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Blow-up conditions of nonlinear parabolic equations and systems under mixed nonlinear boundary conditions

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Abstract

In this paper, we firstly discuss blow-up phenomena for nonlinear parabolic equations

$$u_t = \nabla \cdot \left[\rho(u)\nabla u\right] + f(x,t,u), \quad \text{in } \Omega \times (0,t^*),$$

under mixed nonlinear boundary conditions $\frac{\partial u}{\partial n} + \theta(z)u = h(z,t,u)$ on $\Gamma_1 \times (0,t^*)$ and $u = 0$ on $\Gamma_2 \times (0,t^*)$, where Ω is a bounded domain and Γ_1 and Γ_2 are disjoint subsets of a boundary $\partial\Omega$. Here, f and h are real-valued C^1-functions and ρ is a positive C^1-function. To obtain the blow-up solutions, we introduce the following blow-up conditions:

$$(C_\rho): \quad \begin{aligned} &(2+\epsilon)\int_0^u \rho(w)f(x,t,w)\,dw \le u\rho(u)f(x,t,u) + \beta_1 u^2 + \gamma_1, \\ &(2+\epsilon)\int_0^u \rho^2(w)h(z,t,w)\,dw \le u\rho^2(u)h(z,t,u) + \beta_2 u^2 + \gamma_2, \end{aligned}$$

for $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, and $u \in \mathbb{R}$ for some constants ϵ, β_1, β_2, γ_1, and γ_2 satisfying

$$\epsilon > 0, \quad \beta_1 + \frac{\lambda_R + 1}{\lambda_S}\beta_2 \le \frac{\rho_m^2 \lambda_R}{2}\epsilon \quad \text{and} \quad 0 \le \beta_2 \le \frac{\rho_m^2 \lambda_S}{2}\epsilon,$$

where $\rho_m := \inf_{s>0}\rho(s)$, λ_R is the first Robin eigenvalue and λ_S is the first Steklov eigenvalue. Lastly, we discuss blow-up solutions for nonlinear parabolic systems.

MSC: 35B44; 35K61

Keywords: Blow-up; Mixed nonlinear boundary; Nonlinear parabolic equation

1 Introduction

In this paper, we firstly discuss blow-up solutions to the nonlinear parabolic equations under mixed nonlinear boundary conditions

$$\begin{cases} u_t = \nabla \cdot [\rho(u)\nabla u] + f(x,t,u), & \text{in } \Omega \times (0,t^*), \\ \frac{\partial u}{\partial n} + \theta(z)u = h(z,t,u), & \text{on } \Gamma_1 \times (0,t^*), \\ u = 0, & \text{on } \Gamma_2 \times (0,t^*), \\ u(\cdot,0) = u_0, & \text{in } \overline{\Omega}. \end{cases} \tag{1}$$

Next, we deal with blow-up solutions to the nonlinear parabolic systems under mixed nonlinear boundary conditions

$$\begin{cases} u_t = \nabla \cdot [\rho_1(u)\nabla u] + f_1(x,t,u,v), & & \text{in } \Omega \times (0,t^*), \\ v_t = \nabla \cdot [\rho_2(v)\nabla v] + f_2(x,t,u,v), & & \text{in } \Omega \times (0,t^*), \\ \frac{\partial u}{\partial n} + \theta(z)u = h_1(x,t,u,v), & \frac{\partial v}{\partial n} + \theta(z)v = h_2(x,t,u,v), & \text{on } \Gamma_1 \times (0,t^*), \\ u = v = 0 & & \text{on } \Gamma_2 \times (0,t^*), \\ u(\cdot,0) = u_0, & v(\cdot,0) = v_0, & \text{in } \overline{\Omega}. \end{cases} \tag{2}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with the smooth boundary $\partial\Omega$ and Γ_1, Γ_2 are disjoint open and closed subsets of $\partial\Omega$, respectively, such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$. Also, t^* is the maximal existence time of the solution u (or the solution pair (u,v)).

Also, we assume that f is a real-valued $C^1(\Omega \times \mathbb{R}^+ \times \mathbb{R})$-function, f_1 and f_2 are real-valued $C^1(\Omega \times \mathbb{R}^+ \times \mathbb{R}^2)$-functions, h is a real-valued $C^1(\partial\Omega \times \mathbb{R}^+ \times \mathbb{R})$-function, h_1 and h_2 are real-valued $C^1(\partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^2)$-functions, ρ, ρ_1, and ρ_2 are positive and nonincreasing $C^2(\mathbb{R}^+)$-functions satisfying

$$\inf_{s>0} \rho(s) > 0, \qquad \inf_{s>0} \rho_1(s) > 0, \quad \text{and} \quad \inf_{s>0} \rho_2(s) > 0,$$

and θ is a nonnegative $C^1(\partial\Omega)$-function, where $\mathbb{R}^+ := (0,\infty)$. Moreover, the initial data u_0 and v_0 are assumed to be nontrivial $C^1(\overline{\Omega})$-functions which are compatible with the boundary conditions.

Equation (1) and system (2) appear in several branches of applied sciences. For example, they represent some ecosystems or chemical reaction models such as heat processes in one or more component mixtures. Also, we can consider the above boundary conditions as a migration during in these processes (see [1, 2] and the references therein).

Some special cases of equation (1) and system (2) have been studied from various perspectives with respect to the blow-up property (see [3–14]). For example, Enache studied the following nonlinear parabolic equations:

$$u_t = \nabla \cdot \left[\rho(u)\nabla u\right] + f(u) \quad \text{in } \Omega \times \left(0,t^*\right) \tag{3}$$

under the Dirichlet boundary conditions and the Robin boundary conditions in [15, 16], respectively, where ρ is a positive and nonincreasing $C^2(\mathbb{R}^+)$-function and f is a nonnegative differentiable function. He obtained the blow-up solutions to equation (3) by using a condition

$$(A_\rho) : (2+\epsilon)\int_0^u f(s)\rho(s)\,ds \le uf(u)\rho(u), \quad u>0,$$

for some $\epsilon > 0$.

In fact, condition (A_ρ) has been frequently used to study the blow-up phenomena of nonlinear parabolic equations and systems. There are lots of research works on the equations and systems in which the functions f and h are replaced by a separable type functions in equation (1) and system (2) (see [17–21]). For the example of the systems, Baghaei and Hesaaraki [20] studied the following nonlinear parabolic systems under the nonlinear boundary conditions:

$$\begin{cases} u_t = \sum_{i,j=1}(a^{ij}(x)u_{x_i})_{x_j} - f_1(u,v), & \text{in } \Omega\times(0,t^*), \\ v_t = \sum_{i,j=1}(a^{ij}(x)v_{x_i})_{x_j} - f_2(u,v), & \text{in } \Omega\times(0,t^*), \\ \sum_{i,j=1} a^{ij}(x)u_{x_i}n_j = h_1(u,v), & \text{on } \partial\Omega\times(0,t^*), \\ \sum_{i,j=1} a^{ij}(x)v_{x_i}n_j = h_2(u,v), & \text{on } \partial\Omega\times(0,t^*), \\ u(\cdot,0)=u_0\ge 0, \qquad v(\cdot,0)=v_0\ge 0, & \text{in } \overline{\Omega}, \end{cases} \tag{4}$$

where f_1, f_2, h_1, and h_2 are nonnegative locally Lipschitz continuous functions. They obtained the blow-up solutions by using the condition

$$(A) : \begin{array}{l} (2+\epsilon_1)F(r,s) \ge rf_1(r,s) + sf_2(r,s), \\ (2+\epsilon_2)H(r,s) \le rh_1(r,s) + sh_2(r,s) \end{array}$$

for some $0 \le \epsilon_1 \le \epsilon_2$ with $\epsilon_2 > 0$, where the functions F and H satisfy

$$\frac{\partial F}{\partial r} = f_1(r,s), \qquad \frac{\partial F}{\partial s} = f_2(r,s), \qquad \frac{\partial H}{\partial r} = h_1(r,s), \quad \text{and} \quad \frac{\partial H}{\partial s} = h_2(r,s).$$

On the other hand, Chung and Choi [22] studied the following nonlinear parabolic equations:

$$u_t = \Delta u + f(u) \quad \text{in } \Omega\times\bigl(0,t^*\bigr) \tag{5}$$

under the Dirichlet boundary condition, where f is a nonnegative locally Lipschitz function. They improved the blow-up conditions (A_ρ) for $\rho\equiv 1$ such that

$$(C) : (2+\epsilon)\int_0^u f(s)\,ds \le uf(u) + \beta u^2 + \gamma$$

for some constants $\epsilon > 0$, $0 \le \beta \le \frac{\lambda_D}{2}\epsilon$, $\gamma > 0$. Here, λ_D is the first Dirichlet eigenvalue of the Laplace operator Δ.

