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Spatiotemporal dynamics in a delayed diffusive predator–prey system with nonlocal competition in prey and schooling behavior among predators

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Abstract

The nonlocal competition in prey and schooling behavior among predators are incorporated in a delayed diffusive predator–prey model. Our main interest is to study the dynamic properties of the model generated by nonlocal competition and delay. We mainly concentrate on the stability and Hopf bifurcation at the coexisting equilibrium. Compared with the model without nonlocal competition, our results suggest that nonlocal competition can affect the stability of the coexisting equilibrium, and induce the stably spatial bifurcating periodic solutions.

Keywords: Predator–prey system; Delay; Hopf bifurcation

1 Introduction

In the ecological environment, schooling behavior among predators widely exists, such as wolves, African wild dogs, and lions [1–3]. In [4], Cosner et al. proposed the following functional response

$$\varphi(u,v)=\frac{Ce_0uv}{1+hCe_0uv}.$$

The biological meanings of the parameters are given in Table 1. Unlike the traditional functional response (Holling I–III [5]), it is dependent on predator density and increases with prey and predator densities. This functional response can reflect the schooling behavior among predators. Incorporating this functional response, Ryu et al. [6] studied the following model:

$$\begin{cases} \frac{du}{dt}=ru(1-\frac{u}{K})-\frac{Ce_0uv^2}{1+hCe_0uv}, \\ \frac{dv}{dt}=\frac{\epsilon Ce_0uv^2}{1+hCe_0uv}-\mu v. \end{cases} \tag{1.1}$$

Table 1 Biological description of parameters

Parameter	Definition	Parameter	Definition
t	Time variable	x	Spatial variable
u	Prey density	v	Predator density
r	Prey intrinsic growth rate	K	Prey carrying capacity
C	Capture rate	e_0	Encounter rate
h	Handling time	ϵ	Conversion efficiency
μ	Death rate of predators	τ	Gestation delay
d_1	Diffusion coefficient of prey	d_2	Diffusion coefficient of predators

By the scaling

$$\begin{aligned} &rt=\bar{t}, \qquad \frac{u}{K}=\bar{u}, \qquad hCe_0Ky=\bar{v}, \\ &\frac{1}{Ce_0(hK)^2r}=\alpha, \qquad \frac{\epsilon}{rh}=\beta, \qquad \frac{\mu h}{\epsilon}=\gamma, \end{aligned} \tag{1.2}$$

model (1.1) changed to (dropping the bars)

$$\begin{cases} \frac{du}{dt}=u(1-u)-\frac{\alpha uv^2}{1+uv}, \\ \frac{dv}{dt}=\beta(\frac{uv^2}{1+uv}-\gamma v). \end{cases} \tag{1.3}$$

All parameters are positive. They mainly studied the saddle-node, Hopf, and Bogdanov–Takens types of bifurcations at coexisting equilibrium.

In the real world, the living region of prey and predator is inhomogeneous, and diffusion often occurs. Therefore, it is necessary to consider the spatial effect, such as reaction diffusion. Some work shows that space will affect the dynamic properties of the predator–prey model, such as spatial pattern, inhomogeneous periodic solution, etc. [7–10]. In addition, time delays often occur in predator–prey models, such as maturity delay and resource constraint delay. Time delays often cause spatial oscillations, such as periodic solutions [11–14].

The resources in nature are limited, there will be competition within the population, and this competition is usually nonlocal. In [15, 16], the authors modified the $\frac{u}{K}$ as $\frac{1}{K}\int_\Omega G(x,y)u(y,t)\,dy$ to represent the nonlocal competition, where $G(x,y)$ is some kernel function. In [17], Wu and Song studied a diffusive predator–prey model with nonlocal effect and delay, and suggested that steady-state, Hopf, and steady-state Hopf bifurcations may occur. In [18], Geng et al. studied Hopf, Turing, double-Hopf, and Turing–Hopf bifurcations of a diffusive predator–prey model with nonlocal competition. In [19–22], all the authors show that the nonlocal competition may induce stably spatially inhomogeneous bifurcating periodic solutions, which is different from the model without nonlocal competition. Inspired by the above work, we want to analyze the effect of nonlocal competition, time delay, and spatial diffusion on the model (1.1). Consider the following model

$$\begin{cases} \frac{\partial u(x,t)}{\partial t}=d_1\Delta u+u(1-\int_\Omega G(x,y)u(y,t)\,dy)-\frac{\alpha uv^2}{1+uv}, \\ \frac{\partial v(x,t)}{\partial t}=d_2\Delta v+\beta(\frac{u(t-\tau)v^2(t-\tau)}{1+u(t-\tau)v(t-\tau)}-\gamma v), \quad x\in\Omega, t>0 \\ \frac{\partial u(x,t)}{\partial\bar{\nu}}=\frac{\partial v(x,t)}{\partial\bar{\nu}}=0, \quad x\in\partial\Omega, t>0 \\ u(x,\theta)=u_0(x,\theta)\geq 0, \qquad v(x,\theta)=v_0(x,\theta)\geq 0, \quad x\in\bar{\Omega}, \theta\in[-\tau,0]. \end{cases} \tag{1.4}$$

The biological descriptions of the parameters are given in Table 1. $\int_\Omega G(x,y)u(y,t)\,dy$ represents the nonlocal competition effect.

The rest of this paper is organized as follows. In Sect. 2, we study the stability of coexisting equilibrium and the existence of a Hopf bifurcation. In Sect. 3, we study the property of a Hopf bifurcation. In Sect. 4, we give some numerical simulations to illustrate the theoretical results. In Sect. 5, we give a short conclusion.

2 Stability analysis

Choose $\Omega = (0, l\pi)$, and the kernel function $G(x,y) = \frac{1}{l\pi}$. Denote $\mathbb{N}$ as a positive integer set, and $\mathbb{N}_0$ as a nonnegative integer set. $(0,0)$ and $(K,0)$ are boundary equilibria of system (1.4). The existence of positive equilibria of system (1.4) has been studied in [6], that is

Lemma 2.1 ([6]) *The existence of positive equilibria of system* (1.4) *can be divided into three cases*:

- $\alpha > \alpha_{bt} := \frac{4(1-\gamma)}{27\gamma^2}$: *no positive equilibrium.*
- $\alpha = \alpha_{bt}$ *and* $\beta > \gamma$: *one positive equilibrium* $(\frac{2}{3}, \frac{3\gamma}{2(1-\gamma)})$.
- $\alpha < \alpha_{bt}$ *and* $\beta > \gamma$: *two distinct equilibria* (u_1, v_1) *and* (u_2, v_2), *where* $u_1 < \frac{2}{3} < u_2$, *and* $u_{1,2}$ *are two roots of* $u^3 - u^2 + \frac{\alpha\gamma^2}{1-\gamma} = 0$, $v_{1,2} = \frac{\gamma}{(1-\gamma)u_{1,2}}$.

