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A study on controllability of impulsive fractional evolution equations via resolvent operators

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Abstract
In this article, we study the controllability for impulsive fractional integro-differential evolution equation in a Banach space. The discussions are based on the Mönch fixed point theorem as well as the theory of fractional calculus and the (α,β)-resolvent operator, we concern with the term $u'(\cdot)$ and finding a control v such that the mild solution satisfies $u(b)=u_b$ and $u'(b)=u'_b$. Finally, we present an application to support the validity study.

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Keywords: Controllability; Measure of noncompactness; Mild solution; Mönch fixed point theorem

1 Introduction

Fractional differential equations have been applied to various fields successfully, for example, physics, engineering, and finance. Consequently, more and more researchers paid much attention to this subject and have obtained substantial achievements, we refer the reader to [3, 9, 16, 25] and the references therein.

Controllability plays a significant role in the evolution of modern mathematical control theory. This is a qualitative property of dynamical control systems and is of appropriate significance in control theory. Many fundamental problems of control theory such as pole-assignment, stabilizability and optimal control may be solved under the presumption that the system is controllable. The concept of controllability, when it was first introduced by Kalman [15] in 1963, has become an active area of investigation due to its great applications in the field of physics. Controllability problems for different kinds of dynamical systems have been considered in many papers [1, 2, 4–6, 8, 22, 23].

Controllability is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. It has many significant applications, not only in control theory and systems theory, but also in such fields as industrial and chemical process control, reactor control, control of electric bulk power systems, aerospace engineering and recently in quantum system theory.

Controllability is also strongly related to the theory of realization and so-called minimal realization and canonical forms for linear time-invariant control systems such as the Kalman canonical form, the Jordan canonical form and the Luenberger canonical form. Moreover, it is strongly connected with the minimum energy control problem for many classes of linear finite dimensional, infinite dimensional dynamical systems, and delayed systems both deterministic and stochastic.

In recent years, the controllability problems for various linear and nonlinear deterministic and stochastic dynamic systems have been studied in many publications using different method, we refer the reader to [2, 6, 24]. In addition, Kailasavalli et al. [14] acknowledged the existence and controllability of fractional neutral integro-differential systems with SDD with Banach contraction and resolvent operator technique as the main reference. Dabas et al. [10] studied the existence, uniqueness and continuous dependence of a mild solution for an impulsive neutral fractional order differential equation with infinite delay. Recently, Heping Ma and Biu Liu [20] interpreted the exact controllability and continuous dependence of fractional neutral integro-differential equations with state-dependent delay in Banach spaces. Also Yan [27] discussed the approximate controllability of neutral integro-differential delay systems with inclusion type in Hilbert space by using the fixed point theorem of discontinuous multi-valued operators supported by the Dhage fixed point technique with the resolvent operator. Additionally, Yan and Jia [28] explained the approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-delay.

Especially, the controllability of fractional evolution equations is also studied. In 2015, Liang and Yang [19] investigated the exact controllability for the fractional integro-differential evolution equations in Banach spaces E involving noncompact semigroups and nonlocal functions without Lipschitz continuity,

$$\begin{cases} D^\alpha u(t) + Au(t) = f(t,u(t),Gu(t)) + Bv(t), & t \in J, \\ u(0) = \sum_{i=1}^m c_i u(t_i), \end{cases}$$

where D^α denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, $-A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $T(t)$ $(t \geq 0)$ of uniformly bounded linear operator, the control function v is given in $L^2(J,U)$; U is a Banach space, B is a linear bounded operator from U to E; f is a given function and

$$Gu(t) = \int_0^t K(t,s)u(s)\,ds$$

is a Volterra integral operator.

In 2011, Debbouche and Baleanu [11] studied the controllability for the fractional nonlocal impulsive integro-differential control system of the form

$$\begin{cases} \frac{d^\alpha u(t)}{dt^\alpha} + A(t,u(t))u(t) = (Bv)(t) + \Phi(t,f(t,u(\beta(t)),\int_0^t g(t,s,u(\gamma(s)))\,ds), \\ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1,2,\ldots,m, \\ u(0) + h(u) = u_0, \end{cases}$$

the discussions are based on the theory of fractional calculus as well as on the fixed point technique and the (α,u)-resolvent family.

