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A blow-up result for a system of coupled viscoelastic equations with arbitrary positive initial energy

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Abstract

This article is devoted to a study of the blow-up result for a system of coupled viscoelastic wave equations. By establishing a new auxiliary function and using the reduction to absurdity method, we obtain some sufficient conditions on initial data such that the solution blows up in finite time at arbitrarily high initial energy. This work generalizes and improves earlier results in the literature.

Keywords: Viscoelastic wave equation; Relaxation function; Blow up

1 Introduction

In this article, we investigate the blow-up property of the coupled viscoelastic wave equations of the form

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau) d\tau + |u_t|^{m-2}u_t = f_1(u, v), & x \in \Omega, t > 0, \\ |v_t|^\rho v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau) d\tau + |v_t|^{r-2}v_t = f_2(u, v), & x \in \Omega, t > 0, \\ u(x, t) = v(x, t) = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega. \end{cases} \quad (1.1)$$

Here Ω is a bounded domain of R^n with a smooth boundary $\partial\Omega$. $\rho > 0$, g_1 and g_2 are the kernel of memory terms, the nonlinear terms f_1 and f_2 will be specified later. The problem of (1.1) has been considered by many mathematics researchers and results in connection with blow-up and decay have been extensively established.

For single viscoelastic wave equation, Messaoudi [11] discussed the following equation:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau) d\tau + u_t|u_t|^{m-2} = u|u|^{p-2}, & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, t \geq 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \quad (1.2)$$

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where $m \geq 1, p \geq 2$, Ω is a bounded domain. The author studied the interaction between the weak damping term $u_t|u_t|^{m-2}$ and the nonlinear source term $u|u|^{p-2}$, which was first considered by Levine [9, 10] when $m = 1$, and found, under suitable conditions on g and initial data, that the solutions exist globally for any initial data if $m \geq p$ and blow up in finite time with negative initial energy if $p > m$. This blow-up result has been pushed by the same author in [12], to certain solutions with positive initial energy. Recently, Song [19] proved, by using the reduction to absurdity method, that the solutions of Eq. (1.2) blow up in finite time with the initial data have arbitrarily high initial energy. In the case when the nonlinear damping term $u_t|u_t|^{m-2}$ is replaced by the strong damping term $-\Delta u_t$ in Eq. (1.2), Song and Zhong [21] showed, by using the potential well theory introduced by Payne and Sattinger [16], that a blow-up result for solutions with positive initial energy. Later, Song and Xue [20] improved the blow-up result in which the initial data have arbitrarily high initial energy. In [22], Xu and Lian studied a nonlinear wave equation with weak and strong damping terms and logarithmic source term, they established the local existence of weak solution, showed the global existence, energy decay in the framework of potential well and obtained the blow-up of the solution with sub-critical initial energy. Furthermore, they in parallel extend all the conclusions for the sub-critical case to the critical case by scaling technique. Besides, a high energy infinite time blow-up result is established. Within a similar potential well framework, the semilinear pseudo parabolic equation [24] and parabolic system [23] were discussed in depth.

In the same direction, Song [18] discussed the following initial-boundary value problem:

$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t - \tau)\Delta u(\tau) d\tau + u_t|u_t|^{m-2} = u|u|^{p-2}, \quad x \in \Omega, t > 0, \tag{1.3}$$

where Ω is a bounded domain and $\rho > 0, m \geq 1, p > 2$. By modifying the method used in Messaoudi [12], the author proved that the solution blows up in finite time with initial data have positive initial energy. Recently, He and Song [8] pushed the blow-up result to certain solutions with arbitrary positive initial energy. When the nonlinear damping term $u_t|u_t|^{m-2}$ is substituted by the strong damping term $-\Delta u_t$, Hao et al. [7], inspired by the method used in Song [19], proved that solutions with negative initial energy as well as positive initial energy blow-up in finite time provided $p > \rho + 2$, and obtained, by using the perturbed energy functional technique, that solutions exist globally for any initial data provided $p \leq \rho + 2$.

For a blow-up result in systems of hyperbolic equations, the coupled system

$$\begin{cases} u_{tt} - \Delta u + u_t|u_t|^{m-1} = f_1(u, v), & x \in \Omega, t > 0, \\ v_{tt} - \Delta v + v_t|v_t|^{r-1} = f_2(u, v), & x \in \Omega, t > 0, \\ u(x, t) = v(x, t) = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.4}$$

was considered by Agre and Rammaha [2], where $m, r \geq 1$ and Ω is a bounded domain with smooth boundary. The authors found, by using the same method as in [4], that any weak solution blows up in finite time with negative initial energy. Furthermore, Said-Houari [17] extended this blow-up result to positive initial energy.

In the presence of the memory term, Han and Wang [5] considered the following system of viscoelastic equations:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + u_t |u_t|^{m-1} = f_1(u, v), & x \in \Omega, t > 0, \\ v_{tt} - \Delta v + \int_0^t h(t - \tau) \Delta v(\tau) d\tau + v_t |v_t|^{r-1} = f_2(u, v), & x \in \Omega, t > 0, \\ u(x, t) = v(x, t) = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.5}$$

They obtained the local existence, global existence, uniqueness and a blow-up result for certain solutions with negative initial energy. In [13], Messaoudi and Said-Houari extended this blow-up result to certain solutions with positive initial energy. Later, Zhao and Wang [1] proved the finite time blow-up of solutions whose initial data have arbitrarily high initial energy. In the same nature, Mustafa [14] considered a coupled system of nonlinear viscoelastic equations, he proved the well-posedness and established a generalized stability result for this system. More relevant knowledge we refer the reader to the literature [15].

As far as we know, the problem of the blow-up phenomenon for system (1.1) with arbitrary positive initial energy has not been considered. Our aim in this paper is to extend the research method for the blow-up phenomena used in [8] to the couple viscoelastic wave system (1.1), while we should handle the additional difficulty caused by damping term, viscoelastic term and source term. In order to overcome the difficulty, we construct a suitable auxiliary functions $(u, \frac{u_t |u_t|^\rho}{\rho+1}) + (v, \frac{v_t |v_t|^\rho}{\rho+1}) - (\frac{1}{\varepsilon_1^*})^{\frac{\gamma^*}{\gamma^*-1}} \frac{\gamma-1}{\gamma} E(t)$ and combine the reduction to absurdity method to derive contradiction, namely, we find suitable conditions on initial data such that the solution of (1.1) blows up in finite time at arbitrary high initial energy level.

This article is organized as follows. In Sect. 2, we present some material needed for our work. Section 3 is devoted to the blow-up result.

2 Preliminaries

In this section, we give some material needed for our work. Firstly, let us make the following assumptions.

