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Existence of infinitely many solutions of nonlinear fourth-order discrete boundary value problems

Yanshan Chen[1,2] and Zhan Zhou[1,2*]

*Correspondence: zzhou0321@hotmail.com
[1]School of Mathematics and Information Science, Guangzhou University, Guangzhou, P.R. China
[2]Guangzhou Center for Applied Mathematics, Guangzhou University, Guangzhou, P.R. China

Abstract

The fourth-order discrete Dirichlet boundary value problem is also a discrete elastic beam problem. In this paper, the existence of infinitely many solutions to this problem is investigated through the critical point theory. By an important inequality we established and the oscillatory behavior of f either near the origin or at infinity, we obtain the existence of infinitely many solutions, which either converge to zero or unbounded. In the end, two examples are presented to illustrate our results.

Keywords: Infinitely many solutions; Fourth order; Discrete boundary value problem; Critical point theory

1 Introduction

Let $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of integers and real numbers, respectively. Define $\mathbb{Z}(a) = \{a, a+1, \dots\}$ and $\mathbb{Z}(a,b) = \{a, a+1, \dots, b\}$ for any $a, b \in \mathbb{Z}$ with $a \le b$.

In this paper, we consider the following nonlinear fourth-order difference equation:

$$\triangle^2\bigl(p_{k-2}\triangle^2 u_{k-2}\bigr) = \lambda f(k, u_k), \quad k \in \mathbb{Z}(1,T), \tag{1.1}$$

with the Dirichlet boundary value conditions

$$u_{-1} = u_0 = u_{T+1} = u_{T+2} = 0, \tag{1.2}$$

where T is a given positive integer, $\triangle$ is the forward difference operator defined by $\triangle u_k = u_{k+1} - u_k$, $\triangle^2 u_k = \triangle(\triangle u_k)$, $p_k > 0$ for all $k \in \mathbb{Z}(-1,T)$, $f : \mathbb{Z}(1,T) \times \mathbb{R} \to \mathbb{R}$ is continuous in the second variable.

Boundary value problem (1.1) with (1.2) can be regarded as a discrete analogue of the following fourth-order boundary value problem:

$$\begin{cases} (p(t)u''(t))'' = \lambda f(t, u(t)), & t \in [0, l], \\ u(0) = u(1) = u'(0) = u'(l) = 0. \end{cases} \tag{1.3}$$

This problem gives the equilibrium state of a beam under simple bearing forces at both ends [1, 2]. In the mechanics of materials, the deformation of an elastic beam is usually modeled by the fourth-order problem (1.3) and some of its variants. For such issues, Agarwal [3] and Aftabizadeh [4] discussed the existence and uniqueness of solutions, Bonanno studied the multiplicity of solutions [5], and Graef et al. explored the existence of positive solutions [6].

In recent years, due to the wide applications of difference equations [7–9], the discrete elastic beam problems have attracted extensive attention of scholars. The methods include the fixed point theorem [10], invariant sets of descending flow [11], bifurcation techniques [12], etc. In 2003, the critical point theory was first used to prove the existence of periodic and subharmonic solutions of second-order difference equations [13]. Since then this method has been widely used to discuss periodic solutions [14], homoclinic solutions [15–17], and boundary value problems [18–23] for difference equations. In particular, the critical point theory is also used for boundary value problems of fourth-order difference equations [14, 23, 24]. Among them, Cai et al. obtained some sufficient conditions for the existence of at least two nontrivial solutions of the boundary value problem (1.1) with (1.2) for $\lambda = 1$ in [14].

In addition, He and Yu discussed the fourth-order difference equation

$$\triangle^4 u_{k-2} = \lambda a_k g(u_k), \quad k \in \mathbb{Z}(2, T+2), \tag{1.4}$$

with the following boundary value conditions:

$$u_0 = \Delta^2 u_0 = u_{T+2} = \Delta^2 u_T = 0, \tag{1.5}$$

where $a_k > 0$ for any $k \in \mathbb{Z}(2, T+2)$ in [20]. It is clear that (1.4) is a special case of (1.1) when $p_k \equiv 1$ for $k \in \mathbb{Z}(-1, T)$ and f with the form $f(k,u) = a_k g(u)$. By using the fixed point theorem, the existence of positive solutions to the boundary value problem (1.4) with (1.5) is obtained.

