} -A\left(\|u_\lambda'\|_2^2\right)u_\lambda''(x) &= -A\left(\|\gamma w_\lambda'\|_2^2\right)\gamma w_\lambda''(x) \\ &= \gamma^r \lambda w_\lambda(x)^r = \lambda u_\lambda(x)^r. \end{aligned} \tag{3.2}$$

Since $w_r = \lambda^{1/(1-r)} W_r$, we have

$$u_\lambda(x) = t_\lambda^{1/2}\|W_r'\|_2^{-1} W_r(x). \tag{3.3}$$

On the other hand, suppose that u_λ satisfies (3.2). Then by putting $t_\lambda = \|u_\lambda'\|_2^2$, we see that t_λ satisfies (3.1). Therefore, the number of the positive solutions t of the equation

$$e^t = \lambda\|W_r'\|_2^{1-r} t^{(r-1)/2} \tag{3.4}$$

coincide with the number of the solutions of (3.2). So, we solve the equation (3.1). To do this, for $r > 1$, we set

$$R_r := \frac{r-1}{2}\left\{1 - \log(r-1) + \log 2 + 2\log\|W_r'\|_2\right\}. \tag{3.5}$$

Lemma 3.1 *For $r > 1$, let $\lambda_r := e^{R_r}$.*

(i) *If $0 < \lambda < \lambda_r$, then (3.4) has no solution.*

(ii) *If $\lambda = \lambda_r$, then (3.4) has a unique solution t_0.*

(iii) *If $\lambda > \lambda_r$, then (3.4) has exactly two solutions $t_{\lambda,1}$, $t_{\lambda,2}$ with $0 < t_{\lambda,1} < t_0 < t_{\lambda,2}$.*

Proof We put $K_{\lambda,r} := \lambda\|W_r'\|_2^{1-r}$. By (3.4), we have the equation

$$t = g(t) := \frac{r-1}{2}\log t + \log K_{\lambda,r}. \tag{3.6}$$

Since $g'(t) = \frac{r-1}{2t}$, we see that $g'(t_0) = 1$, where $t_0 = \frac{r-1}{2}$. Then the tangent line of $y = g(t)$ at $(t_0, g(t_0))$ is $g(t) - g(t_0) = t - t_0$. We see from this that if $g(t_0) = t_0$, namely,

$$\frac{r-1}{2}\log\frac{r-1}{2} + \log K_{\lambda,r} = \frac{r-1}{2}, \tag{3.7}$$

then the tangent line of $g(t)$ at $t = t_0$ is exactly the line $y = t$. Equation (3.7) implies that

$$\log\lambda = R_r, \tag{3.8}$$

namely, $\lambda = e^{R_r}$. This implies (ii). Since $\log t$ is a concave function w.r.t. $t > 0$, and $\log K_{\lambda,r}$ is increasing function of $\lambda > 0$, if $0 < \lambda < e^{R_r}$ (resp. $\lambda > e^{R_r}$), then we obtain (i) and (iii), respectively. Thus, the proof is complete. □

Proof of Theorem 1.2 By Lemma 3.1, we obtain Theorem 1.2(i), (ii), and (iii). Now, we show (iv). Assume that $u_\lambda(x)$ is a solution of (1.1) for some $\lambda > 0$. We write $A = e^{\|u_\lambda'\|_2^2}$. By (1.1), we have

$$\left\{Au_\lambda''(x) + \lambda u_\lambda(x)^r\right\}u_\lambda'(x) = 0. \tag{3.9}$$

Recall that $\alpha := \|u_\lambda\|_\infty$. Then (3.9) implies that

$$\frac{1}{2}Au_\lambda'(x)^2 + \frac{1}{r+1}\lambda u_\lambda(x)^{r+1} = \frac{1}{r+1}\lambda\alpha^{r+1}. \tag{3.10}$$

We know that $u_\lambda(x)$ is a positive solution of $-u_\lambda''(x) = (\lambda/A)u_\lambda(x)^r$ with the condition $u_\lambda(0) = u_\lambda(1) = 0$. Therefore, by the result of Gidas, Ni, and Nirenberg [6], we know that $u_\lambda(x) = u_\lambda(1-x)$ $(0 \le x \le 1/2)$. By this, (3.10) implies that for $0 \le x \le 1/2$,

$$u_\lambda'(x) = \sqrt{\frac{2\lambda}{(r+1)A}}\sqrt{\alpha^{r+1} - u_\lambda(x)^{r+1}}. \tag{3.11}$$