In 2017, Lian, Fan and Li [18] investigated the approximate controllability for a class of semilinear fractional differential systems of order $1 < \alpha < 2$ of the form

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) + f(t, x(t)) + Bu(t); & t \in J = [0, b]; \\ x(0) + g(x) = x_0 \in E, \qquad x'(0) + h(x) = y_0 \in E, \end{cases}$$

via the resolvent operator.

In 2019, Singh and Pandey [26] studied some controllability results for the abstract second order Sobolev type impulsive delay differential system of the form

$$\begin{cases} \frac{d^2}{dt^2}(Qu(t)) = Au(t) + Bv(t) + F(t, u_t, \int_0^t K(t,s)(u_s)\,ds), & t \in J', \\ u(t) = \varphi(t), \quad t \in [-\sigma, 0], \qquad u'(0) = \chi_0, \\ \Delta u|_{t=t_k} = G_k^1(u_{t_k}), \qquad \Delta u'|_{t=t_k} = G_k^2(u'_{t_k}), \quad k = 1, 2, \ldots, m. \end{cases}$$

On the other hand, in recent years, much attention has been paid to establishing sufficient conditions for the controllability of linear fractional dynamical systems of order $0 < \alpha < 1$ by several authors; see a recent monograph [1, 2, 4, 5, 8, 22, 23] and various papers [2, 6]. However, there is no work that reported on the problem of controllability of nonlinear fractional dynamical system of order $1 < \alpha < 2$, to the best of our knowledge, up until now the controllability for a class of impulsive fractional integro-differential evolution equation with fractional derivative of order $\alpha \in (1, 2]$ has not been investigated in the literature. Motivated by the above mentioned aspects, in this paper, we discuss the controllability for a class of impulsive fractional integro-differential evolution equation of the form

$$\begin{cases} {}^{c}D_{0^+}^{\alpha}u(t) = Au(t) + f(t, u(t), Gu(t), Fu(t)) + Bv(t), & t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \qquad \Delta u'(t_k) = J_k(u'(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) + g(u) = u_0, \qquad u'(0) + h(u) = u_1, \end{cases} \tag{1.1}$$

where ${}^{c}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (1, 2]$ with the lower limit zero, $A : D(A) \subset E \to E$ a closed linear operator and A generates a strongly continuous (α, β)-resolvent family $S_{\alpha,\beta}(t)$ $(t \geq 0)$ of uniformly bounded linear operator on a Banach space E. The state $u(\cdot)$ takes values in E, $J = [0, b]$ $(b > 0)$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, the $\{t_k\}$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$, $m \in N$; the functions $f : J \times E \times E \times E \to E$ and $I_k, J_k : \mathrm{PC}(J, E) \to E$, $k = 1, 2, \ldots, m$, $g, h : \mathrm{PC}(J, E) \to E$ are appropriate functions satisfying certain assumptions that will be specified later. $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively, the control function v is given in $L^2(J, U)$, U is a Banach space, B is a linear bounded operator from U to E, and the operators G and F are given by

$$Gu(t) = \int_0^t K(t,s)u(s)\,ds, \qquad Fu(t) = \int_0^b H(t,s)u(s)\,ds$$

where $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \{(t,s) : 0 \leq s \leq t \leq b\}$, $H \in C(\Delta_0, \mathbb{R}^+)$, $\Delta_0 = \{(t,s) : 0 \leq s, t \leq b\}$. Throughout this work, we always assume that

$$K^* = \sup_{t\in J} \int_0^t K(t,s)\,ds, \qquad H^* = \sup_{t\in J} \int_0^b H(t,s)\,ds.$$

In this paper, we introduce a suitable concept of a mild solution of the system (1.1). Moreover, we investigate the controllability for the system (1.1), by using the Mönch fixed point theorem combined with (α, β)-resolvent operators.

The paper is organized as follows: The second part of the paper some notations and recall some basic known results. The third part we present a controllability result for the problem (1.1) of our concern. And the last section is provided an example to illustrate applications of the obtained results. Concluding part close this article.

