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Implicit nonlinear fractional differential equations of variable order

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Abstract

In this manuscript, we examine both the existence and the stability of solutions to the implicit boundary value problem of Caputo fractional differential equations of variable order. We construct an example to illustrate the validity of the observed results.

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Keywords: Fractional differential equations of variable order; Boundary value problem; Fixed point theorem; Ulam–Hyers stability

1 Introduction

The idea of fractional calculus is to replace the natural numbers in the derivative's order with rational ones. Although it seems an elementary consideration, it has an exciting relevance explaining some physical phenomena. Especially in the last two decades, significant numbers of papers appeared on this topic, some papers deal with the existence of solutions to problems of variable order; see e.g. [3, 4, 9, 10, 12].

In particular, [2] Benchohra *et al.* studied the existence and uniqueness results for the following nonlinear implicit fractional differential equations:

$$\begin{cases} {}^cD^u_{0^+}x(t) = f(t, x(t), {}^cD^u_{0^+}x(t)), & t \in [0, T], 0 < T < +\infty, 1 < u \leq 2, \\ x(0) = x_0, \qquad x(T) = x_1, \end{cases}$$

where f is a given function, $x_0, x_1 \in \Re$, and ${}^cD^u_{0^+}$ is the Caputo fractional derivative of order u.

Inspired by [2] and [3, 4, 9, 10, 12], we deal with the boundary value problem (BVP)

$$\begin{cases} {}^cD^{u(t)}_{0^+}x(t) = f_1(t, x(t), {}^cD^{u(t)}_{0^+}x(t)), & t \in J := [0, T] \\ x(0) = 0, \qquad x(T) = 0, \end{cases} \tag{1}$$

where $u : J \to (1, 2]$, $f_1 : J \times \Re \times \Re \to \Re$ is a continuous function and ${}^cD^{u(t)}_{0^+}$ is the Caputo fractional derivative of variable-order $u(t)$.

In this paper, we shall look for a solution of (1). Further, we study the stability of the obtained solution of (1) in the sense of Ulam–Hyers *(UH)*.

2 Preliminaries

This section introduces some important fundamental definitions that will be needed for obtaining our results in the next sections.

The symbol $C(J,\Re)$ represents the Banach space of continuous functions $x: J \to \Re$ with the norm

$$\|x\| = \operatorname{Sup}\{|x(t)| : t \in J\}.$$

For $-\infty < a_1 < a_2 < +\infty$, we consider the mappings $u(t) : [a_1, a_2] \to (0, +\infty)$ and $v(t) : [a_1, a_2] \to (n-1, n)$, $n \in N$. Then the left Caputo fractional integral (CFI) of variable-order $u(t)$ for the function $f_2(t)$ [7, 8, 11] is

$$I_{a_1^+}^{u(t)} f_2(t) = \int_{a_1}^{t} \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))} f_2(s)\,ds, \quad t > a_1, \tag{2}$$

and the left Caputo fractional derivative (CFD) of variable-order $v(t)$ for the function $f_2(t)$ [7, 8, 11] is

$${}^{c}D_{a_1^+}^{v(t)} f_2(t) = \int_{a_1}^{t} \frac{(t-s)^{n-v(t)-1}}{\Gamma(n-v(t))} f_2^{(n)}(s)\,ds, \quad t > a_1. \tag{3}$$

As anticipated, in the case of $u(t)$ and $v(t)$ being constant, then CFI and CFD coincide with the standard Caputo fractional derivative and integral, respectively; see e.g. [6–8].

Recall the following pivotal observation.

Lemma 2.1 ([6]) *Let $\alpha_1, \alpha_2 > 0$, $a_1 > 0$, $f_2 \in L(a_1, a_2)$, ${}^{c}D_{a_1^+}^{\alpha_1} f_2 \in L(a_1, a_2)$. Then the differential equation*

$${}^{c}D_{a_1^+}^{\alpha_1} f_2 = 0$$

has the unique solution

$$f_2(t) = \omega_0 + \omega_1(t-a_1) + \omega_2(t-a_1)^2 + \cdots + \omega_{n-1}(t-a_1)^{n-1}$$

and

$$I_{a_1^+}^{\alpha_1}\, {}^{c}D_{a_1^+}^{\alpha_1} f_2(t) = f_2(t) + \omega_0 + \omega_1(t-a_1) + \omega_2(t-a_1)^2 + \cdots + \omega_{n-1}(t-a_1)^{n-1}$$

with $n-1 < \alpha_1 \le n$, $\omega_\ell \in \Re$, $\ell = 0, 1, \ldots, n-1$.