By this, we have

$$\begin{aligned} \|u_\lambda'\|_2^2 &= 2\int_0^{1/2}\sqrt{\frac{2\lambda}{(r+1)A}}\sqrt{\alpha^{r+1} - u_\lambda(x)^{r+1}}u_\lambda'(x)\,dx \\ &= 2\int_0^{\alpha}\sqrt{\frac{2\lambda}{(r+1)A}}\sqrt{\alpha^{r+1} - \theta^{r+1}}\,d\theta \\ &= 2\sqrt{\frac{2\lambda}{(r+1)A}}\alpha^{(r+3)/2}\int_0^1\sqrt{1 - s^{r+1}}\,ds \\ &= 2\sqrt{\frac{2\lambda}{(r+1)A}}M_{r,2}\alpha^{(r+3)/2}. \end{aligned} \tag{3.12}$$

By this, we have

$$A\|u_\lambda'\|_2^4 = \frac{8\lambda}{r+1}M_{r,2}^2\alpha^{r+3}. \tag{3.13}$$

By (3.11), we have

$$\begin{aligned} \frac{1}{2} &= \int_0^{1/2} \frac{u_\lambda'(x)}{\sqrt{\frac{2\lambda}{(r+1)A}}\sqrt{\alpha^{r+1} - u_\lambda(x)^{r+1}}}\,dx \\ &= \sqrt{\frac{(r+1)A}{2\lambda}} \int_0^\alpha \frac{1}{\sqrt{\alpha^{r+1} - \theta^{r+1}}}\,d\theta \\ &= \sqrt{\frac{(r+1)A}{2\lambda}}\alpha^{(1-r)/2} \int_0^1 \frac{1}{\sqrt{1 - s^{r+1}}}\,ds \\ &= \sqrt{\frac{(r+1)A}{2\lambda}} L_r \alpha^{(1-r)/2}. \end{aligned} \tag{3.14}$$

By this, we have

$$A = \frac{\lambda}{2(r+1)L_r^2}\alpha^{r-1}. \tag{3.15}$$

By (3.13) and (3.15), we have

$$\|u_\lambda'\|_2^2 = 4M_{r,2}L_r\alpha^2. \tag{3.16}$$

By this and (3.15), we have

$$A = A\left(\|u_\lambda'\|_2^2\right) = A\left(4M_{r,2}L_r\alpha^2\right) = \frac{\lambda}{2(r+1)L_r^2}\alpha^{r-1}. \tag{3.17}$$

This implies that

$$\lambda = 2(r+1)L_r^2 A\left(4M_{r,2}L_r\alpha^2\right)\alpha^{1-r}. \tag{3.18}$$

Thus, the proof of (iv) is complete. □

Now, we prove Theorem 1.2(v).

Lemma 3.2 *As* $\lambda \to \infty$,

$$u_{1,\lambda}(x) = \lambda^{-1/(r-1)}\left\{1 + \frac{1}{r-1}\|W_r'\|_2^2\lambda^{-2/(r-1)}\left(1 + o(1)\right)\right\}\|W_r'\|_2^{-1}W_r(x), \tag{3.19}$$

$$u_{2,\lambda}(x) = \left\{\log\lambda + \frac{r-1}{2}\log(\log\lambda)\left(1 + o(1)\right)\right\}^{1/2}\|W_r'\|_2^{-1}W_r(x). \tag{3.20}$$

Proof We first prove (3.19). Since $t_0 = (r-1)/2$, by (3.6), we see that $t_{\lambda,1} \to 0$ and $t_{\lambda,2} \to \infty$ as $\lambda \to \infty$. By (3.4), we have

$$t_{\lambda,1} = \frac{r-1}{2}\log t_{\lambda,1} + \log\lambda - (r-1)\log\|W_r'\|_2. \tag{3.21}$$

This implies that

$$t_{\lambda,1} = \|W_r'\|_2^2\lambda^{-2/(r-1)}(1+\delta), \tag{3.22}$$

where $\delta \to 0$ as $\lambda \to \infty$. By this and (3.6), we have

$$\|W_r'\|_2 \lambda^{-2/(r-1)}(1+\delta) = \frac{r-1}{2}\left\{-\frac{2}{r-1}\log\lambda + 2\log\|W_r'\|_2^2 + \log(1+\delta)\right\} + \log\lambda - (r-1)\|W_r'\|_2. \tag{3.23}$$