In 2021, the authors [23] studied the blow-up solutions for the nonlinear parabolic equations

$$\bigl(b(u)\bigr)_t = \nabla\cdot\bigl[\rho(u)\nabla u\bigr] + f(x,t,u) \quad \text{in } \Omega\times\bigl(0,t^*\bigr) \tag{6}$$

under mixed boundary conditions, where ρ is a positive and nonincreasing $C^2(\mathbb{R})$-function and f is a nonnegative $C^2(\Omega \times \mathbb{R}^+ \times \mathbb{R})$-function. They obtained the blow-up solutions by using the modified version of condition (C).

It is well known that the blow-up phenomena are greatly influenced by the shape of domains (see [24]). However, most of all blow-up conditions do not depend on the domains and the boundary conditions. Therefore, it is worthwhile to notice that the above condition (C) depends on the domain Ω, since the first eigenvalue of the Laplace operator depends on the domains.

From the above point of view, we obtained the blow-up condition for the solutions to equation (1) as follows:

$$(C_\rho): \begin{array}{l} (2+\epsilon)F(x,t,u) \le u\rho(u)f(x,t,u) + \beta_1 u^2 + \gamma_1, \\ (2+\epsilon)H(z,t,u) \le u\rho^2(u)h(z,t,u) + \beta_2 u^2 + \gamma_2, \end{array}$$

for $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, and $u \in \mathbb{R}$, for some constants ϵ, β_1, β_2, γ_1, and γ_2, satisfying

$$\epsilon > 0, \quad \beta_1 + \frac{\lambda_R + 1}{\lambda_S}\beta_2 \le \frac{\rho_m^2 \lambda_R}{2}\epsilon, \quad \text{and} \quad 0 \le \beta_2 \le \frac{\rho_m^2 \lambda_S}{2}\epsilon,$$

where $F(x,t,u) := \int_0^u \rho(w) f(x,t,w)\,dw$ and $H(z,t,u) := \int_0^u \rho^2(w) h(z,t,w)\,dw$. Here, $\rho_m := \inf_{s>0} \rho(s)$, λ_R is the first eigenvalue of the Robin eigenvalue problem, and λ_S is the first eigenvalue of the Steklov eigenvalue problem.

Because we deal with the function f in the reaction terms and the function h in the boundary terms, it is important to find the blow-up conditions which depend on the domains and the boundary conditions. From this point, it is worth noticing that information on domain and boundary was applied to the blow-up condition (C_ρ) by using the first Robin eigenvalue and Steklov eigenvalue of the Laplace operator, respectively.

In most of the research results on blow-up, functions f and h have been assumed to be nonnegative. In addition, functions of separable types such as $k(t)f(u)$ or $f(u)$ have been considered. However, the functions f and h in this paper are real-valued functions and can be non-separable, which is one of our main purposes.

Our boundary conditions include various boundary conditions such as the Dirichlet boundary condition, the Neumann boundary condition, the Robin boundary conditions, and so on. One of the meanings of our result is a unified approach.

We organize this paper as follows. In Section 2, we deal with the blow-up solutions to equations (1). In Section 3, we discuss the blow-up solutions to systems (2).

2 Blow-up phenomena: nonlinear parabolic equations

In this section, we discuss blow-up solutions to the nonlinear parabolic equations under the mixed nonlinear boundary conditions (1). We introduce the definition of the blow-up.

Definition 2.1 We say that a solution u to equation (1) blows up in finite time $t^* > 0$ whenever $\int_\Omega u^2(x,t)\,dx \to +\infty$ as $t \nearrow t^*$.

Now, we introduce the first Robin eigenvalue and the first Steklov eigenvalue.

Lemma 2.2 (See [25, 26]) *There exist $\lambda_R \geq 0$ and a nonnegative function $\phi_0 \in W^{1,2}(\Omega)$ such that*

$$\begin{cases} -\Delta\phi_0(x) = \lambda_R\phi_0(x), & x \in \Omega, \\ \frac{\partial\phi_0}{\partial n}(z) + \theta(z)\phi_0(z) = 0, & z \in \Gamma_1, \\ \phi_0(x) = 0, & x \in \Gamma_2. \end{cases}$$

Moreover, λ_R is given by

$$\lambda_R := \inf_{\substack{w\in W^{1,2}(\Omega)\\ w\not\equiv 0}} \frac{\int_\Omega |\nabla w|^2\,dx + \int_{\Gamma_1}\theta(z)w^2\,dS}{\int_\Omega w^2\,dx}.$$

Lemma 2.3 (See [25, 27]) *Let $\Gamma_1 \neq \emptyset$. Then there exist $\lambda_S > 0$ and a nonnegative function $\phi_0 \in W^{1,2}(\Omega)$ such that*

$$\begin{cases} \Delta\phi_0(x) = \phi_0(x), & x \in \Omega, \\ \frac{\partial\phi_0}{\partial n}(z) + \theta(z)\phi_0(z) = \lambda_S\phi_0(z), & z \in \Gamma_1, \\ \phi_0(x) = 0, & x \in \Gamma_2. \end{cases}$$

Moreover, λ_S is given by

$$\lambda_S := \inf_{\substack{w\in W^{1,2}(\Omega)\\ w\not\equiv 0}} \frac{\int_\Omega [|\nabla w|^2 + w^2]\,dx + \int_{\Gamma_1}\theta(z)w^2\,dS}{\int_{\Gamma_1} w^2\,dS}.$$

Now, we state the main theorem.

Theorem 2.4 *Suppose that the functions f and h satisfy condition (C_ρ). In addition, we assume that F and H are nondecreasing in t. Moreover, we assume that the function ρ satisfies*

$$\lim_{s\to 0+} s^2\rho(s) = 0. \tag{7}$$

If the initial data u_0 satisfies

$$\begin{aligned} &-\frac{1}{2}\int_\Omega \rho^2(u_0)|\nabla u_0|^2\,dx + \int_\Omega \left[F(x,0,u_0) - \frac{\gamma_1}{2+\epsilon}\right]dx \\ &\quad + \int_{\Gamma_1}\left[H(z,0,u_0) - \theta(z)\int_0^{u_0} s\rho^2(s)\,ds - \frac{\gamma_2}{2+\epsilon}\right]dS > 0, \end{aligned} \tag{8}$$

then every solution u to equation (1) blows up at finite time $t^* > 0$.

