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Nonlocal boundary value problems for ψ -Hilfer fractional-order Langevin equations

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Abstract

In the paper, we study a boundary value problem for a class of ψ -Hilfer fractional-order Langevin equations with multi-point integral boundary conditions. Existence and uniqueness results are established by using well-known fixed point theorems. Examples illustrating the main results are also included.

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1 Introduction

Fractional differential equations are important since their nonlocal property is suitable to characterize memory phenomena in economic, control, and material sciences. For the basic theory of fractional calculus and fractional differential equations we refer to [1–9]. Various types of fractional derivative and integral operator were studied: Riemann–Liouville, Caputo, Hadamard, Erdelyi–Kober, Grünwald–Letnikov, Marchaud and Riesz are just a few examples. Hilfer [10] proposed a generalized Riemann–Liouville (R–L) fractional derivative, the so called *Hilfer fractional derivative* (HFD), which composites R–L fractional derivative and Caputo fractional derivative. HFD is performed, for example, in the theoretical simulation of dielectric relaxation in glass forming materials. Sousa and Oliveira [11] proposed the ψ -HFD and established ψ -Hilfer fractional differential equations. The fundamental discussion about existence and uniqueness of the solution of a nonlinear fractional-order differential equation involving ψ -HFD along with different types of initial conditions have been investigated in [12–18]. Asawasamrit *et al.* [19] studied the nonlocal boundary value problems for fractional-order differential equations with HFD subject to nonlocal integral boundary conditions. In [20], Saengthong *et al.* considered the existence results for Hilfer–Hadamard sequential fractional differential equations with two point boundary conditions. The study of a boundary value problem for fractional-order ψ -HFD was done by Harikrishnan *et al.* [21]. There has been published some significant work about the nonlinear boundary value problems for HFD, out of which we mention only a few that are relating to this article; see [22–24].

The note by Langevin [25] on Brownian motion had a permanent impact on scientific research and is still often cited. The Langevin equation is an effective tool of mathemati-

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cal physics, which can describe processes as regards time evolution of the velocity of the Brownian motion [26–28]. Among the applications of the Langevin equation, one can use modeling gait variability [29], financial markets [30] and described anomalous diffusion [31]. The existence and stability results for Langevin equations with HFD was considered in [32]. Several articles including a number of fractional derivatives have been considered by authors and researchers in the fractional Langevin equation, for instance [33–35]. Recently, three-point boundary value problems for Langevin equation with Hilfer fractional derivative were studied in [36].

In this paper, we consider a boundary value problem of Langevin fractional differential equations with ψ -HFD and nonlocal integral boundary conditions, given by

$$\mathcal{D}^{\chi_1, \beta_1; \psi} (\mathcal{D}^{\chi_2, \beta_2; \psi} + k)x(t) = f(t, x(t)), \quad t \in J := [a, b], \tag{1.1}$$

$$x(a) = 0, \quad x(b) = \sum_{i=1}^m \lambda_i I^{\delta_i; \psi} x(\tau_i), \tag{1.2}$$

where $\mathcal{D}^{\chi_i, \beta_i; \psi}$, $i = 1, 2$ is the ψ -HFD of order χ_i , $0 < \chi_i < 1$ and type β_i , $0 \leq \beta_i \leq 1$, $i = 1, 2$, $1 < \chi_1 + \chi_2 \leq 2$, $k \in \mathbb{R}$, $a \geq 0$, $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $I^{\delta_i; \psi}$ is ψ -Riemann–Liouville fractional integral of order $\delta_i > 0$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \dots, m$ and $0 \leq a \leq \tau_1 < \tau_2 < \dots < \tau_m \leq b$.

We prove two existence results by using Krasnosel’skii’s fixed point theorem and the Leray–Schauder nonlinear alternative and one existence and uniqueness result via the Banach contraction mapping principle. The main results are presented in Sect. 3. Examples illustrating the main results are also constructed. In Sect. 2 we recall some preliminary facts which are needed in the following.

2 Preliminaries

We need some basic definitions and properties of fractional calculus that are used in this article. Let $C = C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into \mathbb{R} with the norm $\|x\| := \sup\{|x(t)| : t \in J\}$.

Let $\psi \in C^1(J, \mathbb{R})$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in J$.