By this and the Taylor expansion, we have

$$\|W_r'\|_2^2 \lambda^{-2/(r-1)} = \frac{r-1}{2}\bigl(1+o(1)\bigr)\log(1+\delta) = \frac{r-1}{2}\bigl(1+o(1)\bigr)\delta. \tag{3.24}$$

By this, we have

$$\delta = \frac{2}{r-1}\|W_r'\|_2^2\bigl(1+o(1)\bigr)\lambda^{-2/(r-1)}. \tag{3.25}$$

This implies that as $\lambda \to \infty$

$$\begin{aligned} t_{\lambda,1}^{1/2} &= \lambda^{-1/(r-1)}\left\{1 + \frac{2}{r-1}\|W_r'\|_2^2\bigl(1+o(1)\bigr)\lambda^{-2/(r-1)}\right\}^{1/2} \\ &= \lambda^{-1/(r-1)}\left\{1 + \frac{1}{r-1}\|W_r'\|_2^2\bigl(1+o(1)\bigr)\lambda^{-2/(r-1)}\right\}. \end{aligned} \tag{3.26}$$

By this and (3.3), we obtain (3.19). Now, we prove (3.20). Since $t_{\lambda,2} \to \infty$ as $\lambda \to \infty$, we have

$$t_{\lambda,2} = \frac{r-1}{2}\log t_{\lambda,2} + \log\lambda - (r-1)\log\|W_r'\|_2. \tag{3.27}$$

This implies that

$$t_{\lambda,2} = (1+\epsilon)\log\lambda, \tag{3.28}$$

where $\epsilon \to 0$ as $\lambda \to \infty$. By this and (3.21), we obtain

$$\begin{aligned} t_{\lambda,2} &= \log\lambda + \epsilon\log\lambda \\ &= \frac{r-1}{2}\bigl\{\log(1+\epsilon) + \log(\log\lambda)\bigr\} + \log\lambda - (r-1)\log\|W_r'\|_2. \end{aligned} \tag{3.29}$$

By this, we obtain

$$\epsilon = \frac{r-1}{2}\bigl(1+o(1)\bigr)\frac{\log(\log\lambda)}{\log\lambda}. \tag{3.30}$$

By this, (3.3) and (3.28), we obtain (3.20). Thus, the proof is complete. □

4 Proof of Theorem 1.3

Let $\lambda > 0$ be fixed. In what follows, C denotes various constants independent of λ. Following the idea of (3.1)–(3.3), we look for the solution of the form

$$u_\lambda(x) = tw_r(x) = t\lambda^{-1/(r-1)}W_r(x). \tag{4.1}$$

If (4.1) is the solution of (1.1) with $A(\|u_\lambda'\|_p^p) = e^{\|u_\lambda'\|_p^p}$ and $B(\|u_\lambda'\|_q^q) = \|u_\lambda'\|_q^q$, then we have

$$-\exp\bigl(t^p\lambda^{-p/(r-1)}\|W_r'\|_p^p\bigr)W_r''(x) = t^{q+r-1}\lambda^{-q/(r-1)}\|W_r'\|_q^q W_r(x)^r. \tag{4.2}$$

This implies that

$$\exp\bigl(t^p\lambda^{-p/(r-1)}\|W_r'\|_p^p\bigr) = t^{q+r-1}\lambda^{-q/(r-1)}\|W_r'\|_q^q. \tag{4.3}$$

We put $s := t^{q+r-1}$. By taking log of the both side of (4.3), we have

$$C_1 s^{p/(q+r-1)} = \log s + C_2, \tag{4.4}$$

where

$$C_1 := \lambda^{-p/(r-1)}\|W_r'\|_p^p, \qquad C_2 := -\frac{q}{r-1}\log\lambda + \log\|W_r'\|_q^q. \tag{4.5}$$

We put

$$g(s) := C_1 s^{p/(q+r-1)} - \log s - C_2. \tag{4.6}$$

We look for $s > 0$ satisfying $g(s) = 0$. To do this, we consider the graph of $g(s)$. We know that

$$g'(s) = \frac{p}{q+r-1}C_1 s^{(p-q-r+1)/(q+r-1)} - \frac{1}{s}. \tag{4.7}$$

By this, we find that $g'(s_0) = 0$, where

$$s_0 := \left(\frac{q+r-1}{pC_1}\right)^{(q+r-1)/p} = \left(\frac{q+r-1}{p\|W_r'\|_p^p}\right)^{(q+r-1)/p}\lambda^{(q+r-1)/(r-1)}. \tag{4.8}$$