This paper aims to establish the existence results of infinite solutions to the boundary value problem (1.1) with (1.2) by the critical point theorem. To this end, we first construct a function space E and establish an important inequality between two norms in E, then, through the oscillation of nonlinear function f at the origin and at infinity, we obtain sufficient conditions for the existence of infinitely many solutions to the elastic beam problem (1.1) with (1.2).

The rest of this article is organized as follows. In Sect. 2, we establish a variational functional J_λ corresponding to the elastic beam problem (1.1) with (1.2) on the function space E. And we find that the critical points of J_λ are actually solutions to problem (1.1) with (1.2). Furthermore, we construct an inequality that plays an important role in proving our main results. The sufficient conditions for the existence of infinite solutions to problem (1.1) with (1.2) are established and proved in Sect. 3. In Sect. 4, we give two examples to illustrate the rationality and applicability of our conclusions.

2 Preliminaries

In this section, we first establish the variational framework associated with problem (1.1) with (1.2). We consider the T-dimensional Banach space

$$E = \{u : \mathbb{Z}(-1, T+2) \to \mathbb{R} : u_{-1} = u_0 = u_{T+1} = u_{T+2} = 0\}$$

endowed with the norm

$$\|u\| := \left(\sum_{k=0}^{T+1} (\Delta^2 u_{k-1})^2\right)^{\frac{1}{2}}. \tag{2.1}$$

For each $u \in E$, define

$$\Phi(u) = \frac{1}{2}\sum_{k=0}^{T+1} p_{k-1}(\Delta^2 u_{k-1})^2, \qquad \Psi(u) = \sum_{k=1}^{T} F(k, u_k),$$

where

$$F(k,u) := \int_0^u f(k,t)\,dt.$$

Define the functional J_λ on E as $J_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for any $u \in E$. Clearly, Φ, $\Psi \in C^1(E,\mathbb{R})$, and we have

$$\begin{aligned}
\langle J_\lambda'(u), v\rangle &= \lim_{t\to 0} \frac{J_\lambda(u+tv) - J_\lambda(u)}{t} \\
&= \frac{dJ_\lambda(u+tv)}{dt}\bigg|_{t=0} \\
&= \frac{d}{dt}\left(\frac{1}{2}\sum_{k=0}^{T+1} p_{k-1}\big(\Delta^2(u_{k-1} + tv_{k-1})\big)^2 - \lambda \sum_{k=1}^{T} F(k, u_k + tv_k)\right)\bigg|_{t=0} \\
&= \sum_{k=0}^{T+1} p_{k-1}\Delta^2 u_{k-1}\Delta^2 v_{k-1} - \lambda\sum_{k=1}^{T} f(k,u_k)v_k \\
&= \big(p_{k-2}\Delta^2 u_{k-2}\Delta v_{k-1}\big)\big|_0^{T+2} - \sum_{k=1}^{T} \Delta\big(p_{k-2}\Delta^2 u_{k-2}\big)\Delta v_{k-1} - \lambda\sum_{k=1}^{T} f(k,u_k)v_k \\
&= \sum_{k=1}^{T}\big(\Delta^2\big(p_{k-2}\Delta^2 u_{k-2}\big) - \lambda f(k,u_k)\big)v_k
\end{aligned}$$

for any $u, v \in E$. This shows that critical points of functional J_λ are solutions to the boundary value problem (1.1) with (1.2).

Now we present the following result obtained by Ricceri in [25], which will be used to find the critical points of the problem (1.1) with (1.2).

Lemma 2.1 *Let E be a real reflexive Banach space. For any $x \in E$, $J_\lambda(x) = \Phi(x) - \lambda\Psi(x)$, where $\lambda \in \mathbb{R}^+$ and Ψ, $\Phi \in C^1(E,\mathbb{R})$ with Φ coercive, that is, $\lim_{\|x\|\to+\infty}\Phi(x) = +\infty$.*

Assume that $\inf_E \Phi < r$, *let*

$$\alpha = \liminf_{r\to+\infty} \phi(r), \qquad \beta = \liminf_{r\to(\inf_E \Phi)^+} \phi(r),$$

where

$$\phi(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{x\in\Phi^{-1}(-\infty,r)} \Psi(x)) - \Psi(u)}{r - \Phi(u)}.$$

When $\alpha = 0$ (*or* $\beta = 0$), *in the sequel, we agree to read* $1/\alpha$ (*or* $1/\beta$) *as* $+\infty$.