Proof For a solution $u(x,t)$, we define functions A and B on $[0,\infty)$ by

$$A(t) := 2\int_\Omega\int_0^{u(x,t)} s\rho(s)\,ds\,dx + 1$$

and

$$
\begin{aligned}
B(t) :=& -\frac{1}{2}\int_\Omega \rho^2\big(u(x,t)\big)\big|\nabla u(x,t)\big|^2\,dx + \int_\Omega \left[F\big(x,t,u(x,t)\big) - \frac{\gamma_1}{2+\epsilon}\right]dx \\
&+ \int_{\Gamma_1}\left[H\big(z,t,u(z,t)\big) - \theta(z)\int_0^{u(z,t)} s\rho^2(s)\,ds - \frac{\gamma_2}{2+\epsilon}\right]dS
\end{aligned}
$$

for each $t \geq 0$. Firstly, we assume that $\Gamma_1 \neq \emptyset$. It follows from integration by parts and the assumption $\rho' \leq 0$ that

$$
\begin{aligned}
A'(t) &= 2\int_\Omega u\rho(u)u_t\,dx \\
&= 2\int_\Omega u\rho(u)\big[\nabla\cdot\big[\rho(u)\nabla u\big] + f(x,t,u)\big]\,dx \\
&= 2\int_{\partial\Omega} u\rho^2(u)\frac{\partial u}{\partial n}\,dS - 2\int_\Omega\big[\rho^2(u)|\nabla u|^2 + \rho(u)\rho'(u)|\nabla u|^2\big]\,dx \\
&\quad + 2\int_\Omega u\rho(u)f(x,t,u)\,dx \\
&\geq -2\int_\Omega \rho^2(u)|\nabla u|^2\,dx + 2\int_\Omega u\rho(u)f(x,t,u)\,dx \\
&\quad + 2\int_{\Gamma_1} u\rho^2(u)\big[h(z,t,u) - \theta(z)u\big]\,dS
\end{aligned} \tag{9}
$$

for all $t \in (0,t^*)$. Applying condition (C_ρ) to inequality (9), we obtain

$$
\begin{aligned}
A'(t) &\geq -2\int_\Omega \rho^2(u)|\nabla u|^2 - 2\int_{\Gamma_1}\theta(z)u^2\rho^2(u)\,dS \\
&\quad + 2\int_\Omega\big[(2+\epsilon)F(x,t,u) - \beta_1u^2 - \gamma_1\big]\,dx \\
&\quad + 2\int_{\Gamma_1}\big[(2+\epsilon)H(z,t,u) - \beta_2u^2 - \gamma_2\big]\,dS \\
&= 2(2+\epsilon)B(t) + \epsilon\int_\Omega \rho^2(u)|\nabla u|^2\,dx + (4+2\epsilon)\int_{\Gamma_1}\left[\theta(z)\int_0^u s\rho^2(s)\,ds\right]dS \\
&\quad - 2\int_{\Gamma_1}\theta(z)u^2\rho^2(u)\,dS - 2\beta_1\int_\Omega u^2\,dx - 2\beta_2\int_{\Gamma_1}u^2\,dS \\
&\geq 2(2+\epsilon)B(t) + \epsilon\left[\int_\Omega \rho^2(u)|\nabla u|^2\,dx + \int_{\Gamma_1}\theta(z)u^2\rho^2(u)\,dS\right] \\
&\quad - 2\beta_1\int_\Omega u^2\,dx - 2\beta_2\int_{\Gamma_1}u^2\,dS
\end{aligned}
$$

for all $t \in (0,t^*)$. Here, the last term can be estimated by using the following inequality:

$$
\int_0^u s\rho^2(s)\,ds = \frac{1}{2}u^2\rho^2(u) - \int_0^u s^2\rho(s)\rho'(s)\,ds \geq \frac{1}{2}u^2\rho^2(u). \tag{10}
$$

Therefore, we obtain from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} A'(t) &\geq 2(2+\epsilon)B(t) + \left(\rho_m^2\epsilon - \frac{2}{\lambda_S}\beta_2\right)\left[\int_\Omega |\nabla u|^2\,dx + \int_{\Gamma_1}\theta(z)u^2\,dS\right] \\ &\quad - \left(2\beta_1 + \frac{2}{\lambda_S}\beta_2\right)\int_\Omega u^2\,dx \\ &\geq 2(2+\epsilon)B(t) + \left(\rho_m^2\lambda_R\epsilon - \frac{2\lambda_R+2}{\lambda_S}\beta_2 - 2\beta_1\right)\int_\Omega u^2\,dx \\ &\geq 2(2+\epsilon)B(t) \end{aligned} \tag{11}$$

for all $t \in (0,t^*)$. On the other hand, it follows from the fact that F and H are nondecreasing in t and integration by parts that

$$\begin{aligned} B'(t) &= -\int_\Omega \left[\rho(u)\rho'(u)|\nabla u|^2 u_t + \rho^2(u)\nabla u \nabla u_t\right]dx \\ &\quad + \int_\Omega \left[\rho(u)f(x,t,u)u_t + \frac{\partial}{\partial t}F(x,t,u)\right]dx \\ &\quad + \int_{\Gamma_1}\left[\rho^2(u)h(z,t,u)u_t + \frac{\partial}{\partial t}H(z,t,u) - \theta(z)u\rho^2(u)u_t\right]dS \\ &\geq \int_\Omega \left[\rho'(u)|\nabla u|^2 + \rho(u)\Delta u\right]\rho(u)u_t\,dx - \int_{\partial\Omega}\rho^2(u)u_t\frac{\partial u}{\partial n}\,dS \\ &\quad + \int_\Omega \rho(u)f(x,t,u)u_t\,dx + \int_{\Gamma_1}\left[\rho^2(u)h(z,t,u)u_t - \theta(z)u\rho^2(u)u_t\right]dS \\ &= \int_\Omega \left[\nabla\cdot\left(\rho(u)\nabla u\right) + f(x,t,u)\right]\rho(u)u_t\,dx \\ &= \int_\Omega \rho(u)u_t^2\,dx \geq 0 \end{aligned} \tag{12}$$

for all $t \in (0,t^*)$. Considering (11), (12), and the initial data condition (8), it is easy to see that $A(t) > 1$, $A'(t) > 0$, $B(t) > 0$, and $B'(t) > 0$ for all $t \in (0,t^*)$. Now we use the Schwarz inequality and (11) to get

$$\begin{aligned} \frac{2+\epsilon}{2}A'(t)B(t) &\leq \frac{2+\epsilon}{2}A'(t)B(t) \\ &\leq \frac{1}{4}\left[A'(t)\right]^2 \\ &\leq \left(\int_\Omega u^2\rho(u)\,dx\right)\left(\int_\Omega \rho(u)u_t^2\,dx\right) \end{aligned} \tag{13}$$

for all $t \in (0,t^*)$. Integrating by parts, the use of $\rho' \leq 0$ and assumption (7) give

$$2\int_0^u s\rho(s)\,ds = s^2\rho(s)|_0^u - \int_0^u s^2\rho'(s)\,ds \geq u^2\rho(u). \tag{14}$$

Combining (13) and (14), we have

$$\frac{2+\epsilon}{2}A'(t)B(t) \le \left(\int_\Omega 2\int_0^u s\rho(s)\,ds\,dx\right)\left(\int_\Omega \rho(u)u_t^2\,dx\right) \\ \le A(t)B'(t) \tag{15}$$

for all $t \in (0, t^*)$. Then we obtain from (15) that

$$\frac{d}{dt}\left[A^{-\frac{2+\epsilon}{2}}(t)B(t)\right] \ge 0$$

for all $t \in (0, t^*)$. It follows that

$$A^{-\frac{2+\epsilon}{2}}(t)A'(t) \ge 2(2+\epsilon)A^{-\frac{2+\epsilon}{2}}(t)B(t) \ge 2(2+\epsilon)A^{-\frac{2+\epsilon}{2}}(0)B(0)$$

for all $t \in (0, t^*)$, which implies that

$$\frac{d}{dt}\left[A^{-\frac{\epsilon}{2}}(t)\right] = -\frac{\epsilon}{2}A^{-\frac{2+\epsilon}{2}}(t)A'(t) \\ \le -\epsilon(2+\epsilon)A^{-\frac{2+\epsilon}{2}}(0)B(0).$$

Integrating from 0 to t, we finally obtain

$$A(t) \ge \left[\frac{1}{A^{-\frac{\epsilon}{2}}(0) - \epsilon(2+\epsilon)A^{-\frac{2+\epsilon}{2}}(0)B(0)t}\right]^{\frac{2}{\epsilon}}. \tag{16}$$

Therefore, the solution u blows up at finite time $0 < t^* \le T$.