Furthermore,

$${}^{c}D_{a_1^+}^{\alpha_1} I_{a_1^+}^{\alpha_1} f_2(t) = f_2(t)$$

and

$$I_{a_1^+}^{\alpha_1} I_{a_1^+}^{\alpha_2} f_2(t) = I_{a_1^+}^{\alpha_2} I_{a_1^+}^{\alpha_1} f_2(t) = I_{a_1^+}^{\alpha_1+\alpha_2} f_2(t).$$

Remark 2.1 ([13, 15, 16]) Note that the semigroup property is not fulfilled for general functions $u(t)$, $v(t)$, i.e.,

$$I_{a_1^+}^{u(t)} I_{a_1^+}^{v(t)} f_2(t) \neq I_{a_1^+}^{u(t)+v(t)} f_2(t).$$

Example 2.1 Let

$$u(t) = t, \quad t \in [0,4], \qquad v(t) = \begin{cases} 2, & t \in [0,1], \\ 3, & t \in]1,4], \end{cases} \qquad f_2(t) = 2, \quad t \in [0,4],$$

$$\begin{aligned} I_{0^+}^{u(t)} I_{0^+}^{v(t)} f_2(t) &= \int_0^t \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))} \int_0^s \frac{(s-\tau)^{v(s)-1}}{\Gamma(v(s))} f_2(\tau)\, d\tau\, ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[\int_0^1 \frac{(s-\tau)}{\Gamma(2)} 2\, d\tau + \int_1^s \frac{(s-\tau)^2}{\Gamma(3)} 2\, d\tau \right] ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds, \end{aligned}$$

and

$$I_{0^+}^{u(t)+v(t)} f_2(t) = \int_0^t \frac{(t-s)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))} f_2(s)\, ds.$$

So, we get

$$\begin{aligned} I_{0^+}^{u(t)} I_{0^+}^{v(t)} f_2(t)|_{t=3} &= \int_0^3 \frac{(3-s)^2}{\Gamma(3)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds \\ &= \frac{21}{10}, \\ I_{0^+}^{u(t)+v(t)} f_2(t)|_{t=3} &= \int_0^3 \frac{(3-s)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))} f_2(s)\, ds \\ &= \int_0^1 \frac{(3-s)^4}{\Gamma(5)} 2\, ds + \int_1^3 \frac{(3-s)^5}{\Gamma(6)} 2\, ds \\ &= \frac{1}{12} \int_0^1 \left(s^4 - 12s^3 + 54s^2 - 108s + 81 \right) ds \\ &\quad + \frac{1}{60} \int_1^3 \left(-s^5 + 15s^4 - 90s^3 + 270s^2 - 405s + 243 \right) ds \\ &= \frac{665}{180}. \end{aligned}$$

Therefore, we obtain

$$I_{0^+}^{u(t)} I_{0^+}^{v(t)} f_2(t)|_{t=3} \neq I_{0^+}^{u(t)+v(t)} f_2(t)|_{t=3}.$$

Lemma 2.2 ([18]) *Let* $u : J \to (1,2]$ *be a continuous function, then, for* $f_2 \in C_\delta(J,\Re) = \{f_2(t) \in C(J,\Re), t^\delta f_2(t) \in C(J,\Re), 0 \le \delta \le 1\}$, *the variable order fractional integral* $I_{0^+}^{u(t)} f_2(t)$ *exists for any points on* J.

Lemma 2.3 ([18]) *Let $u : J \to (1,2]$ be a continuous function, then $I_{0^+}^{u(t)} f_2(t) \in C(J,\Re)$ for $f_2 \in C(J,\Re)$.*

Definition 2.1 ([5, 14, 17]) Let $I \subset \Re$, I is called a generalized interval if it is either an interval, or $\{a_1\}$ or $\emptyset$.

A finite set $\mathcal{P}$ is called a partition of I if each x in I lies in exactly one of the generalized intervals E in $\mathcal{P}$.

A function $g : I \to \Re$ is called piecewise constant with respect to partition $\mathcal{P}$ of I if for any $E \in \mathcal{P}$, g is constant on E.

Theorem 2.1 (Krasnoselskii fixed point theorem [6]) *Let S be a closed, bounded and convex subset of a real Banach space E and let W_1 and W_2 be operators on S satisfying the following conditions*:

(*i*) $W_1(S) + W_2(S) \subset S$,

(*ii*) *W_1 is continuous on S and $W_1(S)$ is a relatively compact subset of E*,

(*iii*) *W_2 is a strict contraction on S, i.e., there exists $k \in [0,1)$, such that*

$$\| W_2(x) - W_2(y) \| \le k\|x - y\|$$

for every $x, y \in S$.

Then there exists $x \in S$ such that $W_1(x) + W_2(x) = x$.