(I) *If* $\alpha < +\infty$, *then for each* $\lambda \in (0, \frac{1}{\alpha})$ *the following alternatives hold: either*

(I_1) J_λ *possesses a global minimum or*

(I_2) *there is a sequence* $\{u_n\}$ *of critical points of* J_λ *such that* $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(H) *If* $\beta < +\infty$, *then for each* $\lambda \in (0, \frac{1}{\beta})$ *the following alternatives hold: either*

(H_1) *there is a global minimum of* Φ, *which is a local minimum of* J_λ, *or*

(H_2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points of* J_λ *with* $\lim_{n\to+\infty} \Phi(u_n) = \inf_E \Phi$, *which weakly converges to a global minimum of* Φ.

Now we give the following inequality, which plays an important role in the proof of our main results.

Lemma 2.2 *For any* $u \in E$, *we have*

$$\max_{k\in\mathbb{Z}(1,T)} \{|u_k|\} \le \frac{(T+1)\sqrt{T+3}}{4\sqrt{2}} \|u\|. \tag{2.2}$$

Proof Let $\tau \in \mathbb{Z}(1,T)$ be such that

$$|u_\tau| = \max_{k\in\mathbb{Z}(1,T)} \{|u_k|\}.$$

Noticing $u_{-1} = u_0 = 0$, we have

$$u_\tau = \sum_{k=1}^{\tau} \Delta u_{k-1} = \sum_{k=1}^{\tau}\sum_{j=1}^{k} \Delta^2 u_{j-2}.$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |u_\tau| &\le \sum_{k=1}^{\tau}\sum_{j=1}^{k} |\Delta^2 u_{j-2}| \\ &\le \left(\frac{\tau(\tau+1)}{2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\tau}\sum_{j=1}^{k} (\Delta^2 u_{j-2})^2\right)^{\frac{1}{2}} \\ &\le \left(\frac{\tau+1}{2}\right)^{\frac{1}{2}} \tau \left(\sum_{k=1}^{\tau} (\Delta^2 u_{k-2})^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Similarly, by the fact that $u_{T+1} = u_{T+2} = 0$, we have

$$|u_\tau| \le \left(\frac{T-\tau+2}{2}\right)^{\frac{1}{2}} (T-\tau+1) \left(\sum_{k=\tau}^{T} (\Delta^2 u_k)^2\right)^{\frac{1}{2}}. \tag{2.4}$$

If

$$\frac{(\tau+1)\tau^2}{2} \sum_{k=1}^{\tau} (\Delta^2 u_{k-2})^2 \le \frac{(T+1)^2(T+3)}{32} \sum_{k=1}^{T+2} (\Delta^2 u_{k-2})^2,$$

then Lemma 2.1 holds. Otherwise,

$$\frac{(\tau+1)\tau^2}{2} \sum_{k=1}^{\tau} (\Delta^2 u_{k-2})^2 > \frac{(T+1)^2(T+3)}{32} \sum_{k=1}^{T+2} (\Delta^2 u_{k-2})^2.$$

Then

$$\sum_{k=1}^{\tau} (\Delta^2 u_{k-2})^2 > \frac{(T+1)^2(T+3)}{16(\tau+1)\tau^2} \sum_{k=1}^{T+2} (\Delta^2 u_{k-2})^2$$

and

$$\sum_{k=\tau+1}^{T+2} (\Delta^2 u_{k-2})^2 \le \left(1 - \frac{(T+1)^2(T+3)}{16(\tau+1)\tau^2}\right) \sum_{k=1}^{T+2} (\Delta^2 u_{k-2})^2.$$