By an elementary calculation, we see that if $0 < s < s_0$ (resp. $s > s_0$), then $g(s)$ is strictly decreasing (resp. strictly increasing) and $g(s_0)$ is the minimum value of $g(s)$. By (4.5), (4.6), (4.8), and direct calculation, we have

$$\begin{aligned} g(s_0) &= -\log\lambda + \frac{q+r-1}{p}\left(1 - \log\frac{q+r-1}{p} + \log\|W_r'\|_p^p - \frac{p}{q+r-1}\log\|W_r'\|_q^q\right) \\ &= -\log\lambda + C_0. \end{aligned} \tag{4.9}$$

We put $\lambda_1 := e^{C_0}$. Then, $g(s_0) > 0$ if $0 < \lambda < \lambda_1$, $g(s_0) = 0$ if $\lambda = \lambda_1$ and $g(s_0) < 0$ if $\lambda > \lambda_1$. Then, we see that if $0 < \lambda < \lambda_1$, then (4.6) (namely, (4.3) and (4.4)) has no solution, and if $\lambda = \lambda_1$, then (4.6) (namely, (4.3) and (4.4)) has a unique solution s_0, and if $\lambda > \lambda_1$, then (4.6) (namely, (4.3) and (4.4)) has exactly two solutions s_1, s_2 with $s_1 < s_0 < s_2$.

We see from the argument above that Theorem 1.3(i), (ii) hold. Moreover, let $t_{\lambda,j} := s_j^{1/(q+r-1)}$ $(j = 1, 2)$. By this and (4.1), we obtain Theorem 1.3(iii).

Now, we consider the case (iv). Since it is difficult to obtain $t_{\lambda,j} := s_j^{1/(q+r-1)}$ $(j = 1, 2)$ exactly, we first establish the asymptotic formula for $t_{\lambda,j}$ for $\lambda \gg 1$.

Lemma 4.1 *Assume that* $\lambda \gg 1$. *Then,*

$$t_{\lambda,2} = \| W_r' \|_q^{-1} \lambda^{1/(r-1)} (\log \lambda)^{1/p} \left\{ 1 + \frac{p^2}{(q+r-1)^3} \frac{\log(\log \lambda)}{\log \lambda} \big(1 + o(1)\big) \right\}. \tag{4.10}$$

Proof We put $s_{\lambda,2} := t_{\lambda,2}^{q+r-1}$. By (4.3), we have

$$\exp\big(s_{\lambda,2}^{p/(q+r-1)} \lambda^{-p/(r-1)} \| W_r' \|_p^p\big) = s_{\lambda,2} \lambda^{-q/(r-1)} \| W_r' \|_q^q. \tag{4.11}$$

By this, we have

$$\frac{q}{r-1} \log \lambda + s_{\lambda,2}^{p/(q+r-1)} \lambda^{-p/(r-1)} \| W_r' \|_p^p = \log s_{\lambda,2} + q \log \| W_r' \|_q. \tag{4.12}$$

Then, three cases should be considered.

Case 1. Assume that there exists a subsequence of $\{\lambda\}$, which is denoted by $\{\lambda\}$ again, such that as $\lambda \to \infty$,

$$\log \lambda \gg s_{\lambda,2}^{p/(q+r-1)} \lambda^{-p/(r-1)} \| W_r' \|_p^p. \tag{4.13}$$

Then, by this and (4.12), we have

$$\frac{q}{r-1} \big(1 + o(1)\big) \log \lambda = \log s_{\lambda,2}. \tag{4.14}$$

This implies that

$$s_{\lambda,2} = \lambda^{q/(r-1)} \big(1 + o(1)\big). \tag{4.15}$$

By this and (4.8), we have $s_0 > s_{\lambda,2}$. This is a contradiction.