Definition 2.2 ([1]) Equation (1) is *(UH)* stable if there exists $c_{f_1} > 0$, such that, for any $\epsilon > 0$ and for every solution $z \in C(J,\Re)$ of the following inequality:

$$\left| {}^{c}D_{0^+}^{u(t)} z(t) - f_1\left(t, z(t), {}^{c}D_{0^+}^{u(t)} z(t)\right) \right| \le \epsilon, \quad t \in J, \tag{4}$$

there exists a solution $x \in C(J,\Re)$ of Eq. (1) with

$$\left| z(t) - x(t) \right| \le c_{f_1} \epsilon, \quad t \in J.$$

3 Existence of solutions

Let us introduce the following assumption:

(H1) Let $n \in N$ be an integer, $\mathcal{P} = \{J_1 := [0, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3], \dots, J_n := (T_{n-1}, T]\}$ be a partition of the interval J, and let $u(t) : J \to (1,2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

$$u(t) = \sum_{\ell=1}^{n} u_\ell I_\ell(t) = \begin{cases} u_1, & \text{if } t \in J_1, \\ u_2 & \text{if } t \in J_2, \\ \vdots & \\ u_n & \text{if } t \in J_n, \end{cases}$$

where $1 < u_\ell \le 2$ are constants, and I_ℓ is the indicator of the interval $J_\ell := (T_{\ell-1}, T_\ell]$, $\ell = 1, 2, \dots, n$, (with $T_0 = 0$, $T_n = T$) such that

$$I_\ell(t) = \begin{cases} 1 & \text{for } t \in J_\ell, \\ 0 & \text{for elsewhere.} \end{cases}$$

For each $\ell \in \{1,2,\dots,n\}$, the symbol $E_\ell = C(J_\ell,\Re)$, indicates the Banach space of continuous functions $x : J_\ell \to \Re$ equipped with the norm

$$\|x\|_{E_\ell} = \sup_{t\in J_\ell}\bigl|x(t)\bigr|.$$

Then, for any $t \in J_\ell$, $\ell = 1,2,\dots,n$, the left Caputo fractional derivative of variable order $u(t)$ for the function $x(t) \in C(J,\Re)$, defined by (3), could be presented as a sum of left Caputo fractional derivatives of constant-orders u_ℓ, $\ell = 1,2,\dots,n$

$$^{c}D_{0^+}^{u(t)}x(t) = \int_0^{T_1}\frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)}x^{(2)}(s)\,ds + \cdots + \int_{T_{\ell-1}}^{t}\frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)}x^{(2)}(s)\,ds. \tag{5}$$

Thus, according to (5), the BVP (1) can be written for any $t \in J_\ell$, $\ell = 1,2,\dots,n$ in the form

$$\int_0^{T_1}\frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)}x^{(2)}(s)\,ds + \cdots + \int_{T_{\ell-1}}^{t}\frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)}x^{(2)}(s)\,ds = f_1\bigl(t,x(t),{}^{c}D_{0^+}^{u(t)}x(t)\bigr). \tag{6}$$

In what follows we shall introduce the solution to the BVP (1).

Definition 3.1 The BVP (1) has a solution, if there are functions x_ℓ, $\ell = 1,2,\dots,n$, so that $x_\ell \in C([0,T_\ell],\Re)$, fulfilling Eq. (6), and $x_\ell(0) = 0 = x_\ell(T_\ell)$.

Let the function $x \in C(J,\Re)$ be such that $x(t) \equiv 0$ on $t \in [0,T_{\ell-1}]$ and such that it solves the integral equation (6). Then (6) is reduced to

$$^{c}D_{T_{\ell-1}^+}^{u_\ell}x(t) = f_1\bigl(t,x(t),{}^{c}D_{T_{\ell-1}^+}^{u_\ell}x(t)\bigr), \quad t \in J_\ell.$$

We shall deal with the following BVP:

$$\begin{cases} ^{c}D_{T_{\ell-1}^+}^{u_\ell}x(t) = f_1(t,x(t),{}^{c}D_{T_{\ell-1}^+}^{u_\ell}x(t)), & t \in J_\ell \\ x(T_{\ell-1}) = 0, \qquad x(T_\ell) = 0. \end{cases} \tag{7}$$

For our purpose, the upcoming lemma will be a corner stone of the solution of the BVP (7).

Lemma 3.1 *Let $\ell \in \{1,2,\dots,n\}$ be a natural number, $f_1 \in C(J_\ell\times\Re\times\Re,\Re)$ and there exists a number $\delta \in (0,1)$ such that $t^\delta f_1 \in C(J_\ell\times\Re\times\Re,\Re)$.*

Then the function $x \in E_\ell$ is a solution of the BVP (7) if and only if x solves the integral equation

$$x(t) = -(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}y(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell}y(t), \tag{8}$$

where

$$y(t) = f_1\bigl(t,-(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}y(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell}y(t),y(t)\bigr), \quad t \in J_\ell.$$

Proof We presume that $x \in E_\ell$ is solution of the BVP (7) and we take ${}^cD^{u_\ell}_{T^+_{\ell-1}} x(t) = y(t)$. Employing the operator $I^{u_\ell}_{T^+_{\ell-1}}$ to both sides of (7) and regarding Lemma 2.1, we find

$$x(t) = \omega_1 + \omega_2(t - T_{\ell-1}) + I^{u_\ell}_{T^+_{\ell-1}} y(t), \quad t \in J_\ell.$$

By $x(T_{\ell-1}) = 0$, we get $\omega_1 = 0$.