By (2.4), we have

$$|u_\tau|^2 \le \frac{(T-\tau+2)}{2} (T-\tau+1)^2 \left(1 - \frac{(T+1)^2(T+3)}{16(\tau+1)\tau^2}\right) \|u\|^2.$$

We now show that

$$\frac{(T-\tau+2)}{2} (T-\tau+1)^2 \left(1 - \frac{(T+1)^2(T+3)}{16(\tau+1)\tau^2}\right) \le \frac{(T+1)^2(T+3)}{32}.$$

In fact, we consider the function $v : [1, T] \to \mathbb{R}$ given by

$$v(s) = \frac{1}{s^2(s+1)} + \frac{1}{(T-s+1)^2(T-s+2)}.$$

Since

$$v'(s) = -\frac{3s^2+2s}{s^4(s+1)^2} + \frac{3(T-s+1)^2 + 2(T-s+1)}{(T-s+1)^4(T-s+2)^2}$$

is increasing in $[1, T]$, and we see that there exists unique $s = \frac{T+1}{2}$ such that

$$v'\left(\frac{T+1}{2}\right) = 0, \quad \text{and}$$

$$v'(s) < 0 \quad \text{for } s \in \left[1, \frac{T+1}{2}\right), \qquad v'(s) > 0 \quad \text{for } s \in \left(\frac{T+1}{2}, T\right].$$

Therefore, v attains its minimum at $s = \frac{T+1}{2}$, that is,

$$\frac{1}{s^2(s+1)} + \frac{1}{(T-s+1)^2(T-s+2)} \geq \frac{2}{(\frac{T+1}{2})^2(\frac{T+1}{2}+1)} = \frac{16}{(T+1)^2(T+3)}$$

for $s \in [1, T]$. Since $\tau \in \mathbb{Z}(1, T)$, we have

$$\frac{1}{\tau^2(\tau+1)} + \frac{1}{(T-\tau+1)^2(T-\tau+2)} \geq \frac{16}{(T+1)^2(T+3)},$$

which is the same as (2.2). □

3 Main results

In this section, we give our main results. Let

$$\mu = \limsup_{x\to+\infty} \frac{\sum_{k=1}^{T} F(k,x)}{x^2} \tag{3.1}$$

and

$$p_* = \min\{p_k, k \in \mathbb{Z}(1,T)\}, \qquad p^* = \max\{p_k, k \in \mathbb{Z}(1,T)\}.$$

We have the following result.

Theorem 3.1 *Suppose that there are two real sequences $\{\omega_n\}$, $\{c_n\}$ with $\omega_n > 0$ and $\lim_{n\to+\infty} \omega_n = +\infty$ such that*

$$\frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)c_n^2 < \frac{16 p_* \omega_n^2}{(T+1)^2(T+3)} \quad \text{for } n \in \mathbb{Z}(1) \tag{3.2}$$

and

$$\rho < \frac{2\mu}{p_{-1} + p_0 + p_{T-1} + p_T}, \tag{3.3}$$

where

$$\rho = \liminf_{n\to\infty} \frac{\sum_{k=1}^{T} \max_{|x|\leq\omega_n} F(k,x) - \sum_{k=1}^{T} F(k,c_n)}{\frac{16 p_* \omega_n^2}{(T+1)^2(T+3)} - \frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)c_n^2}.$$

Then, for each $\lambda \in (\frac{p_{-1}+p_0+p_{T-1}+p_T}{2\mu}, \frac{1}{\rho})$, problem (1.1) with (1.2) admits an unbounded sequence of solutions.

Proof It is obvious that

$$\lim_{\|u\|\to+\infty} \Phi(u) = \lim_{\|u\|\to+\infty} \frac{1}{2} \sum_{k=0}^{T+1} p_{k-1}\left(\Delta^2 u_{k-1}\right)^2 \geq \lim_{\|u\|\to+\infty} \frac{p_*}{2} \|u\|^2 = +\infty,$$

which means that $\Phi(u)$ is coercive.