Case 2. Assume that there exists a subsequence of $\{\lambda\}$, which is denoted by $\{\lambda\}$ again, such that as $\lambda \to \infty$,

$$\log \lambda \ll s_{\lambda,2}^{p/(q+r-1)} \lambda^{-p/(r-1)} \| W_r' \|_p^p. \tag{4.16}$$

Then, by (4.12), we have

$$s_{\lambda,2}^{p/(q+r-1)} \lambda^{-p/(r-1)} \| W_r' \|_p^p = \big(1 + o(1)\big) \log s_{\lambda,2}. \tag{4.17}$$

This implies that

$$\begin{aligned} s_{\lambda,2} &= \big(\| W_r' \|_p^{-p} \big(1 + o(1)\big) \lambda^{p/(r-1)} \log s_{\lambda,2} \big)^{(q+r-1)/p} \\ &= \| W_r' \|_p^{-(q+r-1)} \big(1 + o(1)\big) \lambda^{(q+r-1)/(r-1)} (\log s_{\lambda,2})^{(q+r-1)/p} \\ &= \| W_r' \|_p^{-(q+r-1)} \big(1 + o(1)\big) \lambda^{(q+r-1)/(r-1)} \left(\frac{q+r-1}{r-1} \log \lambda \right)^{(q+r-1)/p} \\ &= \left(\frac{q+r-1}{r-1} \right)^{(q+r-1)/p} \| W_r' \|_p^{-(q+r-1)} \lambda^{(q+r-1)/(r-1)} (\log \lambda)^{(q+r-1)/p} \big(1 + o(1)\big). \end{aligned} \tag{4.18}$$

By this, (4.12), and (4.18), we have

$$\frac{q}{r-1}\log\lambda + \frac{q+r-1}{r-1}\bigl(1+o(1)\bigr)\log\lambda = \frac{q+r-1}{r-1}\bigl(1+o(1)\bigr)\log\lambda. \tag{4.19}$$

This is a contradiction.

Case 3. Therefore, there exists a subsequence of $\{\lambda\}$, which is denoted by $\{\lambda\}$ again, such that as $\lambda \to \infty$,

$$C^{-1} < \frac{\log\lambda}{s_{\lambda,2}^{p/(q+r-1)}\lambda^{-p/(r-1)}\|W_r'\|_p^p} \le C. \tag{4.20}$$

By this and taking a subsequence of $\{\lambda\}$ again if necessary, we see that there exists a constant $C_4 > 0$ such that as $\lambda \to \infty$,

$$s_{\lambda,2} = C_4\lambda^{(q+r-1)/(r-1)}(\log\lambda)^{(q+r-1)/p}(1+\delta_1), \tag{4.21}$$

where $\delta_1 \to 0$ as $\lambda \to \infty$. By this and (4.12), we have

$$\begin{aligned}&\frac{q}{r-1}\log\lambda + s_\lambda^{p/(q+r-1)}\lambda^{-p/(r-1)}\|W_r'\|_p^p \\ &\quad= \log C_4 + \frac{q+r-1}{r-1}\log\lambda + \frac{q+r-1}{p}\log(\log\lambda) + \log(1+\delta_0) + q\log\|W_r'\|_q.\end{aligned} \tag{4.22}$$

This implies that

$$s_{\lambda,2}^{p/(q+r-1)}\lambda^{-p/(r-1)}\|W_r'\|_p^p = (1+\delta_1)\log\lambda, \tag{4.23}$$

where $\delta_1 \to 0$ as $\lambda \to \infty$. This implies that

$$s_{\lambda,2} = \|W_r'\|_p^{-(q+r-1)}\lambda^{(q+r-1)/(r-1)}(\log\lambda)^{(q+r-1)/p}(1+\delta_0). \tag{4.24}$$

Namely, $C_4 = \|W_r'\|_p^{-(q+r-1)}$. By (4.22) and (4.23), we have

$$\delta_1\log\lambda = \log C_4 + \frac{q+r-1}{p}\log(\log\lambda) + \log(1+\delta_0) + q\log\|W_r'\|_q. \tag{4.25}$$

By this, we have

$$\delta_1 = \frac{q+r-1}{p}\frac{\log(\log\lambda)}{\log\lambda}\bigl(1+o(1)\bigr). \tag{4.26}$$