On the other hand, if $\Gamma_1 = \emptyset$, then it is trivial that the function h cannot affect the solution u. In this case, we can easily obtain the blow-up solution by following the above proof, by using the condition

$$(C_\rho) : (2+\epsilon)F(x,t,u) \le u\rho(u)f(x,t,u) + \beta_1 u^2 + \gamma_1$$

for some constants ϵ, β_1, and γ_1, satisfying $\epsilon > 0$ and $0 \le \beta_1 \le \frac{\rho_m^2\lambda_R}{2}\epsilon$. □

Remark 2.5

(i) We can easily obtain that

$$A(t) \le \rho(0)\int_\Omega u^2\,dx$$

i.e. $\lim_{t\to t^*} A(t) = \infty$ implies $\lim_{t\to t^*}\int_\Omega u^2\,dx = \infty$.

(ii) The upper bound T of the blow-up time t^* can be obtained from inequality (16):

$$T = \frac{A(0)}{\epsilon(2+\epsilon)B(0)}.$$

Now, we discuss nonnegative functions or nonpositive functions since, in fact, if the functions f and h have the same signs on $\Omega \times \mathbb{R}^+ \times \mathbb{R}$ and $\partial\Omega \times \mathbb{R}^+ \times \mathbb{R}$, respectively, then we can improve the blow-up condition (C_ρ).

Theorem 2.6 *Suppose that the function F is nonpositive. Also, we assume that the functions f and h satisfy the conditions*

$$(C_\rho):\quad \begin{aligned} &(2+\epsilon_1)F(x,t,u) \le u\rho(u)f(x,t,u) + \beta_1 u^2 + \gamma_1,\\ &(2+\epsilon_2)H(z,t,u) \le u\rho^2(u)h(z,t,u) + \beta_2 u^2 + \gamma_2, \end{aligned}$$

for all $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, $u > 0$, for some constants $\epsilon_1, \epsilon_2, \beta_1, \beta_2, \gamma_1, \gamma_2$, satisfying

$$0 \le \epsilon_1 \le \epsilon_2, \quad \epsilon_2 > 0, \qquad \beta_1 + \frac{\lambda_R + 1}{\lambda_S}\beta_2 \le \frac{\rho_m^2 \lambda_R}{2}\epsilon_2, \quad \text{and} \quad 0 \le \beta_2 \le \frac{\rho_m^2 \lambda_S}{2}\epsilon_2.$$

In addition, we assume that F and H are nondecreasing in t. Moreover, we assume that the function ρ satisfies

$$\lim_{s\to 0+} s^2 \rho(s) = 0.$$

If the initial data u_0 satisfies

$$\begin{aligned} &-\frac{1}{2}\int_\Omega \rho^2(u_0)|\nabla u_0|^2\,dx + \int_\Omega \left[F(x,0,u_0) - \frac{\gamma_1}{2+\epsilon_2}\right]dx\\ &\quad + \int_{\Gamma_1}\left[H(z,0,u_0) - \theta(z)\int_0^{u_0} s\rho^2(s)\,ds - \frac{\gamma_2}{2+\epsilon_2}\right]dS > 0, \end{aligned}$$

then every solution u to equation (1) blows up at finite time $t^* > 0$.

Proof The proof is basically similar to the proof of Theorem 2.4. Therefore, we state the main difference of the proof. For a solution $u(x,t)$, we define functions A and B on $[0,\infty)$ by

$$A(t) := 2\int_\Omega \int_0^{u(x,t)} s\rho(s)\,ds\,dx + 1$$

and

$$\begin{aligned} B(t) := &-\frac{1}{2}\int_\Omega \rho^2\bigl(u(x,t)\bigr)\bigl|\nabla u(x,t)\bigr|^2\,dx + \int_\Omega \left[F\bigl(x,t,u(x,t)\bigr) - \frac{\gamma_1}{2+\epsilon_2}\right]dx\\ &+ \int_{\Gamma_1}\left[H\bigl(z,t,u(z,t)\bigr) - \theta(z)\int_0^{u(z,t)} s\rho^2(s)\,ds - \frac{\gamma_2}{2+\epsilon_2}\right]dS \end{aligned}$$

for each $t \ge 0$. Now, applying condition (C_ρ) to (9), we obtain

$$\begin{aligned} A'(t) \ge &-2\int_\Omega \rho^2(u)|\nabla u|^2 - 2\int_{\Gamma_1}\theta(z)u^2\rho^2(u)\,dS\\ &+ 2\int_\Omega \bigl[(2+\epsilon_1)F(x,t,u) - \beta_1 u^2 - \gamma_1\bigr]\,dx\\ &+ 2\int_{\Gamma_1}\bigl[(2+\epsilon_2)H(z,t,u) - \beta_2 u^2 - \gamma_2\bigr]\,dS\\ \ge\ & 2(2+\epsilon_2)B(t) + \epsilon\int_\Omega \rho^2(u)|\nabla u|^2\,dx + (4+2\epsilon)\int_{\Gamma_1}\left[\theta(z)\int_0^u s\rho^2(s)\,ds\right]dS \end{aligned}$$

$$
\begin{aligned}
& -2\int_{\Gamma_1}\theta(z)u^2\rho^2(u)\,dS - 2\beta_1\int_\Omega u^2\,dx - 2\beta_2\int_{\Gamma_1}u^2\,dS \\
\geq{}& 2(2+\epsilon_2)B(t) + \epsilon\left[\int_\Omega \rho^2(u)|\nabla u|^2\,dx + \int_{\Gamma_1}\theta(z)u^2\rho^2(u)\,dS\right] \\
& -2\beta_1\int_\Omega u^2\,dx - 2\beta_2\int_{\Gamma_1}u^2\,dS
\end{aligned}
$$

for all $t \geq 0$. Hence, we have from Lemma 2.2 and Lemma 2.3 that

$$A'(t) \geq 2(2+\epsilon_2)B(t).$$

Also, by the same argument as inequality (12), we can obtain

$$B'(t) \geq \int_\Omega \rho(u)u_t^2\,dx \geq 0.$$

Therefore, we can easily obtain in a similar way as the proof of Theorem 2.4 that

$$A(t) \geq \left[\frac{1}{A^{-\frac{\epsilon_2}{2}}(0) - \epsilon_2(2+\epsilon_2)A^{-\frac{2+\epsilon_2}{2}}(0)B(0)t}\right]^{\frac{2}{\epsilon_2}}.$$

