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A note on global existence of strong solution to the 3D micropolar equations with a damping term

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Abstract

This paper studies the Cauchy problem of the 3D incompressible micropolar equations with a damping term $\sigma|u|^{\beta-1}u$ ($\sigma>0, 1\leq\beta<3$). It is shown that the strong solutions exist globally for any $1\leq\beta<3$.

Keywords: Micropolar equations; Damping; Global regularity

1 Introduction

We consider the Cauchy problem of the 3D incompressible micropolar equations with a nonlinear damping term $\sigma|u|^{\beta-1}u$ ($\sigma>0, 1\leq\beta<3$) (see [5]):

$$\begin{cases} u_t+u\cdot\nabla u-(\nu+\kappa)\Delta u+\sigma|u|^{\beta-1}u+\nabla p=2\kappa\nabla\times w, \\ w_t+u\cdot\nabla w+4\kappa w-\gamma\Delta w-\mu\nabla \operatorname{div} w=2\kappa\nabla\times u, \\ \operatorname{div} u=0, \\ u(x,0)=u_0(x), \qquad w(x,0)=w_0(x), \end{cases} \tag{1.1}$$

where $u\in\mathbb{R}^3$, $w\in\mathbb{R}^3$, $p\in\mathbb{R}$ are the velocity field of fluid, the field of microrotation representing the angular velocity of the rotation of the fluid particles and the scalar pressure, respectively. The parameter ν is the kinematic viscosity; κ is the microrotation viscosity; γ and μ are the angular viscosities; σ is the damping coefficient.

When $w=0$ and $\kappa=0$, the system (1.1) is reduced to the incompressible damped Navier–Stokes equations which was studied firstly by Cai and Jiu [1]. They proved that the corresponding equations admit a global weak solution for any $\beta\geq 1$ and a global strong solutions for $\beta\geq\frac{7}{2}$. Moreover, the uniqueness was shown for any $\frac{7}{2}\leq\beta\leq 5$. We refer to [3, 4, 6–8] for more results on the Navier–Stokes equations with a damping term.

Recently, the Cauchy problem (1.1) was considered by Ye [5]. It was proved that system (1.1) admits global strong solution for any $\beta\geq 3$. In this paper, we aim to study existence of global solutions under some smallness condition of the initial data for any $1\leq\beta<3$. Before stating our main results, we firstly state the local strong solutions to (1.1), which can be proved by the similar technique as in [2]. Thus, we omit the details.

Theorem 1.1 *Suppose that* $1 \le \beta < 3$, $(u_0, w_0) \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$. *Then there exist a small positive time* T_0 *and a unique strong solution* (u, w) *to the Cauchy problem* (1.1) *in* $\mathbb{R}^3 \times (0, T_0]$.

Now, our main results read as follows.

Theorem 1.2 *Assume that* $(u_0, w_0) \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$. $(u, w)(x, t)$ *is the corresponding local strong to* (1.1). *For* $1 \le \beta < 3$, *let* $T^* > 0$ *be a maximal existence time of the solution. If* $T^* < \infty$, *then*

$$\lim_{T \to T^*} \|u\|_{L^s(0,T;L^r)} = \infty, \quad \text{with } \frac{2}{s} + \frac{3}{r} \le 1, 3 < r < \infty. \tag{1.2}$$

Theorem 1.3 *Suppose* $(u_0, w_0) \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$. *If* $1 \le \beta < 3$, *then there exists a small positive constant* ε_0 *depending only on* μ, γ, σ, κ *and* ν, *such that if*

$$\left(\|w_0\|_{L^2}^2 + \|u_0\|_{L^2}^2\right)\left(\|\nabla w_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2\right) \le \varepsilon_0, \tag{1.3}$$

the Cauchy problem (1.1) *admits a unique global strong solution, satisfying*

$$\begin{aligned}
&(u, w) \in L^\infty\left(0, T; H^1\left(\mathbb{R}^3\right)\right) \cap L^2\left(0, T; H^2\left(\mathbb{R}^3\right)\right), \\
&|u|^{\frac{\beta-1}{2}} \nabla u \in L^2\left(0, T; L^2\left(\mathbb{R}^3\right)\right), \qquad \nabla |u|^{\frac{\beta+1}{2}} \in L^2\left(0, T; L^2\left(\mathbb{R}^3\right)\right).
\end{aligned}$$

Remark 1 When $w = 0$ and $\kappa = 0$, Theorem 1.1 and 1.2 generalize the previous results for the 3D Navier–Stokes equations with a damping term (see [7, 8]).