By this, (4.23), and the Taylor expansion, we have

$$\begin{aligned}s_{\lambda,2} &= \|W_r'\|_p^{-(q+r-1)}\lambda^{(q+r-1)/(r-1)}(\log\lambda)^{(q+r-1)/p}(1+\delta_1)^{(q+r-1)/p} \\ &= \|W_r'\|_p^{-(q+r-1)}\lambda^{(q+r-1)/(r-1)}(\log\lambda)^{(q+r-1)/p} \\ &\quad\times\left\{1+\left(\frac{p}{q+r-1}\right)^2\frac{\log(\log\lambda)}{\log\lambda}\bigl(1+o(1)\bigr)\right\}.\end{aligned} \tag{4.27}$$

Indeed, we see from (4.27) that $s_0 < s_{\lambda,2}$. Therefore, by (4.27), we obtain (4.10). Thus, the proof is complete. □

Lemma 4.2 *Assume that* $\lambda \gg 1$. *Then*

$$t_{1,\lambda} = \lambda^{q/((r-1)(q+r-1))} \|W_r'\|_q^{-q/(q+r-1)} \times \left\{1 + \frac{1}{q+r-1}\|W_\lambda'\|_p^p \|W_\lambda'\|_q^{pq/(q+r-1)} \lambda^{-p/(q+r-1)}\bigl(1+o(1)\bigr)\right\}. \tag{4.28}$$

Proof Since $s_{\lambda,1} < s_0$, we find from Lemma 4.1 that as $\lambda \to \infty$,

$$s_{\lambda,1}^{p/(q+r-1)} \lambda^{-p/(r-1)} \|W_r'\|_p^p \ll \log \lambda. \tag{4.29}$$

By this and (4.12), we have

$$\bigl(1+o(1)\bigr)\frac{q}{r-1}\log\lambda = \log s_\lambda. \tag{4.30}$$

This implies that

$$s_{\lambda,1} = \lambda^{q/(r-1)(1+o(1))}. \tag{4.31}$$

By this and (4.8), we see that $s = s_{\lambda,1}$ is determined by (4.31). By (4.31), we have

$$s_{\lambda,1}^{p/(q+r-1)} \lambda^{-p/(r-1)} \|W_r'\|_p^p \le C s_1^{-p(r-1)(1+o(1))/(q(q+r-1))} \to 0 \tag{4.32}$$

as $\lambda \to \infty$. Now, we calculate s_1. By (4.30) and (4.12), we have

$$\frac{q}{r-1}\log\lambda = \log s_{\lambda,1} + \bigl(1+o(1)\bigr)q\log\|W_r'\|_q. \tag{4.33}$$

By this, for $\lambda \gg 1$, we have

$$\lambda = \|W_r'\|_q^{r-1} s_1^{(r-1)/q}(1+\eta), \tag{4.34}$$

where $\eta \to 0$ as $\lambda \to \infty$. By this, (4.12), and the Taylor expansion, we have

$$\frac{q}{r-1}\bigl(1+o(1)\bigr)\eta + s_{\lambda,1}^{p/(q+r-1)}\lambda^{-p/(r-1)}\|W_r'\|_p^p = 0. \tag{4.35}$$

By this and (4.34), we have

$$\eta = -\frac{r-1}{q}\|W_\lambda'\|_p^p \|W_\lambda'\|_q^{-pq/(q+r-1)}\lambda^{-p/(q+r-1)}\bigl(1+o(1)\bigr). \tag{4.36}$$

By this, (4.20), and the Taylor expansion, we have

$$\begin{aligned} s_{\lambda,1} &= \lambda^{q/(r-1)}\|W_r'\|_q^{-q}\left(1 - \frac{q}{r-1}\eta\right) \\ &= \lambda^{q/(r-1)}\|W_r'\|_q^{-q}\bigl\{1 + \|W_\lambda'\|_p^p\|W_\lambda'\|_q^{-pq/(q+r-1)}\lambda^{-p/(q+r-1)}\bigl(1+o(1)\bigr)\bigr\}. \end{aligned} \tag{4.37}$$

By this, we have

$$t_{\lambda,1} = \lambda^{q/((r-1)(q+r-1))} \| W_r' \|_q^{-q/(q+r-1)} \times \left\{ 1 + \frac{1}{q+r-1} \| W_\lambda' \|_p^p \| W_\lambda' \|_q^{-pq/(q+r-1)} \lambda^{-p/(q+r-1)} \bigl(1+o(1)\bigr) \right\}. \tag{4.38}$$

