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The existence of solutions for Sturm–Liouville differential equation with random impulses and boundary value problems

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Abstract

In this article, we consider the existence of solutions to the Sturm–Liouville differential equation with random impulses and boundary value problems. We first study the Green function of the Sturm–Liouville differential equation with random impulses. Then, we get the equivalent integral equation of the random impulsive differential equation. Based on this integral equation, we use Dhage's fixed point theorem to prove the existence of solutions to the equation, and the theorem is extended to the general second order nonlinear random impulsive differential equations. Then we use the upper and lower solution method to give a monotonic iterative sequence of the generalized random impulsive Sturm–Liouville differential equations and prove that it is convergent. Finally, we give two concrete examples to verify the correctness of the results.

Keywords: Random impulsive differential equation; Green function; Upper and lower solution; Fixed point theorem; Boundary value problems

1 Introduction

Impulsive dynamical systems are an emerging field drawing attention from both theoretical and applied disciplines. They are often typically described by ordinary differential equations with instantaneous state jumps [24, 29]. And the impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Since many evolution processes, optimal control models in economics, mechanics, electricity, several fields in engineering stimulated neural networks, frequency modulated systems, and some motions of missiles or aircrafts are characterized by the impulsive dynamical behavior, the study of impulsive systems, especially the impulsive differential, is of great importance, for details, see [12, 17, 24, 27, 34]. But real systems are often subject to not only impulse effect but also noise perturbations. Taking into account the stochastic effects, the models are better described as random impulsive differential equations (*RIDEs*) rather than impulsive differential equations or stochastic differential equations. Hence the study of *RIDEs* has received some attention [2, 30]. Recently,

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a large number of important results about the impulsive differential equation have been reported in [3, 4, 6, 13–15, 18, 23, 24, 28–31, 35, 36, 41]. For example, in [6], Gowrisankar et al. investigated the existence and stability of mild solutions of the first order semilinear differential equation with random impulse (1.1) using the contraction principle.

$$
\begin{cases}
x'(t) = Ax(t) + f(t, x_t), & t \neq \xi_k, t \geq t_0, \\
x(\xi_k) = b_k(\tau_k)x(\xi_k^-), & k = 1, 2, \ldots, \\
x_{t_0} = \varphi,
\end{cases}
\tag{1.1}
$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$ with domain $D(A) \subseteq X$. $x_t(s) = x(t + s)$ and ξ_k is the random pulse time point. In addition, many scholars have also studied the properties of random impulsive differential equations. Radhakrishnan et al. [25] studied the existence of solutions for quasilinear random impulsive neutral differential evolution equation by using the analytic semigroup theory and Schauder fixed point theorem. Niu et al. [23] studied the existence and Hyers–Ulam stability of the random impulsive differential equation with the initial condition. Zhang et al. [40] studied the existence and exponential stability of random impulsive fractional differential equations using the Leray–Schauder fixed point theorem.

Moreover, Sturm–Liouville differential equations play an important role in the study of differential equations. For any second order homogeneous linear differential equation

$$
P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0,
$$

we can multiply both sides of the equation by the integral factor $\mu(x) = \frac{1}{P(x)}e^{\int \frac{Q(x)}{P(x)}\,dx}$, so that the equation becomes a Sturm–Liouville differential equation

$$
\left(\mu(x)P(x)y'(x)\right)' + \mu(x)R(x)y(x) = 0.
$$

Therefore, the study of Sturm–Liouville type differential equations is of great significance, and some scholars have conducted in-depth research [22, 37, 38].

Besides, in recent years, the boundary value problems of different order differential equations have emerged as an important area of research, since these problems have applications in various disciplines of science and engineering such as control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, aerodynamics, and damping [5, 36]. Many researchers studied the existence and stability theory for differential equations with a variety of boundary conditions, for instance, see the papers [1, 9–11, 16, 19–21, 26, 33, 42]. For example, Hua, Cong, and Cheng [8] studied equation (1.2) in 2012, which is the existence and uniqueness of solutions for the periodic-integrable boundary value problem of second order differential equations

$$
\begin{cases}
(p(t)x'(t))' + f(t, x(t)) = 0, \\
x(0) = x(T), \\
\int_0^T x(s)\,ds = 0,
\end{cases}
\tag{1.2}
$$

where $p(t) \in C(R, R)$ is a given T-periodic function in $t \in R$, and $p(t) > 0$. $f \in C(R \times R, R)$ is T-periodic in t.

Finally, we found that the current main research results are focused on ordinary impulsive differential equations, and few scholars have studied random impulsive differential equations. Therefore, based on the importance of random impulsive differential equations, we have carried out research on them. Besides, we found that many researchers investigated the normal impulsive differential equations and the boundary value problem of the equations. But there are fewer people who studied the boundary value problem of Sturm–Liouville type differential equations with random impulses and the upper and lower solutions of this kind of equation. In this paper, we discuss the Sturm–Liouville type differential equations with random impulses and boundary value problems, and we derive the Green function (the researchers gave the Green function of the normal impulsive differential equations [4, 7]) of the equations with the random impulses which has never been studied in the past. At the same time, we use the upper and lower solution method to construct the monotone iterative sequence converging to the maximum and minimum solutions of the equation and prove their convergence.

The rest of the paper is organized as follows: in Sect. 2, we introduce some notations and necessary preliminaries to give an idea of some important definitions and lemmas. And the Green function of the random impulsive differential equations is derived. In Sect. 3, we use Dhage's fixed point theorem to study the existence of the solutions of equation (2.1), and then the existence of solutions of general second order nonlinear random impulsive differential equations with boundary value problems is given. In Sect. 4, we use the upper and lower solution method to give the monotonic iterative convergent sequence of the generalized Sturm–Liouville differential equation with random impulses. Finally, two practical examples are given in Sect. 5 to verify the correctness of the theorem.

2 Preliminaries

In this article, we investigate the solution of the following equation:

$$
\begin{cases}
Lx = f(t, x(t)), & t \in J, t \neq \xi_k, k = 1, 2, 3, \ldots, \\
x(\xi_k^+) = b_k(\tau_k) x(\xi_k^-), & k = 1, 2, 3, \ldots, \\
a_{11} x(0) - a_{12} x'(0) = a_{21} x(1) + a_{22} x'(1) = 0,
\end{cases}
\tag{2.1}
$$

where L is the Sturm–Liouville operator defined as $Lu(t) = -(p(t)u'(t))' + q(t)u(t)$. Let X be a Banach space and Ω be a sample space. Assume that τ_k is a random variable defined from Ω to $D_k := (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < \infty$. Furthermore, assume that τ_i and τ_j are independent from each other when $i \neq j$ for $i, j = 1, 2, \ldots$ $u(t)$ is a stochastic process taking values in X. For the sake of simplicity, we denote $J = [0, 1]$. $C = C(J, R)$ is the set of all the stochastic processes mapping J into R. $f : [J, C] \to R$ is a continuous function. $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \ldots$, and $\xi_0 = 0$. Obviously, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots$, i.e., ξ_k forms a strictly increasing sequence. $b_k : D_k \to R$ for each $k = 1, 2, \ldots$ The convergence is under the meaning of the orbit, $u(\xi_k^-) = \lim_{t \uparrow \xi_k} u(t)$. $p(t)$ and $q(t)$ are positive continuous functions mapping $J = [0, 1]$ into $R^+ = [0, +\infty]$. a_{ij} are positive constants for $i, j = 1, 2$.

Denote by $\{W_t, t \in [0, 1]\}$ the simple counting process generated by $\{\xi_k\}_{k \in N}$, it is to say that $\{W_t \geq n\} = \{\xi_n \leq t\}$ and denote by $\mathcal{F}$ the $\sigma\text{-}algebra$ generated by $\{W_t, t \in J\}$, then $\{\Omega, \mathcal{F}, P\}$ is a probability space. Define by $L_p = L_p(\Omega, \mathcal{F}, R)$ the Hilbert space of all $\mathcal{F}\text{-}measurable$, pth integrable random variables with values in R.