Define

$$r_n = \frac{16p_*\omega_n^2}{(T+1)^2(T+3)}.$$

If $u \in E$ and $\Phi(u) < r_n$, then we have the following inequality:

$$\frac{1}{2}p_* \sum_{k=0}^{T+1} \left(\Delta^2 u_{k-1}\right)^2 < r_n.$$

Considering Lemma 2.2, for any $k \in \mathbb{Z}(1,T)$, we have

$$|u_k|^2 \leq \frac{(T+1)^2(T+3)}{32} \sum_{k=0}^{T+1} \left(\Delta^2 u_{k-1}\right)^2 < \omega_n^2.$$

Furthermore, according to the definition of ϕ, we have

$$\phi(r_n) \leq \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\sum_{k=1}^{T} \max_{|x|\leq\omega_n} F(k,x) - \sum_{k=1}^{T} F(k,u_k)}{\frac{16p_*\omega_n^2}{(T+1)^2(T+3)} - \Phi(u)}. \tag{3.4}$$

For any $n \in \mathbb{Z}(1)$, take $(q_n)_k = c_n$ for $k \in \mathbb{Z}(1,T)$ and $(q_n)_{-1} = (q_n)_0 = (q_n)_T = (q_n)_{T+1} = 0$, then $q_n \in E$ and

$$\Phi(q_n) = \frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)c_n^2 \leq r_n$$

by exploiting (3.2). Therefore, from (3.4), we have

$$\begin{aligned}\phi(r_n) &\leq \frac{\sum_{k=1}^{T} \max_{|x|\leq\omega_n} F(k,x) - \sum_{k=1}^{T} F(k,(q_n)_k)}{\frac{16p_*\omega_n^2}{(T+1)^2(T+3)} - \Phi(q_n)} \\ &= \frac{\sum_{k=1}^{T} \max_{|x|\leq\omega_n} F(k,x) - \sum_{k=1}^{T} F(k,c_n)}{\frac{16p_*\omega_n^2}{(T+1)^2(T+3)} - \frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)c_n^2}.\end{aligned}$$

Moreover, combining (3.3), it is clear that $\alpha \leq \liminf_{n\to+\infty} \phi(r_n) \leq \rho < +\infty$.

We assert that J_λ is unbounded from below. In fact, when $\mu < +\infty$, since

$$2\lambda\mu > p_{-1} + p_0 + p_{T-1} + p_T,$$

there exists $\varepsilon_0 > 0$ such that

$$2\lambda(\mu - \varepsilon_0) > p_{-1} + p_0 + p_{T-1} + p_T.$$

From (3.1), we know that there exists a positive sequence $\{a_n\}$ with $\lim_{n\to+\infty} a_n = +\infty$ such that

$$\sum_{k=1}^{T} F(k,a_n) \geq (\mu - \varepsilon_0)a_n^2.$$

For each $n \in \mathbb{Z}(1)$, define $\upsilon_n \in E$ with $(\upsilon_n)_k = a_n$ for $k \in \mathbb{Z}(1, T)$, then we have the following inequality:

$$\begin{aligned} J_\lambda(\upsilon_n) &= \frac{1}{2}\sum_{k=0}^{T+1} p_{k-1}\bigl(\Delta^2(\upsilon_n)_{k-1}\bigr)^2 - \lambda\sum_{k=1}^{T} F\bigl(k, (\upsilon_n)_k\bigr) \\ &\le \frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)a_n^2 - \lambda(\mu - \varepsilon_0)a_n^2 \\ &= \frac{1}{2}\bigl(p_{-1} + p_0 + p_{T-1} + p_T - 2\lambda(\mu - \varepsilon_0)\bigr)a_n^2. \end{aligned} \tag{3.5}$$

The above inequality implies $\lim_{n\to+\infty} J_\lambda(\upsilon_n) = -\infty$. If $\mu = +\infty$, it can be seen that there is a sequence of positive number $\{\bar{a}_n\}$ with $\lim_{n\to+\infty} \bar{a}_n = +\infty$ such that

$$\sum_{k=1}^{T} F(k, \bar{a}_n) \ge \frac{p_{-1} + p_0 + p_{T-1} + p_T}{\lambda}\bar{a}_n^2$$

from the definition of μ. Define $\bar{\upsilon}_n \in E$ as $(\bar{\upsilon}_n)_k = \bar{a}_n$ for $k \in \mathbb{Z}(1, T)$, then

$$\begin{aligned} J_\lambda(\bar{\upsilon}_n) &= \frac{1}{2}\sum_{k=0}^{T+1} p_{k-1}\bigl(\Delta^2(\bar{\upsilon}_n)_{k-1}\bigr)^2 - \lambda\sum_{k=1}^{T} F\bigl(k, (\bar{\upsilon}_n)_k\bigr) \\ &\le -\frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)\bar{a}_n^2 \to -\infty \quad \text{as } n \to +\infty. \end{aligned} \tag{3.6}$$