Definition 2.1 ([11]) Let $\chi > 0$ ($\chi \in \mathbb{R}$), $f \in L_1(J, \mathbb{R})$. Then the ψ -R-L fractional integral of a function f with respect to ψ is defined by

$$I^{\chi; \psi} f(t) = \frac{1}{\Gamma(\chi)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\chi-1} f(s) ds.$$

Definition 2.2 ([11]) Let $n - 1 < \chi < n \in \mathbb{N}$ and $f \in C^n(J, \mathbb{R})$. Then the ψ -HFD $\mathcal{D}^{\chi, \beta; \psi}(\cdot)$ of a function f of order χ and type $0 \leq \beta \leq 1$, is defined by

$$\mathcal{D}^{\chi, \beta; \psi} f(t) = I^{\beta(n-\chi); \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I^{(1-\beta)(n-\chi); \psi} f(t).$$

Lemma 2.1 ([11]) If $f \in C^n[a, b]$, $n - 1 < \chi < n$ and $0 \leq \beta \leq 1$, then

$$(1) \quad I^{\chi; \psi} \mathcal{D}^{\chi, \beta; \psi} f(t) = f(t) - \sum_{k=1}^n \frac{(\psi(t) - \psi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} f^{[n-k]}_{\psi} I^{(1-\beta)(n-\chi); \psi} f(a),$$

where $f_{\psi}^{[n-k]}f(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^{n-k} f(t)$ and $\gamma = \chi + \beta(n - \chi)$.

(2) $\mathcal{D}^{\chi, \beta; \psi} I^{\chi; \psi} f(t) = f(t)$.

Next, we derive an equivalent fractional integral equation for a linear variant of the boundary value problem (1.1)–(1.2).

Lemma 2.2 *Let $a \geq 0$, $0 < \chi_i < 1$, $\gamma_i = \chi_i + \beta_i(1 - \chi_i)$, $i = 1, 2$, $1 < \chi_1 + \chi_2 \leq 2$, and $h \in C([a, b], \mathbb{R})$. Then the function x is a solution of the boundary value problem:*

$$\mathcal{D}^{\chi_1, \beta_1; \psi} (\mathcal{D}^{\chi_2, \beta_2; \psi} + k)x(t) = h(t), \tag{2.1}$$

$$x(a) = 0, \quad x(b) = \sum_{i=1}^m \lambda_i I^{\delta_i; \psi} x(\tau_i), \tag{2.2}$$

if and only if

$$x(t) = I^{\chi_1 + \chi_2; \psi} h(t) - k I^{\chi_2; \psi} x(t) + \frac{(\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{\Lambda \Gamma(\gamma_1 + \chi_2)} \left[\sum_{i=1}^m \lambda_i I^{\chi_1 + \chi_2 + \delta_i; \psi} h(\tau_i) - k \sum_{i=1}^m \lambda_i I^{\chi_2 + \delta_i; \psi} x(\tau_i) - I^{\chi_1 + \chi_2; \psi} h(b) + k I^{\chi_2; \psi} x(b) \right], \tag{2.3}$$

where it is assumed that

$$\Lambda = \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{\Gamma(\gamma_1 + \chi_2)} - \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\gamma_1 + \chi_2 + \delta_i)} (\psi(\tau_i) - \psi(a))^{\gamma_1 + \chi_2 + \delta_i - 1} \neq 0.$$

Proof Applying the ψ -R-L fractional integral of order χ_1 to both sides of (2.1) and using Lemma 2.1 we obtain

$$\mathcal{D}^{\chi_2, \beta_2; \psi} x(t) + kx(t) = I^{\chi_1; \psi} h(t) + \frac{c_0}{\Gamma(\gamma_1)} (\psi(t) - \psi(a))^{\gamma_1 - 1}, \tag{2.4}$$

where c_0 is an arbitrary constant and $\gamma_1 = \chi_1 + \beta_1(1 - \chi_1)$.

Applying the ψ -R-L fractional integral of order χ_2 to both sides of (2.4), we obtain

$$I^{\chi_2; \psi} \mathcal{D}^{\chi_2, \beta_2; \psi} x(t) = I^{\chi_1 + \chi_2; \psi} h(t) - k I^{\chi_2; \psi} x(t) + \frac{c_0}{\Gamma(\gamma_1 + \chi_2)} (\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1}. \tag{2.5}$$

Applying Lemma 2.1 to (2.5), we get

$$x(t) = I^{\chi_1 + \chi_2; \psi} h(t) - k I^{\chi_2; \psi} x(t) + \frac{c_0}{\Gamma(\gamma_1 + \chi_2)} (\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1} + \frac{c_1}{\Gamma(\gamma_2)} (\psi(t) - \psi(a))^{\gamma_2 - 1}. \tag{2.6}$$

Using $x(a) = 0$ in (2.6), we obtain $c_1 = 0$, and hence we get

$$x(t) = I^{\chi_1 + \chi_2; \psi} h(t) - k I^{\chi_2; \psi} x(t) + \frac{c_0}{\Gamma(\gamma_1 + \chi_2)} (\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1}. \tag{2.7}$$

Next, by combining the second condition $x(b) = \sum_{i=1}^m \lambda_i I^{\delta_i; \psi} x(\tau_i)$ of (2.2) with (2.7), we obtain

$$c_0 = \frac{1}{\Lambda} \left[\sum_{i=1}^m \lambda_i I^{\chi_1 + \chi_2 + \delta_i; \psi} h(\tau_i) - k \sum_{i=1}^m \lambda_i I^{\chi_2 + \delta_i; \psi} x(\tau_i) - I^{\chi_1 + \chi_2; \psi} h(b) + k I^{\chi_2; \psi} x(b) \right].$$

Substituting c_0 in (2.7) we get (2.3).