$\| \cdot \|$ is any norm of R, and the expectation of the random variable x is defined as $E(x) = \int_\Omega x \, dP < \infty$. Then we introduce the space $PC = PC(J, L_p) := \{u(t) : u(t) \text{ is strongly measurable}, pth \text{ integrable random process from } J \text{ into } L_p, \text{ and } u(t) \text{ is continuously differentiable when } t \in J \setminus \{\xi_1, \xi_2, \dots\} \text{ and left continuous when } t \in J\}$. It is easy to see that PC is a Banach space with the norm

$$\left\| u(t) \right\|_{PC} = \left(\sup_{t \in J} E \left\| u(t) \right\|^2 \right)^{\frac{1}{2}}.$$

We use the following notations: $P_{cl}(R) = \{Y \subseteq R: Y \text{ is a closed set}\}$, $P_{bd}(R) = \{Y \subseteq R: Y \text{ is a bounded set}\}$, $P_{cp}(R) = \{Y \subseteq R: Y \text{ is a compact set}\}$, $P_{cv}(R) = \{Y \subseteq R: Y \text{ is a convex set}\}$.

Definition 2.1
- The operator A is called upper semi-continuous (u.s.c.) on R if, for each open set V of R containing $A(x_0)$, there exists an open neighborhood N of x_0 such that $A(N) \subseteq V$.
- A is closed graph if there exists a sequence $x_n \to x^*$, $y_n \to y^*$, $y_n = Ax_n$, then we can imply $y^* = Ax^*$.
- A is called a completely continuous operator if A is a bounded linear operator and for every $x_n \rightharpoonup x^*$, we can get $Ax_n \to Ax^*$.
- A is called a compact operator if A is a linear operator and $\overline{A(V)}$ is compact for every $V \in P_{bd}(R)$.

Lemma 2.1 ([32]) *Suppose that $(\mathcal{X}, \| \cdot \|)$ is a normed linear space, then the set A is compact if and only if it is self-column compact.*

Lemma 2.2 (Resonance theorem; [39]) *Suppose that $\mathcal{X}$ is a Banach space, $\mathcal{Y}$ is a linear normed space (B^* space), if W is a subset of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ such that $\sup_{A \in W} \|Ax\| < \infty$, $\forall x \in \mathcal{X}$, then there exists a constant M such that $\|A\| \le M$, $\forall A \in W$.*

Theorem 2.1 *If A is a compact operator, then A is a completely continuous operator.*

Proof Suppose $x_n \rightharpoonup x^*$, we use proof by contradiction. If Ax_n does not converge to $y = Ax^*$, then there exist $\varepsilon_0 > 0$ and $\{n_i\}$ such that $\|Ax_{n_i} - Ax^*\| \ge \varepsilon_0$. From Lemma 2.2, we can know that $\{x_n\}$ is bounded. Combining that A is compact, we can get a subsequence from $\{x_{n_i}\}$, we write it as $\{x_{n_{i_k}}\}$ such that $Ax_{n_{i_k}} \to z$, but for every $y^* \in \mathcal{Y}^*$ ($\mathcal{Y}^*$ is the conjugate space of $\mathcal{Y}$), $\langle y^*, Ax_{n_{i_k}} - y \rangle = \langle A^* y^*, x_{n_{i_k}} - x^* \rangle \to 0$, which implies $Ax_{n_{i_k}} \rightharpoonup y$, then $y = z$, which is a contradiction. The proof is completed. □

Definition 2.2 Suppose that T is a linear operator from $\mathcal{X}$ to $\mathcal{Y}$. $D(T)$ is the definitional domain of T, T is called closed if $x_n \in D(T)$, $x_n \to x$, and $Tx_n \to y$, then we can imply $x \in D(T)$ and $y = Tx$.

Remark 2.1　From the closed graph theorem, we can easily know that if A is a completely continuous operator, then A is u.s.c. if and only if A is a closed graph.

Theorem 2.2 (Dhage's fixed point theorem) *Let X be a Banach space, $A : X \to P_{bd,cl,cv}(R)$, $B : X \to P_{cl,cv}(R)$ are two operators satisfying:*

(i) *A is contraction,*

(ii) *B is u.s.c. and completely continuous.*

Then either

(a) *the operator inclusion $x \in Ax + Bx$ has a solution for $\lambda = 1$ or*

(b) *the set $U = \{u \in X : u \in \lambda Au + \lambda Bu, 0 \le \lambda \le 1\}$ is unbounded.*

Theorem 2.3 *The solution of equation* (2.1) *is equivalent to the solution of the following integral equation*:

$$
u(t) = \int_0^1 G(t,s)\big(-q(s)u(s) + f\big(s,u(s)\big)\big)\,ds
$$

$$
+ \sum_{k=1}^{\infty} \big[W(t,k)u(\xi_k)\big]I_A(\xi_1, \xi_2, \ldots, \xi_k, \ldots), \tag{2.2}
$$

where A is the set of all the sample orbits and $I_A(x)$ is the index function defined as

$$
I_A(t) = \begin{cases} 1 & t \in A, \\ 0 & t \notin A, \end{cases} \tag{2.3}
$$

$$
G(t,s) = \begin{cases} \frac{1}{|Q|}(a_{12} + a_{11}p(0)\int_0^s \frac{1}{p(\tau)}\,d\tau)(a_{22}\frac{1}{p(1)} + a_{21}\int_t^1 \frac{1}{p(\tau)}\,d\tau) & s < t, \\ \frac{1}{|Q|}(a_{12} + a_{11}p(0)\int_0^t \frac{1}{p(\tau)}\,d\tau)(a_{22}\frac{1}{p(1)} + a_{21}\int_s^1 \frac{1}{p(\tau)}\,d\tau) & s \ge t, \end{cases} \tag{2.4}
$$

$$
W(t,k) = \begin{cases} \frac{1}{|Q|}a_{11}p(0)(a_{21}\int_t^1 \frac{1}{p(\tau)}\,d\tau + a_{22}\frac{1}{p(1)})(b_k(\tau_k) - 1) & 0 < \xi_k < t < 1, \\ -\frac{1}{|Q|}a_{21}p(0)(a_{11}\int_0^t \frac{1}{p(\tau)}\,d\tau + a_{12}\frac{1}{p(0)})(b_k(\tau_k) - 1) & 0 < t \le \xi_k < 1, \end{cases} \tag{2.5}
$$

and

$$
Q = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & a_{21}\int_0^1 \frac{p(0)}{p(s)}\,ds + a_{22}\frac{p(0)}{p(1)} \end{bmatrix}. \tag{2.6}
$$

Proof Suppose that $\xi_1, \xi_2, \ldots$ is a sample orbit. Thus, when $t \in (0, \xi_1]$, we have

$$
\big(p(t)u'(t)\big)' = q(t)u(t) - f\big(t, u(t)\big),
$$

$$
u'(t) = \frac{1}{p(t)}p(0)u'(0) + \frac{1}{p(t)}\int_0^t q(s)u(s)\,ds - \frac{1}{p(t)}\int_0^t f\big(s,u(s)\big)\,ds,
$$

$$
u(t) = u(0) + p(0)u'(0)\int_0^t \frac{1}{p(s)}\,ds + \int_0^t \frac{1}{p(\tau)}\int_0^\tau q(s)u(s)\,ds\,d\tau
$$

$$
- \int_0^t \frac{1}{p(\tau)}\int_0^\tau f\big(s,u(s)\big)\,ds\,d\tau.
$$

When $t \in (\xi_1, \xi_2]$, in the same way, we have

$$
u'(t) = \frac{1}{p(t)}p(\xi_1)u'(\xi_1) + \frac{1}{p(t)}\int_{\xi_1}^t q(s)u(s)\,ds - \frac{1}{p(t)}\int_{\xi_1}^t f\big(s,u(s)\big)\,ds.
$$

The first derivative of this function has no impulses, so we can know

$$
u(t) = u\big(\xi_1^+\big) + p(\xi_1)u'(\xi_1) \int_{\xi_1}^t \frac{1}{p(s)}\,ds + \int_{\xi_1}^t \frac{1}{p(\tau)} \int_{\xi_1}^\tau q(s)u(s)\,ds\,d\tau
$$

$$
- \int_{\xi_1}^t \frac{1}{p(\tau)} \int_{\xi_1}^\tau f\big(s, u(s)\big)\,ds\,d\tau
$$

$$
= u\big(\xi_1^+\big) + \left[p(0)u'(0) + \int_0^{\xi_1} q(s)u(s)\,ds - \int_0^{\xi_1} f\big(s, u(s)\big)\,ds \right] \int_{\xi_1}^t \frac{1}{p(s)}\,ds
$$

$$
+ \int_{\xi_1}^t \frac{1}{p(\tau)} \int_{\xi_1}^\tau q(s)u(s)\,ds\,d\tau - \int_{\xi_1}^t \frac{1}{p(\tau)} \int_{\xi_1}^\tau f\big(s, u(s)\big)\,ds\,d\tau.
$$