2.1 Model with nonlocal competition

Make the following hypothesis

$$(\mathbf{H_0}) \qquad \alpha \le \alpha_{bt}, \qquad \gamma < 1.$$

If $(\mathbf{H_0})$ holds, then system (1.4) has one or two coexisting equilibria. Hereinafter, for brevity, we denote $E_*(u_*, v_*)$ as the coexisting equilibrium. Linearize system (1.4) at $E_*(u_*, v_*)$

$$\frac{\partial u}{\partial t}\begin{pmatrix} u(x,t) \\ u(x,t) \end{pmatrix} = D\begin{pmatrix} \Delta u(t) \\ \Delta v(t) \end{pmatrix} + L_1\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} + L_2\begin{pmatrix} u(x,t-\tau) \\ v(x,t-\tau) \end{pmatrix} + L_3\begin{pmatrix} \hat{u}(x,t) \\ \hat{v}(x,t) \end{pmatrix}, \tag{2.1}$$

where

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} a_1 & a_2 \\ 0 & -\beta\gamma \end{pmatrix},$$
$$L_2 = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} -u_* & 0 \\ 0 & 0 \end{pmatrix},$$

and $a_1 = \frac{u_* v_*^3 \alpha}{(1+u_* v_*)^2} > 0$, $a_2 = -\frac{u_* v_* (2+u_* v_*)\alpha}{(1+u_* v_*)^2} < 0$, $b_1 = \frac{v_*^2 \beta}{(1+u_* v_*)^2} > 0$, $b_2 = \frac{u_* v_* (2+u_* v_*)\beta}{(1+u_* v_*)^2} > 0$, $\hat{u} = \frac{1}{l\pi}\int_0^{l\pi} u(y,t)\,dy$. The characteristic equations are

$$\lambda^2 + A_n\lambda + B_n + (C_n - b_2\lambda)e^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0, \tag{2.2}$$

where

$$A_0 = u_* - a_1 + \beta\gamma, \qquad B_0 = (u_* - a_1)\beta\gamma, \qquad C_0 = -b_2(u_* - a_1) - a_2 b_1,$$
$$A_n = (d_1 + d_2)\frac{n^2}{l^2} + \beta\gamma - a_1, \qquad B_n = d_1 d_2 \frac{n^4}{l^4} + (\beta\gamma d_1 - a_1 d_2)\frac{n^2}{l^2} - a_1\beta\gamma, \tag{2.3}$$
$$C_n = -b_2 d_1 \frac{n^2}{l^2} + a_1 b_2 - a_2 b_1, \quad n \in \mathbb{N}.$$

When $\tau = 0$, the characteristic equations (2.2) are

$$\lambda^2 + (A_n - b_2)\lambda + B_n + C_n = 0, \quad n \in \mathbb{N}_0, \tag{2.4}$$

where

$$\begin{cases} A_0 - b_2 = -a_1 + u_* - b_2, \qquad B_0 + C_0 = (a_1 - u_*)(b_2 - \beta\gamma) - a_2 b_1, \\ A_n - b_2 = \frac{n^2}{l^2}(d_1 + d_2) - a_1 + \beta\gamma - b_2, \\ B_n + C_n = d_1 d_2 \frac{n^4}{l^4} + \frac{n^2}{l^2}[-d_2 a_1 + d_1(\beta\gamma - b_2)] \\ \qquad + a_1(b_2 - \beta\gamma) - a_2 b_1, \quad n \in \mathbb{N}. \end{cases} \tag{2.5}$$

Make the following hypothesis

$$(\mathbf{H_1}) \qquad A_n - b_2 > 0, \qquad B_n + C_n > 0, \quad \text{for } n \in \mathbb{N}_0.$$

Theorem 2.1 *For system* (1.4), *assume* $\tau = 0$ *and* $(\mathbf{H_0})$ *hold. Then,* $E_*(u_*, v_*)$ *is locally asymptotically stable under* $(\mathbf{H_1})$.

Proof If $(\mathbf{H_1})$ holds, we can obtain that the characteristic roots of (2.4) all have negative real parts. Then, $E_*(u_*, v_*)$ is locally asymptotically stable. □

Let $\mathrm{i}\omega$ ($\omega > 0$) be a solution of Eq. (2.2), then

$$-\omega^2 + \mathrm{i}\omega A_n + B_n + (C_n - b_2\mathrm{i}\omega)(\cos\omega\tau - \mathrm{i}\sin\omega\tau) = 0.$$

We can obtain $\cos\omega\tau = \frac{\omega^2(b_2 A_n + C_n) - B_n C_n}{C_n^2 + d^2\omega^2}$, $\sin\omega\tau = \frac{\omega(A_n C_n + B_n b_2 - b_2\omega^2)}{C_n^2 + b_2^2\omega^2}$. This leads to

$$\omega^4 + \omega^2\left(A_n^2 - 2B_n - b_2^2\right) + B_n^2 - C_n^2 = 0. \tag{2.6}$$

Let $z = \omega^2$, then (2.6) becomes

$$z^2 + z\left(A_n^2 - 2B_n - b_2^2\right) + B_n^2 - C_n^2 = 0, \tag{2.7}$$

and the roots of (2.7) are $z^{\pm} = \frac{1}{2}[-P_n \pm \sqrt{P_n^2 - 4Q_n R_n}]$, where $P_n = A_k^2 - 2B_k - b_2^2$, $Q_n = B_n + C_n$, and $R_n = B_n - C_n$. If $(\mathbf{H_0})$ and $(\mathbf{H_1})$ hold, $Q_n > 0$ ($n \in \mathbb{N}_0$). By direct calculation, we have

$$\begin{aligned}
P_0 &= (a_1 - u_*)^2 + \beta^2\gamma^2 - b_2^2, \\
P_k &= \left(a_1 - d_1\frac{k^2}{l^2}\right)^2 + \left(d_2\frac{n^2}{l^2} + \beta\gamma\right)^2 - b_2^2, \\
R_0 &= a_2b_1 - (a_1 - u_*)(b_2 + \beta\gamma) \\
R_k &= d_1d_2\frac{k^4}{l^4} + (b_2d_1 - a_1d_2 + d_1\beta\gamma)\frac{k^2}{l^2} + a_2b_1 - a_1b_2 - a_1\beta\gamma, \quad \text{for } k \in \mathbb{N}.
\end{aligned}$$

Define

$$\begin{aligned}
\mathbb{W}_1 &= \{n | R_n < 0, n \in \mathbb{N}_0\}, \\
\mathbb{W}_2 &= \left\{n | R_n > 0, P_n < 0, P_n^2 - 4Q_nR_n > 0, n \in \mathbb{N}_0\right\}, \\
\mathbb{W}_3 &= \left\{n | R_n > 0, P_n^2 - 4Q_nR_n < 0, n \in \mathbb{N}_0\right\},
\end{aligned}$$

and

$$\begin{aligned}
\omega_n^{\pm} &= \sqrt{z_n^{\pm}}, \\
\tau_n^{j,\pm} &= \begin{cases} \frac{1}{\omega_n^{\pm}} \arccos(V_{\cos}^{(n,\pm)}) + 2j\pi, & V_{\sin}^{(n,\pm)} \geq 0, \\ \frac{1}{\omega_n^{\pm}}[2\pi - \arccos(V_{\cos}^{(n,\pm)})] + 2j\pi, & V_{\sin}^{(n,\pm)} < 0, \end{cases} \\
V_{\cos}^{(n,\pm)} &= \frac{(\omega_n^{\pm})^2(b_2A_n + C_n) - B_nC_n}{C_n^2 + b_2^2(\omega_n^{\pm})^2}, \\
V_{\sin}^{(n,\pm)} &= \frac{\omega_n^{\pm}(A_nC_n + B_nb_2 - b_2(\omega_n^{\pm})^2)}{C_n^2 + b_2^2(\omega_n^{\pm})^2}.
\end{aligned} \tag{2.8}$$

We have the following lemma.