Let $x(t)$ satisfy $x(T_\ell) = 0$. So, we observe that

$$\omega_2 = -(T_\ell - T_{\ell-1})^{-1} I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell).$$

Then we find

$$x(t) = -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(t),$$

where

$$y(t) = f_1\bigl(t, -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(t), y(t)\bigr), \quad t \in J_\ell.$$

Conversely, let $x \in E_\ell$ be a solution of the integral equation (8). Regarding the continuity of the function $t^\delta f_1$ and Lemma 2.1, we deduce that x is the solution of the BVP (7).

We will prove the existence result for the BVP (7). This result is based on Theorem 2.1. □

Theorem 3.1 *Let the conditions of Lemma 3.1 be satisfied and there exist constants* $K, L > 0$, *such that* $t^\delta |f_1(t, y_1, z_1) - f_1(t, y_2, z_2)| \leq K|y_1 - y_2| + L|z_1 - z_2|$, *for any* $y_i, z_i \in \Re$, $i = 1, 2$, $t \in J_\ell$. *and the inequality*

$$\frac{2(T_\ell - T_{\ell-1})^{u_\ell - 1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \left(2K \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} + L\right) < 1, \tag{9}$$

holds.

Then the BVP (7) possesses at least one solution in E_ℓ.

Proof We construct the operators

$$W_1, W_2 : E_\ell \to E_\ell$$

as follows:

$$W_1 y(t) = -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell), \qquad W_2 y(t) = I^{u_\ell}_{T^+_{\ell-1}} y(t), \tag{10}$$

where

$$y(t) = f_1\bigl(t, -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(t), y(t)\bigr), \quad t \in J_\ell.$$

It follows from the properties of fractional integrals and from the continuity of the function $t^\delta f_1$ that the operators $W_1, W_2 : E_\ell \to E_\ell$ defined in (10) are well defined.

Let

$$R_\ell \geq \frac{\frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}}{1 - \frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta}-T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(2K\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + L)},$$

where

$$f^\star = \sup_{t\in J_\ell} \bigl|f_1(t,0,0)\bigr|.$$

We consider the set

$$B_{R_\ell} = \bigl\{y \in E_\ell, \|y\|_{E_\ell} \leq R_\ell\bigr\}.$$

Clearly B_{R_ℓ} is nonempty, closed, convex and bounded.

Now, we demonstrate that W_1, W_2 satisfy the assumption of Theorem 2.1. We shall prove it in four phases.

STEP 1: Claim: $W_1(B_{R_\ell}) + W_2(B_{R_\ell}) \subseteq (B_{R_\ell})$.

For $y \in B_{R_\ell}$, we have

$$\begin{aligned}
&\bigl|(W_1y)(t) + (W_2y)(t)\bigr| \\
&\quad\leq \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\bigl|f_1\bigl(s,-(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}y(T_\ell) \\
&\qquad + I_{T_{\ell-1}^+}^{u_\ell}y(s),y(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t - s)^{u_\ell-1}\bigl|f_1\bigl(s,-(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}y(T_\ell) \\
&\qquad + I_{T_{\ell-1}^+}^{u_\ell}y(s),y(s)\bigr)\bigr|\,ds \\
&\quad\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\bigl|f_1\bigl(s,-(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}y(T_\ell) \\
&\qquad + I_{T_{\ell-1}^+}^{u_\ell}y(s),y(s)\bigr)\bigr|\,ds \\
&\quad\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\bigl|f_1\bigl(s,-(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}y(T_\ell) \\
&\qquad + I_{T_{\ell-1}^+}^{u_\ell}y(s),y(s)\bigr) - f_1(s,0,0)\bigr|\,ds \\
&\qquad + \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\bigl|f_1(s,0,0)\bigr|\,ds \\
&\quad\leq \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}s^{-\delta}\bigl(K\bigl|-(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}y(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell}y(s)\bigr| \\
&\qquad + L\bigl|y(s)\bigr|\bigr)\,ds + \frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \\
&\quad\leq \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\bigl(K\bigl|I_{T_{\ell-1}^+}^{u_\ell}y(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell}y(s)\bigr| + L\bigl|y(s)\bigr|\bigr)\,ds \\
&\qquad + \frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\bigl(2K\bigl\|I^{u_\ell}_{T^+_{\ell-1}}y\bigr\|_{E_\ell} + L\|y\|_{E_\ell}\bigr)\int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\,ds + \frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} \\
&\leq \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\left(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + L\right)R_\ell + \frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} \\
&\leq R_\ell,
\end{aligned}
$$

which means that $W_1(B_{R_\ell}) + W_2(B_{R_\ell}) \subseteq B_{R_\ell}$.

STEP 2: Claim: W_1 is continuous.