We can easily prove the converse by direct computation. The proof is completed. \square

Fixed point theorems play a major role in establishing the existence theory for the problem (1.1)–(1.2). We collect here some well-known fixed point theorems used in this paper.

Lemma 2.3 (Banach contraction principle [37]) *Let D be a non-empty closed subset of a Banach space E . Then any contraction mapping T from D into itself has a unique fixed point.*

Lemma 2.4 (Krasnosel’skiĭ’s fixed point theorem [38]) *Let \mathcal{M} be a closed, bounded, convex, and non-empty subset of a Banach space. Let A, B be the operators such that (i) $Ax + By \in \mathcal{M}$ whenever $x, y \in \mathcal{M}$; (ii) A is compact and continuous; (iii) B is contraction mapping. Then there exists $z \in \mathcal{M}$ such that $z = Az + bz$.*

Lemma 2.5 (Leray–Schauder nonlinear alternative [39]) *Let E be a Banach space, C a closed, convex subset of E , U an open subset of C and $0 \in U$. Suppose that $\mathcal{D} : \overline{U} \rightarrow C$ is a continuous, compact (that is, $\mathcal{D}(\overline{U})$ is a relatively compact subset of C) map. Then either*

- (i) \mathcal{D} has a fixed point in \overline{U} , or
- (ii) there is a $x \in \partial U$ (the boundary of U in C) and $v \in (0, 1)$ with $x = v\mathcal{D}(x)$.

3 Main results

In this section, we investigate the existence and uniqueness of solution for the boundary value problem (1.1)–(1.2).

By Lemma 2.2 we define an operator $\mathcal{N} : \mathcal{C} \rightarrow \mathcal{C}$ by

$$\begin{aligned} (\mathcal{N}x)(t) &= I^{\chi_1 + \chi_2; \psi} f(t, x(t)) - k I^{\chi_2; \psi} x(t) \\ &+ \frac{(\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{\Lambda \Gamma(\gamma_1 + \chi_2)} \left[\sum_{i=1}^m \lambda_i I^{\chi_1 + \chi_2 + \delta_i; \psi} f(\tau_i, x(\tau_i)) \right. \\ &\left. - k \sum_{i=1}^m \lambda_i I^{\chi_2 + \delta_i; \psi} x(\tau_i) - I^{\chi_1 + \chi_2; \psi} f(b, x(b)) + k I^{\chi_2; \psi} x(b) \right], \quad t \in J. \end{aligned} \tag{3.1}$$

For the sake of brevity, we set

$$\begin{aligned} \Omega_1 &= \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \\ &\times \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_1 + \chi_2 + \delta_i}}{\Gamma(\chi_1 + \chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} \right] \end{aligned} \tag{3.2}$$

and

$$\Omega_2 = |k| \left\{ \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \right. \\ \left. \times \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_2 + \delta_i}}{\Gamma(\chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} \right] \right\}. \tag{3.3}$$

3.1 Existence results

We prove our first existence result for the boundary value problem (1.1)–(1.2) by using Krasnosel’skii’s fixed point theorem [38].

Theorem 3.1 *Assume that:*

(H1) $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $|f(t, x(t))| \leq \varphi(t)$, $\forall (t, x) \in J \times \mathbb{R}$, with $\varphi \in C(J, \mathbb{R})$.

(H2) $\Omega_2 < 1$, where Ω_2 is given by (3.3).

Then there exists at least one solution for the boundary value problem (1.1)–(1.2) on J .