Combining these two equations, we have

$$
u(t) = u(0) + u\big(\xi_1^+\big) - u(\xi_1) + p(0)u'(0) \int_0^t \frac{1}{p(s)}\,ds
$$

$$
+ \int_{\xi_1}^t \frac{1}{p(s)}\,ds \int_0^{\xi_1} q(s)u(s)\,ds + \int_{\xi_1}^t \frac{1}{p(\tau)} \int_{\xi_1}^\tau q(s)u(s)\,ds\,d\tau
$$

$$
+ \int_0^{\xi_1} \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau - \left[\int_{\xi_1}^t \frac{1}{p(s)}\,ds \int_0^{\xi_1} f\big(s, u(s)\big)\,ds \right.
$$

$$
\left. + \int_{\xi_1}^t \frac{1}{p(\tau)} \int_{\xi_1}^\tau f\big(s, u(s)\big)\,ds\,d\tau + \int_0^{\xi_1} \frac{1}{p(\tau)} \int_0^\tau f\big(s, u(s)\big)\,ds\,d\tau \right].
$$

Combining with the identity

$$
\int_{\xi_1}^t \frac{1}{p(s)}\,ds \int_0^{\xi_1} q(s)u(s)\,ds = \int_{\xi_1}^t \frac{1}{p(\tau)} \int_0^{\xi_1} q(s)u(s)\,ds\,d\tau,
$$

we can get

$$
u(t) = u(0) + p(0)u'(0) \int_0^t \frac{1}{p(s)}\,ds + \int_0^t \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau
$$

$$
- \int_0^t \frac{1}{p(\tau)} \int_0^\tau f\big(s, u(s)\big)\,ds\,d\tau + u\big(\xi_1^+\big) - u(\xi_1),
$$

$$
u'(t) = \frac{1}{p(t)} p(0)u'(0) + \frac{1}{p(t)} \int_0^t q(s)u(s)\,ds - \frac{1}{p(t)} \int_0^t f\big(s, u(s)\big)\,ds.
$$

Suppose when $t \in (\xi_k, \xi_{k+1}]$

$$
u(t) = u(0) + p(0)u'(0) \int_0^t \frac{1}{p(s)}\,ds + \int_0^t \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau
$$

$$
- \int_0^t \frac{1}{p(\tau)} \int_0^\tau f\big(s, u(s)\big)\,ds\,d\tau + \sum_{n=1}^k \big[u\big(\xi_n^+\big) - u(\xi_n)\big].
$$

Then, using the same way, we can get when $t \in (\xi_{k+1}, \xi_{k+2}]$,

$$u(t) = u(0) + p(0)u'(0) \int_0^t \frac{1}{p(s)}\, ds + \int_0^t \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\, ds\, d\tau$$

$$- \int_0^t \frac{1}{p(\tau)} \int_0^\tau f\big(s, u(s)\big)\, ds\, d\tau + \sum_{n=1}^{k+1}\big[u\big(\xi_n^+\big) - u(\xi_n)\big].$$

So, using the mathematical induction, we can get

$$u(t) = \sum_{n=0}^{\infty}\bigg\{ u(0) + p(0)u'(0) \int_0^t \frac{1}{p(s)}\, ds + \int_0^t \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\, ds\, d\tau$$

$$- \int_0^t \frac{1}{p(\tau)} \int_0^\tau f\big(s, u(s)\big)\, ds\, d\tau + \sum_{0 < \xi_k < t}\big[u\big(\xi_k^+\big) - u(\xi_k)\big]\bigg\} I_{(\xi_n, \xi_{n+1}]}(t)$$

and

$$u'(t) = \frac{1}{p(t)}p(0)u'(0) + \frac{1}{p(t)}\int_0^t q(s)u(s)\, ds - \frac{1}{p(t)}\int_0^t f\big(s, u(s)\big)\, ds.$$

It is easy to see that

$$u(1) = u(0) + p(0)u'(0) \int_0^1 \frac{1}{p(s)}\, ds + \int_0^1 \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\, ds\, d\tau$$

$$- \int_0^1 \frac{1}{p(\tau)} \int_0^\tau f\big(s, u(s)\big)\, ds\, d\tau + \sum_{k=1}^{\infty}\big[u\big(\xi_k^+\big) - u(\xi_k)\big],$$

$$u'(1) = \frac{1}{p(1)}p(0)u'(0) + \frac{1}{p(1)}\int_0^1 q(s)u(s)\, ds - \frac{1}{p(1)}\int_0^1 f\big(s, u(s)\big)\, ds.$$

Plug in the boundary value conditions, we can get

$$a_{11}u(0) - a_{12}u'(0) = 0,$$

$$a_{21}u(0) + \bigg[a_{21}p(0) \int_0^1 \frac{1}{p(s)}\, ds + a_{22}\frac{p(0)}{p(1)}\bigg]u'(0)$$

$$= -\bigg[a_{21}\bigg(\int_0^1 \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\, ds\, d\tau$$

$$- \int_0^1 \frac{1}{p(\tau)} \int_0^\tau f\big(s, u(s)\big)\, ds\, d\tau + \sum_{k=1}^{\infty}\big[u\big(\xi_k^+\big) - u(\xi_k)\big]\bigg)$$

$$+ a_{22}\bigg(\frac{1}{p(1)}\int_0^1 q(s)u(s)\, ds - \frac{1}{p(1)}\int_0^1 f\big(s, u(s)\big)\, ds\bigg)\bigg].$$

Then we define the matrix Q as

$$Q = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & a_{21}\int_0^1 \frac{p(0)}{p(s)}\, ds + a_{22}\frac{p(0)}{p(1)} \end{bmatrix}.$$

So, the solution of the above equation group is

$$u(0) = \frac{1}{|Q|} \begin{vmatrix} -a_{12} & 0 \\ a_{21} \int_0^1 \frac{p(0)}{p(s)}\,ds + a_{22}\frac{p(0)}{p(1)} & \zeta \end{vmatrix}, \qquad u'(0) = \frac{1}{|Q|} \begin{vmatrix} 0 & a_{11} \\ \zeta & a_{21} \end{vmatrix},$$

where ζ is defined as

$$\zeta = \left[a_{21}\left(\int_0^1 \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau \right.\right.$$

$$\left. - \int_0^1 \frac{1}{p(\tau)} \int_0^\tau f(s,u(s))\,ds\,d\tau + \sum_{k=1}^\infty \left[u(\xi_k^+) - u(\xi_k) \right] \right)$$

$$\left. + a_{22}\left(\frac{1}{p(1)} \int_0^1 q(s)u(s)\,ds - \frac{1}{p(1)} \int_0^1 f(s,u(s))\,ds \right) \right].$$

So, we can write $u(t)$ as

$$u(t) = \frac{1}{|Q|} \left\{ -a_{12}a_{21}\left(\int_0^1 \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau \right.\right.$$

$$\left. - \int_0^1 \frac{1}{p(\tau)} \int_0^\tau f(s,u(s))\,ds\,d\tau + \sum_{k=1}^\infty \left[u(\xi_k^+) - u(\xi_k) \right] \right)$$

$$- a_{12}a_{22}\left(\frac{1}{p(1)} \int_0^1 q(s)u(s)\,ds - \frac{1}{p(1)} \int_0^1 f(s,u(s))\,ds \right)$$

$$- \int_0^t \frac{1}{p(s)}\,ds\, p(0)\left[a_{11}a_{21}\left(\int_0^1 \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau \right.\right.$$

$$\left. - \int_0^1 \frac{1}{p(\tau)} \int_0^\tau f(s,u(s))\,ds\,d\tau + \sum_{k=1}^\infty \left[u(\xi_k^+) - u(\xi_k) \right] \right)$$

$$\left.\left. + a_{11}a_{22}\left(\frac{1}{p(1)} \int_0^1 q(s)u(s)\,ds - \frac{1}{p(1)} \int_0^1 f(s,u(s))\,ds \right) \right] \right\}$$

$$+ \int_0^t \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau - \int_0^t \frac{1}{p(\tau)} \int_0^\tau f(s,u(s))\,ds\,d\tau$$

$$+ \sum_{0 < \xi_k < t} \left[u(\xi_k^+) - u(\xi_k) \right]$$

$$= \frac{1}{|Q|} \left[-a_{12}a_{21} \int_0^1 \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau \right.$$

$$- a_{12}a_{22}\frac{1}{p(1)} \int_0^1 q(s)u(s)\,ds$$

$$- a_{11}a_{21}p(0) \int_0^t \frac{1}{p(s)}\,ds \int_0^1 \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau$$