(A1) $g_i : R_+ \rightarrow R_+$ (for $i = 1, 2$) are non-increasing differentiable functions satisfying

$$g_i(0) > 0, \quad 1 - \int_0^\infty g_i(\tau) d\tau = l_i > 0.$$

(A2) For nonlinear terms, we assume that

$$0 < \rho < +\infty, n = 1, 2, \quad 0 < \rho \leq \frac{2}{n-2}, \quad n \geq 3,$$

$$2 \leq m, r < +\infty, n = 1, 2, \quad 2 \leq m, r \leq \frac{2n}{n-2}, \quad n \geq 3.$$

(A3) For the functions f_1 and f_2 , we note that

$$f_1(u, v) = [a|u + v|^{2(p+1)}(u + v) + b|u|^p u |v|^{p+2}],$$

$$f_2(u, v) = [a|u + v|^{2(p+1)}(u + v) + b|v|^p v|u|^{p+2}], \quad a, b > 0,$$

where

$$-1 < p < +\infty, n = 1, 2, \quad -1 < p \leq \frac{3-n}{n-2}, \quad n \geq 3.$$

It is easy to verify that

$$uf_1(u, v) + vf_2(u, v) = 2(p + 2)F(u, v), \quad \forall (u, v) \in \mathbb{R}^2,$$

where

$$F(u, v) = \frac{1}{2(p + 2)} [a|u + v|^{2(p+2)} + 2b|uv|^{p+2}].$$

At present, we state the following local existence theorem which can be proved by combining the arguments in [3, 6]. Here we omit the proof.

Theorem 2.1 *Suppose that (A1), (A2), (A3) hold and $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$ are given, then system (1.1) possesses a unique local solution (u, v) such that*

$$\begin{aligned} (u, v) &\in C([0, T]; H_0^1(\Omega)) \times C([0, T]; H_0^1(\Omega)), \\ (u_t, v_t) &\in C([0, T]; L^2(\Omega)) \cap L^m(\Omega \times [0, T]) \times C([0, T]; L^2(\Omega)) \cap L^r(\Omega \times [0, T]), \end{aligned}$$

for the maximum existence time $T > 0$, where $T \in (0, \infty]$.

The energy of the system (1.1) is

$$\begin{aligned} E(t) &= \frac{1}{\rho + 2} (\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}) + \frac{1}{2} \left(1 - \int_0^t g_1(\tau) d\tau \right) \|\nabla u\|_2^2 - \int_{\Omega} F(u, v) dx \\ &\quad + \frac{1}{2} [(g_1 \circ \nabla u) + (g_2 \circ \nabla v)] + \frac{1}{2} \left(1 - \int_0^t g_2(\tau) d\tau \right) \|\nabla v\|_2^2, \end{aligned}$$

where

$$(g \circ \nabla v)(t) = \int_0^t g(t - \tau) \|v(t) - v(\tau)\|_2^2 d\tau.$$

Lemma 2.2 *Assume (A1) holds, and let (u, v) be a solution of system (1.1), then $E(t)$ is non-increasing, namely*

$$\frac{dE(t)}{dt} \leq -\|u_t\|_m^m - \|v_t\|_r^r \leq 0, \quad \forall t \geq 0. \tag{2.1}$$

Proof Multiplying the first two equations in system (1.1) by u_t, v_t , respectively, and then integrating over Ω , we get

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{\rho + 2} (\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}) + \frac{1}{2} (\|\nabla u\|_2^2 + \|\nabla v\|_2^2) - \int_{\Omega} F(u, v) dx \right] \\ = -\|u_t\|_m^m - \|v_t\|_r^r + \int_0^t g_1(t - \tau) \int_{\Omega} \nabla u_t(t) \cdot \nabla u(\tau) dx d\tau \end{aligned}$$

$$+ \int_0^t g_2(t - \tau) \int_{\Omega} \nabla v_t(t) \cdot \nabla v(\tau) \, dx \, d\tau, \tag{2.2}$$

where

$$\begin{aligned} & \int_0^t g_1(t - \tau) \int_{\Omega} \nabla u_t(t) \cdot \nabla u(\tau) \, dx \, d\tau \\ &= \int_0^t g_1(t - \tau) \int_{\Omega} \nabla u_t(t) \cdot [\nabla u(\tau) - \nabla u(t)] \, dx \, d\tau \\ & \quad + \int_0^t g_1(t - \tau) \int_{\Omega} \nabla u_t(t) \cdot \nabla u(t) \, dx \, d\tau \\ &= -\frac{1}{2} \int_0^t g_1(t - \tau) \left(\frac{d}{dt} \int_{\Omega} |\nabla u(\tau) - \nabla u(t)|^2 \, dx \right) d\tau \\ & \quad + \int_0^t g_1(\tau) \left(\frac{d}{dt} \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 \, dx \right) d\tau \\ &= -\frac{1}{2} \frac{d}{dt} \left[\int_0^t g_1(t - \tau) \int_{\Omega} |\nabla u(\tau) - \nabla u(t)|^2 \, dx \, d\tau \right] \\ & \quad + \frac{1}{2} \frac{d}{dt} \left[\int_0^t g_1(\tau) \int_{\Omega} |\nabla u(t)|^2 \, dx \, d\tau \right] \\ & \quad + \frac{1}{2} \int_0^t g_1'(t - \tau) \int_{\Omega} |\nabla u(\tau) - \nabla u(t)|^2 \, dx \, d\tau - \frac{1}{2} g_1(t) \int_{\Omega} |\nabla u(t)|^2 \, dx. \end{aligned} \tag{2.3}$$

Similarly

$$\begin{aligned} & \int_0^t g_2(t - \tau) \int_{\Omega} \nabla v_t(t) \cdot \nabla v(\tau) \, dx \, d\tau \\ &= -\frac{1}{2} \frac{d}{dt} \left[\int_0^t g_2(t - \tau) \int_{\Omega} |\nabla v(\tau) - \nabla v(t)|^2 \, dx \, d\tau \right] \\ & \quad + \frac{1}{2} \frac{d}{dt} \left[\int_0^t g_2(\tau) \int_{\Omega} |\nabla v(t)|^2 \, dx \, d\tau \right] \\ & \quad + \frac{1}{2} \int_0^t g_2'(t - \tau) \int_{\Omega} |\nabla v(\tau) - \nabla v(t)|^2 \, dx \, d\tau - \frac{1}{2} g_2(t) \int_{\Omega} |\nabla v(t)|^2 \, dx. \end{aligned} \tag{2.4}$$