Proof We will show that the operator \mathcal{N} defined by (3.1) satisfies the assumptions of Krasnosel’skii’s fixed point theorem. We split the operator \mathcal{N} into the sum of two operators \mathcal{N}_1 and \mathcal{N}_2 on the closed ball $\mathcal{B}_\rho = \{x \in C : \|x\| \leq \rho\}$ with $\rho \geq (\|\varphi\| \Omega_1) / (1 - \Omega_2)$, $\sup_{t \in J} \varphi(t) = \|\varphi\|$, where

$$(\mathcal{N}_1 x)(t) = I^{\chi_1 + \chi_2; \psi} f(t, x(t)) + \frac{(\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{\Lambda \Gamma(\gamma_1 + \chi_2)} \\ \times \left[\sum_{i=1}^m \lambda_i I^{\chi_1 + \chi_2 + \delta_i; \psi} f(\tau_i, x(\tau_i)) - I^{\chi_1 + \chi_2; \psi} f(b, x(b)) \right], \quad t \in J, \tag{3.4}$$

and

$$(\mathcal{N}_2 x)(t) = -k I^{\chi_2; \psi} x(t) + \frac{(\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{\Lambda \Gamma(\gamma_1 + \chi_2)} \\ \times \left[-k \sum_{i=1}^m \lambda_i I^{\chi_2 + \delta_i; \psi} x(\tau_i) + k I^{\chi_2; \psi} x(b) \right], \quad t \in J. \tag{3.5}$$

For any $x, y \in \mathcal{B}_\rho$, we have

$$|(\mathcal{N}_1 x)(t) + (\mathcal{N}_2 y)(t)| \\ \leq \sup_{t \in J} \left\{ I^{\chi_1 + \chi_2} |f(t, x(t))| + |k| I^{\chi_2; \psi} |y(t)| + \frac{(\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \right. \\ \times \left[\sum_{i=1}^m |\lambda_i| I^{\chi_1 + \chi_2 + \delta_i; \psi} |f(\tau_i, x(\tau_i))| + I^{\chi_1 + \chi_2; \psi} |f(b, x(b))| \right. \\ \left. \left. + |k| \sum_{i=1}^m |\lambda_i| I^{\chi_2 + \delta_i; \psi} |y(\tau_i)| + |k| I^{\chi_2; \psi} |y(b)| \right] \right\}$$

$$\begin{aligned} &\leq \|\varphi\| \left\{ \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \right. \\ &\quad \times \left. \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_1 + \chi_2 + \delta_i}}{\Gamma(\chi_1 + \chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} \right] \right\} \\ &\quad + \|\gamma\| |k| \left\{ \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \right. \\ &\quad \times \left. \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_2 + \delta_i}}{\Gamma(\chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} \right] \right\} \\ &\leq \|\varphi\| \Omega_1 + \rho \Omega_2 \leq \rho, \end{aligned}$$

and hence $\|\mathcal{N}_1 x + \mathcal{N}_2 y\| \leq \rho$, which implies that $\mathcal{N}_1 x + \mathcal{N}_2 y \in \mathcal{B}_\rho$. By using (H2) it is easy to prove that \mathcal{N}_2 is a contraction mapping.

The operator \mathcal{N}_1 is continuous, since f is continuous. It is uniformly bounded on \mathcal{B}_ρ as

$$\|\mathcal{N}_1 x\| \leq \Omega_1 \|\varphi\|. \tag{3.6}$$

Now, we prove that the operator \mathcal{N}_1 is compact. Setting $\sup_{(t,x) \in J \times \mathcal{B}_\rho} |f(t,x)| = \bar{f} < \infty$, we obtain

$$\begin{aligned} &|(\mathcal{N}_1 x)(t_2) - (\mathcal{N}_1 x)(t_1)| \\ &\leq \frac{\bar{f}}{\Gamma(\chi_1 + \chi_2)} \left| \int_a^{t_1} [\psi'(s)(\psi(t_2) - \psi(s))^{\chi_1 + \chi_2 - 1} - \psi'(s)(\psi(t_1) - \psi(s))^{\chi_1 + \chi_2 - 1}] ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\chi_1 + \chi_2 - 1} ds \right| \\ &\quad + \frac{(\psi(t_2) - \psi(a))^{\gamma_1 + \chi_2 - 1} - (\psi(t_1) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \\ &\quad \times \left[\sum_{i=1}^m |\lambda_i| \bar{f} \frac{(\psi(\tau_i) - \psi(a))^{\chi_1 + \chi_2 + \delta_i}}{\Gamma(\chi_1 + \chi_2 + \delta_i + 1)} + \bar{f} \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} \right] \\ &\leq \frac{\bar{f}}{\Gamma(\chi_1 + \chi_2 + 1)} [2(\psi(t_2) - \psi(t_1))^{\chi_1 + \chi_2} \\ &\quad + |(\psi(t_2) - \psi(a))^{\chi_1 + \chi_2} - (\psi(t_1) - \psi(a))^{\chi_1 + \chi_2}|] \\ &\quad + \frac{(\psi(t_2) - \psi(a))^{\gamma_1 + \chi_2 - 1} - (\psi(t_1) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \\ &\quad \times \left[\sum_{i=1}^m |\lambda_i| \bar{f} \frac{(\psi(\tau_i) - \psi(a))^{\chi_1 + \chi_2 + \delta_i}}{\Gamma(\chi_1 + \chi_2 + \delta_i + 1)} + \bar{f} \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} \right], \end{aligned}$$

which tends to zero as $t_2 - t_1 \rightarrow 0$, independently of $x \in \mathcal{B}_\rho$. Thus, \mathcal{N}_1 is equicontinuous and hence \mathcal{N}_1 is relatively compact on \mathcal{B}_ρ . By the Arzelà–Ascoli theorem, \mathcal{N}_1 is compact on \mathcal{B}_ρ . It follows by Krasnosel’skii’s fixed point theorem (Lemma 2.4) that the problem (1.1)–(1.2) has at least one solution on J . The proof is complete. \square