Lemma 2.2 *Assume* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$ *hold, then the following results hold.*

- *Eq.* (2.2) *has a pair of purely imaginary roots* $\pm i\omega_n^+$ *at* $\tau_n^{j,+}$ *for* $j \in \mathbb{N}_0$ *and* $n \in \mathbb{W}_1$.
- *Eq.* (2.2) *has two pairs of purely imaginary roots* $\pm i\omega_n^{\pm}$ *at* $\tau_n^{j,\pm}$ *for* $j \in \mathbb{N}_0$ *and* $n \in \mathbb{W}_2$.
- *Eq.* (2.2) *has no purely imaginary root for* $n \in \mathbb{W}_3$.

Lemma 2.3 *Assume* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$ *hold. Then,* $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,+}} > 0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,-}} < 0$ *for* $n \in \mathbb{W}_1 \cup \mathbb{W}_2$ *and* $j \in \mathbb{N}_0$.

Proof By (2.2), we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + A_n - b_2e^{-\lambda\tau}}{(C_n - b_2\lambda)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Then,

$$\begin{aligned}\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_n^{j,\pm}} &= \mathrm{Re}\left[\frac{2\lambda + A_n - b_2e^{-\lambda\tau}}{(C_n - b_2\lambda)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}\right]_{\tau=\tau_n^{j,\pm}} \\ &= \left[\frac{1}{C_n^2 + b_2^2\omega^2}\left(2\omega^2 + A_n^2 - 2B_n - b_2^2\right)\right]_{\tau=\tau_n^{j,\pm}} \\ &= \pm\left[\frac{1}{C_n^2 + b_2^2\omega^2}\sqrt{\left(A_n^2 - 2B_n - b_2^2\right)^2 - 4\left(B_n^2 - C_n^2\right)}\right]_{\tau=\tau_n^{j,\pm}}.\end{aligned}$$

Therefore, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,+}} > 0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,-}} < 0$. □

Denote $\tau_* = \min\{\tau_n^0 | n \in \mathbb{W}_1 \cup \mathbb{W}_2\}$. Note that $\tau = \tau_m^{j,+}(\tau = \tau_m^{j,-})$ may be equal to $\tau = \tau_n^{j,+}(\tau = \tau_n^{j,-})$, for some $m \neq n$. In this case, high codimensional bifurcation will occur. In this paper, we do not consider this case. Then, we have the following theorem.

Theorem 2.2 *Assume* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$ *hold, then the following statements are true for system* (1.4).

- $E_*(u_*, v_*)$ *is locally asymptotically stable for* $\tau > 0$ *when* $\mathbb{W}_1 \cup \mathbb{W}_2 = \varnothing$.
- $E_*(u_*, v_*)$ *is locally asymptotically stable for* $\tau \in [0, \tau_*)$ *when* $\mathbb{W}_1 \cup \mathbb{W}_2 \neq \varnothing$.
- $E_*(u_*, v_*)$ *is unstable for* $\tau \in (\tau_*, \tau_* + \varepsilon)$ *for some* $\varepsilon > 0$ *when* $\mathbb{W}_1 \cup \mathbb{W}_2 \neq \varnothing$.
- *Hopf bifurcation occurs at* (u_*, v_*) *when* $\tau = \tau_n^{j,+}$ $(\tau = \tau_n^{j,-})$, $j \in \mathbb{N}_0$, $n \in \mathbb{W}_1 \cup \mathbb{W}_2$.

2.2 Model without nonlocal competition

The model (1.4) without nonlocal competition is

$$\begin{cases}\frac{\partial u(x,t)}{\partial t} = d_1\Delta u + u(1-u) - \frac{\alpha uv^2}{1+uv}, \\ \frac{\partial v(x,t)}{\partial t} = d_2\Delta v + \beta(\frac{u(t-\tau)v^2(t-\tau)}{1+u(t-\tau)v(t-\tau)} - \gamma v), \quad x \in \Omega, t > 0 \\ \frac{\partial u(x,t)}{\partial \vec{\nu}} = \frac{\partial v(x,t)}{\partial \vec{\nu}} = 0, \quad x \in \partial\Omega, t > 0 \\ u(x,\theta) = u_0(x,\theta) \geq 0, \qquad v(x,\theta) = v_0(x,\theta) \geq 0, \quad x \in \bar{\Omega}, \theta \in [-\tau, 0].\end{cases} \tag{2.9}$$

Linearizing system (1.4) at (u_*, v_*) gives:

$$\begin{pmatrix}\frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t}\end{pmatrix} = d\Delta\begin{pmatrix}u(t) \\ v(t)\end{pmatrix} + L_1'\begin{pmatrix}u(t) \\ v(t)\end{pmatrix} + L_2'\begin{pmatrix}u(t-\tau) \\ v(t-\tau)\end{pmatrix}, \tag{2.10}$$

where

$$L_1' = \begin{pmatrix}a_1 - u_* & a_2 \\ 0 & -\beta\gamma\end{pmatrix}, \qquad L_2' = \begin{pmatrix}0 & 0 \\ b_1 & b_2\end{pmatrix}.$$

The characteristic equations of (2.10) are

$$\lambda^2 + \lambda A_n' + B_n' + \left(C_n' - b_2\lambda\right)e^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0, \tag{2.11}$$

where

$$A'_n = (d_1 + d_2)\frac{n^2}{l^2} + \beta\gamma + u_* - a_1,$$
$$B'_n = d_1 d_2 \frac{n^4}{l^4} + (\beta\gamma d_1 + d_2 u_* - a_1 d_2)\frac{n^2}{l^2} + (u_* - a_1)\beta\gamma,$$
$$C'_n = -b_2 d_1 \frac{n^2}{l^2} + (u_* - a_1)b_2 - a_2 b_1, \quad n \in \mathbb{N}_0.$$

When $\tau = 0$, the characteristic Eq. (2.11) reduces to the following equation:

$$\lambda^2 + (A'_n - b_2)\lambda + B'_n + C'_n = 0, \quad n \in \mathbb{N}_0, \tag{2.12}$$

where

$$\begin{cases} A'_n - b_2 = \frac{n^2}{l^2}(d_1 + d_2) - a_1 + u_* + \beta\gamma - b_2, \\ B'_n + C'_n = d_1 d_2 \frac{n^4}{l^4} + \frac{n^2}{l^2}[d_2(u_* - a_1) + d_1(\beta\gamma - b_2)] \\ \qquad + (a_1 - u_*)(b_2 - \beta\gamma) - a_2 b_1. \end{cases} \tag{2.13}$$

Make the following hypothesis

$$(\mathbf{H_2}) \qquad A'_n - b_2 > 0, \qquad B'_n + C'_n > 0, \quad n \in \mathbb{N}_0. \tag{2.14}$$

Theorem 2.3 *For system* (2.9), *assume* $\tau = 0$ *and* ($\mathbf{H_0}$) *holds. Then,* $E_*(u_*, v_*)$ *is locally asymptotically stable under* ($\mathbf{H_2}$).