2 Preliminaries

Let E and E_1 be two Banach space. For any Banach space E, the norm of E is defined by $\|\cdot\|_E$. The space of all bounded linear operator from E to E_1 is denoted by $\mathcal{L}(E, E_1)$ and $\mathcal{L}(E, E)$ is written as $\mathcal{L}(E)$. We denote by $C(J, E)$ the Banach space of all continuous E-value function on interval J the norm $\|u\|_C = \max_{t\in J} \|u(t)\|$. We use $\|f\|_{L^p}$ to denote the $L^p(J, E)$ norm of f whenever $f \in L^p(J, E)$ for some p with $1 \leq p < \infty$. We consider the following spaces:

Let $\mathrm{PC}(J, E) = \{u : J \to E, u(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \dots, m\}$. Evidently, $\mathrm{PC}(J, E)$ is a Banach space with the norm $\|u\|_{\mathrm{PC}} = \sup_{t\in J}\{\|u(t)\| : u \in \mathrm{PC}(J, E)\}$.

Let $\mathrm{PC}^1(J, E)$ be the spaces of all functions $u \in \mathrm{PC}(J, E)$, which are continuously differentiable on J', and the lateral derivatives

$$u'_R(t) = \lim_{s\to 0+} \frac{u(t+s) - u(t^+)}{s} \quad \text{and} \quad u'_L(t) = \lim_{s\to 0-} \frac{u(t+s) - u(t^-)}{s}$$

are continuous on $[0, b)$ and $(0, b]$, respectively. Furthermore, for $u \in \mathrm{PC}^1(J, E)$, we denote by $u'(t)$ the left derivative at $t \in (0, b]$, and by $u'(0)$, the right derivative at zero. It is easy to see that the space $\mathrm{PC}^1(J, E)$ is a Banach space with the norm

$$\|u\|_{\mathrm{PC}^1} = \max\Big\{\sup_{t\in J}\big\|u(t)\big\|, \sup_{t\in J}\big\|u'(t)\big\|\Big\}.$$

In the following, let us recall some well-known definitions. For more details, see [16].

Definition 2.1 The fractional integral of order γ with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$I_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)\,ds, \quad t > 0, \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f : [0,\infty) \to \mathbb{R}$ can be written as

$$^{L}D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} \, ds, \quad t > 0, n-1 < \gamma < n.$$

Definition 2.3 The Caputo fractional derivative of order γ for a function $f : [0,\infty) \to \mathbb{R}$ can be written as

$$^{c}D_t^{\gamma} f(t) = {}^{L}D_t^{\gamma} \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \quad t > 0, n-1 < \gamma < n,$$

where $n = [\gamma] + 1$ and $[\gamma]$ denotes the integer part of γ.

Lemma 2.1 ([3]) *For $q > 0$, the general solution of the fractional differential equation $^{c}D_t^{q} u(t) = 0$ is given by*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, $n = [q] + 1$ and $[q]$ denotes the integer part of the real number q.

Now, we review some definitions and lemmas on fractional calculus. For $\beta \geq 0$, let

$$g_\beta(t) = \begin{cases} \frac{t^{\beta-1}}{\Gamma(\beta)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function . The finite convolution of f and g is denoted by $(f * g)(t) = \int_0^t f(t-s) g(s) \, ds$.

A strongly continuous family $\{T(t)\}_{t\geq 0} \subseteq B(E)$ is said to be exponentially bounded if there are constants $M \geq 0$ and $\omega \in \mathbb{R}$, such that

$$\| T_\alpha(t) \| \leq M e^{\omega t}, \quad t \geq 0.$$

Definition 2.4 ([9]) Let $A : D(A) \subseteq E \to E$ be closed linear operators defined on a Banach space E and $\alpha, \beta > 0$. Let $\rho(A)$ be the resolvent set of A, we say that the A is the generator of an (α,β)-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha,\beta} : [0,\infty) \to \mathcal{L}(E)$ such that $S_{\alpha,\beta}(t)$ is exponentially bounded, $\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$, and for all $u \in E$,

$$\lambda^{\alpha-\beta} \left(\lambda^\alpha I - A \right)^{-1} u = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t) u \, dt, \quad \operatorname{Re}\lambda > \omega. \tag{2.1}$$

In this case, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the (α,β)-resolvent family (also called the (α,β)-resolvent operator function) generated by A.