Example 3.1 Consider the boundary value problem of the Langevin equation with the ψ -HFD of the form

$$\mathcal{D}^{\frac{1}{2}, \frac{1}{3}; e^t} \left(\mathcal{D}^{\frac{3}{4}, \frac{1}{2}; e^t} + \frac{1}{12} \right) x(t) = \frac{2e^{\frac{1}{2}-t}}{2t + 15} \left(\frac{|x(t)|}{1 + |x(t)|} \right) + \frac{1}{3}, \quad t \in \left[\frac{1}{2}, \frac{5}{2} \right], \tag{3.7}$$

$$x\left(\frac{1}{2}\right) = 0, \quad x\left(\frac{5}{2}\right) = \frac{1}{3} I^{\frac{1}{2}; e^t} x(1) + \frac{1}{2} I^{\frac{3}{2}; e^t} x\left(\frac{3}{2}\right) + \frac{3}{5} I^{\frac{5}{2}; e^t} x(2). \tag{3.8}$$

Here $\chi_1 = 1/2, \chi_2 = 3/4, \beta_1 = 1/3, \beta_2 = 1/2, k = 1/12, a = 1/2, b = 5/2, m = 3, \delta_1 = 1/2, \delta_2 = 3/2, \delta_3 = 5/2, \tau_1 = 1, \tau_2 = 3/2, \tau_3 = 2, \lambda_1 = 1/3, \lambda_2 = 1/2, \lambda_3 = 3/5$ and $\psi(t) = e^t$. We see that $\chi_1 + \chi_2 = 5/4 \in (1, 2]$. Using the given data, we get $\gamma_1 = 2/3, \gamma_2 = 7/8, |\Lambda| \approx 17.47973707, \Omega_1 \approx 24.38179472$ and $\Omega_2 \approx 0.9594635717$.

Observe that $\Omega_2 < 1$ and we obtain

$$|f(t, x)| = \left| \frac{2e^{\frac{1}{2}-t}}{2t + 15} \left(\frac{|x|}{1 + |x|} \right) + \frac{1}{3} \right| \leq \frac{2e^{\frac{1}{2}-t}}{2t + 15} + \frac{1}{3} := \phi(t).$$

Thus all the assumptions of Theorem 3.1 are satisfied. Therefore by its conclusion, the boundary value problem (3.7)–(3.8) has at least one solution on $[1/2, 5/2]$.

Our next existence result is based on the Leray–Schauder nonlinear alternative (Lemma 2.5).

Theorem 3.2 *Suppose that (H2) holds. In addition we assume that:*

- (H3) $|f(t, x)| \leq p(t)\phi(|x|)$ for each $(t, x) \in J \times \mathbb{R}$ where $\phi : [0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function and $p \in C(J, \mathbb{R}^+)$.
- (H4) There exists a constant $\mathcal{M} > 0$, such that

$$\frac{(1 - \Omega_2)\mathcal{M}}{\phi(\mathcal{M})\|p\|_{\Omega_1}} > 1, \tag{3.9}$$

where Ω_1, Ω_2 are given by (3.2) and (3.3), respectively.

Then there exists at least one solution for the boundary value problem (1.1)–(1.2) on J .

Proof We will prove that the operator \mathcal{N} defined by (3.1) satisfies the hypothesis of the Leray–Schauder nonlinear alternative. For this purpose, we first establish that operator \mathcal{N} maps bounded sets (balls) into a bounded set in \mathcal{C} . Let $\mathcal{B}_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ be a bounded ball with radius r in \mathcal{C} . Then, for $t \in J$, we have