Let iω ($\omega > 0$) be a solution of Eq. (2.10), and $z = \omega^2$. Similarly, we can obtain $z^{\pm}_{n,w} = \frac{1}{2}[-P'_n \pm \sqrt{(P'_n)^2 - 4Q'_n R'_n}]$, where $P'_n = (A'_k)^2 - 2B'_k - b_2^2$, $Q'_n = B'_n + C'_n$, and $R'_n = B'_n - C'_n$. If ($\mathbf{H_0}$) and ($\mathbf{H_2}$) hold, $Q'_n > 0$ ($n \in \mathbb{N}_0$). By direct calculation, we have

$$P'_0 = P_0, R'_0 = R_0,$$
$$P'_k = \left(a_1 - u_* - d_1 \frac{k^2}{l^2}\right)^2 + \left(d_2 \frac{n^2}{l^2} + \beta\gamma\right)^2 - b_2^2,$$
$$R'_k = d_1 d_2 \frac{k^4}{l^4} + \left[d_2(u_* - a_1) + d_1(b_2 + \beta\gamma)\right]\frac{k^2}{l^2} + b_2 u_* + (u_* - a_1)\beta\gamma, \quad \text{for } k \in \mathbb{N}.$$

Define

$$\mathbb{W}'_1 = \{n | R'_n < 0, n \in \mathbb{N}_0\},$$
$$\mathbb{W}'_2 = \{n | R'_n > 0, P'_n < 0, (P'_n)^2 - 4Q'_n R'_n > 0, n \in \mathbb{N}_0\},$$
$$\mathbb{W}'_3 = \{n | R'_n > 0, (P'_n)^2 - 4Q'_n R'_n < 0, n \in \mathbb{N}_0\},$$

and

$$\begin{aligned}
&\omega_{n,w}^{\pm} = \sqrt{z_{n,w}^{\pm}},\\
&\tau_{n,w}^{j,\pm} = \begin{cases} \frac{1}{\omega_{n,w}^{\pm}} \arccos(V_{\cos,w}^{(n,\pm)}) + 2j\pi, & V_{\sin,w}^{(n,\pm)} \geq 0,\\ \frac{1}{\omega_{n,w}^{\pm}} [2\pi - \arccos(V_{\cos,w}^{(n,\pm)})] + 2j\pi, & V_{\sin,w}^{(n,\pm)} < 0, \end{cases}\\
&V_{\cos,w}^{(n,\pm)} = \frac{(\omega_{n,w}^{\pm})^2 (b_2 A'_n + C'_n) - B'_n C'_n}{(C'_n)^2 + b_2^2 (\omega_{n,w}^{\pm})^2},\\
&V_{\sin,w}^{(n,\pm)} = \frac{\omega_{n,w}^{\pm} (A'_n C'_n + B'_n b_2 - b_2 (\omega_{n,w}^{\pm})^2)}{(C'_n)^2 + b_2^2 (\omega_{n,w}^{\pm})^2}.
\end{aligned} \tag{2.15}$$

We have the following lemma.

Lemma 2.4 *Assume* ($\mathbf{H_0}$) *and* ($\mathbf{H_1}$) *hold, then the following results hold.*

- *Eq.* (2.11) *has a pair of purely imaginary roots* $\pm i\omega_{n,w}^{+}$ *at* $\tau_{n,w}^{j,+}$ *for* $j \in \mathbb{N}_0$ *and* $n \in \mathbb{W}'_1$.
- *Eq.* (2.11) *has two pairs of purely imaginary roots* $\pm i\omega_{n,w}^{\pm}$ *at* $\tau_{n,w}^{j,\pm}$ *for* $j \in \mathbb{N}_0$ *and* $n \in \mathbb{W}'_2$.
- *Eq.* (2.11) *has no purely imaginary root for* $n \in \mathbb{W}'_3$.

Lemma 2.5 *Assume* ($\mathbf{H_0}$) *and* ($\mathbf{H_1}$) *hold. Then,* $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_{n,w}^{j,+}} > 0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_{n,w}^{j,-}} < 0$ *for* $n \in \mathbb{W}'_1 \cup \mathbb{W}'_2$ *and* $j \in \mathbb{N}_0$.

Proof By (2.11), we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + A'_n - b_2 e^{-\lambda\tau}}{(C'_n - b_2\lambda)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Then,

$$\begin{aligned}
&\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_{n,w}^{j,\pm}}\\
&\quad = \mathrm{Re}\left[\frac{2\lambda + A'_n - b_2 e^{-\lambda\tau}}{(C'_n - b_2\lambda)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}\right]_{\tau=\tau_{n,w}^{j,\pm}}\\
&\quad = \left[\frac{1}{(C'_n)^2 + b_2^2\omega^2}\left(2\omega^2 + \left(A'_n\right)^2 - 2B'_n - b_2^2\right)\right]_{\tau=\tau_{n,w}^{j,\pm}}\\
&\quad = \pm\left[\frac{1}{(C'_n)^2 + b_2^2\omega^2}\sqrt{\left(\left(A'_n\right)^2 - 2B'_n - b_2^2\right)^2 - 4\left(\left(B'_n\right)^2 - \left(C'_n\right)^2\right)}\right]_{\tau=\tau_{n,w}^{j,\pm}}.
\end{aligned}$$

Therefore, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_{n,w}^{j,+}} > 0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_{n,w}^{j,-}} < 0$. □

Denote $\tau'_* = \min\{\tau_n^0 | n \in \mathbb{W}'_1 \cup \mathbb{W}'_2\}$. We have the following theorem.

Theorem 2.4 *Assume* ($\mathbf{H_0}$) *and* ($\mathbf{H_1}$) *hold, then the following statements are true for system* (2.9).

- $E_*(u_*, v_*)$ *is locally asymptotically stable for* $\tau > 0$ *when* $\mathbb{W}'_1 \cup \mathbb{W}'_2 = \varnothing$.
- $E_*(u_*, v_*)$ *is locally asymptotically stable for* $\tau \in [0, \tau'_*)$ *when* $\mathbb{W}'_1 \cup \mathbb{W}'_2 \neq \varnothing$.
- $E_*(u_*, v_*)$ *is unstable for* $\tau \in (\tau'_*, \tau_* + \varepsilon)$ *for some* $\varepsilon > 0$ *when* $\mathbb{W}'_1 \cup \mathbb{W}'_2 \neq \varnothing$.
- *Hopf bifurcation occurs at* (u_*, v_*) *when* $\tau = \tau_{n,w}^{j,+}$ $(\tau = \tau_{n,w}^{j,-})$, $j \in \mathbb{N}_0$, $n \in \mathbb{W}'_1 \cup \mathbb{W}'_2$.