Hence, the solution u blows up at finite time $0 < t^* \leq T$. Furthermore, the upper bound T of the blow-up time satisfies

$$T = \frac{A(0)}{\epsilon_2(2+\epsilon_2)B(0)}. \qquad \square$$

Theorem 2.7 *Suppose that the function H is nonpositive. Also, we assume that the functions f and h satisfy the conditions*

$$
(C_\rho):\ \begin{aligned}
&(2+\epsilon_1)F(x,t,u) \leq u\rho(u)f(x,t,u) + \beta_1u^2 + \gamma_1, \\
&(2+\epsilon_2)H(z,t,u) \leq u\rho^2(u)h(z,t,u) + \beta_2u^2 + \gamma_2,
\end{aligned}
$$

for all $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, $u > 0$, for some constants ϵ_1, ϵ_2, β_1, β_2, γ_1, γ_2, satisfying

$$\epsilon_1 \geq \epsilon_2, \quad \epsilon_1 > 0, \qquad \beta_1 + \frac{\lambda_R+1}{\lambda_S}\beta_2 \leq \frac{\rho_m^2\lambda_R}{2}\epsilon_1, \quad \textit{and} \quad 0 \leq \beta_2 \leq \frac{\rho_m^2\lambda_S}{2}\epsilon_1.$$

In addition, we assume that F and H are nondecreasing in t. Moreover, we assume that the function ρ satisfies

$$\lim_{s\to 0+} s^2\rho(s) = 0.$$

If the initial data u_0 satisfies

$$
\begin{aligned}
& -\frac{1}{2}\int_\Omega \rho^2(u_0)|\nabla u_0|^2\,dx + \int_\Omega\left[F(x,0,u_0) - \frac{\gamma_1}{2+\epsilon_1}\right]dx \\
& + \int_{\Gamma_1}\left[H(z,0,u_0) - \theta(z)\int_0^{u_0} s\rho^2(s)\,ds - \frac{\gamma_2}{2+\epsilon_1}\right]dS > 0,
\end{aligned}
$$

then every solution u to equation (1) *blows up at finite time* $0 < t^* \leq T$ *with*

$$T = \frac{A(0)}{\epsilon_1(2+\epsilon_1)B(0)}.$$

Proof The proof is basically similar to the proof of Theorem 2.4 and Corollary 2.6. Therefore, one can easily complete this proof by following the proofs. □

Since t is the one of variables of the reaction term f, we can expect that condition (C_ρ) may depend on t. From this point of view, we obtain the following condition $(C_\rho)'$. In fact, condition $(C_\rho)'$ is the generalized version of condition (C_ρ):

$$(C_\rho)': \begin{array}{l} \big(2+\epsilon(t)\big)F(x,t,u) \leq u\rho(u)f(x,t,u) + \beta_1(t)u^2 + \gamma_1(x,t), \\ \big(2+\epsilon(t)\big)H(z,t,u) \leq u\rho^2(u)h(z,t,u) + \beta_2(t)u^2 + \gamma_2(z,t), \end{array}$$

for all $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, $u \in \mathbb{R}$, for some differentiable functions ϵ, β_1, β_2, γ_1, γ_2, satisfying

$$\inf_{s>0} \epsilon(s) > 0, \quad \beta_1(t) + \frac{\lambda_R + 1}{\lambda_S}\beta_2(t) \leq \frac{\rho_m^2\lambda_R}{2}\epsilon(t), \quad \text{and} \quad 0 \leq \beta_2(t) \leq \frac{\rho_m^2\lambda_S}{2}\epsilon(t)$$

for $t > 0$.

Theorem 2.8 *Let* $\Gamma_1 \neq \emptyset$. *Suppose that the functions f and h satisfy condition* $(C_\rho)'$. *In addition, we assume that*

$$F(x,t,u) - \frac{\gamma_1(x,t)}{2+\epsilon(t)} \quad \textit{and} \quad H(z,t,u) - \frac{\gamma_2(z,t)}{2+\epsilon(t)} \quad \textit{are nondecreasing in } t. \tag{17}$$

Moreover, we assume that the function ρ *satisfies*

$$\lim_{s\to 0+} s^2\rho(s) = 0.$$

If the initial data u_0 *satisfies*

$$\begin{aligned} &-\frac{1}{2}\int_\Omega \rho^2(u_0)|\nabla u_0|^2\,dx + \int_\Omega \left[F(x,0,u_0) - \frac{\gamma_1(x,0)}{2+\epsilon(0)}\right]dx \\ &\quad + \int_{\Gamma_1}\left[H(z,0,u_0) - \theta(z)\int_0^{u_0} s\rho^2(s)\,ds - \frac{\gamma_2(z,0)}{2+\epsilon(0)}\right]dS > 0, \end{aligned} \tag{18}$$

then every solution u to equation (1) *blows up at finite time* $0 < t^* \leq T$ *with*

$$T = \frac{A(0)}{\epsilon_m(2+\epsilon_m)B(0)},$$

where $\epsilon_m := \inf_{s>0}\epsilon(s)$.

Proof For a solution $u(x,t)$, we define functions A and B on $[0,\infty)$ by

$$A(t) := 2\int_\Omega\int_0^{u(x,t)} s\rho(s)\,ds\,dx + 1$$

and

$$B(t) := -\frac{1}{2}\int_\Omega \rho^2\big(u(x,t)\big)\big|\nabla u(x,t)\big|^2\,dx + \int_\Omega \left[F\big(x,t,u(x,t)\big) - \frac{\gamma_1(x,t)}{2+\epsilon(t)}\right]dx + \int_{\Gamma_1}\left[H\big(z,t,u(z,t)\big) - \theta(z)\int_0^{u(z,t)} s\rho^2(s)\,ds - \frac{\gamma_2(z,t)}{2+\epsilon(t)}\right]dS$$

for each $t \geq 0$. Then the proof is basically similar to the proof of Theorem 2.4. Applying condition $(C_\rho)'$ to inequality (9), we can obtain

$$\begin{aligned}
A'(t) &\geq -2\int_\Omega \rho^2(u)|\nabla u|^2 - 2\int_{\Gamma_1}\theta(z)u^2\rho^2(u)\,dS \\
&\quad + 2\int_\Omega \big[\big(2+\epsilon(t)\big)F(x,t,u) - \beta_1(t)u^2 - \gamma_1(x,t)\big]\,dx \\
&\quad + 2\int_{\Gamma_1}\big[\big(2+\epsilon(t)\big)H(z,t,u) - \beta_2(t)u^2 - \gamma_2(z,t)\big]\,dS \\
&= 2\big[2+\epsilon(t)\big]B(t) + \epsilon(t)\int_\Omega \rho^2(u)|\nabla u|^2\,dx \\
&\quad + \big[4+2\epsilon(t)\big]\int_{\Gamma_1}\theta(z)\int_0^u s\rho^2(s)\,ds\,dS - 2\int_{\Gamma_1}\theta(z)u^2\rho^2(u)\,dS \\
&\quad - 2\beta_1(t)\int_\Omega u^2\,dx - 2\beta_2(t)\int_{\Gamma_1}u^2\,dS \\
&\geq 2\big[2+\epsilon(t)\big]B(t) + \epsilon(t)\left[\int_\Omega \rho^2(u)|\nabla u|^2\,dx + \int_{\Gamma_1}\theta(z)u^2\rho^2(u)\,dS\right] \\
&\quad - 2\beta_1(t)\int_\Omega u^2\,dx - 2\beta_2(t)\int_{\Gamma_1}u^2\,dS
\end{aligned}$$

for all $t \in (0,t^*)$. Here, the last term can be estimated by using inequality (10). Therefore, we obtain from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned}
A'(t) &\geq 2\big[2+\epsilon(t)\big]B(t) + \left[\rho_m^2\epsilon(t) - \frac{2}{\lambda_S}\beta_2(t)\right]\left[\int_\Omega |\nabla u|^2\,dx + \int_{\Gamma_1}\theta(z)u^2\,dS\right] \\
&\quad - \left[2\beta_1(t) + \frac{2}{\lambda_S}\beta_2(t)\right]\int_\Omega u^2\,dx \\
&\geq 2\big[2+\epsilon(t)\big]B(t) + \left[\rho_m^2\lambda_R\epsilon(t) - \frac{2\lambda_R+2}{\lambda_S}\beta_2(t) - 2\beta_1(t)\right]\int_\Omega u^2\,dx \\
&\geq 2\big[2+\epsilon(t)\big]B(t)
\end{aligned} \tag{19}$$