By inserting (2.3) and (2.4) into (2.2), and combining (A1), we can obtain

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{\rho + 2} (\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}) + \frac{1}{2} \left(1 - \int_0^t g_1(\tau) \, d\tau \right) \|\nabla u\|_2^2 \right. \\ & \quad + \left. \left(1 - \int_0^t g_2(\tau) \, d\tau \right) \|\nabla v\|_2^2 \right. \\ & \quad + \left. \frac{1}{2} [(g_1 \circ \nabla u) + (g_2 \circ \nabla v)] - \int_{\Omega} F(u, v) \, dx \right] \\ &= -\|u_t\|_m^m - \|v_t\|_r^r + \frac{1}{2} [(g_1' \circ \nabla u) + (g_2' \circ \nabla v)] - \frac{1}{2} g_1(t) \|\nabla u\|_2^2 - \frac{1}{2} g_2(t) \|\nabla v\|_2^2, \end{aligned}$$

namely

$$\begin{aligned}
 E'(t) &= -\|u_t\|_m^m - \|v_t\|_r^r + \frac{1}{2}[(g'_1 \circ \nabla u) + (g'_2 \circ \nabla v)] \frac{1}{2}g_1(t)\|\nabla u\|_2^2 - \frac{1}{2}g_2(t)\|\nabla v\|_2^2 \\
 &\leq -\|u_t\|_m^m - \|v_t\|_r^r \leq 0.
 \end{aligned}
 \tag{2.5}$$

□

Lemma 2.3 ([1], Lemma 2) *There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(p+2)}(|u|^{2(p+2)} + |v|^{2(p+2)}) \leq F(u, v) \leq \frac{c_1}{2(p+2)}(|u|^{2(p+2)} + |v|^{2(p+2)}).$$

Lemma 2.4 *Assume $0 < \pi < \alpha < \gamma$ hold, then we have the following inequality:*

$$\|u\|_\alpha^\alpha \leq \|u\|_\pi^\pi + \|u\|_\gamma^\gamma.$$

Proof (1) When $|u| > 1$, then $\int_\Omega |u|^\alpha dx \leq \int_\Omega |u|^\gamma dx \leq \int_\Omega |u|^\pi dx + \int_\Omega |u|^\gamma dx$ is true.

(2) When $|u| \leq 1$, then $\int_\Omega |u|^\alpha dx \leq \int_\Omega |u|^\pi dx \leq \int_\Omega |u|^\pi dx + \int_\Omega |u|^\gamma dx$ is true, where $|\cdot|$ represents the absolute value. □

Lemma 2.5 *Assume $0 < \varrho < \varsigma$ hold, then we have the following inequality:*

$$\|u\|_\varrho^\varrho < \|u\|_\varsigma^\varsigma + 1.$$

Proof (1) Assume $\|u\|_\varsigma > 1$ hold, then $\|u\|_\varrho^\varrho < \|u\|_\varsigma^\varsigma < \|u\|_\varsigma^\varsigma + 1$ is true.

(2) Assume $\|u\|_\varsigma \leq 1$ hold, then $\|u\|_\varrho^\varrho < \|u\|_\varsigma^\varsigma + 1$ is true. □

In order to obtain our main result, we need the following lemma which presents the same one of (He and Song [8] Lemma 2.2) with suitable modification.

Lemma 2.6 ([8], Lemma 2.2) *Assume that $2 < \rho + 2 < \min\{m, r\}$ and $\max\{m, r\} < 2(p + 2)$. Assume further that (A1)–(A3) hold and g_i ($i = 1, 2$) satisfying (3.1). If there exists a number $t_0 \geq 0$ such that $E(t_0) < 0$, then the solution of the system (1.1) blows up in finite time.*

3 Blow-up result

In this section, we discuss the blow-up phenomenon.

Theorem 3.1 *Assume the conditions (A1)–(A3) and $2 < \rho + 2 < \min\{m, r\}$, $\max\{m, r\} < 2(p + 2)$ hold, and g_i satisfies the condition*

$$\max \left\{ \int_0^\infty g_1(\tau) d\tau, \int_0^\infty g_2(\tau) d\tau \right\} < \frac{p+1}{p+1 + \frac{1}{4(p+2)}},
 \tag{3.1}$$

for $i = 1, 2$. Let (u, v) be the solution of system (1.1), satisfying

$$\int_\Omega u(0) \frac{u_t(0)|u_t(0)|^\rho}{\rho+1} dx + \int_\Omega v(0) \frac{v_t(0)|v_t(0)|^\rho}{\rho+1} dx > ME(0) > 0,
 \tag{3.2}$$

then (u, v) blows up in finite time, where

$$M = \left(\frac{1}{\varepsilon_1^*}\right)^{\frac{\gamma^*}{\gamma^*-1}} \frac{\gamma - 1}{\gamma}$$

$\gamma = \max\{m, r\}$, $\gamma^* = \min\{m, r\}$, $\varepsilon_1^* > 0$ is a constant such that $\left(\frac{1}{\varepsilon_1}\right)^{\frac{\gamma^*}{\gamma^*-1}} \frac{\gamma-1}{\gamma} \geq \frac{2(p+2)(1-\varepsilon)}{\beta}$, $\varepsilon \in (0, 1)$ is a small enough constant such that

$$\kappa_1(\varepsilon) = ((p + 2)(1 - \varepsilon) - 1)l_1 - \frac{1}{4(p + 2)(1 - \varepsilon)}(1 - l_1) > 0,$$

$$\kappa_2(\varepsilon) = ((p + 2)(1 - \varepsilon) - 1)l_2 - \frac{1}{4(p + 2)(1 - \varepsilon)}(1 - l_2) > 0,$$

$$\alpha = \min\left\{\kappa_1(\varepsilon)\lambda - \frac{\varepsilon_1^m}{m}, \kappa_2(\varepsilon)\lambda - \frac{\varepsilon_1^r}{r}, c_0\varepsilon - \frac{\varepsilon_1^m}{m}, c_0\varepsilon - \frac{\varepsilon_1^r}{r}\right\},$$

$$\beta = \min\left\{\frac{1}{\rho + 1} + \frac{2(p + 2)(1 - \varepsilon)}{\rho + 2}, \alpha\right\},$$

and λ is the first eigenvalue of $-\Delta$.