3 Property of Hopf bifurcation

By the work [23, 24], we study the property of Hopf bifurcation. For fixed $j \in \mathbb{N}_0$ and $n \in \mathbb{W}_1 \cup \mathbb{W}_2$, we denote $\tilde{\tau} = \tau_n^{j,\pm}$. Let $\bar{u}(x,t) = u(x,\tau t) - u_*$ and $\bar{v}(x,t) = v(x,\tau t) - v_*$. Dropping the bar, (1.4) can be written as

$$\begin{cases} \frac{\partial u}{\partial t} = \tau[d_1\Delta u + (u+u_*)(1 - \frac{1}{l\pi}\int_0^{l\pi}(u(y,t)+u_*)\,dy) - \frac{\alpha(u+u_*)(v+v_*)^2}{1+(u+u_*)(v+v_*)}], \\ \frac{\partial v}{\partial t} = \tau[d_2\Delta v + \frac{\beta(u(t-1)+u_*)(v(t-1)+v_*)^2}{1+(u(t-1)+u_*)(v(t-1)+v_*)} - \beta\gamma(v+v_*)]. \end{cases} \tag{3.1}$$

We rewrite system (3.1) as the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = \tau[d_1\Delta u + a_1 u + a_2 v - u_*\hat{u} + \alpha_1 u^2 - u\hat{u} + \alpha_2 uv + \alpha_3 v^2 \\ \quad + \alpha_4 u^3 + \alpha_5 u^2 v + \alpha_6 uv^2 \\ \quad + \alpha_7 v^3] + h.o.t., \\ \frac{\partial v}{\partial t} = \tau[d_2\Delta v + b_1 u(t-1) + b_2 v(t-1) - \beta\gamma v \\ \quad + \beta_1 u^2(t-1) + \beta_2 u(t-1)v(t-1) + \beta_3 u^2(t-1) \\ \quad + \beta_4 u^3(t-1) + \beta_5 u^2(t-1)v(t-1)] \\ \quad + \beta_6 u(t-1)v^2(t-1) + \beta_7 v^3(t-1)] + h.o.t., \end{cases} \tag{3.2}$$

where $\alpha_1 = \frac{v_*^3\alpha}{(1+u_*v_*)^3}$, $\alpha_2 = -\frac{2v_*\alpha}{(1+u_*v_*)^3}$, $\alpha_3 = -\frac{u_*\alpha}{(1+u_*v_*)^3}$, $\alpha_4 = -\frac{v_*^4\alpha}{(1+u_*v_*)^4}$, $\alpha_5 = \frac{v_*^2\alpha}{(1+u_*v_*)^4}$, $\alpha_6 = \frac{(-1+2u_*v_*)\alpha}{(1+u_*v_*)^4}$, $\alpha_7 = \frac{u_*^2\alpha}{(1+u_*v_*)^4}$, $\beta_1 = -\frac{v_*^3\beta}{(1+u_*v_*)^3}$, $\beta_2 = \frac{2v_*\beta}{(1+u_*v_*)^3}$, $\beta_3 = \frac{u_*\beta}{(1+u_*v_*)^3}$, $\beta_4 = \frac{v_*^4\beta}{(1+u_*v_*)^4}$, $\beta_5 = -\frac{3v_*^2\beta}{(1+u_*v_*)^4}$, $\beta_6 = \frac{(1-2u_*v_*)\beta}{(1+u_*v_*)^4}$, $\beta_7 = -\frac{6u_*^2\beta}{(1+u_*v_*)^4}$.

Define the real-valued Sobolev space $X := \{(u,v)^T : u,v \in H^2(0,l\pi), (u_x,v_x)|_{x=0,l\pi} = 0\}$, the complexification of X $X_{\mathbb{C}} := X \oplus \mathrm{i}X = \{x_1 + \mathrm{i}x_2 | x_1, x_2 \in X\}$, and the inner product $\langle \tilde{u}, \tilde{v}\rangle := \int_0^{l\pi} \overline{u_1} v_1\,dx + \int_0^{l\pi} \overline{u_2} v_2\,dx$ for $\tilde{u} = (u_1,u_2)^T$, $\tilde{v} = (v_1,v_2)^T$, $\tilde{u},\tilde{v} \in X_{\mathbb{C}}$. The phase space $\mathscr{C} := C([-1,0],X)$ is with the sup norm, then we can write $\phi_t \in \mathscr{C}$, $\phi_t(\theta) = \phi(t+\theta)$ or $-1 \le \theta \le 0$. Denote $\beta_n^{(1)}(x) = (\gamma_n(x),0)^T$, $\beta_n^{(2)}(x) = (0,\gamma_n(x))^T$, and $\beta_n = \{\beta_n^{(1)}(x), \beta_n^{(2)}(x)\}$, where $\{\beta_n^{(i)}(x)\}$ is an orthonormal basis of X. We define the subspace of $\mathscr{C}$ as $\mathbb{B}_n := \operatorname{span}\{\langle \phi(\cdot), \beta_n^{(j)}\rangle \beta_n^{(j)} | \phi \in \mathscr{C}, j = 1,2\}$, $n \in \mathbb{N}_0$. There exists a 2×2 matrix function $\eta^n(\sigma,\tilde{\tau})$ $-1 \le \sigma \le 0$, such that $-\tilde{\tau} D\frac{n^2}{l^2}\phi(0) + \tilde{\tau}L(\phi) = \int_{-1}^0 d\eta^n(\sigma,\tau)\phi(\sigma)$ for $\phi \in \mathscr{C}$. The bilinear form on $\mathscr{C}^* \times \mathscr{C}$ is defined by

$$(\psi,\phi) = \psi(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\sigma} \psi(\xi-\sigma)\,d\eta^n(\sigma,\tilde{\tau})\phi(\xi)\,d\xi, \tag{3.3}$$

for $\phi \in \mathscr{C}$, $\psi \in \mathscr{C}^*$. Define $\tau = \tilde{\tau} + \mu$, then the system undergoes a Hopf bifurcation at $(0,0)$ when $\mu = 0$, with a pair of purely imaginary roots $\pm \mathrm{i}\omega_{n_0}$. Let A denote the infinitesimal generators of the semigroup, and A^* be the formal adjoint of A under the bilinear form (3.3). Define the following function

$$\delta(n_0) = \begin{cases} 1 & n_0 = 0, \\ 0 & n_0 \in \mathbb{N}. \end{cases} \tag{3.4}$$

Choose $\eta_{n_0}(0,\tilde{\tau}) = \tilde{\tau}[(-n_0^2/l^2)D + L_1 + L_3\delta(n_{n_0})]$, $\eta_{n_0}(-1,\tilde{\tau}) = -\tilde{\tau}L_2$, $\eta_{n_0}(\sigma,\tilde{\tau}) = 0$ for $-1 < \sigma < 0$. Let $p(\theta) = p(0)e^{\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}$ $(\theta \in [-1,0])$, $q(\vartheta) = q(0)e^{-\mathrm{i}\omega_{n_0}\tilde{\tau}\vartheta}$ $(\vartheta \in [0,1])$ be the eigenfunctions of $A(\tilde{\tau})$ and A^* corresponds to $i\omega_{n_0}\tilde{\tau}$, respectively. We can choose $p(0) = (1,p_1)^T$,