$$\begin{aligned} |(\mathcal{N}x)(t)| &\leq \sup_{t \in J} \left\{ I^{\chi_1 + \chi_2; \psi} |f(t, x(t))| + |k| I^{\chi_2; \psi} |x(t)| \right. \\ &\quad + \frac{(\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{\Lambda \Gamma(\gamma_1 + \chi_2)} \left[\sum_{i=1}^m |\lambda_i| I^{\chi_1 + \chi_2 + \delta_i; \psi} |f(\tau_i, x(\tau_i))| \right. \\ &\quad \left. \left. + |k| \sum_{i=1}^m |\lambda_i| I^{\chi_2 + \delta_i; \psi} |x(\tau_i)| + I^{\chi_1 + \chi_2; \psi} |f(b, x(b))| + |k| I^{\chi_2; \psi} |x(b)| \right] \right\} \\ &\leq \|p\| \phi(\|x\|) \left\{ \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \right\} \end{aligned}$$

$$\begin{aligned}
 & \times \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_1 + \chi_2 + \delta_i}}{\Gamma(\chi_1 + \chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} \right] \\
 & + \|x\| |k| \left\{ \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \right\} \\
 & \times \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_2 + \delta_i}}{\Gamma(\chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} \right] \\
 & = \|p\| \phi(\|x\|) \Omega_1 + \|x\| \Omega_2,
 \end{aligned}$$

and consequently

$$\|\mathcal{N}x\| \leq \|p\| \phi(r) \Omega_1 + \Omega_2 r. \tag{3.10}$$

Next, we will show that \mathcal{N} maps bounded sets into equicontinuous sets of \mathcal{C} . Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in \mathcal{B}_r$. Then we have

$$\begin{aligned}
 & |(\mathcal{N}x)(t_2) - (\mathcal{N}x)(t_1)| \\
 & \leq \frac{\|p\| \psi(r)}{\Gamma(\chi_1 + \chi_2 + 1)} [2(\psi(t_2) - \psi(t_1))^{\chi_1 + \chi_2} \\
 & + |(\psi(t_2) - \psi(a))^{\chi_1 + \chi_2} - (\psi(t_1) - \psi(a))^{\chi_1 + \chi_2}|] \\
 & + \frac{(\psi(t_2) - \psi(a))^{\gamma_1 + \chi_2 - 1} - (\psi(t_1) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \\
 & \times \left[\sum_{i=1}^m |\lambda_i| \|p\| \psi(r) \frac{(\psi(\tau_i) - \psi(a))^{\chi_1 + \chi_2 + \delta_i}}{\Gamma(\chi_1 + \chi_2 + \delta_i + 1)} + \|p\| \psi(r) \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} \right] \\
 & + \frac{|k|r}{\Gamma(\chi_2 + 1)} (\psi(t_2) - \psi(t_1))^{\chi_2} + \frac{(\psi(t_2) - \psi(a))^{\gamma_1 + \chi_2 - 1} - (\psi(t_1) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Lambda| \Gamma(\gamma_1 + \chi_2)} \\
 & \times \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_2 + \delta_i}}{\Gamma(\chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} \right] r.
 \end{aligned}$$

As $t_1 \rightarrow t_2$, the right hand side of the above inequality tends to zero independently of $x \in \mathcal{B}_r$. The set $\mathcal{N}\mathcal{B}_r$ is equicontinuous. Thus, we have proved that $\mathcal{N}\mathcal{B}_r$ is relatively compact. By the Arzelà–Ascoli theorem, \mathcal{N} is completely continuous.

Finally, we show that the set of all solutions to equations $x = \omega \mathcal{N}x$ is bounded for $\omega \in (0, 1)$. Following the computation in the first step, we obtain

$$|x(t)| \leq \phi(\|x\|) \|p\| \Omega_1 + \|x\| \Omega_2, \quad t \in [a, b],$$

which yields

$$\frac{(1 - \Omega_2) \|x\|}{\phi(\|x\|) \|p\| \Omega_1} \leq 1.$$

According to (H4), there exists $\mathcal{M} > 0$ satisfying $\|x\| \neq \mathcal{M}$. Introduce the set

$$U = \{x \in C(J, \mathbb{R}) : \|x\| < \mathcal{M}\}, \tag{3.11}$$

and notice that $\overline{U} \rightarrow C$ is continuous and completely continuous. From the choice of U there is no $x \in \partial U$, such that $x = \omega \mathcal{N}x$ for some $\omega \in (0, 1)$. As a result of the nonlinear alternative of the Leray–Schauder type (Theorem 2.5) we deduce that \mathcal{N} has a fixed point $x \in \overline{U}$, which is a solution of the boundary value problem (1.1)–(1.2). This completes the proof. \square