2 The proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. In what follows, C denotes a generic positive constant depending only on μ, γ, σ, ν, κ and β. Let (u, w) be a strong solution of (1.1) on $\mathbb{R}^3 \times (0, T)$ described in Theorem 1.1. As aforementioned, we shall prove Theorem 1.2 by contradiction arguments. So, from now on we assume otherwise that

$$\int_0^{T^*} \|u\|_{L^r}^s \, dt = M_0 < \infty, \tag{2.1}$$

with $\frac{2}{s} + \frac{3}{r} \le 1$, $3 < r < \infty$.

First, multiplying $(1.1)_1$ and $(1.1)_2$ by u and w, respectively, integrating (by parts) the resulting equations over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left(|u|^2 + |w|^2\right) dx + \sigma \int |u|^{\beta+1} \, dx + (\nu + \kappa) \int |\nabla u|^2 \, dx \\
&\quad + \gamma \int |\nabla w|^2 \, dx + 4\kappa \int |w|^2 \, dx + \mu |\operatorname{div} w|^2 \, dx \\
&\quad = -\int u \cdot \nabla u \cdot u \, dx - \int u \cdot \nabla w \cdot w \, dx + 2\kappa \int (\nabla \times w) \cdot u \, dx + 2\kappa \int (\nabla \times u) \cdot w \, dx \\
&\quad = 2\kappa \int (\nabla \times w) \cdot u \, dx + 2\kappa \int (\nabla \times u) \cdot w \, dx
\end{aligned}$$

$$= 4\kappa \int (\nabla \times u) \cdot w \, dx \le \kappa \|\nabla u\|_{L^2}^2 + 4\kappa \|w\|_{L^2}^2, \tag{2.2}$$

which integrate with respect to t,

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \int_0^t \left(2\nu \|\nabla u\|_{L^2}^2 + 2\gamma \|\nabla w\|_{L^2}^2\right) dx \le \|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2. \tag{2.3}$$

Next, multiplying $(1.1)_1$ by $-\Delta u$ and integrating (by parts) the resulting equations over $\mathbb{R}^3$. By Hölder's, Young's and the Gagliardo–Nirenberg inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 \, dx + \sigma\beta \int |u|^{\beta-1}|\nabla u|^2 \, dx + (\nu+\kappa)\int |\Delta u|^2 \, dx \\
&= \int u \cdot \nabla u \cdot \Delta u \, dx - 2\kappa \int (\nabla \times w) \cdot \Delta u \, dx \\
&\le \frac{\nu+\kappa}{4}\|\Delta u\|_{L^2}^2 + \frac{2}{\nu+\kappa}\int |u|^2|\nabla u|^2 \, dx + \frac{8\kappa^2}{\nu+\kappa}\|\nabla w\|_{L^2}^2 \\
&\le \frac{\nu+\kappa}{4}\|\Delta u\|_{L^2}^2 + \frac{C}{\nu+\kappa}\|u\|_{L^r}^2 \|\nabla u\|_{L^{\frac{2r}{r-2}}}^2 + \frac{8\kappa^2}{\nu+\kappa}\|\nabla w\|_{L^2}^2 \\
&\le \frac{\nu+\kappa}{4}\|\Delta u\|_{L^2}^2 + \frac{8\kappa^2}{\nu+\kappa}\|\nabla w\|_{L^2}^2 + \frac{C}{\nu+\kappa}\|u\|_{L^r}^2 \|\nabla u\|_{L^2}^{\frac{2(r-3)}{r}} \|\Delta u\|_{L^2}^{\frac{6}{r}} \\
&\le \frac{\nu+\kappa}{2}\|\Delta u\|_{L^2}^2 + \frac{8\kappa^2}{\nu+\kappa}\|\nabla w\|_{L^2}^2 + \frac{C}{(\nu+\kappa)^2}\|u\|_{L^r}^{\frac{2r}{r-3}} \|\nabla u\|_{L^2}^2
\end{aligned} \tag{2.4}$$

with arbitrary $r > 3$.

Similarly, multiplying $(1.1)_2$ by $-\Delta w$ and integrating (by parts) the resulting equations over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla w|^2 \, dx + \gamma \int |\Delta w|^2 \, dx + 4\kappa \int |\nabla w|^2 \, dx + \mu |\nabla \mathrm{div} w|^2 \, dx \\
&= \int u \cdot \nabla w \cdot \Delta w \, dx - 2\kappa \int (\nabla \times u) \cdot \Delta w \, dx \\
&\le \frac{\gamma}{4}\|\Delta w\|_{L^2}^2 + \frac{2}{\gamma}\int |u|^2|\nabla w|^2 \, dx + \frac{8\kappa^2}{\gamma}\|\nabla u\|_{L^2}^2 \\
&\le \frac{\gamma}{2}\|\Delta w\|_{L^2}^2 + \frac{8\kappa^2}{\gamma}\|\nabla u\|_{L^2}^2 + \frac{C}{\gamma^2}\|u\|_{L^r}^{\frac{2r}{r-3}} \|\nabla w\|_{L^2}^2.
\end{aligned} \tag{2.5}$$