$q(0) = M(1, q_2)$, where $p_1 = \frac{1}{a_2}(\mathrm{i}\omega_{n_0} + d_1 n_0^2/l^2 - a_1 + u_*\delta(n_0))$, $q_2 = a_2/(\mathrm{i}\omega_{n_0} - b_2 e^{\mathrm{i}\tau\omega_{n_0}} + \beta\gamma + \frac{d_2 n^2}{l^2})$, and $M = (1 + p_1 q_2 + \tilde{\tau} q_2(b_1 + b_2 p_1)e^{-\mathrm{i}\omega_{n_0}\tilde{\tau}})^{-1}$. Then, (3.1) can be rewritten in an abstract form

$$\begin{aligned}\frac{dU(t)}{dt} &= (\tilde{\tau} + \mu)D\Delta U(t) + (\tilde{\tau} + \mu)\big[L_1(U_t) + L_2 U(t-1) + L_3\hat{U}(t)\big]\\ &\quad + F(U_t, \hat{U}_t, \mu),\end{aligned} \tag{3.5}$$

where

$$F(\phi, \mu) = (\tilde{\tau} + \mu)\begin{pmatrix} \alpha_1\phi_1(0)^2 - \phi_1(0)\hat{\phi}_1(0) + \alpha_2\phi_1(0)\phi_2(0) \\ + \alpha_3\phi_2(0)^2 + \alpha_4\phi_1^3(0) + \alpha_5\phi_1^2(0)\phi_2(0) \\ + \alpha_6\phi_1(0)\phi_2^2(0) + \alpha_7\phi_2^3(0) \\ \beta_1\phi_1^2(-1) + \beta_2\phi_1(-1)\phi_2(-1) + \beta_3\phi_2^2(-1) \\ + \beta_4\phi_1^3(-1) + \beta_4\phi_1^2(-1)\phi_2(-1) \\ + \beta_6\phi_1(-1)\phi_2^2(-1) + \beta_7\phi_2^3(-1) \end{pmatrix}, \tag{3.6}$$

respectively, for $\phi = (\phi_1, \phi_2)^T \in \mathscr{C}$ and $\hat{\phi}_1 = \frac{1}{l\pi}\int_0^{l\pi}\phi\,dx$. Then, the space $\mathscr{C}$ can be decomposed as $\mathscr{C} = P \oplus Q$, where $P = \{zp\gamma_{n_0}(x) + \bar{z}\bar{p}\gamma_{n_0}(x) | z \in \mathbb{C}\}$, $Q = \{\phi \in \mathscr{C} | (q\gamma_{n_0}(x), \phi) = 0 \text{ and } (\bar{q}\gamma_{n_0}(x), \phi) = 0\}$. Then, system (3.6) can be rewritten as $U_t = z(t)p(\cdot)\gamma_{n_0}(x) + \bar{z}(t)\bar{p}(\cdot)\gamma_{n_0}(x) + \omega(t, \cdot)$ and $\hat{U}_t = \frac{1}{l\pi}\int_0^{l\pi} U_t\,dx$, where

$$z(t) = \big(q\gamma_{n_0}(x), U_t\big), \qquad \omega(t, \theta) = U_t(\theta) - 2\,\mathrm{Re}\big\{z(t)p(\theta)\gamma_{n_0}(x)\big\}. \tag{3.7}$$

Then, we have $\dot{z}(t) = \mathrm{i}\omega) n_0\tilde{\tau}z(t) + \bar{q}(0)\langle F(0, U_t), \beta_{n_0}\rangle$. There exists a center manifold $\mathcal{C}_0$ and ω can be written as follows near $(0, 0)$:

$$\omega(t, \theta) = \omega\big(z(t), \bar{z}(t), \theta\big) = \omega_{20}(\theta)\frac{z^2}{2} + \omega_{11}(\theta)z\bar{z} + \omega_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.8}$$

Then, restrict the system to the center manifold: $\dot{z}(t) = \mathrm{i}\omega_{n_0}\tilde{\tau}z(t) + g(z, \bar{z})$. Denote $g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots$. By direct computation, we have

$$\begin{aligned} &g_{20} = 2\tilde{\tau}M(\varsigma_1 + q_2\varsigma_2)I_3, \qquad g_{11} = \tilde{\tau}M(\varrho_1 + q_2\varrho_2)I_3, \qquad g_{02} = \bar{g}_{20},\\ &g_{21} = 2\tilde{\tau}M\big[(\kappa_{11} + q_2\kappa_{21})I_2 + (\kappa_{12} + q_2\kappa_{22})I_4\big],\end{aligned}$$

where $I_2 = \int_0^{l\pi}\gamma_{n_0}^2(x)\,dx$, $I_3 = \int_0^{l\pi}\gamma_{n_0}^3(x)\,dx$, $I_4 = \int_0^{l\pi}\gamma_{n_0}^4(x)\,dx$, $\varsigma_1 = -\delta\mathrm{n} + \alpha_1 + \alpha_2\xi + \alpha_3\xi^2$, $\varsigma_2 = e^{-2i\tau\omega_n}(\beta_1 + \xi(\beta_2 + \beta_3\xi))$, $\varrho_1 = \frac{1}{4}(2\alpha_1 - 2\delta\mathrm{n} + \alpha_2\bar{\xi} + \alpha_2\xi + 2\alpha_3\bar{\xi}\xi)$, $\varrho_2 = \frac{1}{4}(2\beta_1 + 2\beta_3\bar{\xi}\xi + \beta_2(\bar{\xi} + \xi))$, $\kappa_{11} = 2W_{11}^{(1)}(0)(-1 + 2\alpha_1 - \delta\mathrm{n} + \alpha_2\xi) + 2W_{11}^{(2)}(0)(\alpha_2 + 2\alpha_3\xi) + W_{20}^{(1)}(0)(-1 + 2\alpha_1 - \delta\mathrm{n} + \alpha_2\bar{\xi}) + W_{20}^{(2)}(0)(\alpha_2 + 2\alpha_3\bar{\xi})$, $\kappa_{12} = \frac{1}{2}(3\alpha_4 + \alpha_5(\bar{\xi} + 2\xi) + \xi(2\alpha_6\bar{\xi} + \alpha_6\xi + 3\alpha_7\bar{\xi}\xi))$, $\kappa_{21} = 2e^{-i\tau\omega_n}W_{11}^{(1)}(-1)(2\beta_1 + \beta_2\xi) + 2e^{-i\tau\omega_n}W_{11}^{(2)}(-1)(\beta_2 + 2\beta_3\xi) + e^{i\tau\omega_n}W_{20}^{(1)}(-1)(2\beta_1 + \beta_2\bar{\xi}) + e^{i\tau\omega_n}W_{20}^{(2)}(-1)(\beta_2 + 2\beta_3\bar{\xi})$, $\kappa_{22} = \frac{1}{2}e^{-i\tau\omega_n}(3\beta_4 + \beta_5(\bar{\xi} + 2\xi) + \xi(2\beta_6\bar{\xi} + \beta_6\xi + 3\beta_7\bar{\xi}\xi))$.