Proof Suppose that (u, v) is a global solution of system (1.1). Multiplying the first two equations of system (1.1) by u and v , respectively, and integrating over Ω , we obtain

$$\begin{aligned} & (|u_t|^\rho u_{tt}, u) + \|\nabla u\|_2^2 + \int_\Omega \int_0^t g_1(t - \tau)\Delta u(\tau) d\tau u(t) dx + \int_\Omega uu_t|u_t|^{m-2} dx \\ & = \int_\Omega uf_1(u, v) dx, \end{aligned} \tag{3.3}$$

$$\begin{aligned} & (|v_t|^\rho v_{tt}, v) + \|\nabla v\|_2^2 + \int_\Omega \int_0^t g_2(t - \tau)\Delta v(\tau) d\tau v(t) dx + \int_\Omega vv_t|v_t|^{m-2} dx \\ & = \int_\Omega vf_2(u, v) dx. \end{aligned} \tag{3.4}$$

Taking the derivative of $(u, \frac{u_t|u_t|^\rho}{\rho+1})$ and $(v, \frac{v_t|v_t|^\rho}{\rho+1})$, respectively, and combining (3.3) and (3.4), we have

$$\begin{aligned} \frac{d}{dt} \left(u, \frac{u_t|u_t|^\rho}{\rho + 1}\right) &= \frac{1}{\rho + 1} \|u_t(t)\|_{\rho+2}^{\rho+2} - \|\nabla u\|_2^2 + \int_0^t g_1(t - \tau) \int_\Omega \nabla u(\tau)\nabla u(t) dx d\tau \\ &\quad - \int_\Omega uu_t|u_t|^{m-2} dx + \int_\Omega uf_1(u, v) dx, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \frac{d}{dt} \left(v, \frac{v_t|v_t|^\rho}{\rho + 1}\right) &= \frac{1}{\rho + 1} \|v_t(t)\|_{\rho+2}^{\rho+2} - \|\nabla v\|_2^2 + \int_0^t g_2(t - \tau) \int_\Omega \nabla v(\tau)\nabla v(t) dx d\tau \\ &\quad - \int_\Omega vv_t|v_t|^{m-2} dx + \int_\Omega vf_2(u, v) dx. \end{aligned} \tag{3.6}$$

For the third term on the right side of (3.5), we get

$$\int_0^t g_1(t - \tau) \int_\Omega \nabla u(\tau)\nabla u(t) dx d\tau = \int_0^t g_1(t - \tau) \int_\Omega \nabla u(t)\nabla(u(\tau) - u(t)) dx d\tau$$

$$\begin{aligned}
 & + \int_0^t g_1(t - \tau) \|\nabla u(t)\|_2^2 d\tau \\
 & = \int_0^t g_1(t - \tau) \int_{\Omega} \nabla u(t)(\nabla u(\tau) - \nabla u(t)) dx d\tau \\
 & + \int_0^t g_1(\tau) d\tau \|\nabla u(t)\|_2^2. \tag{3.7}
 \end{aligned}$$

In the same way, we have

$$\begin{aligned}
 \int_0^t g_2(t - \tau) \int_{\Omega} \nabla v(\tau)\nabla v(t) dx d\tau & = \int_0^t g_2(t - \tau) \int_{\Omega} \nabla v(t)(\nabla v(\tau) - \nabla v(t)) dx d\tau \\
 & + \int_0^t g_2(\tau) d\tau \|\nabla v(t)\|_2^2. \tag{3.8}
 \end{aligned}$$

Then, inserting (3.7) and (3.8) into (3.5) and (3.6), respectively,

$$\begin{aligned}
 & \frac{d}{dt} \left(u, \frac{u_t|u_t|^\rho}{\rho + 1} \right) \\
 & = \frac{1}{\rho + 1} \|u_t(t)\|_{\rho+2}^{\rho+2} - \|\nabla u\|_2^2 + \int_0^t g_1(t - \tau) \int_{\Omega} \nabla u(t)(\nabla u(\tau) - \nabla u(t)) dx d\tau \\
 & + \int_0^t g_1(\tau) d\tau \|\nabla u(t)\|_2^2 - \int_{\Omega} uu_t|u_t|^{m-2} dx + \int_{\Omega} uf_1(u, v) dx, \tag{3.9}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{d}{dt} \left(v, \frac{v_t|v_t|^\rho}{\rho + 1} \right) \\
 & = \frac{1}{\rho + 1} \|v_t(t)\|_{\rho+2}^{\rho+2} - \|\nabla v\|_2^2 + \int_0^t g_2(t - \tau) \int_{\Omega} \nabla v(t)(\nabla v(\tau) - \nabla v(t)) dx d\tau \\
 & + \int_0^t g_2(\tau) d\tau \|\nabla v(t)\|_2^2 - \int_{\Omega} vv_t|v_t|^{r-2} dx + \int_{\Omega} vf_2(u, v) dx. \tag{3.10}
 \end{aligned}$$

For the third term on the right side of (3.9), applying the Cauchy inequality, we obtain

$$\begin{aligned}
 & \int_0^t g_1(t - \tau) \int_{\Omega} \nabla u(t)(\nabla u(\tau) - \nabla u(t)) dx d\tau \\
 & \geq -\frac{2(p + 2)(1 - \varepsilon)}{2} (g_1 \circ \nabla u)(t) - \frac{1}{4(p + 2)(1 - \varepsilon)} \int_0^t g_1(\tau) d\tau \|\nabla u\|_2^2, \tag{3.11}
 \end{aligned}$$

where $\varepsilon \in (0, 1)$, similarly

$$\begin{aligned}
 & \int_0^t g_2(t - \tau) \int_{\Omega} \nabla v(t)(\nabla v(\tau) - \nabla v(t)) dx d\tau \\
 & \geq -\frac{2(p + 2)(1 - \varepsilon)}{2} (g_2 \circ \nabla v)(t) - \frac{1}{4(p + 2)(1 - \varepsilon)} \int_0^t g_2(\tau) d\tau \|\nabla v\|_2^2. \tag{3.12}
 \end{aligned}$$

Inserting (3.11) and (3.12) into (3.9) and (3.10), respectively, we can get

$$\frac{d}{dt} \left(u, \frac{u_t|u_t|^\rho}{\rho + 1} \right) \geq \frac{1}{\rho + 1} \|u_t(t)\|_{\rho+2}^{\rho+2} - \|\nabla u\|_2^2 - \frac{2(p + 2)(1 - \varepsilon)}{2} (g_1 \circ \nabla u)(t)$$

$$\begin{aligned}
 & + \int_0^t g_1(\tau) d\tau \|\nabla u(t)\|_2^2 - \int_{\Omega} uu_t|u_t|^{m-2} dx + \int_{\Omega} uf_1(u, v) dx \\
 & - \frac{1}{4(p+2)(1-\varepsilon)} \int_0^t g_1(\tau) d\tau \|\nabla u\|_2^2, \tag{3.13} \\
 \frac{d}{dt} \left(v, \frac{v_t|v_t|^\rho}{\rho+1} \right) & \geq \frac{1}{\rho+1} \|v_t(t)\|_{\rho+2}^{\rho+2} - \|\nabla v\|_2^2 - \frac{2(p+2)(1-\varepsilon)}{2} (g_2 \circ \nabla v)(t) \\
 & + \int_0^t g_2(\tau) d\tau \|\nabla v(t)\|_2^2 - \int_{\Omega} vv_t|v_t|^{r-2} dx + \int_{\Omega} vf_2(u, v) dx \\
 & - \frac{1}{4(p+2)(1-\varepsilon)} \int_0^t g_2(\tau) d\tau \|\nabla v\|_2^2. \tag{3.14}
 \end{aligned}$$