$$- \frac{p(0)}{p(1)}a_{11}a_{22} \int_0^1 q(s)u(s)\,ds$$

$$\left. + |Q| \int_0^t \frac{1}{p(\tau)} \int_0^\tau q(s)u(s)\,ds\,d\tau \right]$$

$$+ \frac{1}{|Q|}\left[a_{12}a_{21} \int_0^1 \frac{1}{p(\tau)} \int_0^\tau f\bigl(s,u(s)\bigr)\,ds\,d\tau \right.$$

$$+ a_{12}a_{22} \frac{1}{p(1)} \int_0^1 f\bigl(s,u(s)\bigr)\,ds$$

$$+ a_{11}a_{21}p(0) \int_0^t \frac{1}{p(s)}\,ds \int_0^1 \frac{1}{p(\tau)} \int_0^\tau f\bigl(s,u(s)\bigr)\,ds\,d\tau$$

$$+ a_{11}a_{22}\frac{p(0)}{p(1)} \int_0^t \frac{1}{p(s)}\,ds \int_0^1 f\bigl(s,u(s)\bigr)\,ds$$

$$\left. - |Q| \int_0^t \frac{1}{p(\tau)} \int_0^\tau f\bigl(s,u(s)\bigr)\,ds\,d\tau \right]$$

$$+ \frac{1}{|Q|}\left[-a_{12}a_{21} \sum_{k=1}^\infty \bigl[u(\xi_k^+) - u(\xi_k)\bigr] \right.$$

$$- a_{11}a_{21}p(0) \int_0^t \frac{1}{p(s)}\,ds \sum_{k=1}^\infty \bigl[u(\xi_k^+) - u(\xi_k)\bigr]$$

$$\left. + |Q| \sum_{0<\xi_k<t} \bigl[u(\xi_k^+) - u(\xi_k)\bigr] \right]$$

$$= - \int_0^1 G(t,s)q(s)u(s)\,ds + \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds + \sum_{k=1}^\infty W(t,k)u(\xi_k).$$

And we have completed the proof of Theorem 2.3.   □

Remark 2.2 We can see that $G(t,s)$ is a positive continuous function of t and s.

Remark 2.3 Using the same way, we can prove that the solution of the equation

$$\begin{cases} -x''(t) = f(t,x(t),x'(t)), & t \in J \setminus \{\xi_1, \xi_2, \ldots\}, \\ x(\xi_k^+) = b_k(\tau_k)x(\xi_k), & k = 1,2,3,\ldots, \\ a_{11}x(0) - a_{12}x'(0) = a_{21}x(1) + a_{22}x'(1) = 0, \end{cases} \qquad (2.7)$$

is equivalent to the solution of the following integral equation:

$$x(t) = \int_0^1 G(t,s)f\bigl(s,x(s),x'(s)\bigr)\,ds + \sum_{k=1}^\infty \bigl[W(t,k)x(\xi_k)\bigr]I_A(\xi_1, \xi_2, \ldots, \xi_k, \ldots), \qquad (2.8)$$

where

$$G(t,s) = \begin{cases} \frac{1}{|Q|}(a_{11}s + a_{12})(a_{21}(1-t) + a_{22}) & s < t, \\ \frac{1}{|Q|}(a_{11}t + a_{12})(a_{21}(1-s) + a_{22}) & s \ge t, \end{cases} \qquad (2.9)$$

$$W(t,k) = \begin{cases} \frac{1}{|Q|}a_{11}(a_{21}(1-t) + a_{22})(b_k(\tau_k) - 1) & 0 < \xi_k < t < 1, \\ -\frac{1}{|Q|}a_{21}(a_{11}t + a_{12})(b_k(\tau_k) - 1) & 0 < t \le \xi_k < 1, \end{cases} \qquad (2.10)$$

and

$$|Q| = \begin{vmatrix} a_{11} & -a_{12} \\ a_{21} & a_{21} + a_{22} \end{vmatrix} = a_{11}a_{21} + a_{11}a_{22} + a_{12}a_{21}. \tag{2.11}$$

Definition 2.3 Define the operator $\Lambda : PC(J, L_2) \to PC(J, L_2)$ such that

$$\Lambda u = \int_0^1 G(t,s)\big[f\big(s,u(s)\big) - q(s)u(s)\big]\,\mathrm{d}s + \sum_{k=1}^\infty \big[W(t,k)u(\xi_k)\big]I_A(\xi_1, \xi_2, \ldots, \xi_k, \ldots). \tag{2.12}$$

Theorem 2.4 (Operator decomposition theorem) *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces, T is an operator from $\mathcal{X}$ to $\mathcal{Y}$. $u(t) : \Omega \to \mathcal{X}$ is a functional, $\{\Omega_i\}$ is a division of Ω, it is to say that $\bigcup_{i \in I} \Omega_i = \Omega$, and Ω_i and Ω_j have no element in common for every $i, j \in I$, $i \neq j$, where I is an arbitrary set. Then $u(t)$ is the fixed point of T if and only if, for every $i \in I$, $u_i(t)$ is the fixed point of T, where the definition domain of $u_i(t)$ is Ω_i and $u_i(t) \equiv u(t)$ when $t \in \Omega_i$.*

3 The existence of solutions

In this section, we list the following basic assumptions of this paper and prove our main results.

(H_1): There exists a constant M such that, for each $u_1, u_2 \in PC(J, L_2)$,

$$\big\| f\big(t, u_1(t)\big) - f\big(t, u_2(t)\big) \big\| \leq M \big\| u_1(t) - u_2(t) \big\|.$$

(H_2): There exists a constant η_2 such that

$$\sup_{k,\omega} \frac{\| b_k(\tau_k) - 1 \|}{E\| b_k(\tau_k) - 1 \|} \leq \eta_2 < \infty.$$

(H_3): $\{ E\| b_k(\tau_k) - 1 \| \}$ is a convergent series, and

$$\sum_{k=1}^\infty \sup_\omega \big\| b_k(\tau_k) - 1 \big\| \leq \eta_3 < \infty,$$

where η_3 is a constant.

(H_4): There exists a constant M_f such that, for each $u \in PC(J, L_2)$ and $t \in J$,

$$\big\| f\big(t, u(t)\big) \big\|_{PC} \leq M_f \big(\big\| u(t) \big\|_{PC} + 1 \big).$$

(H_5): Suppose

$$\eta_0 = \frac{1}{|Q|} \left(a_{12} + a_{11}p(0) \int_0^1 \frac{1}{p(\tau)}\,\mathrm{d}\tau \right) \left(a_{22}\frac{1}{p(1)} + a_{21} \int_0^1 \frac{1}{p(\tau)}\,\mathrm{d}\tau \right),$$

$$\mathcal{C}_1 = \max\left\{ \frac{1}{|Q|} a_{11}p(0) \left(a_{21} \int_0^1 \frac{1}{p(\tau)}\,\mathrm{d}\tau + a_{22}\frac{1}{p(1)} \right), \right.$$

$$\left. \frac{1}{|Q|} a_{21}p(0) \left(a_{11} \int_0^1 \frac{1}{p(\tau)}\,\mathrm{d}\tau + a_{12}\frac{1}{p(0)} \right) \right\},$$

$$M_q = \sup_{t \in J} \|q(t)\|,$$

then they should satisfy the following equalities:

$$4\eta_0^2 M_q^2 + 4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2 < 1$$

and

$$\eta_0 M < 1.$$

Theorem 3.1 *If conditions* $(H_1) \sim (H_5)$ *are met, equation* (2.1) *has a solution* $u(t)$ *in* $PC(J, L_2)$ *which satisfies*

$$\|u(t)\|_{PC} \le \frac{4\eta_0^2 M_f^2 + 2\eta_0 M_f \sqrt{1 - 4\eta_0^2 M_q^2 - 2\mathcal{C}_1^2 \eta_2 \eta_3^2}}{1 - (4\eta_0^2 M_q^2 + 4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2)}.$$

Proof First of all, we can decompose the operator Λ into A and B, that is to say $\Lambda u(t) = Au(t) + Bu(t)$ for every $u(t) \in PC(J, L_2)$. The operators A and B are defined as

$$Au = \int_0^1 G(t,s) f(s, u(s)) \, ds,$$

$$Bu = -\int_0^1 G(t,s) q(s) u(s) \, ds + \sum_{k=1}^{\infty} [W(t,k) x(\xi_k)] I_A(\xi_1, \xi_2, \ldots, \xi_k, \ldots).$$

It is easy to see that B is a linear operator, and we can easily prove that the solution of equation (2.1) is equivalent to the fixed point of the operator $\Lambda = A + B$. Then we will prove Theorem 3.1 in six steps.