Now, we compute $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in [-1, 0]$ to give g_{21}. By (3.7), we have

$$\dot{\omega} = \dot{U}_t - \dot{z}p\gamma_{n_0}(x) - \dot{\bar{z}}\bar{p}\gamma_{n_0}(x) = A\omega + H(z, \bar{z}, \theta), \tag{3.9}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.10}$$

Compared the coeffcients of (3.8) with (3.9), we have

$$(A - 2\mathrm{i}\omega_{n_0}\tilde{\tau}I)\omega_{20} = -H_{20}(\theta), \qquad A\omega_{11}(\theta) = -H_{11}(\theta). \tag{3.11}$$

Then, we have

$$\begin{aligned} \omega_{20}(\theta) &= \frac{-g_{20}}{\mathrm{i}\omega_{n_0}\tilde{\tau}}p(0)e^{\mathrm{i}\omega_{n_0}\tilde{\tau}\theta} - \frac{\bar{g}_{02}}{3\mathrm{i}\omega_{n_0}\tilde{\tau}}\bar{p}(0)e^{-\mathrm{i}\omega_{n_0}\tilde{\tau}\theta} + E_1 e^{2\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}, \\ \omega_{11}(\theta) &= \frac{g_{11}}{\mathrm{i}\omega_{n_0}\tilde{\tau}}p(0)e^{\mathrm{i}\omega_{n_0}\tilde{\tau}\theta} - \frac{\bar{g}_{11}}{\mathrm{i}\omega_{n_0}\tilde{\tau}}\bar{p}(0)e^{-\mathrm{i}\omega_{n_0}\tilde{\tau}\theta} + E_2, \end{aligned} \tag{3.12}$$

where $E_1 = \sum_{n=0}^{\infty} E_1^{(n)}$, $E_2 = \sum_{n=0}^{\infty} E_2^{(n)}$,

$$\begin{aligned} E_1^{(n)} &= \left(2\mathrm{i}\omega_{n_0}\tilde{\tau}I - \int_{-1}^{0} e^{2\mathrm{i}\omega_{n_0}\tilde{\tau}\theta}\,d\eta_{n_0}(\theta,\tilde{\tau})\right)^{-1}\langle\tilde{F}_{20},\beta_n\rangle, \\ E_2^{(n)} &= -\left(\int_{-1}^{0} d\eta_{n_0}(\theta,\tilde{\tau})\right)^{-1}\langle\tilde{F}_{11},\beta_n\rangle, \quad n \in \mathbb{N}_0, \\ \langle\tilde{F}_{20},\beta_n\rangle &= \begin{cases} \frac{1}{l\pi}\hat{F}_{20}, & n_0 \neq 0, n = 0, \\ \frac{1}{2l\pi}\hat{F}_{20}, & n_0 \neq 0, n = 2n_0, \\ \frac{1}{l\pi}\hat{F}_{20}, & n_0 = 0, n = 0, \\ 0, & \text{other}, \end{cases} \\ \langle\tilde{F}_{11},\beta_n\rangle &= \begin{cases} \frac{1}{l\pi}\hat{F}_{11}, & n_0 \neq 0, n = 0, \\ \frac{1}{2l\pi}\hat{F}_{11}, & n_0 \neq 0, n = 2n_0, \\ \frac{1}{l\pi}\hat{F}_{11}, & n_0 = 0, n = 0, \\ 0, & \text{other}, \end{cases} \end{aligned} \tag{3.13}$$

and $\hat{F}_{20} = 2(\varsigma_1,\varsigma_2)^T$, $\hat{F}_{11} = 2(\varrho_1,\varrho_2)^T$.

Thus, we can obtain

$$\begin{aligned} c_1(0) &= \frac{\mathrm{i}}{2\omega_n\tilde{\tau}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{1}{2}g_{21}, \qquad \mu_2 = -\frac{\mathrm{Re}(c_1(0))}{\mathrm{Re}(\lambda'(\tilde{\tau}))}, \\ T_2 &= -\frac{1}{\omega_{n_0}\tilde{\tau}}\left[\mathrm{Im}\big(c_1(0)\big) + \mu_2\,\mathrm{Im}\big(\lambda'(\tau_n^j)\big)\right], \qquad \beta_2 = 2\,\mathrm{Re}\big(c_1(0)\big). \end{aligned} \tag{3.14}$$

Theorem 3.1 *For any critical value τ_n^j ($n \in \mathbb{S}$, $j \in \mathbb{N}_0$), we have the following results*:

- *When $\mu_2 > 0$ (resp., <0), the Hopf bifurcation is forward (resp., backward).*
- *When $\beta_2 < 0$ (resp., >0), the bifurcating periodic solutions on the center manifold are orbitally asymptotically stable (resp., unstable).*
- *When $T_2 > 0$ (resp., $T_2 < 0$), the period increases (resp., decreases).*

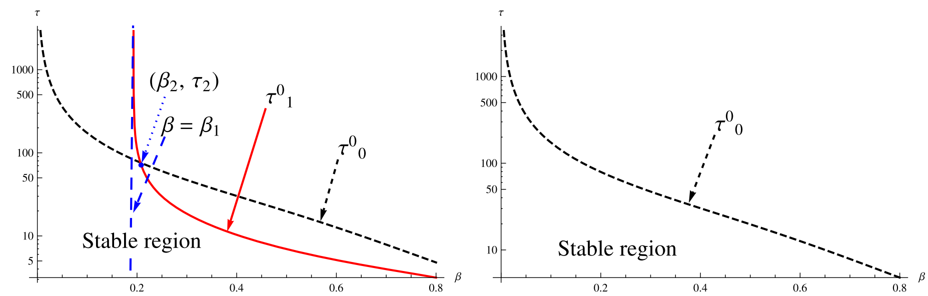

Figure 1 Bifurcation diagram of system (1.4) for β and τ. (Left: model (1.4). Right: model (2.9))

Table 2 Some parameters for model (1.4) with different ξ

β	τ_*	μ_2	β_2	T_2
0.18	89.9073	1.5764×10^7	−565.8943	1467.4131
0.5	10.3066	257.2119	−0.4447	1.3340

4 Numerical simulations

To study the effect of nonlocal competition, we also give numerical simulations for models (1.4) and (2.9). Fix the following parameters:

$$\alpha = 1.2, \qquad \gamma = 0.25, \qquad d_1 = 0.1, \qquad d_2 = 0.3, \qquad l = 0.8.$$

System (1.4) has two positive equilibria $E_1 \approx (0.4126, 0.8079)$ and $E_2 \approx (0.8670, 0.3845)$. Since E_2 is always unstable, we mainly analyze the stability of E_1. We can obtain the bifurcation diagrams of systems (1.4) and (2.9) with β (Fig. 1), where $\beta_1 \approx 0.1929$ and $(\beta_2, \tau_2) \approx (0.2064, 80.1787)$. We also compute some parameters for model (1.4) with different β (Table 2).

From Fig. 1, we can see that increasing β is no benefit to the stability of coexisting equilibrium. For the model (2.9), a spatially inhomogeneous periodic solution curve does not exist. For the model (1.4), when $0 < \beta < \beta_1$, the stability of the coexistence equilibrium E_1 is similar to model (2.9). When $\beta > \beta_1$, the spatially inhomogeneous periodic solution curve (τ_1^0) exists, and is larger than the spatially homogeneous periodic solution curve (τ_0^0) for $\beta_1 < \beta < \beta_2$. This means that the spatially homogeneous periodic solution will appear first, and the spatially inhomogeneous periodic solution is usually unstable. However, when $\beta > \beta_2$, the spatially inhomogeneous periodic solution curve (τ_1^0) is smaller than the spatially homogeneous periodic solution curve (τ_0^0). This means that the spatially inhomogeneous periodic solution will appear first, and may be asymptotically stable.

Choose $\beta = 0.18$, when $\tau < \tau_* \approx 89.9073$, the coexistence equilibrium E_1 is asymptotically stable for models (1.4) and (2.9) (Fig. 2). When $\tau > \tau_*$, the coexistence equilibrium E_1 is unstable and the spatial homogeneous periodic solution appears for models (1.4) and (2.9) (Fig. 3).