Adding (3.13) and (3.14), we derive that

$$\begin{aligned}
 & \frac{d}{dt} \left(u, \frac{u_t|u_t|^\rho}{\rho+1} \right) + \frac{d}{dt} \left(v, \frac{v_t|v_t|^\rho}{\rho+1} \right) \\
 & \geq \frac{1}{\rho+1} (\|u_t(t)\|_{\rho+2}^{\rho+2} + \|v_t(t)\|_{\rho+2}^{\rho+2}) + 2(p+2) \int_{\Omega} F(u, v) dx \\
 & - \left(1 - \int_0^t g_1(\tau) d\tau \right) \|\nabla u\|_2^2 - \left(1 - \int_0^t g_2(\tau) d\tau \right) \|\nabla v\|_2^2 \\
 & - \frac{2(p+2)(1-\varepsilon)}{2} ((g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)) \\
 & - \frac{1}{4(p+2)(1-\varepsilon)} \left(\int_0^t g_1(\tau) d\tau \|\nabla u\|_2^2 + \int_0^t g_2(\tau) d\tau \|\nabla v\|_2^2 \right) \\
 & - \int_{\Omega} uu_t|u_t|^{m-2} dx - \int_{\Omega} vv_t|v_t|^{r-2} dx. \tag{3.15}
 \end{aligned}$$

Adding $2(p+2)(1-\varepsilon)E(t)$ on the right side of (3.15), we can get

$$\begin{aligned}
 & \frac{d}{dt} \left(u, \frac{u_t|u_t|^\rho}{\rho+1} \right) + \frac{d}{dt} \left(v, \frac{v_t|v_t|^\rho}{\rho+1} \right) \\
 & \geq \left(\frac{1}{\rho+1} + \frac{2(p+2)(1-\varepsilon)}{\rho+2} \right) (\|u_t(t)\|_{\rho+2}^{\rho+2} + \|v_t(t)\|_{\rho+2}^{\rho+2}) \\
 & + 2(p+2)\varepsilon \int_{\Omega} F(u, v) - 2(p+2)(1-\varepsilon)E(t) \\
 & + ((p+2)(1-\varepsilon) - 1) \left(1 - \int_0^t g_1(\tau) d\tau \right) \|\nabla u\|_2^2 \\
 & + ((p+2)(1-\varepsilon) - 1) \left(1 - \int_0^t g_2(\tau) d\tau \right) \|\nabla v\|_2^2 \\
 & - \frac{1}{4(p+2)(1-\varepsilon)} \left(\int_0^t g_1(\tau) d\tau \|\nabla u\|_2^2 + \int_0^t g_2(\tau) d\tau \|\nabla v\|_2^2 \right) \\
 & - \int_{\Omega} uu_t|u_t|^{m-2} dx - \int_{\Omega} vv_t|v_t|^{r-2} dx. \tag{3.16}
 \end{aligned}$$

For the last two terms on the right side of (3.16), applying the Hölder inequality and the Young inequality, we arrive at

$$\left| \int_{\Omega} |u_t|^{m-2} u_t u \, dx \right| \leq \frac{\varepsilon_1^m \|u\|_m^m}{m} + \left(\frac{1}{\varepsilon_1} \right)^{\frac{m}{m-1}} \frac{(m-1) \|u_t\|_m^m}{m}, \tag{3.17}$$

where $\varepsilon_1 > 0$. By Lemma 2.4 and the conditions of the theorem, one can deduce that $\|u\|_m^m \leq \|u\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)}$, then the inequality (3.17) can be rewritten as

$$\left| \int_{\Omega} |u_t|^{m-2} u_t u \, dx \right| \leq \frac{\varepsilon_1^m}{m} (\|u\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)}) + \left(\frac{1}{\varepsilon_1} \right)^{\frac{m}{m-1}} \frac{(m-1) \|u_t\|_m^m}{m}, \tag{3.18}$$

similarly, we have

$$\left| \int_{\Omega} |v_t|^{r-2} v_t v \, dx \right| \leq \frac{\varepsilon_1^r}{r} (\|v\|_2^2 + \|v\|_{2(p+2)}^{2(p+2)}) + \left(\frac{1}{\varepsilon_1} \right)^{\frac{r}{r-1}} \frac{(r-1) \|v_t\|_r^r}{r}. \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.16), then we have

$$\begin{aligned} & \frac{d}{dt} \left(u, \frac{u_t |u_t|^\rho}{\rho+1} \right) + \frac{d}{dt} \left(v, \frac{v_t |v_t|^\rho}{\rho+1} \right) + \left(\frac{1}{\varepsilon_1} \right)^{\frac{m}{m-1}} \frac{(m-1) \|u_t\|_m^m}{m} + \left(\frac{1}{\varepsilon_1} \right)^{\frac{r}{r-1}} \frac{(r-1) \|v_t\|_r^r}{r} \\ & \geq \left(\frac{1}{\rho+1} + \frac{2(p+2)(1-\varepsilon)}{\rho+2} \right) (\|u_t(t)\|_{\rho+2}^{\rho+2} + \|v_t(t)\|_{\rho+2}^{\rho+2}) \\ & \quad + 2(p+2)\varepsilon \int_{\Omega} F(u, v) - 2(p+2)(1-\varepsilon)E(t) \\ & \quad + ((p+2)(1-\varepsilon) - 1) \left(1 - \int_0^t g_1(\tau) \, d\tau \right) \|\nabla u\|_2^2 \\ & \quad + ((p+2)(1-\varepsilon) - 1) \left(1 - \int_0^t g_2(\tau) \, d\tau \right) \|\nabla v\|_2^2 \\ & \quad - \frac{1}{4(p+2)(1-\varepsilon)} \left(\int_0^t g_1(\tau) \, d\tau \|\nabla u\|_2^2 + \int_0^t g_2(\tau) \, d\tau \|\nabla v\|_2^2 \right) \\ & \quad - \frac{\varepsilon_1^r}{r} (\|v\|_2^2 + \|v\|_{2(p+2)}^{2(p+2)}) - \frac{\varepsilon_1^m}{m} (\|u\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)}). \end{aligned} \tag{3.20}$$