By combining (3.5) with (3.6), we can conclude that condition (I_1) of Lemma 2.1 does not hold. Therefore, the functional J_λ has a sequence of critical points with $\lim_{n\to+\infty} \Phi(u_n) = +\infty$, which means that the problem (1.1) with (1.2) admits an unbounded sequence of solutions. □

Corollary 3.2 *If there is a sequence of positive numbers $\{\tilde{\omega}_n\}$ with $\tilde{\omega}_n \to +\infty$ as $n \to +\infty$ such that*

$$\tilde{\rho} < \frac{2\mu}{p_{-1} + p_0 + p_{T-1} + p_T}, \tag{3.7}$$

where

$$\tilde{\rho} = \liminf_{n\to\infty} \frac{(T+1)^2(T+3)\sum_{k=1}^{T} \max_{|x|\le\tilde{\omega}_n} F(k, x)}{16p_*\tilde{\omega}_n^2},$$

then, for each $\lambda \in (\frac{p_{-1}+p_0+p_{T-1}+p_T}{2\mu}, \frac{1}{\tilde{\rho}})$, problem (1.1) with (1.2) admits an unbounded sequence of nontrivial solutions.

Proof Taking $c_n = 0$ for all $n \in \mathbb{Z}(1)$, it can be easily proved by Theorem 3.1. □

In particular, if the nonlinear function f in (1.1) with the form $f(k, u) = a_k g(u)$, where $a_k > 0$ for $k \in \mathbb{Z}(1, T)$, and $p_k \equiv 1$ for $k \in \mathbb{Z}(-1, T)$. Then (1.1) reads

$$\Delta^2\bigl(p_{k-2}\Delta^2 u_{k-2}\bigr) = \lambda a_k g(u_k), \quad k \in \mathbb{Z}(1, T). \tag{3.8}$$

Define

$$\bar{\mu} = \limsup_{x\to+\infty} \frac{\bar{G}(x)}{x^2},$$

where

$$\bar{G}(x) = \int_0^x g(s)\,ds.$$

Then we have the following.

Corollary 3.3 *Suppose that there are two real sequences* $\{\bar{\omega}_n\}$, $\{\bar{c}_n\}$ *with* $\bar{\omega}_n > 0$ *and* $\lim_{n\to+\infty} \bar{\omega}_n = +\infty$ *such that*

$$\frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)\bar{c}_n^2 < \frac{16 p_* \bar{\omega}_n^2}{(T+1)^2(T+3)} \quad \textit{for } n \in \mathbb{Z}(1) \tag{3.9}$$

and

$$\bar{\rho} < \frac{2\bar{\mu}}{p_{-1} + p_0 + p_{T-1} + p_T}, \tag{3.10}$$

where

$$\bar{\rho} = \liminf_{n\to\infty} \frac{\max_{|x|\le\bar{\omega}_n} \bar{G}(x) - \bar{G}(\bar{c}_n)}{\frac{16 p_* \bar{\omega}_n^2}{(T+1)^2(T+3)} - \frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)\bar{c}_n^2}.$$

Then, for each $\lambda \in \frac{1}{\sum_{k=1}^T a_k}(\frac{p_{-1}+p_0+p_{T-1}+p_T}{2\bar{\mu}}, \frac{1}{\bar{\rho}})$, *problem* (3.8) *with* (1.2) *admits an unbounded sequence of nontrivial solutions.*

Now, we discuss the existence of infinitely many solutions to the boundary value problem (1.1) with (1.2) by using the oscillatory behavior of the nonlinear function at the origin.