Lemma 2.2 (*i*) *The operator $S_{\alpha,2}(t) : \mathbb{R}_+ \to \mathcal{L}(E)$ associated with $S_{\alpha,1}$ is defined by*

$$S_{\alpha,2}(t) = \int_0^t S_{\alpha,1}(s) \, ds, \quad t \geq 0. \tag{2.2}$$

(ii) The resolvent family $S_{\alpha,\alpha}(t):\mathbb{R}_+\to\mathcal{L}(E)$ associated with the solution operator $S_{\alpha,1}$ is defined by

$$S_{\alpha,\alpha}(t)=I_t^{\alpha-1}S_{\alpha,1}(t), \tag{2.3}$$

$$S_{\alpha,\alpha-1}(t)=I_t^{\alpha-2}S_{\alpha,1}(t). \tag{2.4}$$

Proof (i). By (2.1), we have

$$\lambda^{\alpha-1}\left(\lambda^\alpha I-A\right)^{-1}u=\int_0^\infty e^{-\lambda t}S_{\alpha,1}(t)u\,dt,\quad \operatorname{Re}\lambda>\omega, u\in E, \tag{2.5}$$

$$\lambda^{\alpha-2}\left(\lambda^\alpha I-A\right)^{-1}u=\int_0^\infty e^{-\lambda t}S_{\alpha,2}(t)u\,dt,\quad \operatorname{Re}\lambda>\omega, u\in E. \tag{2.6}$$

Thus, by (2.5), (2.6), we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda t}S_{\alpha,2}(t)u\,dt&=\lambda^{\alpha-2}\left(\lambda^\alpha I-A\right)^{-1}u=\frac{1}{\lambda}\lambda^{\alpha-1}\left(\lambda^\alpha I-A\right)^{-1}u\\&=\frac{1}{\lambda}\int_0^\infty e^{-\lambda t}S_{\alpha,1}(t)u\,dt\\&=\int_0^\infty e^{-\lambda t}(g_1*S_{\alpha,1})(t)u\,dt=\int_0^\infty e^{-\lambda t}\left(\int_0^t S_{\alpha,1}(s)\,ds\right)u\,dt.\end{aligned}$$

(ii). By (2.1), we have

$$\left(\lambda^\alpha I-A\right)^{-1}u=\int_0^\infty e^{-\lambda t}S_{\alpha,\alpha}(t)u\,dt,\quad \operatorname{Re}\lambda>\omega, u\in E. \tag{2.7}$$

Thus, by (2.7), we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda t}S_{\alpha,\alpha}(t)u\,dt&=\left(\lambda^\alpha I-A\right)^{-1}u=\frac{1}{\lambda^{\alpha-1}}\lambda^{\alpha-1}\left(\lambda^\alpha I-A\right)^{-1}u\\&=\frac{1}{\lambda^{\alpha-1}}\int_0^\infty e^{-\lambda t}S_{\alpha,1}(t)u\,dt\\&=\int_0^\infty e^{-\lambda t}(g_{\alpha-1}*S_{\alpha,1})(t)u\,dt\\&=\int_0^\infty e^{-\lambda t}\left(I_t^{\alpha-1}S_{\alpha,1}(t)\right)u\,dt.\end{aligned}$$

(iii). By (2.1), we have

$$\lambda\left(\lambda^\alpha I-A\right)^{-1}u=\int_0^\infty e^{-\lambda t}S_{\alpha,\alpha-1}(t)u\,dt,\quad \operatorname{Re}\lambda>\omega, u\in E. \tag{2.8}$$

Thus, by (2.8), we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda t}S_{\alpha,\alpha-1}(t)u\,dt&=\lambda\left(\lambda^\alpha I-A\right)^{-1}u=\frac{1}{\lambda^{\alpha-2}}\lambda^{\alpha-1}\left(\lambda^\alpha I-A\right)^{-1}u\\&=\frac{1}{\lambda^{\alpha-2}}\int_0^\infty e^{-\lambda t}S_{\alpha,1}(t)u\,dt\end{aligned}$$

$$= \int_0^\infty e^{-\lambda t} (g_{\alpha-2} * S_{\alpha,1})(t) u \, dt$$

$$= \int_0^\infty e^{-\lambda t} \big(I_t^{\alpha-2} S_{\alpha,1}(t) \big) u \, dt. \qquad \square$$

Lemma 2.3 *Let A be the infinitesimal generator of the strongly continuous (α,β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ in E. Then*

(i): $\frac{d}{dt} S_{\alpha,1}(t)u = A S_{\alpha,\alpha}(t)u$, *for $t \in J$ and $u \in E$;*