Example 3.2 Consider the boundary value problem of the Langevin equation with the ψ -HFD of the form

$$\begin{aligned} & \mathcal{D}^{\frac{2}{3}, \frac{4}{3}; \log t} \left(\mathcal{D}^{\frac{4}{5}, \frac{2}{3}; \log t} + \frac{1}{18} \right) x(t) \\ &= \frac{3}{3t + 1049} \left(x^2(t) + \frac{1}{2} e^{-x^2(t)} + \frac{1}{2} \right), \quad t \in \left[\frac{1}{3}, \frac{5}{3} \right], \end{aligned} \tag{3.12}$$

$$x\left(\frac{1}{3}\right) = 0, \quad x\left(\frac{5}{3}\right) = \frac{1}{5} I^{\frac{3}{7}; \log t} x\left(\frac{2}{3}\right) + \frac{2}{5} I^{\frac{4}{7}; \log t} x(1) + \frac{3}{5} I^{\frac{5}{7}; \log t} x\left(\frac{4}{3}\right). \tag{3.13}$$

Here $\chi_1 = 2/3, \chi_2 = 4/5, \beta_1 = 4/5, \beta_2 = 2/3, k = 1/18, a = 1/3, b = 5/3, m = 3, \delta_1 = 3/7, \delta_2 = 4/7, \delta_3 = 5/7, \tau_1 = 2/3, \tau_2 = 1, \tau_3 = 4/3, \lambda_1 = 1/5, \lambda_2 = 2/5, \lambda_3 = 3/5$ and $\psi(t) = \log t$. Observe that $\chi_1 + \chi_2 = 22/15 \in (1, 2]$.

With these values we found $\gamma_1 = \gamma_2 = 14/15, |\Lambda| \approx 0.2902990904, \Omega_1 \approx 14.08506960$ and $\Omega_2 \approx 0.91444444033 < 1$. Notice that

$$\begin{aligned} |f(t, x)| &= \left| \frac{3}{3t + 1049} \left(x^2 + \frac{1}{2} e^{-x^2} + \frac{1}{2} \right) \right| \\ &\leq \left(\frac{3}{3t + 1049} \right) (x^2 + 1) := p(t)\phi(x), \end{aligned}$$

which fulfills the hypothesis (H3) with $\|p\| \leq 1/350$ and $\phi(\mathcal{M}) = \mathcal{M}^2 + 1$. Also (H4) is satisfied for $\mathcal{M} \in (0.7025154750, 1.423456188)$. Hence by Theorem 3.2, the problem (3.12)–(3.13) has at least one solution on $[1/3, 5/3]$.

3.2 Uniqueness result

We shall use the Banach contraction mapping principle to prove the uniqueness of the solutions of the boundary value problem (1.1)–(1.2).

Theorem 3.3 *Assume that:*

(H5) *There exists a positive constant $\mathcal{L} > 0$ such that $|f(t, x) - f(t, y)| \leq \mathcal{L}|x - y|$, for each $t \in J$ and $x, y \in \mathbb{R}$.*

Then the boundary value problem (1.1)–(1.2) has a unique solution on J , provided that

$$\mathcal{L}\Omega_1 + \Omega_2 < 1, \tag{3.14}$$

where the constants Ω_1, Ω_2 are defined by (3.2) and (3.3) respectively.

Proof We want to prove that the operator $\mathcal{N} : C \rightarrow C$ defined by (3.1) has a fixed point. This fixed point is then a solution of the problem (1.1)–(1.2). Set $\sup_{t \in J} |f(t, 0)| = \mathcal{K} < \infty$,

$\mathcal{B}_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ and choose $r \geq (\mathcal{K}\Omega_1)/(1 - \mathcal{L}\Omega_1 - \Omega_2)$. Then we will show that $\mathcal{N}\mathcal{B}_r \subset \mathcal{B}_r$. For any $x \in \mathcal{B}_r$, we have

$$\begin{aligned}
 & |(\mathcal{N}x)(t)| \\
 & \leq \sup_{t \in J} \left\{ I^{\chi_1 + \chi_2; \psi} (|f(t, x(t)) - f(t, 0)| + |f(t, 0)|) + |k| I^{\chi_2; \psi} |x(t)| \right. \\
 & \quad + \frac{(\psi(t) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{\Delta \Gamma(\gamma_1 + \chi_2)} \left[\sum_{i=1}^m |\lambda_i| I^{\chi_1 + \chi_2 + \delta_i; \psi} (|f(\tau_i, x(\tau_i)) - f(\tau_i, 0)| + |f(\tau_i, 0)|) \right. \\
 & \quad + |k| \sum_{i=1}^m |\lambda_i| I^{\chi_2 + \delta_i; \psi} |x(\tau_i)| + I^{\chi_1 + \chi_2; \psi} (|f(b, x(b)) - f(b, 0)| + |f(b, 0)|) \\
 & \quad \left. \left. + |k| I^{\chi_2; \psi} |x(b)| \right] \right\} \\
 & \leq (\mathcal{L}\|x\| + \mathcal{K}) \left(\frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Delta| \Gamma(\gamma_1 + \chi_2)} \right. \\
 & \quad \times \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_1 + \chi_2 + \delta_i}}{\Gamma(\chi_1 + \chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} \right] \\
 & \quad + \|x\| \left(|k| \left\{ \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Delta| \Gamma(\gamma_1 + \chi_2)} \right. \right. \\
 & \quad \left. \left. \times \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_2 + \delta_i}}{\Gamma(\chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} \right] \right\} \right) \\
 & \leq (\mathcal{L}r + \mathcal{K})\Omega_1 + r\Omega_2 < r.
 \end{aligned}$$

Therefore $\|\mathcal{N}x\| \leq r$ which implies that $\mathcal{N}\mathcal{B}_r \subset \mathcal{B}_r$.