Theorem 3.4 *Suppose that there are two real sequences* $\{z_n\}$ *and* $\{\bar{z}_n\}$, *where* $\bar{z}_n > 0$ *and* $\lim_{n\to+\infty} \bar{z}_n = 0$, *such that*

$$\frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)z_n^2 < \frac{16 p_* \bar{z}_n^2}{(T+1)^2(T+3)} \quad \text{for } n \in \mathbb{Z}(1) \tag{3.11}$$

and

$$\varrho < \frac{2\mu}{p_{-1} + p_0 + p_{T-1} + p_T}, \tag{3.12}$$

where

$$\varrho = \liminf_{n\to\infty} \frac{\sum_{k=1}^T \max_{|x|\le\bar{z}_n} F(k,x) - \sum_{k=1}^T F(k,z_n)}{\frac{16 p_* \bar{z}_n^2}{(T+1)^2(T+3)} - \frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T)z_n^2}.$$

Then, for each $\lambda \in (\frac{p_{-1}+p_0+p_{T-1}+p_T}{2\mu}, \frac{1}{\varrho})$, *problem* (1.1) *with* (1.2) *has a sequence of nontrivial solutions that converges to* 0.

The proof of Theorem 3.4 is similar to that of Theorem 3.1, so we omit it.

Corollary 3.5 *Suppose that there is a sequence* $\{\tilde{z}_n\}$ *where* $\tilde{z}_n > 0$ *and* $\lim_{n\to+\infty} \tilde{z}_n = 0$ *such that*

$$\bar{\varrho} < \frac{2\mu}{p_{-1} + p_0 + p_{T-1} + p_T}, \tag{3.13}$$

where

$$\bar{\varrho} = \liminf_{n\to\infty} \frac{(T+1)^2(T+3)\sum_{k=1}^{T} \max_{|x|\le\tilde{z}_n} F(k,x)}{16 p_* \tilde{z}_n^2}.$$

Then, for each $\lambda \in (\frac{p_{-1}+p_0+p_{T-1}+p_T}{2\mu}, \frac{1}{\bar{\varrho}})$, *problem* (1.1) *with* (1.2) *has a sequence of nontrivial solutions that converges to* 0.

Considering the boundary value problem (3.8) with (1.2), we have the following result when the nonlinear function g oscillates at the origin.

Corollary 3.6 *Suppose there are two real sequences* $\{b_n\}$, $\{\bar{b}_n\}$ *with* $\bar{b}_n > 0$ *and* $\lim_{n\to+\infty} \bar{b}_n = 0$ *such that*

$$\frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T) b_n^2 < \frac{16 p_* \bar{b}_n^2}{(T+1)^2(T+3)} \quad \textit{for } n \in \mathbb{Z}(1) \tag{3.14}$$

and

$$\sigma < \frac{2\bar{\mu}}{p_{-1} + p_0 + p_{T-1} + p_T}, \tag{3.15}$$

where

$$\sigma = \liminf_{n\to\infty} \frac{\max_{|x|\le\bar{b}_n} \bar{G}(x) - \bar{G}(b_n)}{\frac{16 p_* \bar{b}_n^2}{(T+1)^2(T+3)} - \frac{1}{2}(p_{-1} + p_0 + p_{T-1} + p_T) b_n^2}.$$

Then, for each $\lambda \in \frac{1}{\sum_{k=1}^{T} a_k}(\frac{p_{-1}+p_0+p_{T-1}+p_T}{2\bar{\mu}}, \frac{1}{\sigma})$, *problem* (3.8) *with* (1.2) *admits a sequence of nontrivial solutions that converges to* 0.