We presume that the sequence (y_n) converges to y in E_ℓ and $t \in J_\ell$. Then

$$
\begin{aligned}
&\bigl|(W_1y_n)(t) - (W_1y)(t)\bigr| \\
&\leq \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\
&\quad\times \int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\bigl|f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}y_n(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}}y_n(s), y_n(s)\bigr) \\
&\quad - f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}}y(s), y(s)\bigr)\bigr|\,ds \\
&\leq \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\
&\quad\times \int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}s^{-\delta}\bigl(K\bigl|-(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1})I^{u_\ell}_{T^+_{\ell-1}}\bigl(y_n(T_\ell) - y(T_\ell)\bigr) \\
&\quad + I^{u_\ell}_{T^+_{\ell-1}}\bigl(y_n(s) - y(s)\bigr)\bigr| + L\bigl|\bigl(y_n(s) - y(s)\bigr)\bigr|\bigr) \\
&\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\bigl(K\bigl|I^{u_\ell}_{T^+_{\ell-1}}\bigl(y_n(T_\ell) - y(T_\ell)\bigr) \\
&\quad + I^{u_\ell}_{T^+_{\ell-1}}\bigl(y_n(s) - y(s)\bigr)\bigr| + L\bigl|\bigl(y_n(s) - y(s)\bigr)\bigr|\bigr) \\
&\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\bigl(2K\bigl\|I^{u_\ell}_{T^+_{\ell-1}}(y_n - y)\bigr\|_{E_\ell} + L\|y_n - y\|_{E_\ell}\bigr)\int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\,ds \\
&\leq \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\left(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + L\right)\|y_n - y\|_{E_\ell},
\end{aligned}
$$

i.e., we obtain

$$
\bigl\|(W_1y_n) - (W_1y)\bigr\|_{E_\ell} \to 0 \quad \text{as } n \to \infty.
$$

Ergo, the operator W_1 is a continuous on E_ℓ.

STEP 3: W_1 is compact

Now, we will show that $W_1(B_{R_\ell})$ is relatively compact, meaning that W_1 is compact. Clearly $W_1(B_{R_\ell})$ is uniformly bounded because by Step 1, we have $W_1(B_{R_\ell}) = \{W_1(y) : y \in B_{R_\ell}\} \subset W_1(B_{R_\ell}) + W_2(B_{R_\ell}) \subseteq (B_{R_\ell})$ thus for each $y \in B_{R_\ell}$ we have $\|W_1(y)\|_{E_\ell} \leq R_\ell$, which means that $W_1(B_{R_\ell})$ is bounded. It remains to show that $W_1(B_{R_\ell})$ is equicontinuous.

For $t_1, t_2 \in J_\ell, t_1 < t_2$ and $y \in B_{R_\ell}$, we have

$$
\bigl|(W_1y)(t_2) - (W_1y)(t_1)\bigr|
$$

$$
\begin{aligned}
&= \Bigg| -\frac{(T_\ell - T_{\ell-1})^{-1}(t_2 - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) \\
&\quad + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\big)\, ds + \frac{(T_\ell - T_{\ell-1})^{-1}(t_1 - T_{\ell-1})}{\Gamma(u_\ell)} \\
&\quad \times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\big)\, ds \Bigg| \\
&\le \frac{(T_\ell - T_{\ell-1})^{-1}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \\
&\quad \times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} \big| f_1\big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\big)\big|\, ds \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \\
&\quad \times \int_{T_{\ell-1}}^{T_\ell} \big| f_1\big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s), y(s)\big) - f_1(s,0,0)\big|\, ds \\
&\quad + \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \int_{T_{\ell-1}}^{T_\ell} \big| f_1(s,0,0)\big|\, ds \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \\
&\quad \times \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} \big(K \big| -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s)\big| + L\big|y(s)\big|\big)\, ds \\
&\quad + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \\
&\quad \times \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} \big(K \big| I^{u_\ell}_{T^+_{\ell-1}} y(T_\ell) + I^{u_\ell}_{T^+_{\ell-1}} y(s)\big| + L\big|y(s)\big|\big)\, ds \\
&\quad + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \big(2K \big\| I^{u_\ell}_{T^+_{\ell-1}} y\big\|_{E_\ell} + L\|y\|_{E_\ell}\big) \int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\, ds \\
&\quad + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-2}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \\
&\quad \times \left(2K \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} + L\right) \|y\|_{E_\ell} \\
&\quad + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \big((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\big) \\
&\le \Bigg[\frac{(T_\ell - T_{\ell-1})^{u_\ell-2}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \left(2K \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} + L\right) \|y\|_{E_\ell} \\
&\quad + \frac{f^\star (T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \Bigg]
\end{aligned}
$$

$$\times \bigl((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\bigr).$$

Hence $\|(W_1y)(t_2) - (W_1y)(t_1)\|_{E_\ell} \to 0$ as $|t_2 - t_1| \to 0$. It implies that $W_1(B_{R_\ell})$ is equicontinuous.