Now for $x, y \in \mathcal{C}$, we have

$$\begin{aligned}
 & |(\mathcal{N}x)(t) - (\mathcal{N}y)(t)| \\
 & \leq \left(\frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Delta| \Gamma(\gamma_1 + \chi_2)} \right. \\
 & \quad \times \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_1 + \chi_2 + \delta_i}}{\Gamma(\chi_1 + \chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_1 + \chi_2}}{\Gamma(\chi_1 + \chi_2 + 1)} \right] \\
 & \quad + |k| \left(\left\{ \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1 + \chi_2 - 1}}{|\Delta| \Gamma(\gamma_1 + \chi_2)} \right. \right. \\
 & \quad \left. \left. \times \left[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\tau_i) - \psi(a))^{\chi_2 + \delta_i}}{\Gamma(\chi_2 + \delta_i + 1)} + \frac{(\psi(b) - \psi(a))^{\chi_2}}{\Gamma(\chi_2 + 1)} \right] \right\} \right) \|x - y\| \\
 & = (\mathcal{L}\Omega_1 + \Omega_2)\|x - y\|.
 \end{aligned}$$

Therefore $\|(\mathcal{N}x) - (\mathcal{N}y)\| \leq (\mathcal{L}\Omega_1 + \Omega_2)\|x - y\|$. As $\mathcal{L}\Omega_1 + \Omega_2 < 1$, \mathcal{N} is a contraction. By the Banach contraction mapping principle (Lemma 2.3) we deduce that \mathcal{N} has a fixed

point, which is the unique solution of the boundary value problem (1.1)–(1.2). The proof is finished. \square

Example 3.3 Consider the boundary value problem of the Langevin equation with the ψ -HFD of the form

$$\mathcal{D}^{\frac{3}{7}, \frac{2}{7}; t^2+1} \left(\mathcal{D}^{\frac{5}{7}, \frac{4}{7}; t^2+1} + \frac{1}{13} \right) x(t) = \frac{2}{4t+31} \left(\frac{x^2(t) + 2|x(t)|}{1 + |x(t)|} \right) + \frac{1}{4}, \quad t \in \left[\frac{1}{4}, \frac{5}{4} \right], \quad (3.15)$$

$$x\left(\frac{1}{4}\right) = 0, \quad x\left(\frac{5}{4}\right) = \frac{3}{8} I^{\frac{2}{3}; t^2+1} x\left(\frac{1}{2}\right) + \frac{5}{8} I^{\frac{4}{3}; t^2+1} x\left(\frac{3}{4}\right) + \frac{7}{8} I^{\frac{5}{3}; t^2+1} x(1). \quad (3.16)$$

Here $\chi_1 = 3/7$, $\chi_2 = 5/7$, $\beta_1 = 2/7$, $\beta_2 = 4/7$, $k = 1/13$, $a = 1/4$, $b = 5/4$, $m = 3$, $\delta_1 = 2/3$, $\delta_2 = 4/3$, $\delta_3 = 5/3$, $\tau_1 = 1/2$, $\tau_2 = 3/4$, $\tau_3 = 1$, $\lambda_1 = 3/8$, $\lambda_2 = 5/8$, $\lambda_3 = 7/8$ and $\psi(t) = t^2 + 1$. Note that $\chi_1 + \chi_2 = 8/7 \in (1, 2]$.

With these values we found $\gamma_1 = 29/49$, $\gamma_2 = 43/49$, $|\Lambda| \approx 0.6572399879$, $\Omega_1 \approx 4.725942806$ and $\Omega_2 \approx 0.3825052846 < 1$ and $\mathcal{L} = 1/8$ as

$$|f(t, x) - f(t, y)| \leq \frac{1}{8} |x - y|, \quad \text{for any } x, y \in \mathbb{R}.$$

Since $\mathcal{L}\Omega_1 + \Omega_2 \approx 0.9732481354 < 1$, by Theorem 3.3, the problem (3.15)–(3.16) has a unique solution on $[1/4, 5/4]$.

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Authors’ contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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