Take $\gamma = \max\{m, r\}$, $\gamma^* = \min\{m, r\}$. Combining with (2.5), we know $E'(t) \leq -\|u_t\|_m^m - \|v_t\|_r^r$, that is, $-E'(t) \geq \|u_t\|_m^m + \|v_t\|_r^r$, then (3.20) can be rewritten as

$$\begin{aligned} & \frac{d}{dt} \left(\left(u, \frac{u_t |u_t|^\rho}{\rho+1} \right) + \left(v, \frac{v_t |v_t|^\rho}{\rho+1} \right) - \left(\frac{1}{\varepsilon_1} \right)^{\frac{\gamma^*}{\gamma^*-1}} \frac{\gamma-1}{\gamma} E(t) \right) \\ & \geq \frac{d}{dt} \left(\left(u, \frac{u_t |u_t|^\rho}{\rho+1} \right) + \left(v, \frac{v_t |v_t|^\rho}{\rho+1} \right) \right) + \left(\frac{1}{\varepsilon_1} \right)^{\frac{\gamma^*}{\gamma^*-1}} \frac{\gamma-1}{\gamma} (\|u_t\|_m^m + \|v_t\|_r^r) \\ & \geq \frac{d}{dt} \left(u, \frac{u_t |u_t|^\rho}{\rho+1} \right) + \frac{d}{dt} \left(v, \frac{v_t |v_t|^\rho}{\rho+1} \right) + \left(\frac{1}{\varepsilon_1} \right)^{\frac{m}{m-1}} \frac{(m-1) \|u_t\|_m^m}{m} \\ & \quad + \left(\frac{1}{\varepsilon_1} \right)^{\frac{r}{r-1}} \frac{(r-1) \|v_t\|_r^r}{r} \end{aligned}$$

$$\begin{aligned}
 &\geq \left(\frac{1}{\rho+1} + \frac{2(p+2)(1-\varepsilon)}{\rho+2} \right) (\|u_t(t)\|_{\rho+2}^{\rho+2} + \|v_t(t)\|_{\rho+2}^{\rho+2}) \\
 &\quad + 2(p+2)\varepsilon \int_{\Omega} F(u, v) \, dx - 2(p+2)(1-\varepsilon)E(t) \\
 &\quad + ((p+2)(1-\varepsilon) - 1) \left(1 - \int_0^t g_1(\tau) \, d\tau \right) \|\nabla u\|_2^2 \\
 &\quad + ((p+2)(1-\varepsilon) - 1) \left(1 - \int_0^t g_2(\tau) \, d\tau \right) \|\nabla v\|_2^2 \\
 &\quad - \frac{1}{4(p+2)(1-\varepsilon)} \left(\int_0^t g_1(\tau) \, d\tau \|\nabla u\|_2^2 + \int_0^t g_2(\tau) \, d\tau \|\nabla v\|_2^2 \right) \\
 &\quad - \frac{\varepsilon_1^r}{r} (\|v\|_2^2 + \|v\|_{2(p+2)}^{2(p+2)}) - \frac{\varepsilon_1^m}{m} (\|u\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)}). \tag{3.21}
 \end{aligned}$$

Combining Lemma 2.3 and the Poincaré inequality, we can deduce

$$\begin{aligned}
 &\frac{d}{dt} \left(\left(u, \frac{u_t|u_t|^\rho}{\rho+1} \right) + \left(v, \frac{v_t|v_t|^\rho}{\rho+1} \right) - \left(\frac{1}{\varepsilon_1} \right)^{\frac{\gamma^*}{\gamma^*-1}} \frac{\gamma-1}{\gamma} E(t) \right) \\
 &\geq \left(\frac{1}{\rho+1} + \frac{2(p+2)(1-\varepsilon)}{\rho+2} \right) (\|u_t(t)\|_{\rho+2}^{\rho+2} + \|v_t(t)\|_{\rho+2}^{\rho+2}) \\
 &\quad + c_0\varepsilon (\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}) - 2(p+2)(1-\varepsilon)E(t) \\
 &\quad + \left[((p+2)(1-\varepsilon) - 1) \left(1 - \int_0^t g_1(\tau) \, d\tau \right) - \frac{1}{4(p+2)(1-\varepsilon)} \int_0^t g_1(\tau) \, d\tau \right] \lambda \|u\|_2^2 \\
 &\quad + \left[((p+2)(1-\varepsilon) - 1) \left(1 - \int_0^t g_2(\tau) \, d\tau \right) - \frac{1}{4(p+2)(1-\varepsilon)} \int_0^t g_2(\tau) \, d\tau \right] \lambda \|v\|_2^2 \\
 &\quad - \frac{\varepsilon_1^r}{r} (\|v\|_2^2 + \|v\|_{2(p+2)}^{2(p+2)}) - \frac{\varepsilon_1^m}{m} (\|u\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)}) \\
 &\geq \left(\frac{1}{\rho+1} + \frac{2(p+2)(1-\varepsilon)}{\rho+2} \right) (\|u_t(t)\|_{\rho+2}^{\rho+2} + \|v_t(t)\|_{\rho+2}^{\rho+2}) \\
 &\quad + c_0\varepsilon (\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}) - 2(p+2)(1-\varepsilon)E(t) \\
 &\quad + \left[((p+2)(1-\varepsilon) - 1)l_1 - \frac{1}{4(p+2)(1-\varepsilon)}(1-l_1) \right] \lambda \|u\|_2^2 \\
 &\quad + \left[((p+2)(1-\varepsilon) - 1)l_2 - \frac{1}{4(p+2)(1-\varepsilon)}(1-l_2) \right] \lambda \|v\|_2^2 \\
 &\quad - \frac{\varepsilon_1^r}{r} (\|v\|_2^2 + \|v\|_{2(p+2)}^{2(p+2)}) - \frac{\varepsilon_1^m}{m} (\|u\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)}), \tag{3.22}
 \end{aligned}$$

where λ is the first eigenvalue of $-\Delta$, now we take

$$\begin{aligned}
 \kappa_1(\varepsilon) &= ((p+2)(1-\varepsilon) - 1)l_1 - \frac{1}{4(p+2)(1-\varepsilon)}(1-l_1), \\
 \kappa_2(\varepsilon) &= ((p+2)(1-\varepsilon) - 1)l_2 - \frac{1}{4(p+2)(1-\varepsilon)}(1-l_2).
 \end{aligned}$$