STEP 4: W_2 is a strict contraction

For $x(t), y(t) \in E_\ell$, we obtain

$$\begin{aligned}
&\bigl|(W_2x)(t) - (W_2y)(t)\bigr| \\
&= \biggl|\frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} x(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell} x(s), x(s)\bigr)\,ds \\
&\quad - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} y(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell} y(s), y(s)\bigr)\,ds\biggr| \\
&\le \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} \bigl|f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} x(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell} x(s), x(s)\bigr) \\
&\quad - f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} y(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell} y(s), y(s)\bigr)\bigr|\,ds \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} s^{-\delta}\bigl(K\bigl|(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1})\bigl(I_{T_{\ell-1}^+}^{u_\ell}(x-y)(T_\ell)\bigr) \\
&\quad + \bigl(I_{T_{\ell-1}^+}^{u_\ell}(x-y)(s)\bigr)\bigr| + L\bigl|(x-y)(s)\bigr|\bigr)\,ds \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} s^{-\delta}\bigl(K\bigl|\bigl(I_{T_{\ell-1}^+}^{u_\ell}(x-y)(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell}(x-y)(s)\bigr| + L\bigl|(x-y)(s)\bigr|\bigr)\,ds \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \bigl(2K\|\bigl(I_{T_{\ell-1}^+}^{u_\ell}(x-y)\|_{E_\ell} + L\|x-y\|_{E_\ell}\bigr) \int_{T_{\ell-1}}^{t} s^{-\delta}\,ds \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \biggl(2K\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + L\biggr)\|x-y\|_{E_\ell}.
\end{aligned}$$

Consequently by (9), the operator W_2 is a strict contraction.

Therefore, all conditions of Theorem 2.1 are fulfilled and thus there exists $\widetilde{x_\ell} \in B_{R_\ell}$, such that $W_1\widetilde{x_\ell} + W_2\widetilde{x_\ell} = \widetilde{x_\ell}$, which is a solution of the BVP (7). Since $B_{R_\ell} \subset E_\ell$, the claim of Theorem 3.1 is proved.

Now, we will prove the existence result for the BVP (1).

Introduce the following assumption:

(H2) Let $f_1 \in C(J \times \Re \times \Re, \Re)$ and there exists a number $\delta \in (0,1)$ such that $t^\delta f_1 \in C(J \times \Re \times \Re, \Re)$ and there exist constants $K, L > 0$, such that $t^\delta |f_1(t, y_1, z_1) - f_1(t, y_2, z_2)| \le K|y_1 - y_2| + L|z_1 - z_2|$, for any $y_1, y_2, z_1, z_2 \in \Re$ and $t \in J$. □

Theorem 3.2 *Let the conditions* ($H1$), ($H2$) *and inequality* (9) *be satisfied for all* $\ell \in \{1, 2, \ldots, n\}$.

Then the problem (1) *possesses at least one solution in* $C(J, \Re)$.

Proof For any $\ell \in \{1, 2, \ldots, n\}$ according to Theorem 3.1 the BVP (7) possesses at least one solution $\widetilde{x_\ell} \in E_\ell$.

For any $\ell \in \{1,2,\dots,n\}$ we define the function

$$x_\ell = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell. \end{cases}$$

Thus, the function $x_\ell \in C([0,T_\ell],\Re)$ solves the integral equation (6) for $t \in J_\ell$ with $x_\ell(0) = 0, x_\ell(T_\ell) = \widetilde{x}_\ell(T_\ell) = 0$.

Then the function

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} \\ \vdots \\ x_n(t) = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \end{cases} \tag{11}$$

is a solution of the BVP (1) in $C(J,\Re)$. □

4 Ulam–Hyers stability

Theorem 4.1 *Let the conditions* (*H*1), (*H*2) *and inequality* (9) *be satisfied. Then BVP* (1) *is* (UH) *stable.*

Proof Let $\epsilon > 0$ an arbitrary number and the function $z(t)$ from $z \in C(J_\ell,\Re)$ satisfy inequality (4).

For any $\ell \in \{1,2,\dots,n\}$ we define the functions $z_1(t) \equiv z(t), t \in [0,T_1]$ and for $\ell = 2,3,\dots,n$:

$$z_\ell(t) = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ z(t), & t \in J_\ell. \end{cases}$$

For any $\ell \in \{1,2,\dots,n\}$ according to equality (5) for $t \in J$ we get

$${}^cD^{u(t)}_{T_{\ell-1}^+} z_\ell(t) = \int_{T_{\ell-1}}^t \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} z^{(2)}(s)\, ds.$$