for all $t \in (0,t^*)$. On the other hand, we have from condition (17) and integration by parts that

$$\begin{aligned}
B'(t) &= -\int_\Omega \big[\rho(u)\rho'(u)|\nabla u|^2 u_t + \rho^2(u)\nabla u\nabla u_t\big]\,dx \\
&\quad + \int_\Omega \left[\rho(u)f(x,t,u)u_t + \frac{\partial}{\partial t}\left(F(x,t,u) - \frac{\gamma_1(x,t)}{2+\epsilon(t)}\right)\right]dx \\
&\quad + \int_{\Gamma_1}\left[\rho^2(u)h(z,t,u)u_t - \theta(z)u\rho^2(u)u_t + \frac{\partial}{\partial t}\left(H(z,t,u) - \frac{\gamma_2(z,t)}{2+\epsilon(t)}\right)\right]dS
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_\Omega \bigl[\rho'(u)|\nabla u|^2 + \rho(u)\Delta u\bigr]\rho(u)u_t\,dx - \int_{\partial\Omega} \rho^2(u)u_t \frac{\partial u}{\partial n}\,dS \\
&\quad + \int_\Omega \rho(u)f(x,t,u)u_t\,dx + \int_{\Gamma_1} \bigl[\rho^2(u)h(z,t,u)u_t - \theta(z)u\rho^2(u)u_t\bigr]\,dS \\
&= \int_\Omega \bigl[\nabla\cdot\bigl(\rho(u)\nabla u\bigr) + f(x,t,u)\bigr]\rho(u)u_t\,dx \\
&= \int_\Omega \rho(u)u_t^2\,dx \geq 0
\end{aligned} \tag{20}
$$

for all $t \in (0, t^*)$. Considering (19), (20), and the initial data condition (18), it is easy to see that $A(t) > 1$, $A'(t) > 0$, $B(t) > 0$, and $B'(t) > 0$ for all $t \in (0, t^*)$. Now we use the Schwarz inequality and (19) to get

$$
\begin{aligned}
\frac{2+\epsilon_m}{2}A'(t)B(t) &\leq \frac{2+\epsilon(t)}{2}A'(t)B(t) \\
&\leq \frac{1}{4}\bigl[A'(t)\bigr]^2 \\
&\leq \left(\int_\Omega u^2\rho(u)\,dx\right)\left(\int_\Omega \rho(u)u_t^2\,dx\right)
\end{aligned} \tag{21}
$$

for all $t \in (0, t^*)$, where $\epsilon_m := \inf_{s>0}\epsilon(s)$. Applying (14) to (21), we have

$$
\begin{aligned}
\frac{2+\epsilon_m}{2}A'(t)B(t) &\leq \frac{2+\epsilon(t)}{2}A'(t)B(t) \\
&\leq \left(\int_\Omega 2\int_0^u s\rho(s)\,ds\,dx\right)\left(\int_\Omega \rho(u)u_t^2\,dx\right) \\
&\leq A(t)B'(t)
\end{aligned} \tag{22}
$$

for all $t \in (0, t^*)$. Then we obtain from (22) that

$$
\frac{d}{dt}\bigl[A^{-\frac{2+\epsilon_m}{2}}(t)B(t)\bigr] \geq 0
$$

for all $t \in (0, t^*)$. Hence, we can obtain the following in a similar way as the proof of Theorem 2.4:

$$
A(t) \geq \left[\frac{1}{A^{\frac{-\epsilon_m}{2}}(0) - \epsilon_m(2+\epsilon_m)A^{-\frac{2+\epsilon_m}{2}}(0)B(0)t}\right]^{\frac{2}{\epsilon_m}}.
$$

Therefore, the solution u blows up at finite time $0 < t^* \leq T$. Furthermore, the upper bound T of the blow-up time satisfies

$$
T = \frac{A(0)}{\epsilon_m(2+\epsilon_m)B(0)}. \qquad \square
$$

Remark 2.9 Let us assume that $\epsilon'(t) \le 0$, $t > 0$. Then we can obtain another upper bound of the blow-up time. More precisely, we obtain from (22) and the fact $A(t) > 1$ that

$$\begin{aligned}
\frac{d}{dt}\bigl[A^{-\frac{2+\epsilon(t)}{2}}(t)B(t)\bigr] &= -\frac{2+\epsilon(t)}{2}A^{-1-\frac{2+\epsilon(t)}{2}}(t)A'(t)B(t) + A^{-\frac{2+\epsilon(t)}{2}}B'(t) \\
&\quad - \frac{\epsilon'(t)}{2}A^{-\frac{2+\epsilon(t)}{2}}(t)B(t)\ln A(t) \\
&\ge -\frac{\epsilon'(t)}{2}A^{-\frac{2+\epsilon(t)}{2}}(t)B(t)\ln A(t) \ge 0
\end{aligned}$$

for all $t \in (0, t^*)$. It follows that

$$A^{-\frac{2+\epsilon(t)}{2}}(t)A'(t) \ge 2\bigl(2+\epsilon(t)\bigr)A^{-\frac{2+\epsilon(t)}{2}}(t)B(t) \ge 2\bigl(2+\epsilon(t)\bigr)A^{-\frac{2+\epsilon(0)}{2}}(0)B(0)$$

for all $t \in (0, t^*)$, which implies that

$$\begin{aligned}
\frac{d}{dt}\bigl[A^{-\frac{\epsilon_m}{2}}(t)\bigr] &= -\frac{\epsilon_m}{2}A^{-\frac{2+\epsilon_m}{2}}(t)A'(t) \\
&\le -\frac{\epsilon_m}{2}A^{-\frac{2+\epsilon(t)}{2}}(t)A'(t) \\
&\le -\epsilon_m A^{-\frac{2+\epsilon(0)}{2}}(0)B(0)\bigl[2+\epsilon(t)\bigr].
\end{aligned}$$

Integrating from 0 to t, we finally obtain

$$A(t) \ge \left[\frac{1}{A^{-\frac{\epsilon_m}{2}}(0) - \epsilon_m A^{-\frac{2+\epsilon(0)}{2}}(0)B(0)\int_0^t [2+\epsilon(s)]\,ds}\right]^{\frac{2}{\epsilon_m}}.$$

Therefore, the solution u blows up at finite time $0 < t^* \le T$. Furthermore, the upper bound T of the blow-up time satisfies

$$\int_0^T \bigl[2+\epsilon(s)\bigr]\,ds = \frac{A^{\frac{2+\epsilon(0)-\epsilon_m}{2}}(0)}{\epsilon_m B(0)}.$$

Remark 2.10 Condition (C) was discussed by Chung and Choi (see [22]). From a careful reading of their analysis, we can obtain that

$$(C_\rho) \text{ holds if and only if } \quad \begin{aligned}
F(x,t,u) &= u^{2+\epsilon}g_1(x,t,u) + \frac{\beta_1}{\epsilon}u^2 + \frac{\gamma_1}{2+\epsilon}, \\
H(z,t,u) &= u^{2+\epsilon}g_2(z,t,u) + \frac{\beta_2}{\epsilon}u^2 + \frac{\gamma_2}{2+\epsilon},
\end{aligned}$$

for some real-valued continuous functions g_1 and g_2 which are nondecreasing in u.