(ii): $\frac{d}{dt} S_{\alpha,2}(t)u = S_{\alpha,1}(t)u$, *for $t \in J$ and $u \in E$;*

(iii): $\frac{d}{dt} S_{\alpha,\alpha}(t)u = S_{\alpha,\alpha-1}(t)u$, *for $t \in J$ and $u \in E$.*

Proof Since A is the infinitesimal generator of the strongly continuous (α,β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$. So, for all $b > 0$, the series $t^{\alpha-1}\sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(\alpha+\alpha k)}$, $\sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(1+\alpha k)}$, $t\sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(2+\alpha k)}$ are uniformly convergent on $[0,b]$. Thus, for $t \in [0,b]$, we have

$$\begin{aligned}
\frac{d}{dt} S_{\alpha,1}(t)u &= \left[\sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(1+\alpha k)} u \right]' = At^{\alpha-1} \sum_{k=1}^\infty \frac{A^{k-1} t^\alpha (k-1)}{\Gamma(\alpha+\alpha(k-1))} u \\
&= At^{\alpha-1} \sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(\alpha+\alpha k)} u = A S_{\alpha,\alpha}(t)u, \\
\frac{d}{dt} S_{\alpha,2}(t)u &= \left[t\sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(2+\alpha k)} u \right]' = \left[\sum_{k=0}^\infty \frac{A^k t^{k\alpha+1}}{\Gamma(2+\alpha k)} u \right]' \\
&= \sum_{k=0}^\infty \frac{A^k (1+\alpha k) t^{\alpha k}}{(1+\alpha k)\Gamma(1+\alpha k)} u = \sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(1+\alpha k)} u = S_{\alpha,1}(t)u,
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt} S_{\alpha,\alpha}(t)u &= \left[t^{\alpha-1} \sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(\alpha+\alpha k)} u \right]' = \left[\sum_{k=0}^\infty \frac{A^k t^{\alpha k+\alpha-1}}{\Gamma(\alpha+\alpha k)} u \right]' \\
&= \sum_{k=1}^\infty \frac{A^k (\alpha k+\alpha-1) t^{\alpha k+\alpha-2}}{\Gamma(\alpha+\alpha k)} u = \sum_{k=0}^\infty \frac{A^k (\alpha k+\alpha-1) t^{\alpha k+\alpha-2}}{(\alpha k+\alpha-1)\Gamma(\alpha-1+\alpha k)} u \\
&= t^{\alpha-2} \sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(\alpha-1+\alpha k)} u = t^{\alpha-2} E_{\alpha,\alpha-1}\big(At^\alpha\big) u = S_{\alpha,\alpha-1}(t)u.
\end{aligned} \qquad \square$$

Lemma 2.4 ([9]) *The operators $S_{\alpha,1}(t)$, $S_{\alpha,2}(t)$, $S_{\alpha,\alpha}(t)$ and $S_{\alpha,\alpha-1}(t)$ have the following properties.*

(*i*)*The operators $S_{\alpha,1}(t)$, $S_{\alpha,2}(t)$, $S_{\alpha,\alpha}(t)$ and $S_{\alpha,\alpha-1}(t)$ are strongly continuous for all $t \geq 0$.*

(*ii*) *If $S_{\alpha,\beta}(t)$ $(t \geq 0)$ is an equicontinuous (α,β)-resolvent family, then $S_{\alpha,1}(t)$, $S_{\alpha,2}(t)$, $S_{\alpha,\alpha}(t)$ and $S_{\alpha,\alpha-1}(t)$ are also equicontinuous in E for $t > 0$.*

Now, we can formulate some basic properties of operators $S_{\alpha,1}(t)$, $S_{\alpha,2}(t)$, $S_{\alpha,\alpha-1}(t)$, and $S_{\alpha,\alpha}(t)$.