Taking the (CFI) $I^{u_\ell}_{T_{\ell-1}^+}$ of both sides of the inequality (4), we obtain

$$\begin{aligned} \Bigg| z_\ell(t) &+ \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\ &\times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f_1\big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T_{\ell-1}^+} z_\ell(T_\ell) + I^{u_\ell}_{T_{\ell-1}^+} z_\ell(s), z_\ell(s)\big)\, ds \\ &- \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^t (t-s)^{u_\ell - 1} f_1\big(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I^{u_\ell}_{T_{\ell-1}^+} z_\ell(T_\ell) \\ &+ I^{u_\ell}_{T_{\ell-1}^+} z_\ell(s), z_\ell(s)\big)\, ds \Bigg| \end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon \int_{T_{\ell-1}}^{t} \frac{(t-s)^{u_\ell-1}}{\Gamma(u_\ell)}\,ds \\
&\leq \epsilon \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}.
\end{aligned}
$$

According to Theorem 3.2, BVP (1) has a solution $x \in C(J,\Re)$ defined by $x(t) = x_\ell(t)$ for $t \in J_\ell, \ell = 1,2,\dots,n$, where

$$
x_\ell = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \tag{12}
$$

and $\widetilde{x}_\ell \in E_\ell$ is a solution of (7). According to Lemma 3.1 the integral equation

$$
\begin{aligned}
\widetilde{x}_\ell(t) = {}& -\frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\
&\times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(T_\ell) + I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\bigr)\,ds \\
&+ \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(T_\ell) \\
&+ I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\bigr)\,ds
\end{aligned} \tag{13}
$$

holds.

Let $t \in J_\ell, \ell = 1,2,\dots,n$. Then by Eqs. (12) and (13) we get

$$
\begin{aligned}
&\bigl|z(t) - x(t)\bigr| \\
&\quad= \bigl|z(t) - x_\ell(t)\bigr| \\
&\quad= \bigl|z_\ell(t) - \widetilde{x}_\ell(t)\bigr| \\
&\quad= \biggl| z_\ell(t) + \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\
&\qquad\times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(T_\ell) \\
&\qquad+ I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\bigr)\,ds \\
&\qquad- \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(T_\ell) \\
&\qquad+ I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\bigr)\,ds \biggr| \\
&\qquad+ \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \\
&\qquad\times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} \bigl| f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} z_\ell(T_\ell) \\
&\qquad+ I_{T_{\ell-1}^+}^{u_\ell} z_\ell(s), z_\ell(s)\bigr)\,ds - f_1\bigl(s, -(T_\ell - T_{\ell-1})^{-1}(s - T_{\ell-1}) I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(T_\ell) \\
&\qquad+ I_{T_{\ell-1}^+}^{u_\ell} \widetilde{x}_\ell(s), \widetilde{x}_\ell(s)\bigr)\bigr|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\big|f_1\big(s,-(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}z_\ell(T_\ell)\\
&+I_{T_{\ell-1}^+}^{u_\ell}z_\ell(s),z_\ell(s)\big)\,ds\\
&-f_1\big(s,-(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})I_{T_{\ell-1}^+}^{u_\ell}\widetilde{x}_\ell(T_\ell)+I_{T_{\ell-1}^+}^{u_\ell}\widetilde{x}_\ell(s),\widetilde{x}_\ell(s)\big)\big|\,ds\\
\le{}&\epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\frac{(T_\ell-T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\\
&\times\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}s^{-\delta}\big(K\big|(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})\big(I_{T_{\ell-1}^+}^{u_\ell}\big(z_\ell(T_\ell)-\widetilde{x}_\ell(T_\ell)\big)\big)\\
&+\big(I_{T_{\ell-1}^+}^{u_\ell}\big(z_\ell(s)-\widetilde{x}_\ell(s)\big)\big)\big|+L\big|\big(z_\ell(s)-\widetilde{x}_\ell(s)\big)\big|\big)\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}s^{-\delta}\big(K\big|(T_\ell-T_{\ell-1})^{-1}(s-T_{\ell-1})\big(I_{T_{\ell-1}^+}^{u_\ell}\big(z_\ell(T_\ell)-\widetilde{x}_\ell(T_\ell)\big)\big)\\
&+\big(I_{T_{\ell-1}^+}^{u_\ell}\big(z_\ell(s)-\widetilde{x}_\ell(s)\big)\big)\big|+L\big|\big(z_\ell(s)-\widetilde{x}_\ell(s)\big)\big|\big)\,ds\\
\le{}&\epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\\
&\times\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\big(K\big|\big(I_{T_{\ell-1}^+}^{u_\ell}\big(z_\ell(T_\ell)-\widetilde{x}_\ell(T_\ell)\big)\big)+\big(I_{T_{\ell-1}^+}^{u_\ell}\big(z_\ell(s)-\widetilde{x}_\ell(s)\big)\big)\big|\\
&+L\big|\big(z_\ell(s)-\widetilde{x}_\ell(s)\big)\big|\big)\,ds\\
\le{}&\epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\\
&+\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\big(2K\big\|I_{T_{\ell-1}^+}^{u_\ell}(z_\ell-\widetilde{x}_\ell)\big\|_{E_\ell}+L\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}\big)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\,ds\\
\le{}&\epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\\
&\times\left(2K\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}+L\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}\right)\\
\le{}&\epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\\
&\times\left(2K\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+L\right)\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}\\
\le{}&\epsilon\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+\mu\|z-x\|,
\end{aligned}
$$

where

$$
\mu=\max_{\ell=1,2,\dots,n}\frac{2(T_\ell-T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\left(2K\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}+L\right).
$$