3 Blow-up phenomena: nonlinear parabolic systems

In this section, we discuss blow-up solutions to the nonlinear parabolic systems under the mixed nonlinear boundary conditions (2). In this section, we assume that, for the functions f_1, f_2, h_1, and h_2, there exist functions F and H such that

$$\frac{\partial}{\partial r}F(x,t,r,s) = \rho_1(r)f_1(x,t,r,s), \qquad \frac{\partial}{\partial s}F(x,t,r,s) = \rho_2(s)f_2(x,t,r,s)$$

and

$$\frac{\partial}{\partial r}H(z,t,r,s) = \rho_1^2(r)h_1(x,t,r,s), \qquad \frac{\partial}{\partial s}H(z,t,r,s) = \rho_2^2(s)h_2(x,t,r,s).$$

Now, we introduce a condition for functions f_1, f_2, h_1, and h_2 as follows:

$$(C_\rho): \begin{aligned} (2+\epsilon)F(x,t,u,v) &\le u\rho_1(u)f_1(x,t,u,v) + v\rho_2(v)f_2(x,t,u,v) \\ &\quad + \beta_1 u^2 + \beta_2 v^2 + \gamma_1, \\ (2+\epsilon)H(z,t,u,v) &\le u\rho_1^2(u)h_1(z,t,u,v) + v\rho_2^2(v)h_2(z,t,u,v) \\ &\quad + \beta_3 u^2 + \beta_4 v^2 + \gamma_2 \end{aligned}$$

for all $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, $u \in \mathbb{R}$, and $v \in \mathbb{R}$, for some constants ϵ, β_1, β_2, β_3, β_4, γ_1, γ_2, satisfying

$$\begin{aligned} &\epsilon > 0, \quad \beta_1 + \frac{\lambda_R + 1}{\lambda_S}\beta_3 \le \frac{\rho_{1,m}^2 \lambda_R}{2}\epsilon, \qquad \beta_2 + \frac{\lambda_R + 1}{\lambda_S}\beta_4 \le \frac{\rho_{2,m}^2 \lambda_R}{2}\epsilon, \\ &0 \le \beta_3 \le \frac{\rho_{1,m}^2 \lambda_S}{2}\epsilon, \quad \text{and} \quad 0 \le \beta_4 \le \frac{\rho_{2,m}^2 \lambda_S}{2}\epsilon, \end{aligned}$$

where $\rho_{1,m} := \inf_{s>0} \rho_1(s)$ and $\rho_{2,m} := \inf_{s>0} \rho_2(s)$.

Now, we discuss the blow-up solutions to system (2).

Theorem 3.1 *Let $\Gamma_1 \neq \emptyset$. Suppose that the functions f_1, f_2 satisfy conditions (C_ρ). In addition, we assume that F and H are nondecreasing in t. Moreover, we assume that the functions ρ_1 and ρ_2 satisfy*

$$\lim_{s\to 0+} s^2 \rho_1(s) = \lim_{s\to 0+} s^2 \rho_2(s) = 0. \tag{23}$$

If the initial data u_0 satisfies

$$\begin{aligned} &-\frac{1}{2}\int_\Omega \left[\rho_1^2(u_0)|\nabla u_0|^2 + \rho_2^2(v_0)|\nabla v_0|^2\right] dx + \int_\Omega \left[F(x,0,u_0,v_0) - \frac{\gamma_1}{2+\epsilon}\right] dx \\ &\quad + \int_{\Gamma_1} \left[H(z,0,u_0,v_0) - \theta(z)\left[\int_0^{u_0} s\rho_1^2(s) + \int_0^{v_0} s\rho_2^2(s)\,ds\right] - \frac{\gamma_2}{2+\epsilon}\right] dS > 0, \end{aligned} \tag{24}$$

then every solution pair (u,v) to system (2) blows up at finite time t^*.

Proof First of all, we define functionals A and B by

$$A(t) := 2\int_\Omega \left[\int_0^{u(x,t)} s\rho_1(s)\,ds + \int_0^{v(x,t)} s\rho_2(s)\,ds\right] dx + 1$$

and

$$
\begin{aligned}
B(t) := &-\frac{1}{2}\int_\Omega \left[\rho_1^2\big(u(x,t)\big)\left|\nabla u(x,t)\right|^2 + \rho_2^2\big(v(x,t)\big)\left|\nabla v(x,t)\right|^2 + \right] dx \\
&+ \int_\Omega \left[F\big(x,t,u(x,t),v(x,t)\big) - \frac{\gamma_1}{2+\epsilon}\right] dx \\
&+ \int_{\Gamma_1} \left[H\big(z,t,u(x,t),v(x,t)\big) - \frac{\gamma_2}{2+\epsilon}\right] dS \\
&- \int_{\Gamma_1} \theta(z)\left[\int_0^{u(z,t)} s\rho_1^2(s)\,ds + \int_0^{v(z,t)} s\rho_2^2(s)\,ds\right] dS.
\end{aligned}
$$

In fact, the proof is basically similar to the case of Theorem 2.4. We have from integration by parts and the assumptions $\rho_1' \le 0$, $\rho_2' \le 0$ that

$$
\begin{aligned}
A'(t) &= 2\int_\Omega \left[u\rho_1(u)u_t + v\rho_2(v)v_t\right] dx \\
&\ge -2\int_\Omega \left[\rho_1^2(u)|\nabla u|^2 + \rho_2^2(v)|\nabla v|^2\right] dx \\
&\quad + 2\int_\Omega \left[u\rho_1(u)f_1(x,t,u,v) + v\rho_2(v)f_2(x,t,u,v)\right] dx \\
&\quad + 2\int_{\Gamma_1} \left[u\rho_1^2(u)\left[h_1(z,t,u,v) - \theta(z)u\right] + v\rho_2^2(v)\left[h_2(z,t,u,v) - \theta(z)u\right]\right] dS
\end{aligned}
$$

for $t \in (0,t^*)$. We use condition (C_ρ) to obtain

$$
\begin{aligned}
A'(t) \ge\ & 2(2+\epsilon)B(t) + \epsilon\int_\Omega \left[\rho_1^2(u)|\nabla u|^2 + \rho_2^2(v)|\nabla v|^2\right] dx \\
&+ \epsilon\int_{\Gamma_1} \theta(z)\left[u^2\rho_1^2(u) + v^2\rho_2^2(v)\right] dS \\
&- 2\beta_1\int_\Omega u^2\,dx - 2\beta_2\int_\Omega v^2\,dx - 2\beta_3\int_{\Gamma_1} u^2\,dS - 2\beta_4\int_{\Gamma_1} v^2\,dS
\end{aligned}
$$

for all $t \in (0,t^*)$. Here, this term can be obtained by similar way to inequality (10) in the proof of Theorem 2.4. Therefore, we obtain from Lemma 2.2 and Lemma 2.3 that

$$
\begin{aligned}
A'(t) &\ge 2(2+\epsilon)B(t) + \left[\rho_{1,m}^2\lambda_R\epsilon - \frac{2\lambda_R+2}{\lambda_S}\beta_3 - 2\beta_1\right]\int_\Omega u^2\,dx \\
&\quad + \left[\rho_{2,m}^2\lambda_R\epsilon - \frac{2\lambda_R+2}{\lambda_S}\beta_4 - 2\beta_2\right]\int_\Omega u^2\,dx \\
&\ge 2(2+\epsilon)B(t)
\end{aligned}
\tag{25}
$$

for all $t \in (0,t^*)$. On the other hand, it follows from similar way to (12) that