Then (3.22) can be rewritten as

$$\begin{aligned} & \frac{d}{dt} \left(\left(u, \frac{u_t |u_t|^\rho}{\rho + 1} \right) + \left(v, \frac{v_t |v_t|^\rho}{\rho + 1} \right) - \left(\frac{1}{\varepsilon_1} \right)^{\frac{\gamma^*}{\gamma^* - 1}} \frac{\gamma - 1}{\gamma} E(t) \right) \\ & \geq \left(\frac{1}{\rho + 1} + \frac{2(p + 2)(1 - \varepsilon)}{\rho + 2} \right) (\|u_t(t)\|_{\rho + 2}^{\rho + 2} + \|v_t(t)\|_{\rho + 2}^{\rho + 2}) \\ & \quad + \left(\kappa_1(\varepsilon)\lambda - \frac{\varepsilon_1^m}{m} \right) \|u\|_2^2 + \left(\kappa_2(\varepsilon)\lambda - \frac{\varepsilon_1^r}{r} \right) \|v\|_2^2 \\ & \quad + \left(c_0\varepsilon - \frac{\varepsilon_1^r}{r} \right) \|v\|_{2(p+2)}^{2(p+2)} + \left(c_0\varepsilon - \frac{\varepsilon_1^m}{m} \right) \|u\|_{2(p+2)}^{2(p+2)} \\ & \quad - 2(p + 2)(1 - \varepsilon)E(t). \end{aligned} \tag{3.23}$$

By the condition

$$\max \left\{ \int_0^\infty g_1(\tau) d\tau, \int_0^\infty g_2(\tau) d\tau \right\} < \frac{p + 1}{p + 1 + \frac{1}{4(p+2)}},$$

we can obtain

$$\begin{aligned} & ((p + 2) - 1)l_1 - \frac{1}{4(p + 2)}(1 - l_1) > 0, \\ & ((p + 2) - 1)l_2 - \frac{1}{4(p + 2)}(1 - l_2) > 0. \end{aligned}$$

Then we choose ε small enough such that

$$\begin{aligned} \kappa_1(\varepsilon) &= ((p + 2)(1 - \varepsilon) - 1)l_1 - \frac{1}{4(p + 2)(1 - \varepsilon)}(1 - l_1) > 0, \\ \kappa_2(\varepsilon) &= ((p + 2)(1 - \varepsilon) - 1)l_2 - \frac{1}{4(p + 2)(1 - \varepsilon)}(1 - l_2) > 0. \end{aligned}$$

And we pick ε_1 small enough such that

$$\begin{aligned} & \min \left\{ \kappa_1(\varepsilon)\lambda - \frac{\varepsilon_1^m}{m}, \kappa_2(\varepsilon)\lambda - \frac{\varepsilon_1^r}{r} \right\} > 0, \\ & \min \left\{ c_0\varepsilon - \frac{\varepsilon_1^m}{m}, c_0\varepsilon - \frac{\varepsilon_1^r}{r} \right\} > 0. \end{aligned} \tag{3.24}$$

Then we choose

$$\begin{aligned} \alpha &= \min \left\{ \kappa_1(\varepsilon)\lambda - \frac{\varepsilon_1^m}{m}, \kappa_2(\varepsilon)\lambda - \frac{\varepsilon_1^r}{r}, c_0\varepsilon - \frac{\varepsilon_1^m}{m}, c_0\varepsilon - \frac{\varepsilon_1^r}{r} \right\}, \\ \beta &= \min \left\{ \frac{1}{\rho + 1} + \frac{2(p + 2)(1 - \varepsilon)}{\rho + 2}, \alpha \right\}. \end{aligned}$$

Using Lemma 2.4, (3.23) can be deduced as

$$\begin{aligned}
 & \frac{d}{dt} \left(\left(u, \frac{u_t |u_t|^\rho}{\rho + 1} \right) + \left(v, \frac{v_t |v_t|^\rho}{\rho + 1} \right) - \left(\frac{1}{\varepsilon_1} \right)^{\frac{\gamma^*}{\gamma^* - 1}} \frac{\gamma - 1}{\gamma} E(t) \right) \\
 & \geq \left(\frac{1}{\rho + 1} + \frac{2(p + 2)(1 - \varepsilon)}{\rho + 2} \right) (\|u_t(t)\|_{\rho+2}^{\rho+2} + \|v_t(t)\|_{\rho+2}^{\rho+2}) \\
 & \quad + \alpha (\|u\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)}) + \alpha (\|v\|_2^2 + \|v\|_{2(p+2)}^{2(p+2)}) \\
 & \quad - 2(p + 2)(1 - \varepsilon)E(t) \\
 & \geq \beta (\|u_t(t)\|_{\rho+2}^{\rho+2} + \|v_t(t)\|_{\rho+2}^{\rho+2}) \\
 & \quad + \beta (\|u\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)}) + \beta (\|v\|_2^2 + \|v\|_{2(p+2)}^{2(p+2)}) \\
 & \quad - 2(p + 2)(1 - \varepsilon)E(t) \\
 & \geq \beta (\|u_t(t)\|_{\rho+2}^{\rho+2} + \|v_t(t)\|_{\rho+2}^{\rho+2}) + \beta \|u\|_{\rho+2}^{\rho+2} + \beta \|v\|_{\rho+2}^{\rho+2} - 2(p + 2)(1 - \varepsilon)E(t). \tag{3.25}
 \end{aligned}$$

By applying the Hölder and Young inequalities, we can deduce that

$$\left(u, \frac{u_t |u_t|^\rho}{\rho + 1} \right) \leq \|u\|_{\rho+2}^{\rho+2} + \|u_t\|_{\rho+2}^{\rho+2},$$

then we have

$$\begin{aligned}
 & \frac{d}{dt} \left(\left(u, \frac{u_t |u_t|^\rho}{\rho + 1} \right) + \left(v, \frac{v_t |v_t|^\rho}{\rho + 1} \right) - \left(\frac{1}{\varepsilon_1} \right)^{\frac{\gamma^*}{\gamma^* - 1}} \frac{\gamma - 1}{\gamma} E(t) \right) \\
 & \geq \beta \left(\left(u, \frac{u_t |u_t|^\rho}{\rho + 1} \right) + \left(v, \frac{v_t |v_t|^\rho}{\rho + 1} \right) - \frac{2(p + 2)(1 - \varepsilon)}{\beta} E(t) \right). \tag{3.26}
 \end{aligned}$$