Then

$$
\|z-x\|(1-\mu)\le\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\epsilon.
$$

We obtain, for each $t \in J_\ell$,

$$\left|z(t) - x(t)\right| \le \|z - x\| \le \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{(1-\mu)\Gamma(u_\ell + 1)}\epsilon := c_{f_1}\epsilon.$$

Therefore, the BVP (1) is *(UH)* stable. □

5 Example

Let us consider the following fractional boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{u(t)}x(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |x(t)| + |{}^cD_{0^+}^{u(t)}x(t)|)}, & t \in J := [0, 2], \\ x(0) = 0, \quad x(2) = 0. \end{cases} \tag{14}$$

Let

$$f_1(t, y, z) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + y + z)}, \quad (t, y, z) \in [0, 2] \times [0, +\infty) \times [0, +\infty).$$

$$u(t) = \begin{cases} \frac{3}{2}, & t \in J_1 := [0, 1], \\ \frac{9}{5}, & t \in J_2 :=]1, 2]. \end{cases} \tag{15}$$

Then we have

$$\begin{aligned} &t^{\frac{1}{3}}\left|f_1(t, y_1, z_1) - f_1(t, y_2, z_2)\right| \\ &\quad = \left|\frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)}\left(\frac{1}{1 + y_1 + z_1} - \frac{1}{1 + y_2 + z_2}\right)\right| \\ &\quad \le \frac{e^{-t}(|y_1 - y_2| + |z_1 - z_2|)}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + y_1 + z_1)(1 + y_2 + z_2)} \\ &\quad \le \frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)}\left(|y_1 - y_2| + |z_1 - z_2|\right) \\ &\quad \le \frac{1}{(e+5)}|y_1 - y_2| + \frac{1}{(e+5)}|z_1 - z_2|. \end{aligned}$$

Hence the condition (H2) holds with $\delta = \frac{1}{3}$ and $K = L = \frac{1}{e+5}$.

By (15), according to (7) we consider two auxiliary BVPs for Caputo fractional differential equations of constant order,

$$\begin{cases} {}^cD_{0^+}^{\frac{3}{2}}x(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |x(t)| + |{}^cD^{\frac{3}{2}}x(t)|)}, & t \in J_1, \\ x(0) = 0, \quad x(1) = 0 \end{cases} \tag{16}$$

and

$$\begin{cases} {}^cD_{1^+}^{\frac{9}{5}}x(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1 + |x(t)| + |{}^cD^{\frac{9}{5}}x(t)|)}, & t \in J_2, \\ x(1) = 0, \quad x(2) = 0. \end{cases} \tag{17}$$

Next, we prove that the condition (9) is fulfilled for $\ell = 1$. Indeed,

$$\begin{aligned} \frac{2({T_1}^{1-\delta} - {T_0}^{1-\delta})(T_1 - T_0)^{u_1-1}}{(1-\delta)\Gamma(u_1)} \left(\frac{2K(T_1 - T_0)^{u_1}}{\Gamma(u_1+1)} + L \right) &= \frac{1}{\frac{2}{3}(e+5)\Gamma(\frac{3}{2})} \left(\frac{2}{\Gamma(\frac{5}{2})} + 1 \right) \\ &\simeq 0.3664 < 1. \end{aligned}$$

Accordingly the condition (9) is achieved. By Theorem 3.1, the problem (16) has a solution $\widetilde{x}_1 \in E_1$.

We prove that the condition (9) is fulfilled for $\ell = 2$. Indeed,

$$\begin{aligned} \frac{2({T_2}^{1-\delta} - {T_1}^{1-\delta})(T_2 - T_1)^{u_2-1}}{(1-\delta)\Gamma(u_2)} \left(\frac{2K(T_2 - T_1)^{u_2}}{\Gamma(u_2+1)} + L \right) &= \frac{2^{\frac{2}{3}} - 1}{\frac{2}{3}\Gamma(\frac{9}{5})} \frac{1}{e+5} \left(\frac{2}{\Gamma(\frac{14}{5})} + 1 \right) \\ &\simeq 0.2682 < 1. \end{aligned}$$

Thus, the condition (9) is satisfied.

According to Theorem 3.1, the BVP (17) possesses a solution $\widetilde{x}_2 \in E_2$.

Then, by Theorem 3.2, the BVP (14) has a solution

$$x(t) = \begin{cases} \widetilde{x}_1(t), & t \in J_1, \\ x_2(t), & t \in J_2, \end{cases}$$

where

$$x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2(t), & t \in J_2. \end{cases}$$

According to Theorem 4.1, BVP (14) is *(UH)* stable.

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