Step (1):

A is a single-valued operator, so $Au \in P_{cl,cv}(R)$. Then we prove that, for every $u \in \mathcal{B}_q = \{u(t) : \|u(t)\|_{PC} \le q\}$, $\|Au(t)\|_{PC} \le \eta_1$, where η_1 is a constant.

$$G(t,s) \le \frac{1}{|Q|} \left(a_{12} + a_{11} p(0) \int_0^1 \frac{1}{p(\tau)} \, d\tau \right) \left(a_{22} \frac{1}{p(1)} + a_{21} \int_0^1 \frac{1}{p(\tau)} \, d\tau \right) = \eta_0.$$

Hence,

$$\begin{aligned}
E\|Au\|^2 &\le E \left(\int_0^1 \|G(t,s) f(s, u(s))\| \, ds \right)^2 \\
&\le E \left[\int_0^1 \|G(t,s)\|^2 \, ds \int_0^1 \|f(s, u(s))\|^2 \, ds \right] \\
&\le \eta_0^2 E \left[\int_0^1 \|f(s, u(s))\|^2 \, ds \right] \\
&\le \eta_0^2 \int_0^1 \|f(s, u(s))\|_{PC}^2 \, ds \\
&\le \eta_0^2 M_f^2 (q+1)^2 = \eta_1^2.
\end{aligned}$$

Step (2):

We prove that A is a contraction.

$$\|Au_1 - Au_2\|^2 = \left\| \int_0^1 G(t,s)\big[f\big(s,u_1(s)\big) - f\big(s,u_2(s)\big)\big]\,\mathrm{d}s \right\|^2$$

$$\le M^2\eta_0^2 \int_0^1 \|u_1(s) - u_2(s)\|^2\,\mathrm{d}s,$$

$$\mathrm{E}\|Au_1 - Au_2\|^2 \le M^2\eta_0^2 \int_0^1 \int_\Omega \|u_1(s) - u_2(s)\|^2\,\mathrm{d}\mu\,\mathrm{d}s$$

$$\le M^2\eta_0^2 \sup_{t\in J} \mathrm{E}\|u_1(t) - u_2(t)\|^2.$$

So, we get

$$\|Au_1 - Au_2\|_{PC} \le M\eta_0 \|u_1(t) - u_2(t)\|_{PC},$$

and we can easily know that A is a contraction.

Step (3):

It is easy to see that B is a single-valued operator, so for each $u(t) \in PC(J, L_2)$, $Bu \in P_{cl,cv}(R)$. Next, we prove, for every $u(t) \in \mathcal{B}_q$, that Bu is bounded.

$$Bu(t) = -\int_0^1 G(t,s)q(s)u(s)\,\mathrm{d}s + \sum_{k=1}^\infty \big[W(t,k)x(\xi_k)\big]I_A(\xi_1, \xi_2, \dots, \xi_k, \dots),$$

$$\|Bu\|^2 \le 2\int_0^1 \|G(t,s)q(s)u(s)\|^2\,\mathrm{d}s + 2\left\| \sum_{k=1}^\infty W(t,k)u(\xi_k) \right\|^2$$

$$\le 2\eta_0^2 \int_0^1 \|q(s)u(s)\|^2\,\mathrm{d}s + 2\left\| \sum_{k=1}^\infty W(t,k)u(\xi_k) \right\|^2.$$

And based on the second mean value theorem of integrals, we know, for each k, that there exists $\omega_k \in \Omega$ such that

$$\mathrm{E}\left\| \sum_{k=1}^\infty W(t,k)u(\xi_k) \right\|^2$$

$$= \int_\Omega \left\| \sum_{k=1}^\infty W(t,k)u(\xi_k) \right\|^2\,\mathrm{d}\mu$$

$$= \int_\Omega \left\| \sum_{k=1}^\infty W(t,k)u(\xi_k) \right\|\,\mathrm{d}\mu \left\| \sum_{k=1}^\infty W(t,k)(\omega_k)u\big(\xi_k(\omega_k)\big) \right\|$$

$$\le C_1^2 \int_\Omega \left\| \sum_{k=1}^\infty \big(b_k(\tau_k) - 1\big)u(\xi_k) \right\|\,\mathrm{d}\mu \left\| \sum_{k=1}^\infty \big(b_k(\tau_k) - 1\big)u\big(\xi_k(\omega_k)\big) \right\|$$

$$\le C_1^2\eta_3 \left[\sum_{k=1}^\infty \frac{\sup_{\omega_k} \|b_k(\tau_k(\omega_k)) - 1\|}{\int_\Omega \|b_k(\tau_k) - 1\|\,\mathrm{d}\mu} \int_\Omega \|b_k(\tau_k) - 1\|\,\mathrm{d}\mu \int_\Omega \|u(\xi_k)\|\,\mathrm{d}\mu \right]$$

$$\times \max_k \|u\big(\xi_k(\omega_k)\big)\|$$

$$\leq C_1^2 \eta_2 \eta_3^2 \max_k \int_\Omega \|u(\xi_k)\| \, d\mu \, \|u(\xi_k(\omega_k))\|$$

$$\leq C_1^2 \eta_2 \eta_3^2 \max_k \int_\Omega \|u(\xi_k)\|^2 \, d\mu$$

$$\leq C_1^2 \eta_2 \eta_3^2 \|u(\xi_k)\|_{PC}^2.$$

Based on the above equation, we can easily know that Bu is bounded for each $u(t) \in \mathcal{B}_q$.

Step (4):

Define $B(\mathcal{B}_q) = \{Bu(t) : u(t) \in \mathcal{B}_q\}$, then we prove that $B(\mathcal{B}_q)$ is equicontinuous. Based on Theorem 2.4, we only need to prove for each $(\xi_k, \xi_{k+1}]$, $k = 0, 1, 2, \ldots$ and every $\varepsilon > 0$ that there exists $\delta > 0$ such that, for any $t_1, t_2 \in (\xi_k, \xi_{k+1}]$, $\|t_1 - t_2\| < \delta$, we have $\|Bu(t_1) - Bu(t_2)\|_{PC} < \varepsilon$ for every $u(t) \in \mathcal{B}_q$.

$$\big\|(Bu)(t_1) - (Bu)(t_2)\big\|^2$$

$$= \left\| \int_0^1 \big[G(t_1, s) - G(t_2, s)\big]\big(-q(s)u(s)\big) \, ds \right.$$

$$\left. + \sum_{n=1}^\infty \big[W(t_1, n) - W(t_2, n)\big]u(\xi_n)I_A(\cdot) \right\|^2$$

$$\leq 2\left(\int_0^1 \|G(t_1, s) - G(t_2, s)\| \, \|q(s)u(s)\| \, ds \right)^2$$

$$+ 2\left\| \sum_{n=1}^\infty \big[W(t_1, n) - W(t_2, n)\big]u(\xi_n) \right\|^2,$$

so we have

$$E\big\|(Bu)(t_1) - (Bu)(t_2)\big\|^2$$

$$\leq 2M_q^2 q^2 \int_0^1 \|G(t_1, s) - G(t_2, s)\|^2 \, ds$$

$$+ 2\sup_{n,\omega} \frac{\|W(t_1, n) - W(t_2, n)\|}{\int_\Omega \|W(t_1, n) - W(t_2, n)\| \, d\mu} \|u(t)\|_{PC}^2$$

$$\times \left(\sum_{n=1}^\infty E\|W(t_1, n) - W(t_2, n)\| \right)\left(\sum_{n=1}^\infty \sup_\omega \|W(t_1, n) - W(t_2, n)\| \right),$$

where

$$\sum_{n=1}^\infty \|W(t_1, k) - W(t_2, k)\|$$

$$\leq \left\| \frac{1}{|Q|} a_{11} a_{21} p(0) \int_{t_1}^{t_2} \frac{1}{p(\tau)} \, d\tau \sum_{n=1}^k \big(b_n(\tau_n) - 1\big) \right\|$$

$$+ \left\| \frac{1}{|Q|} a_{11} a_{21} p(0) \int_{t_1}^{t_2} \frac{1}{p(\tau)} \, d\tau \sum_{n=k+1}^\infty \big(b_n(\tau_n) - 1\big) \right\|$$

$$\leq \left\| \frac{1}{|Q|} a_{11} a_{21} p(0) \int_{t_1}^{t_2} \frac{1}{p(\tau)} \, d\tau \right\| \sum_{n=1}^{\infty} \left\| b_n(\tau_n) - 1 \right\|,$$

hence

$$\sum_{n=1}^{\infty} \mathrm{E} \left\| W(t_1, n) - W(t_2, n) \right\| \leq \eta_3 \left\| \frac{1}{|Q|} a_{11} a_{21} p(0) \int_{t_1}^{t_2} \frac{1}{p(\tau)} \, d\tau \right\|.$$