Thus, we obtain (4.28). □

Proof of Theorem 1.3 By Lemma 4.2, for $\lambda \gg 1$, we obtain

$$u_{1,\lambda}(x) = \lambda^{-1/(q+r-1)} \| W_r' \|_q^{-q/(q+r-1)} \times \left\{ 1 + \frac{1}{q+r-1} \| W_\lambda' \|_p^p \| W_\lambda' \|_q^{pq/(q+r-1)} \lambda^{-p/(q+r-1)} \bigl(1+o(1)\bigr) \right\} W_r(x). \tag{4.39}$$

This implies (1.14). Further, by Lemma 4.1, we obtain

$$u_{2,\lambda}(x) = \| W_r' \|_q^{-1} (\log \lambda)^{1/p} \left\{ 1 + \frac{p^2}{(q+r-1)^3} \frac{\log(\log \lambda)}{\log \lambda} \bigl(1+o(1)\bigr) \right\} W_r(x). \tag{4.40}$$

This implies (1.16). To obtain (1.15) and (1.17), we just put $x = 1/2$ in (4.39) and (4.40). Thus, the proof is complete. □

Appendix

Let $r > 1$. We first show (1.6), which was proved in [14] for completeness. We apply the time map argument to (1.3), cf. [12]. Since (1.3) is autonomous, we have

$$W_r(x) = W_r(1-x), \quad x \in [0, 1/2], \tag{A.1}$$

$$W_r'(x) > 0, \quad x \in [0, 1/2), \tag{A.2}$$

$$\xi := \| W_r \|_\infty = \max_{0 \le x \le 1} W_r(x) = W_r(1/2). \tag{A.3}$$

By (1.3), for $0 \le x \le 1$, we have

$$\left\{ W_r''(x) + W_r(x)^r \right\} W_r'(x) = 0. \tag{A.4}$$

By this and (A.3), we have

$$\frac{1}{2} W_r'(x)^2 + \frac{1}{r+1} W_r(x)^{r+1} = \text{constant} = \frac{1}{r+1} W_p(1/2)^{r+1} = \frac{1}{r+1} \xi^{r+1}. \tag{A.5}$$

By this and (A.2), for $0 \le x \le 1/2$, using $\theta = \xi s$, we have

$$W_r'(x) = \sqrt{\frac{2}{r+1} \bigl(\xi^{r+1} - W_r(x)^{r+1} \bigr)}. \tag{A.6}$$

By (A.1) and (A.6), we have

$$\frac{1}{2} = \int_0^{1/2} 1\,dx = \int_0^{1/2} \frac{W_r'(x)}{\sqrt{\frac{2}{r+1}(\xi^{r+1} - W_r(x)^{r+1})}}\,dx \tag{A.7}$$

$$= \sqrt{\frac{r+1}{2}} \int_0^{\xi} \frac{1}{\sqrt{\xi^{r+1} - \theta^{r+1}}}\,d\theta$$

$$= \sqrt{\frac{r+1}{2}}\xi^{(1-r)/2} \int_0^1 \frac{1}{\sqrt{1 - s^{r+1}}}\,ds$$

$$= \sqrt{\frac{r+1}{2}}\xi^{(1-r)/2} L_r.$$

By this, we have

$$\xi = \bigl(2(r+1)\bigr)^{1/(r-1)} L_r^{2/(r-1)}. \tag{A.8}$$

This implies (1.6). We next show (1.5). By (A.1), (A.3), and (A.6), we have

$$\|W_r'\|_m^m = 2\int_0^{1/2} W_r'(x)^{m-1} W_r'(x)\,dx \tag{A.9}$$

$$= 2\left(\frac{2}{r+1}\right)^{(m-1)/2} \int_0^{1/2} \bigl(\xi^{r+1} - W_r(x)^{r+1}\bigr)^{(m-1)/2} W_r'(x)\,dx$$

$$= 2^{(m+1)/2}(r+1)^{-(m-1)/2} \int_0^{\xi} \bigl(\xi^{r+1} - \theta^{r+1}\bigr)^{(m-1)/2}\,d\theta$$

$$= 2^{(m+1)/2}(r+1)^{-(m-1)/2} \xi^{(m-1)(r+1)/2+1} \int_0^1 \bigl(1 - s^{r+1}\bigr)^{(m-1)/2}\,ds$$

$$= 2^{mr/(r-1)}(r+1)^{m/(r-1)} L_r^{(mr+m-r+1)/(r-1)} M_{r,m}.$$

This implies (1.5).

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