It is easy to see that

$$\left(\frac{1}{\varepsilon_1} \right)^{\frac{\gamma^*}{\gamma^* - 1}} \frac{\gamma - 1}{\gamma} \rightarrow +\infty, \quad \varepsilon_1 \rightarrow 0^+,$$

and $\frac{2(p+2)(1-\varepsilon)}{\beta}$ is a positive constant, hence there exists a constant ε_1^* such that

$$\left(\frac{1}{\varepsilon_1} \right)^{\frac{\gamma^*}{\gamma^* - 1}} \frac{\gamma - 1}{\gamma} \geq \frac{2(p + 2)(1 - \varepsilon)}{\beta}.$$

Therefore, we have

$$\begin{aligned}
 & \frac{d}{dt} \left(\left(u, \frac{u_t |u_t|^\rho}{\rho + 1} \right) + \left(v, \frac{v_t |v_t|^\rho}{\rho + 1} \right) - \left(\frac{1}{\varepsilon_1^*} \right)^{\frac{\gamma^*}{\gamma^* - 1}} \frac{\gamma - 1}{\gamma} E(t) \right) \\
 & \geq \beta \left(\left(u, \frac{u_t |u_t|^\rho}{\rho + 1} \right) + \left(v, \frac{v_t |v_t|^\rho}{\rho + 1} \right) - \left(\frac{1}{\varepsilon_1^*} \right)^{\frac{\gamma^*}{\gamma^* - 1}} \frac{\gamma - 1}{\gamma} E(t) \right). \tag{3.27}
 \end{aligned}$$

Take

$$H(t) = \left(u, \frac{u_t|u_t|^\rho}{\rho+1}\right) + \left(v, \frac{v_t|v_t|^\rho}{\rho+1}\right) - \left(\frac{1}{\varepsilon_1^*}\right)^{\frac{\gamma^*}{\gamma^*-1}} \frac{\gamma-1}{\gamma} E(t),$$

from (3.2), we know

$$H(0) = \left(u(0), \frac{u_t(0)|u_t(0)|^\rho}{\rho+1}\right) + \left(v(0), \frac{v_t(0)|v_t(0)|^\rho}{\rho+1}\right) - \left(\frac{1}{\varepsilon_1^*}\right)^{\frac{\gamma^*}{\gamma^*-1}} \frac{\gamma-1}{\gamma} E(0) > 0.$$

By calculating $H'(t) \geq \beta H(t)$, we can get

$$H(t) \geq e^{\beta t} H(0), \quad \forall t \geq 0. \tag{3.28}$$

Since (u, v) shows global existence, by Lemma 2.2 and Lemma 2.6, we have $0 < E(t) \leq E(0)$, $t \in [0, +\infty)$, then

$$\begin{aligned} \|u\|_{\rho+2}^{\rho+2} + \|u_t\|_{\rho+2}^{\rho+2} + \|v\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2} &\geq \left(u, \frac{u_t|u_t|^\rho}{\rho+1}\right) + \left(v, \frac{v_t|v_t|^\rho}{\rho+1}\right) \\ &\geq e^{\beta t} H(0). \end{aligned} \tag{3.29}$$

Using the Hölder inequality, Lemma 2.2 and Lemma 2.6, we have

$$\begin{aligned} &\|u(t)\|_{\rho+2} + \|v\|_{\rho+2} \\ &\leq \|u(0)\|_{\rho+2} + \|v(0)\|_{\rho+2} + \int_0^t \|u_t(\tau)\|_{\rho+2} d\tau + \int_0^t \|v_t(\tau)\|_{\rho+2} d\tau \\ &\leq \|u(0)\|_{\rho+2} + \|v(0)\|_{\rho+2} + C_1 \int_0^t \|u_t(\tau)\|_m d\tau + C_2 \int_0^t \|v_t(\tau)\|_r d\tau \\ &\leq \|u(0)\|_{\rho+2} + \|v(0)\|_{\rho+2} + C_1 t^{\frac{m-1}{m}} \left(\int_0^t \|u_t(\tau)\|_m d\tau\right)^{\frac{1}{m}} \\ &\quad + C_2 t^{\frac{r-1}{r}} \left(\int_0^t \|v_t(\tau)\|_r d\tau\right)^{\frac{1}{r}} \\ &\leq \|u(0)\|_{\rho+2} + \|v(0)\|_{\rho+2} + C_1 t^{\frac{m-1}{m}} (E(0) - E(t))^{\frac{1}{m}} + C_2 t^{\frac{r-1}{r}} (E(0) - E(t))^{\frac{1}{r}} \\ &\leq \|u(0)\|_{\rho+2} + \|v(0)\|_{\rho+2} + C_1 t^{\frac{m-1}{m}} (E(0))^{\frac{1}{m}} + C_2 t^{\frac{r-1}{r}} (E(0))^{\frac{1}{r}}, \end{aligned} \tag{3.30}$$

where C_1 and C_2 are positive constants. By combining (3.29) and (3.30), we know $\|u\|_{\rho+2}^{\rho+2} + \|v\|_{\rho+2}^{\rho+2}$ shows polynomial growth and $\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}$ shows exponential growth. By (2.5) and $E(t)$ being nonnegative, we can deduce

$$\int_0^t \|u_t(\tau)\|_m^m d\tau + \int_0^t \|v_t(\tau)\|_r^r d\tau \leq E(0), \tag{3.31}$$

and thanks to Lemma 2.5 and the assumption $2 < \rho + 2 < \min\{m, r\}$, we have

$$\begin{aligned} \|u_t(\tau)\|_m^{\rho+2} &< \|u_t(\tau)\|_m^m + 1, \\ \|v_t(\tau)\|_r^{\rho+2} &< \|v_t(\tau)\|_r^r + 1. \end{aligned} \quad (3.32)$$

By using the Sobolev embedding theorem and combining (3.31) and (3.32), we can get

$$\begin{aligned} &\int_0^t \|u_t(\tau)\|_{\rho+2}^{\rho+2} d\tau + \int_0^t \|v_t(\tau)\|_{\rho+2}^{\rho+2} d\tau \\ &\leq C \left(\int_0^t \|u_t(\tau)\|_m^{\rho+2} d\tau + \int_0^t \|v_t(\tau)\|_r^{\rho+2} d\tau \right) \\ &\leq C \left(\int_0^t (\|u_t(\tau)\|_m^m + 1) d\tau + \int_0^t (\|v_t(\tau)\|_r^r + 1) d\tau \right) \\ &\leq CE(0) + 2Ct, \end{aligned}$$

which contradicts with $\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}$ showing exponential growth. Hence the theorem is proved. \square

Acknowledgements

The author is very grateful to the referees' valuable suggestions, which improved this work a lot.

Funding

Not applicable.

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The author declares that they have no competing interests.

Authors' contributions

The author declares that the author read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 6 February 2021 Accepted: 8 June 2021 Published online: 15 July 2021

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