So, combining with the above equations, we have

$$\mathrm{E} \left\| (Bu)(t_1) - (Bu)(t_2) \right\|^2$$

$$\leq 2 M_q^2 q^2 \int_0^1 \left\| G(t_1, s) - G(t_2, s) \right\|^2 ds$$

$$+ 2 \frac{1}{|Q|^2} a_{11}^2 a_{21}^2 p^2(0) \sup_{k,\omega} \frac{\| b_k(\tau_k) - 1 \|}{\int_\Omega \| b_k(\tau_k) - 1 \| \, d\mu} \| u(t) \|_{PC}^2 \left(\int_{t_1}^{t_2} \frac{1}{p(\tau)} \, d\tau \right)^2$$

$$\times \left(\sum_{k=1}^{\infty} E \| b_k(\tau_k) - 1 \| \, d\mu \right) \left(\sum_{n=1}^{\infty} \sup_\omega \| b_k(\tau_k) - 1 \| \right)$$

$$\leq 2 M_q^2 q^2 \int_0^1 \left\| G(t_1, s) - G(t_2, s) \right\|^2 ds$$

$$+ 2 \eta_2 \eta_3^2 \frac{1}{|Q|^2} a_{11}^2 a_{21}^2 p^2(0) \left(\int_{t_1}^{t_2} \frac{1}{p(\tau)} \, d\tau \right)^2 \| u(t) \|_{PC}^2.$$

It is easy to see that $\mathrm{E} \| (Bu)(t_1) - (Bu)(t_2) \|^2 \to 0$ as $|t_1 - t_2| \to 0$, so we have proved that the set is equicontinuous. From *step (3)* to *step (4)*, combining with the Arzela–Ascoli theorem, we can easily know that $B(\mathcal{B}_q)$ is sequentially compact and $\overline{B(\mathcal{B}_q)}$ is self-listed. Using Lemma 2.1, we can know that $\overline{B(\mathcal{B}_q)}$ is compact. Combining with Theorem 2.1, we have proved that B is completely continuous.

Step (5):

We prove that B is u.s.c. Based on Remark 2.1, we only need to prove that for each $u_n(t) \to u^*(t)$ we have $(Bu_n)(t) \to (Bu^*)(t)$.

$$\left\| (Bu_n)(t) - (Bu^*)(t) \right\|^2$$

$$= \left\| \int_0^1 G(t,s) \left[-q(s) \left(u_n(s) - u^*(s) \right) \right] ds + \sum_{k=1}^{\infty} W(t,k) \left[u_n(\xi_k) - u^*(\xi_k) \right] I_A(\cdot) \right\|^2$$

$$\leq 2 \eta_0^2 M_q^2 \int_0^1 \left\| u_n(s) - u^*(s) \right\|^2 ds + 2 \mathcal{C}_1^2 \left\| \sum_{k=1}^{\infty} \left(b_k(\tau_k) - 1 \right) \left[u_n(\xi_k) - u^*(\xi_k) \right] \right\|^2.$$

Hence,

$$\mathrm{E} \left\| (Bu_n)(t) - (Bu^*)(t) \right\|^2$$

$$\leq 2 \eta_0^2 M_q^2 \| u_n(s) - u^*(s) \|_{PC}^2$$

$$+ 2 \mathcal{C}_1^2 \sup_{k,\omega} \frac{\| b_k(\tau_k(\omega_k)) - 1 \|}{\int_\Omega \| b_k(\tau_k) - 1 \| \, d\mu} \sum_{k=1}^{\infty} \int_\Omega \| b_k(\tau_k) - 1 \| \, d\mu \int_\Omega \| u_n(\xi_k) - u^*(\xi_k) \| \, d\mu$$

$$\times \max_k \left\| u_n\big(\xi_k(\omega_k)\big) - u^*\big(\xi_k(\omega_k)\big) \right\| \sum_{k=1}^{\infty} \sup_{\omega} \left\| b_k(\tau_k) - 1 \right\|$$

$$\leq 2\eta_0^2 M_q^2 \left\| u_n(s) - u^*(s) \right\|_{PC}^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2 \left\| u_n(s) - u^*(s) \right\|_{PC}^2.$$

So, we have proved that B is u.s.c.

Step (6):

We prove that the set $U = \{u(t) : u(t) \in \lambda(Au)(t) + \lambda(Bu)(t), 0 \leq \lambda \leq 1\}$ is bounded. If $u(t) \in U$, then we have

$$u(t) = \lambda \int_0^1 G(t,s)\big[-q(s)u(s) + f\big(s, u(s)\big)\big]\,ds + \lambda \sum_{k=1}^{\infty} \big[W(t,k)u(\xi_k)\big]I_A(\xi_1, \xi_2, \ldots, \xi_k, \ldots),$$

$$\left\| u(t) \right\|^2 \leq 2\lambda^2 \left(\int_0^1 \left\| G(t,s)\big[-q(s)u(s) + f\big(s, u(s)\big)\big] \right\|\,ds \right)^2$$

$$+ 2\lambda^2 \left(\sum_{k=1}^{\infty} \left\| W(t,k)u(\xi_k) \right\| \right)^2$$

$$\leq 4\lambda^2 \eta_0^2 M_q^2 \int_0^1 \left\| u(s) \right\|^2\,ds + 4\lambda^2 \eta_0^2 \int_0^1 \left\| f\big(s, u(s)\big) \right\|^2\,ds$$

$$+ 2\lambda^2 \mathcal{C}_1^2 \left\| \sum_{k=1}^{\infty} \big(b_k(\tau_k) - 1\big)u(\xi_k) \right\|^2,$$

hence

$$\mathbb{E}\left\| u(t) \right\|^2 \leq 4\lambda^2 \eta_0^2 M_q^2 \left\| u(s) \right\|_{PC}^2 + 4\lambda^2 \eta_0^2 \int_0^1 \mathbb{E}\left\| f\big(s, u(s)\big) \right\|^2\,ds$$

$$+ 2\lambda^2 \mathcal{C}_1^2 \eta_2 \left\| u(\xi_k) \right\|_{PC}^2 \sum_{k=1}^{\infty} \int_{\Omega} \left\| b_k(\tau_k) - 1 \right\|\,d\mu \sum_{k=1}^{\infty} \sup_{\omega} \left\| b_k\big(\tau_k(\omega)\big) - 1 \right\|$$

$$\leq 4\lambda^2 \eta_0^2 M_q^2 \left\| u(s) \right\|_{PC}^2 + 4\lambda^2 \eta_0^2 M_f^2 \big(\left\| u(s) \right\|_{PC} + 1\big)^2$$

$$+ 2\lambda^2 \mathcal{C}_1^2 \eta_2 \eta_3^2 \left\| u(t) \right\|_{PC}^2.$$

By simplifying, we can get

$$\big[1 - \big(4\lambda^2 \eta_0^2 M_q^2 + 4\lambda^2 \eta_0^2 M_f^2 + 2\lambda^2 \mathcal{C}_1^2 \eta_2 \eta_3^2\big)\big]\|u\|_{PC}^2$$

$$- 8\lambda^2 \eta_0^2 M_f^2 \|u\|_{PC} - 4\lambda^2 \eta_0^2 M_f^2 \leq 0.$$

This is a quadratic function, and

$$\Delta = 16\lambda^2 \eta_0^2 M_f^2 \big(1 - 4\lambda^2 \eta_0^2 M_q^2 - 2\lambda^2 \mathcal{C}_1^2 \eta_2 \eta_3^2\big) \geq 0.$$

Then we can get

$$\left\| u(t) \right\|_{PC} \leq \frac{4\lambda^2 \eta_0^2 M_f^2 + 2\lambda\eta_0 M_f \sqrt{1 - 4\lambda^2 \eta_0^2 M_q^2 - 2\lambda^2 \mathcal{C}_1^2 \eta_2 \eta_3^2}}{1 - \big(4\lambda^2 \eta_0^2 M_q^2 + 4\lambda^2 \eta_0^2 M_f^2 + 2\lambda^2 \mathcal{C}_1^2 \eta_2 \eta_3^2\big)}$$

$$= \frac{4\eta_0^2 M_f^2 + 2\eta_0 M_f \sqrt{\frac{1}{\lambda^2} - 4\eta_0^2 M_q^2 - 2\mathcal{C}_1^2 \eta_2 \eta_3^2}}{\frac{1}{\lambda^2} - (4\eta_0^2 M_q^2 + 4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2)}$$

$$\leq \frac{4\eta_0^2 M_f^2 + 2\eta_0 M_f \sqrt{1 - 4\eta_0^2 M_q^2 - 2\mathcal{C}_1^2 \eta_2 \eta_3^2}}{1 - (4\eta_0^2 M_q^2 + 4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2)}.$$

So, the set U is bounded. As a consequence of Theorem 2.2, we deduce that $A + B$ has a fixed point $u(t)$ which is a solution of equation (2.1), and we have completed the proof of Theorem 3.1. $\qquad\square$

Using the same way, we can prove the following theorem.