Adding (2.4) and (2.5), by Gronwall's inequality and (2.3), we have

$$\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \int_0^t \left(\|\Delta u\|_{L^2}^2 + \|\Delta w\|_{L^2}^2\right) \le C\left(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2\right). \tag{2.6}$$

Therefore, if $u \in L^s(0,T;L^r)$ with $\frac{2}{s} + \frac{3}{r} \le 1$, we can take $(u,w)|_{t=T^*}$ as the initial data, then the local strong solutions (u,w) can be extended beyond T^*. This contradicts the assumption that $T^* > 0$ is the maximal existence time. The proof of Theorem 1.2 is complete.

3 The proof of Theorem 1.3

Throughout this section, we denote

$$C_0 := \|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2. \tag{3.1}$$

Let (u,w) be the strong solution to the problem (1.1) on $\mathbb{R}^3 \times (0,T)$, then one has the following estimates. First, we infer from (2.4) and (2.5) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left(|\nabla u|^2 + |\nabla w|^2\right) dx + \sigma\beta \int |u|^{\beta-1}|\nabla u|^2\, dx + (\nu+\kappa)\int |\Delta u|^2\, dx \\
&\quad + \gamma \int |\Delta w|^2\, dx + 4\kappa \int |\nabla w|^2\, dx + \mu|\nabla \mathrm{div} w|^2\, dx \\
&= \int u\cdot\nabla u\cdot\Delta u\, dx + \int u\cdot\nabla w\cdot\Delta w\, dx - 4\kappa\int (\nabla\times w)\cdot\Delta u\, dx \\
&\le 4\kappa\|\nabla w\|_{L^2}^2 + \left(\kappa + \frac{\nu}{4}\right)\|\Delta u\|_{L^2}^2 + \frac{\gamma}{2}\|\Delta w\|_{L^2}^2 \\
&\quad + \frac{1}{\nu}\int |u|^2|\nabla u|^2\, dx + \frac{1}{2\gamma}\int |u|^2|\nabla w|^2\, dx \\
&\le 4\kappa\|\nabla w\|_{L^2}^2 + \left(\kappa + \frac{\nu}{4}\right)\|\Delta u\|_{L^2}^2 + \frac{\gamma}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{\nu}\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 \\
&\quad + \frac{1}{2\gamma}\|u\|_{L^\infty}^2\|\nabla w\|_{L^2}^2 \\
&\le 4\kappa\|\nabla w\|_{L^2}^2 + \left(\kappa + \frac{\nu}{4}\right)\|\Delta u\|_{L^2}^2 + \frac{\gamma}{2}\|\Delta w\|_{L^2}^2 \\
&\quad + C\|u\|_{L^6}\|\Delta u\|_{L^2}\left(\frac{1}{\nu}\|\nabla u\|_{L^2}^2 + \frac{1}{2\gamma}\|\nabla w\|_{L^2}^2\right) \\
&\le 4\kappa\|\nabla w\|_{L^2}^2 + \left(\kappa + \frac{\nu}{2}\right)\|\Delta u\|_{L^2}^2 + \frac{\gamma}{2}\|\Delta w\|_{L^2}^2 + \frac{C}{\nu^3}\|\nabla u\|_{L^2}^6 \\
&\quad + \frac{C}{\gamma^3}\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}^4.
\end{aligned} \tag{3.2}$$

Then we obtain after integrating (3.2) with respect to t

$$\begin{aligned}
&\sup_{0\le s\le t}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) + \int_0^t \left(\|\Delta u\|_{L^2}^2 + \|\Delta w\|_{L^2}^2\right) ds \\
&\quad + \int_0^t \left(\sigma \left\||u|^{\frac{\beta-1}{2}}\nabla u\right\|_{L^2}^2 + \frac{4\sigma(\beta-1)}{(\beta+1)^2}\left\|\nabla |u|^{\frac{\beta+1}{2}}\right\|_{L^2}^2\right) ds \\
&\le \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + C_1\int_0^t \left(\|\nabla u\|_{L^2}^6 + \|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}^4\right) ds \\
&\le \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + C_1 \sup_{0\le s\le t}\left(\|\nabla u\|_{L^2}^4 + \|\nabla w\|_{L^2}^4\right)\int_0^t \|\nabla u\|_{L^2}^2\, ds \\
&\le \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + C_1C_0 \sup_{0\le s\le t}\left(\|\nabla u\|_{L^2}^4 + \|\nabla w\|_{L^2}^4\right).
\end{aligned} \tag{3.3}$$

Next, define the function $A(t)$ as follows:

$$A(t) := \sup_{0\le s\le t}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right). \tag{3.4}$$