4 Examples

Example 4.1 Consider (1.1) with (1.2) when

$$f(k,u) = f(u) = \big(1 + |u|\big)\big(2 + 2\epsilon + 2\sin\big(\epsilon \ln\big(|u| + 1\big)\big) + \epsilon \cos\big(\epsilon \ln\big(|u| + 1\big)\big)\big)$$

for any $k \in \mathbb{Z}(1,T)$ and $\epsilon > 0$. Then, for $u \ge 0$, it can be obtained by direct calculation

$$F(k,u) = F(u) = \int_0^u f(s)\,ds = (1+u)^2\big(1 + \epsilon + \sin\big(\epsilon \ln(u+1)\big) - 1 - \epsilon.$$

Obviously, $f(u) \ge 0$ for $u \in \mathbb{R}$, and $F(u)$ is increasing at $(-\infty, +\infty)$. Take

$$\tilde{\omega}_n = e^{\frac{1}{\epsilon}(\frac{3\pi}{2}+2n\pi)} - 1, \qquad \nu_n = e^{\frac{1}{\epsilon}(\frac{\pi}{2}+2n\pi)} - 1.$$

Then we have $\lim_{n\to+\infty} \nu_n = \lim_{n\to+\infty} \tilde{\omega}_n = +\infty$ and

$$\mu = \limsup_{x\to+\infty} \frac{\sum_{k=1}^{T} F(k,x)}{x^2} \geq \limsup_{n\to+\infty} \frac{\sum_{k=1}^{T} F(k,\nu_n)}{\nu_n^2} = (2+\epsilon)T.$$

In addition,

$$\tilde{\rho} = \liminf_{n\to\infty} \frac{(T+1)^2(T+3)\sum_{k=1}^{T} F(k,\tilde{\omega}_n)}{16p_*\tilde{\omega}_n^2} = \frac{(T+1)^2(T+3)T\epsilon}{16p_*}.$$

Let ϵ be sufficiently small such that

$$\frac{(T+1)^2(T+3)T\epsilon}{16p_*} \leq \frac{(2+\epsilon)T}{p_{-1}+p_0+p_{T-1}+p_T},$$

which implies that (3.7) of Corollary 3.2 holds. Therefore, by Corollary 3.2, for any $\lambda \in \frac{1}{T}(\frac{p_{-1}+p_0+p_{T-1}+p_T}{2+\epsilon}, \frac{16p_*}{(T+1)^2(T+3)\epsilon})$, the boundary value problem (1.1) with (1.2) has an unbounded sequence of solutions.

Example 4.2 Consider (1.1) with (1.2) when

$$f(k,u) = f(u) = \begin{cases} u(2+2\epsilon+2\sin(\epsilon^2\ln|u|)+\epsilon^2\cos(\epsilon^2\ln|u|)), & u \neq 0, \\ 0, & u = 0, \end{cases}$$

for any $k \in \mathbb{Z}(1,T)$ and $\epsilon > 0$. Then, for $u \neq 0$, we have

$$F(k,u) = F(u) = \int_0^u f(s)\,ds = u^2(1+\epsilon+\sin(\epsilon^2\ln|u|)).$$

It can be seen that $f(u) \geq 0$ for $u \geq 0$, $F(u)$ is increasing at $[0,+\infty)$ and $F(-u) = F(u)$. It is easy to get that

$$\mu = \limsup_{x\to+\infty} \frac{\sum_{k=1}^{T} F(k,x)}{x^2} = (2+\epsilon)T.$$

Let $\zeta_n = e^{-\frac{1}{\epsilon^2}(\frac{\pi}{2}+2n\pi)}$, then $\lim_{n\to+\infty} \zeta_n = 0$, $\zeta_n > 0$ for $n \in \mathbb{Z}(1)$. After a simple calculation, we have

$$\bar{\varrho} = \liminf_{n\to\infty} \frac{(T+1)^2(T+3)\sum_{k=1}^{T} F(k,\zeta_n)}{16p_*\zeta_n^2} = \frac{(T+1)^2(T+3)T\epsilon}{16p_*}.$$

Take ϵ be small enough such that

$$\frac{(T+1)^2(T+3)T\epsilon}{16p_*} \leq \frac{(2+\epsilon)T}{p_{-1}+p_0+p_{T-1}+p_T},$$

which means that (3.13) of Corollary 3.5 holds. Hence, from Corollary 3.5, for any $\lambda \in \frac{1}{T}(\frac{p_{-1}+p_0+p_{T-1}+p_T}{2+\epsilon}, \frac{16p_*}{(T+1)^2(T+3)\epsilon})$, the boundary value problem (1.1) with (1.2) admits a sequence of solutions which converges to 0.

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Declarations

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