(H_6): There exists a constant M such that, for each $x_1, x_2 \in PC(J, L_2)$,

$$\left\| f(t, x_1, x_1') - f(t, x_2, x_2') \right\| \leq M \|x_1 - x_2\|.$$

(H_7): There exists a constant M_f such that, for each $u \in PC(J, L_2)$ and $t \in J$,

$$\left\| f(t, x(t), x'(t)) \right\| \leq M_f \left(\left\| x(t) \right\|_{PC} + 1 \right).$$

(H_8): Suppose

$$\eta_0 = \frac{(a_{11} + a_{12})(a_{21} + a_{22})}{a_{11}a_{21} + a_{12}a_{21} + a_{11}a_{22}},$$

$$\mathcal{C}_1 = \frac{\max\{a_{11}(a_{21} + a_{22}), a_{21}(a_{11} + a_{12})\}}{a_{11}a_{21} + a_{12}a_{21} + a_{11}a_{22}},$$

then they should satisfy the following equalities:

$$4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2 < 1,$$

and

$$\eta_0 M < 1.$$

Theorem 3.2 *If conditions* $(H_2) \sim (H_3)$ *and* $(H_6) \sim (H_8)$ *are met, then equation* (2.7) *has a solution* $x(t)$ *in* $PC(J, L_2)$ *which satisfies*

$$\left\| x(t) \right\|_{PC} \leq \frac{4\eta_0^2 M_f^2 + 2\eta_0 M_f \sqrt{1 - 2\mathcal{C}_1^2 \eta_2 \eta_3^2}}{1 - (+4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2)}.$$

4 The upper and lower solutions

In this section, we consider the upper and lower solutions of the following generalized Sturm–Liouville differential equation with random impulses:

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & t \in J', \\ u(\xi_k^+) = b_k(\tau_k)u(\xi_k), & k = 1, 2, 3, \ldots, \\ a_{11}u(0) - a_{12}u'(0) = a_{21}u(1) + a_{22}u'(1) = 0. \end{cases} \tag{4.1}$$

The notation in this equation is the same as the previous definition. First of all, we consider the following linear random impulsive differential equation:

$$
\begin{cases}
-u''(t) = f(t, h(t), h'(t)) - M(u(t) - h(t)), & t \in J', \\
u(\xi_k^+) - u(\xi_k) = (b_k(\tau_k) - 1)h(\xi_k), & k = 1, 2, 3, \ldots, \\
a_{11}u(0) - a_{12}u'(0) = a_{21}u(1) + a_{22}u'(1) = 0,
\end{cases}
\tag{4.2}
$$

where $h(t) \in PC(J, L_2)$ is a stochastic process. Based on Theorem 2.3, we can get that the solution of equation (4.2) is equivalent to the solution of the following integral equation:

$$
u(t) = -M \int_0^1 G(t, s)u(s)\,ds + \int_0^1 G(t, s)\big[f\big(s, h(s), h'(s)\big) + Mh(s)\big]\,ds
$$
$$
+ \sum_{k=1}^{\infty}\big[W(t, k)h(\xi_k)\big]I_A(\xi_1, \xi_2, \ldots, \xi_k, \ldots),
$$

where $G(t, s)$ and $W(t, k)$ are defined as (2.9) and (2.10).

Then we define the operator $\Lambda : PC(J, L_2) \to PC(J, L_2)$ as

$$
\Lambda h = -M \int_0^1 G(t, s)\Lambda h(s)\,ds + \int_0^1 G(t, s)\big[f\big(s, h(s), h'(s)\big) + Mh(s)\big]\,ds
$$
$$
+ \sum_{k=1}^{\infty}\big[W(t, k)h(\xi_k)\big]I_A(\xi_1, \xi_2, \ldots, \xi_k, \ldots),
$$

we can easily prove that $h(t)$ is the solution of the equation if and only if it is the fixed point of the operator Λ.

Definition 4.1 $\omega_0(t) \in PC(J, L_2)$ is called an upper solution of equation (4.1) if $\omega_0(t)$ satisfies the following inequality:

$$
\begin{cases}
-\omega_0''(t) \geq f(t, \omega_0(t), \omega_0'(t)), \\
\omega_0(\xi_k^+) \geq b_k(\tau_k)\omega_0(\xi_k), \\
a_{11}\omega_0(0) - a_{12}\omega_0'(0) \geq x_0, \\
a_{21}\omega_0(1) + a_{22}\omega_0'(1) \geq x_0^*.
\end{cases}
$$

If the above inequalities are reversed, we call it a lower solution of equation (4.1).

Lemma 4.1 ([32]) *Suppose that E is a semi-ordered Banach space. For $x_0, y_0 \in E$, $x_0 \leq y_0$, and $D = [x_0(t), y_0(t)]$, $A : D \to E$ is an operator. Assume that the following conditions are satisfied*:

(i) *A is an increasing operator,*

(ii) *x_0 is the lower solution of A and y_0 is the upper solution of A,*

(iii) *A is a continuous operator,*

(iv) *A(D) is a relatively compact set of columns in E.*

Then A has a maximum fixed point and a minimum fixed point in D. Let x_0 and y_0 be the initial conditions. We then have the iteration sequences

$$x_n = Ax_{n-1}, \qquad y_n = Ay_{n-1}, \quad n = 1, 2, \dots.$$

Thus,

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0,$$

and

$$x_n \to x^*, \qquad y_n \to y^*.$$

(H_9) $v_0(t)$ is the lower solution of equation (4.1) and $\omega_0(t)$ is the upper solution of equation (4.1), and they meet the following inequality:

$$v_0(t) \leq \omega_0(t)$$

for any $t \in J$.

(H_{10})

$$\inf_k \{ b_k(\tau_k) - 1 : k \in N \} > 0.$$

Theorem 4.1 *If conditions (H_2) $\sim$ (H_3), (H_6) $\sim$ (H_{10}) are met, then equation (4.1) has the maximum solution $u^*(t)$ and the minimum solution $u_*(t)$ in $[v_0(t), \omega_0(t)] \cap PC(J, L_2)$. $\omega_n(t) = \Lambda \omega_{n-1}(t)$ uniformly converges to $u^*(t)$, $v_n(t) = \Lambda v_{n-1}(t)$ uniformly converges to $u_*(t)$, where $n = 1, 2, \dots.$*

Proof First of all, we prove that v_0 is the lower solution of Λ. It is to say that we should prove that $v_0(t) \leq \Lambda v_0(t) = v_1(t)$. We can easily prove this when there is no pulse (for more details, see [34]). When the equation is equipped with the random impulses, we have

$$v_0(\xi_k^+) - v_0(\xi_k) \leq (b_k(\tau_k) - 1)v_0(\xi_k) = v_1(\xi_k^+) - v_1(\xi_k),$$

combining with $v_1(\xi_k) \geq v_0(\xi_k)$, we can get $v_1(\xi_k^+) \geq v_0(\xi_k^+)$. So, we have proved $v_0(t) \leq \Lambda v_0(t)$.

Then we prove that it is an increasing operator. It is to say that, for any $h_1(t) \leq h_2(t)$, we have $\Lambda h_1(t) \leq \Lambda h_2(t)$. When there is no pulse, we can easily prove this conclusion (for more details, see [13]). When there are random impulses, we have

$$\Lambda h_1(\xi_k^+) - \Lambda h_1(\xi_k) = (b_k(\tau_k) - 1)h_1(\xi_k)$$
$$\leq (b_k(\tau_k) - 1)h_2(\xi_k) = \Lambda h_2(\xi_k^+) - \Lambda h_2(\xi_k),$$

so we have proved that $\Lambda h_1(t) \leq \Lambda h_2(t)$.