Choose $\beta = 0.5$, when $\tau < \tau_* \approx 10.3066$, the coexistence equilibrium E_1 is asymptotically stable for models (1.4) and (2.9) (Fig. 4). However, when $\tau_* < \tau < \tau_0^0 \approx 19.7545$, for model (1.4) the coexistence equilibrium E_1 is unstable and the spatial homogeneous periodic solution does not exist. The stably spatial inhomogeneous periodic solution appears (Fig. 5 left). However, for the model (2.9), the coexistence equilibrium E_1 is still asymptotically

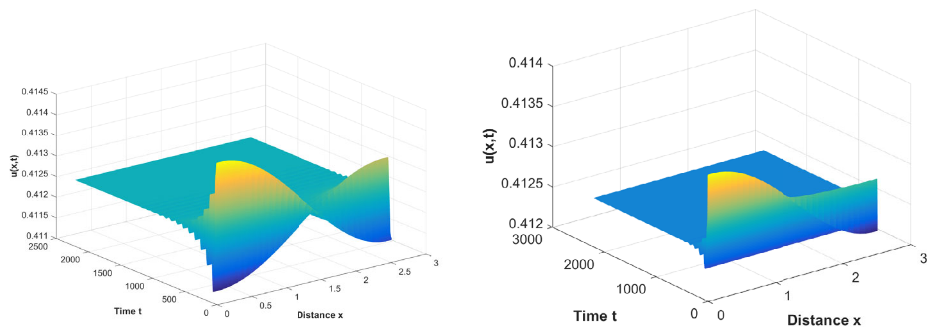

Figure 2 Numerical simulations for prey population with $\beta = 0.18$ $\tau = 22$. (Left: model (1.4). Right: model (2.9))

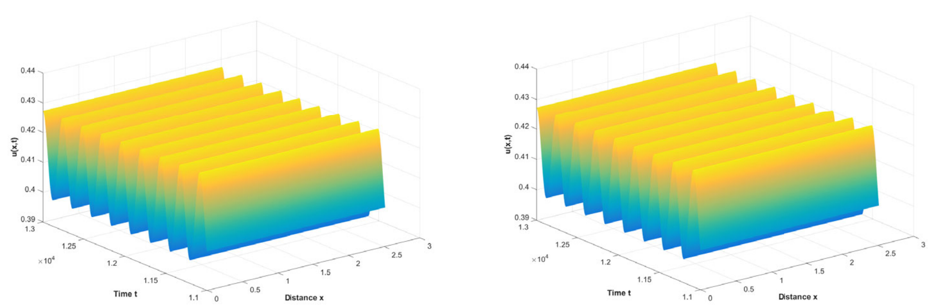

Figure 3 Numerical simulations for prey population with $\beta = 0.18$ $\tau = 22$. (Left: model (1.4). Right: model (2.9))

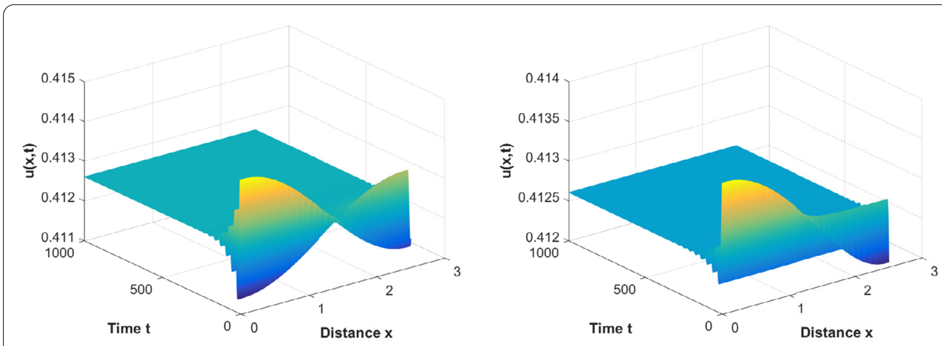

Figure 4 Numerical simulations for prey population with $\beta = 0.5$ $\tau = 5$. (Left: model (1.4). Right: model (2.9))

stable (Fig. 5 right). When $\tau > \tau_0^0$, for model (1.4), the coexistence equilibrium E_1 is unstable and the unstably spatial homogeneous periodic solution exists. The stably spatial inhomogeneous periodic solution still exists (Fig. 6 left). However, for the model (2.9), the coexistence equilibrium (u_*, v_*) is unstable, and the stably spatial homogeneous periodic solution appears (Fig. 6 right).

5 Conclusion

In this paper, we study a delayed diffusive predator–prey model with nonlocal competition in prey and schooling behavior among predators. We mainly study the local stability

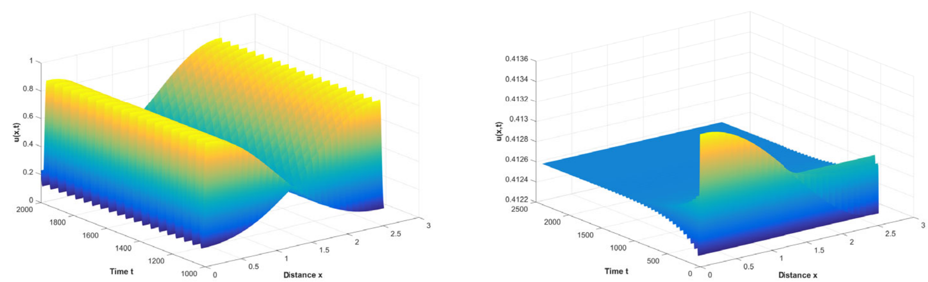

Figure 5 Numerical simulations for prey population with $\beta = 0.5$ $\tau = 12$. (Left: model (1.4). Right: model (2.9))

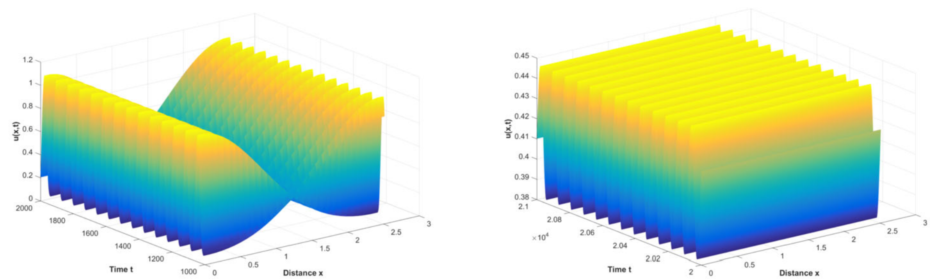

Figure 6 Numerical simulations for prey population with $\beta = 0.5$ $\tau = 22$. (Left: model (1.4). Right: model (2.9))

of coexisting equilibrium and the existence of Hopf bifurcation. We also studied the property of bifurcating periodic solutions by the normal form method and center manifold theorem. Our results show that diffusion and delay can induce a spatially inhomogeneous periodic solution, which is usually unstable. However, the model incorporating nonlocal competition may have a stably spatially inhomogeneous periodic solution.

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Availability of data and materials
Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
All authors contributed to the study conception and design. Material preparation, data collection, and analysis were performed by RY. All authors read and approved the final manuscript.

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