Due to the regularity of u and w, one can deduce that $A(t)$ is a continuous function on $[0,T]$. According to (3.3), we have

$$A(t) \le \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + C_1C_0A^2(t). \tag{3.5}$$

Now, by (3.5), one can prove that

$$A(t) \le 3\big(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2\big). \tag{3.6}$$

In fact, we assume that

$$\big(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2\big)\big(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2\big) \le \varepsilon_0 \tag{3.7}$$

and set

$$T_* := \max\big\{t \in [0,T] : A(t) \le 3\big(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2\big), \forall s \in [0,t]\big\}. \tag{3.8}$$

Then we claim that

$$T = T_*.$$

Otherwise, we have $T_* \in (0,T)$. By the continuity of $A(t)$, it follows from (3.5), (3.7)–(3.8) that

$$\begin{aligned} A(T_*) &\le \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + C_1C_0A^2(T_*) \\ &\le \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + 3C_1C_0\big(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2\big)A(T_*) \\ &\le \|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + \frac{1}{2}A(T_*), \end{aligned} \tag{3.9}$$

here, we choose $\varepsilon_0 = \frac{1}{6C_1}$. Thus, from (3.9) we deduce that

$$A(T_*) \le 2\big(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2\big). \tag{3.10}$$

This contradicts (3.8). Hence, by virtue of the argument of continuity and (3.10), we can easily get the desired (3.6).

Finally, we ready to give the proof of Theorem 1.2. In fact, due to Theorem 1.1, there is a unique local strong solution (u,w) to Eqs. (1.1). Let T^* be the maximal existence time to the solution. We will show that $T^* = \infty$. Otherwise, by contradiction, we take $T^* < \infty$, then, by Theorem 1.2, we get, for any (s,r) with $\frac{2}{s} + \frac{3}{r} \le 1$, $3 < r < \infty$,

$$\int_0^{T^*} \|u\|_{L^r}^s \, dt = \infty, \tag{3.11}$$

which together with Sobolev's inequality $\|u\|_{L^6} \le C\|\nabla u\|_{L^2}$ leads to

$$\int_0^{T^*} \|\nabla u\|_{L^2}^4 \, dt = \infty. \tag{3.12}$$

On the other hand, by Hölder's inequality, the Gagliardo–Nirenberg inequality, (2.3) and (3.7), we get

$$\begin{aligned}\int_0^{T^*} \|\nabla u\|_{L^2}^4\,dt &\le \sup_{0\le t\le T^*}\|\nabla u\|_{L^2}^2\int_0^{T^*}\|\nabla u\|_{L^2}^2\,dt\\ &\le C\bigl(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2\bigr)\bigl(\|\nabla u_0\|_{L^2}^2+\|\nabla w_0\|_{L^2}^2\bigr)\\ &< +\infty, \end{aligned} \tag{3.13}$$

contradicting (3.12). This contradiction shows that $T^* = \infty$, and thus we obtain the global strong solution of (1.1). This ends the proof of Theorem 1.3.

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Authors' contributions
WW and YL designed the methodology. WW wrote the draft and derived the theorem. YL reviewed and revised the paper. All authors read and approved the final manuscript.

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References

1. Cai, X., Jiu, Q.: Weak and strong solutions for the incompressible Navier–Stokes equations with damping. J. Math. Anal. Appl. **343**, 799–809 (2008)
2. Chae, M., Kang, K., Lee, J.: Existence of smooth solutions to coupled chemotaxis-fluid equations. Discrete Contin. Dyn. Syst. **33**, 2271–2297 (2013)
3. Gala, S., Ragusa, M.A.: A new regularity criterion for the Navier–Stokes equations in terms of the two components of the velocity. Electron. J. Qual. Theory Differ. Equ. **2016**, 26 (2016)
4. Gala, S., Ragusa, M.A.: On the regularity criterion for the Navier–Stokes equations in terms of one directional derivative. Asian-Eur. J. Math. **10**, Article ID 1750012 (2017)
5. Ye, Z.: Global existence of strong solution to the 3D micropolar equations with a damping term. Appl. Math. Lett. **83**, 188–193 (2018)
6. Zhang, Z., Wu, X., Lu, M.: On the uniqueness of strong solution to the incompressible Navier–Stokes equations with damping. J. Math. Anal. Appl. **377**, 414–419 (2011)
7. Zhong, X.: Global well-posedness to the incompressible Navier–Stokes equations with damping. Electron. J. Qual. Theory Differ. Equ. **2017**, 62 (2017)
8. Zhou, Y.: Regularity and uniqueness for the 3D incompressible Navier–Stokes equations with damping. Appl. Math. Lett. **25**, 1822–1825 (2012)