Based on the proof of Theorem 3.1, we can easily prove that Λ is a continuous operator and $\Lambda([v_0, \omega_0])$ is a relatively compact set of columns. Combining with Lemma 4.1, we complete the proof of Theorem 4.1. □

In this section, we give some examples to illustrate our main result.

5 Examples

Example 5.1　Consider the following equation:

$$\begin{cases} -\left(\frac{9}{1.5-t}u'(t)\right)' + \frac{9t}{t^2+30}u(t) = \frac{1}{8}te^t \sin u(t), & t \in J \setminus \{\xi_1, \xi_2, \dots\}, \\ x(\xi_k^+) = (\sqrt{\tau_k} + 1)x(\xi_k), & k = 1, 2, \dots, \\ x(0) - \frac{1}{2}x'(0) = \frac{1}{3}x(1) + x'(1) = 0, \end{cases} \tag{5.1}$$

where $\{\tau_k\}$ is a variable sequence, τ_i and τ_j are independent from each other for each $i \neq j$. $\tau_k \sim U(0, \frac{1}{4^k})$, it is to say that the probability density function of τ_k is

$$p(x) = \begin{cases} 4^k & x \in (0, \frac{1}{4^k}), \\ 0 & \text{others.} \end{cases}$$

Then it is easy to see that $\{\sqrt{\tau_k}\}$ is also a variable sequence, and for every $i \neq j$, $\sqrt{\tau_i}$ and $\sqrt{\tau_j}$ are independent. We can easily get the probability density function of $\sqrt{\tau_k}$:

$$p(x) = \begin{cases} 2^{2k+1}x & x \in (0, \frac{1}{2^k}), \\ 0 & \text{others.} \end{cases}$$

Set $\xi_0 = 0$, $\xi_{k+1} = \xi_k + \tau_{k+1}$, obviously, $\{\xi_k\}$ is a process with independent increments, and the impulsive moments $\{\xi_k\}$ form a strictly increasing sequence. And we have

$$\xi_k \leq \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^k} = \frac{1}{3}\left(1 - \frac{1}{4^k}\right) < 1.$$

In this example, we define the norm $\|x\| = |x|$. Then we have

$$\eta_3 = \sum_{k=1}^{\infty} \sup_{\omega_k} \|b_k(\tau_k) - 1\| = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1,$$

$$\eta_2 = \sup_{k,\omega} \frac{\|b_k(\tau_k) - 1\|}{E\|b_k(\tau_k) - 1\|} = \frac{\frac{1}{2^k}}{\frac{1}{3}\frac{1}{2^{k-1}}} = \frac{3}{2},$$

$$|Q| = \begin{vmatrix} a_{11} & -a_{12} \\ a_{21} & a_{21}\int_0^1 \frac{p(0)}{p(s)}\,ds + a_{22}\frac{p(0)}{p(1)} \end{vmatrix} = \frac{13}{18},$$

$$\eta_0 = \frac{1}{|Q|}\left(a_{12} + a_{11}p(0)\int_0^1 \frac{1}{p(\tau)}\,d\tau\right)\left(a_{22}\frac{1}{p(1)} + a_{21}\int_0^1 \frac{1}{p(\tau)}\,d\tau\right) = \frac{35}{26},$$

and

$$C_1 = \max\left\{\frac{1}{|Q|}a_{11}p(0)\left(a_{21}\int_0^1 \frac{1}{p(\tau)}\,d\tau + a_{22}\frac{1}{p(1)}\right),\right.$$

$$\left.\frac{1}{|Q|}a_{21}p(0)\left(a_{11}\int_0^1 \frac{1}{p(\tau)}\,d\tau + a_{12}\frac{1}{p(0)}\right)\right\} = \frac{7}{13}.$$

So we have

$$\eta_0 M = \frac{35}{1872} \approx 0.019 < 1,$$

$$4\eta_0^2 M_q^2 + 4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2 = \frac{739,855,753}{841,928,256} \approx 0.879 < 1.$$

Based on Theorem 3.1, we can know that equation (5.1) has a solution $u(t)$ which satisfies

$$\|u(t)\|_{PC} \leq \frac{4\eta_0^2 M_f^2 + 2\eta_0 M_f \sqrt{1 - 4\eta_0^2 M_q^2 - 2\mathcal{C}_1^2 \eta_2 \eta_3^2}}{1 - (4\eta_0^2 M_q^2 + 4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2)} \approx 0.119.$$

We have completed the proof of the example.

Example 5.2 Now we consider the following second order random impulsive differential equation with boundary value problems:

$$\begin{cases} -33u''(t) = e^{t^2} u(t) \sin u'(t) + t^2 + \frac{1}{2}t, & t \in J \setminus \{\xi_1, \xi_2, \ldots\}, \\ x(\xi_k^+) = (\tau_k + 1)x(\xi_k), & k = 1, 2, \ldots, \\ x(0) - x'(0) = \frac{1}{7}x(1) + \frac{1}{4}x'(1) = 0. \end{cases} \tag{5.2}$$

Here we define the norm $\|u\|_{PC} = \sup_{t \in J} E|u(t)|^2$. $\{\tau_k\}$ is a variable sequence and τ_i and τ_j are independent from each other when $i \neq j$. The probability density function of τ_k is

$$p(x) = \begin{cases} 4^{k+2}x & x \in [0, \frac{1}{2^{k+2}}), \\ 2^{k+3} - 4^{k+2}x & x \in [\frac{1}{2^{k+2}}, \frac{1}{2^{k+1}}), \\ 0 & \text{others.} \end{cases}$$

Suppose $\xi_{k+1} = \xi_k + \tau_{k+1}$, $k = 0, 1, 2, \ldots$. Obviously, we have

$$\xi_k < \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2} < 1.$$

Then we have

$$\eta_3 = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2},$$

$$\eta_2 = \sup_{k,\omega} \frac{\|b_k(\tau_k) - 1\|}{E\|b_k(\tau_k) - 1\|} = \frac{\frac{1}{2^{k+1}}}{\frac{1}{2^{k+2}}} = 2.$$

And we can easily get $M_f = \frac{1}{22}$, $M = \frac{1}{33}$,

$$\eta_0 = \frac{(a_{11} + a_{12})(a_{21} + a_{22})}{a_{11}a_{21} + a_{12}a_{21} + a_{11}a_{22}} = \frac{22}{15},$$

$$\mathcal{C}_1 = \frac{\max\{a_{11}(a_{21} + a_{22}), a_{21}(a_{11} + a_{12})\}}{a_{11}a_{21} + a_{12}a_{21} + a_{11}a_{22}} = \frac{11}{15}.$$

Hence, $\eta_0 M = \frac{2}{45} \approx 0.044 < 1$ and $4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2 = \frac{5}{9} \approx 0.556 < 1$. So, this equation satisfies all the conditions of Theorem 3.2, which shows that the equation has a solution satisfying

$$\left\| u(t) \right\|_{PC} \le \frac{4\eta_0^2 M_f^2 + 2\eta_0 M_f \sqrt{1 - 2\mathcal{C}_1^2 \eta_2 \eta_3^2}}{1 - (+4\eta_0^2 M_f^2 + 2\mathcal{C}_1^2 \eta_2 \eta_3^2)} = \frac{9(1 + \sqrt{26})}{225} \approx 0.244.$$

We can easily prove that $\omega_0(t) = 0$ is an upper solution and $v_0(t) = -\frac{1}{1+\sin t}$ is a lower solution of equation (5.2) and $\omega_0(t) \ge v_0(t)$. So, equation (5.2) satisfies all the conditions of Theorem 4.1. Hence, we can get the extremal solutions of problem (5.2) between v_0 and ω_0 by constructing iterative sequences starting from v_0 and ω_0:

$$v_n(t) = -M \int_0^1 G(t,s) v_n(s) \, ds + \int_0^1 G(t,s) \big[f\big(s, v_{n-1}(s), v'_{n-1}(s)\big) + M v_{n-1}(s) \big] \, ds$$
$$+ \sum_{k=1}^{\infty} \big[W(t,k) v_{n-1}(\xi_k) \big] I_A(\xi_1, \xi_2, \ldots, \xi_k, \ldots),$$
$$\omega_n(t) = -M \int_0^1 G(t,s) \omega_n(s) \, ds + \int_0^1 G(t,s) \big[f\big(s, \omega_{n-1}(s), \omega'_{n-1}(s)\big) + M \omega_{n-1}(s) \big] \, ds$$
$$+ \sum_{k=1}^{\infty} \big[W(t,k) \omega_{n-1}(\xi_k) \big] I_A(\xi_1, \xi_2, \ldots, \xi_k, \ldots).$$

We have completed the proof of the example.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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