$$
\begin{aligned}
B'(t) &= -\int_\Omega \left[\rho_1(u)\rho_1'(u)|\nabla u|^2 u_t + \rho_1^2(u)\nabla u\nabla u_t\right] dx \\
&\quad - \int_\Omega \left[\rho_2(v)\rho_2'(v)|\nabla v|^2 v_t + \rho_2^2(v)\nabla v\nabla v_t\right] dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\Omega}\left[\rho_1(u)f_1(x,t,u,v)u_t+\rho_2(v)f_2(x,t,u,v)v_t+\frac{\partial}{\partial t}F(x,t,u,v)\right]dx\\
&+\int_{\Gamma_1}\left[\rho_1^2(u)h_1(z,t,u,v)u_t+\rho_2^2(v)h_2(z,t,u,v)v_t+\frac{\partial}{\partial t}H(z,t,u,v)\right]dS\\
&-\int_{\Gamma_1}\theta(z)\left[u\rho_1^2(u)u_t+v\rho_2^2(v)v_t\right]dS\\
\geq{}&\int_{\Omega}\left[\rho_1(u)u_t^2+\rho_2(v)v_t^2\right]dx\geq 0
\end{aligned}
\tag{26}
$$

for all $t\in(0,t^*)$. Considering (25), (26), and the initial data condition (24), it is easy to see that $A(t)>1$, $A'(t)>0$, $B(t)>0$, and $B'(t)>0$ for all $t\in(0,t^*)$. Now we use the Schwarz inequality and (25) to get

$$
\begin{aligned}
&\frac{2+\epsilon}{2}A'(t)B(t)\\
&\leq\frac{1}{4}\left[A'(t)\right]^2=\left[\int_{\Omega}\left[u\rho_1(u)u_t+v\rho_2(v)v_t\right]dx\right]^2\\
&\leq\left[\left\|u\rho_1^{\frac{1}{2}}(u)\right\|_{L^2(\Omega)}\left\|\rho_1^{\frac{1}{2}}(u)u_t\right\|_{L^2(\Omega)}+\left\|v\rho_2^{\frac{1}{2}}(v)\right\|_{L^2(\Omega)}\left\|\rho_2^{\frac{1}{2}}(v)v_t\right\|_{L^2(\Omega)}\right]^2\\
&\leq\left(\int_{\Omega}u^2\rho_1(u)\,dx+\int_{\Omega}\rho_1(u)u_t\,dx\right)\left(\int_{\Omega}v^2\rho_2(v)\,dx+\int_{\Omega}\rho_2(v)v_t\,dx\right)
\end{aligned}
$$

for all $t\in(0,t^*)$. Using $\rho_1'\leq 0$, $\rho_2'\leq 0$, and assumption (23), we obtain from similar way to (14) that

$$
\frac{2+\epsilon}{2}A'(t)B(t)<A(t)B'(t)
$$

for all $t\in(0,t^*)$. Therefore, we can obtain

$$
A(t)\geq\left[\frac{1}{A^{-\frac{\epsilon}{2}}(0)-\epsilon(2+\epsilon)A^{-\frac{2+\epsilon}{2}}(0)B(0)t}\right]^{\frac{2}{\epsilon}}.
$$

Hence, the solution pair (u,v) blows up at finite time $0<t^*\leq T$. Furthermore, the upper bound T of the blow-up time satisfies

$$
T=\frac{A(0)}{\epsilon(2+\epsilon)B(0)}. \qquad \square
$$

From the proofs of Theorems 2.4 and 3.1, we obtain the blow-up solution to the following nonlinear parabolic systems under the mixed nonlinear boundary conditions for $k\in\mathbb{N}$:

$$
\begin{cases}
\frac{\partial}{\partial t}u_i=\nabla\cdot(\rho_i(u_i)\nabla u_i)+f_i(x,t,u_1,\ldots,u_k), & \text{in } \Omega\times(0,t^*),\\
\frac{\partial u_i}{\partial n}+\theta(z)u_i=h_i(x,t,u_1,\ldots,u_k), & \text{on } \Gamma_1\times(0,t^*),\\
u_i=0, & \text{on } \Gamma_2\times(0,t^*),\\
u_i(\cdot,0)=\psi_i\geq 0, & \text{in } \overline{\Omega},
\end{cases}
\tag{27}
$$

for $i = 1,\dots,k$. Here, the functions f_i are nonnegative $C^1(\Omega \times \mathbb{R}^+ \times \mathbb{R}^k)$-functions and h_i are nonnegative $C^1(\partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^k)$-functions such that

$$\begin{aligned}
&\frac{\partial}{\partial r_i} F(x,t,r_1,\dots,r_i,\dots,r_k) = f_i(x,t,r_1,\dots,r_i,\dots,r_k)\rho_i(r_i), \\
&\frac{\partial}{\partial r_i} H(z,t,r_1,\dots,r_i,\dots,r_k) = h_i(x,t,r_1,\dots,r_i,\dots,r_k)\rho_i^2(r_i),
\end{aligned}$$

for $i = 1,\dots,k$, and ρ_i are positive $C^1(\mathbb{R})$-functions satisfying

$$\rho_i'(s) \le 0, \quad s > 0, \quad \text{and} \quad \inf_{s>0} \rho(s) > 0,$$

for $i = 1,\dots,k$. Also, ψ_i are nonnegative and nontrivial $C^1(\overline{\Omega})$ functions satisfying the boundary conditions for $i = 1,\dots,k$.

Corollary 3.2 *Let $\Gamma_1 \neq \emptyset$ and $k \in \mathbb{N}$. Suppose that the functions f_i and h_i satisfy the conditions*

$$(C_\rho): \begin{cases}
(2+\epsilon)F(x,t,u_1,\dots,u_k) \le \sum_{j=1}^{k} u_j \rho_j(u_j) f_j(x,t,u_1,\dots,u_k) + \sum_{j=1}^{k} \beta_{1,j} u_j^2 + \gamma_1, \\
(2+\epsilon)H(z,t,u_1,\dots,u_k) \le \sum_{j=1}^{k} u_j \rho_j^2(u_j) h_j(z,t,u_1,\dots,u_k) + \sum_{j=1}^{k} \beta_{2,j} u_j^2 + \gamma_2
\end{cases}$$

for some constants ϵ, $\beta_{1,j}$, $\beta_{2,j}$, γ_1, and γ_2 satisfying

$$\epsilon > 0, \qquad \beta_{1,j} + \frac{\lambda_R + 1}{\lambda_S}\beta_{2,j} \le \frac{\rho_{j,m}^2 \lambda_R \epsilon}{2}, \quad \text{and} \quad 0 \le \beta_{2,j} \le \frac{\rho_{j,m}^2 \lambda_S \epsilon}{2}$$

for $j = 1,\dots,k$, where $\rho_{j,m} := \inf_{s>0} \rho_j(s)$, $j = 1,\dots,k$. In addition, we assume that F and H are nondecreasing in t. Moreover, we assume that the functions ρ_i satisfy

$$\lim_{s \to 0+} s^2 \rho_j(s) = 0$$

for $j = 1,\dots,k$. If the initial data u_0 satisfies

$$\begin{aligned}
&-\frac{1}{2}\int_\Omega \left[\sum_{j=1}^{k} \rho_j^2(\psi_i)|\nabla\psi_i|^2 \right] dx + \int_\Omega \left[F(x,0,\psi_1,\dots,\psi_k) - \frac{\gamma_1}{2+\epsilon} \right] dx \\
&\quad + \int_{\Gamma_1} \left[H(z,0,\psi_1,\dots,\psi_k) - \theta(z) \left[\sum_{j=1}^{k} \int_0^{\psi_i} s\rho_i^2(s) \right] - \frac{\gamma_2}{2+\epsilon} \right] dS > 0,
\end{aligned}$$

then every solution pair $(u_1,\dots,u_k)$ to system (27) blows up at finite time t^*.

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Competing interests
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Authors' contributions
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