Lemma 2.5 *For fixed $t \geq 0$, $S_{\alpha,2}(t)$, $S_{\alpha,\alpha}(t)$ and $S_{\alpha,\alpha-1}(t)$ are linear and bounded operators on E.*

Proof For any fixed $t \geq 0$, it is easy to check that $S_{\alpha,2}(t)$, $S_{\alpha,\alpha}(t)$, $S_{\alpha,\alpha-1}(t)$ are also linear operators since $S_{\alpha,1}(t)$ is a linear operator. For any $u \in E$, by Lemma 2.2, we have

$$\|S_{\alpha,2}(t)u\| = \left\| \int_0^t S_{\alpha,1}(s)u\,ds \right\| \leq Mt\|u\|$$

and

$$\begin{aligned} \|S_{\alpha,\alpha}(t)u\| &= \left\| \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} S_{\alpha,1}(t)u\,ds \right\| = \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}\|u\|, \\ \|S_{\alpha,\alpha-1}(t)u\| &= \left\| \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} S_{\alpha,1}(t)u\,ds \right\| = \frac{Mt^{\alpha-2}}{\Gamma(\alpha-1)}\|u\|. \end{aligned}$$ □

Now, we recall some properties of Hausdorff measure of noncompactness that will be used later.

Definition 2.5 ([7]) The Hausdorff measure of noncompactness α on a bounded subset D of the Banach space E is defined as

$$\begin{aligned} \alpha(D) := \inf\{\epsilon > 0 : &D \text{ can be covered by finite number of} \\ &\text{balls of radius smaller then } \epsilon\}. \end{aligned}$$

Let $\alpha(\cdot)$, $\alpha(\cdot)_{\mathrm{PC}}$ and $\alpha(\cdot)_{\mathrm{PC}^1}$ denote the Hausdorff measure of noncompactness on $C(J,E)$, $\mathrm{PC}(J,E)$ and $\mathrm{PC}^1(J,E)$, respectively. For any $B \subset C(J,E)$ and $t \in J$, set $B(t) = \{u(t) : u \in B\} \subset E$. If B is bounded in $C(J,E)$, then $B(t)$ is bounded in E, and $\alpha(B(t)) \leq \alpha(B)$.

Lemma 2.6 ([17]) *Let E be a Banach space, and let $B \subset E$ be bounded. Then there exists a countable set $B_0 \subset B$, such that $\alpha(B) \leq 2\alpha(B_0)$.*

Lemma 2.7 ([13]) *Let E be a Banach space, and let $B \subset C(J,E)$ be equicontinuous and bounded, then $\alpha(B(t))$ is continuous on J, and $\alpha(B) = \max_{t\in J} \alpha(B(t))$.*

Lemma 2.8 ([7]) *Let $D \subset \mathrm{PC}([a,b],E)$ be bounded and piecewise equicontinuous, then $\alpha_{\mathrm{PC}}(D(t))$ is piecewise continuous for $t \in [a,b]$, and $\alpha_{\mathrm{PC}}(D) = \sup\{\alpha(D) : t \in [a,b]\}$, where $D(t) = \{u(t) : u \in D\}$.*

Lemma 2.9 ([7]) *Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from $[a,b]$ into E such that $\|w_n(t)\| \leq g(t)$ for every $n \geq 1$ and almost all $t \in [a,b]$, where $g \in L^1([a,b],\mathbb{R}^+)$, then the function $h(t) = \alpha\{w_n(t) : n \geq 1\}$ contained in $L^1(J,\mathbb{R}^+)$ satisfies*

$$\alpha\left(\left\{\int_a^t w_n(s)\,ds : n \geq 1\right\}\right) \leq 2\int_a^t h(s)\,ds.$$

Lemma 2.10 ([7]) *Let $D \subset \mathrm{PC}^1([a,b],E)$ be bounded and let the elements of D' be piecewise equicontinuous, then*

$$\alpha_{\mathrm{PC}^1}(D) = \max\Bigl\{\sup_{t\in[a,b]} \alpha\bigl(D(t)\bigr), \sup_{t\in[a,b]} \alpha\bigl(D'(t)\bigr)\Bigr\}, \quad \textit{where } D'(t) = \bigl\{u'(t) : u \in D\bigr\}.$$

Next, we are ready to construct a mild solution for the impulsive system (1.1).

Lemma 2.11 *Assume $A : D(A) \subset E \to E$ is a closed linear operator, A is known as the infinitesimal generator of the (α,β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ on a Banach space E. Then the problem (1.1) has a unique solution $u \in \mathrm{PC}^1(J,E)$ and satisfies the following integral equation*:

$$\begin{aligned} u(t) &= S_{\alpha,1}(t)\bigl(u_0 - g(u)\bigr) + S_{\alpha,2}(t)\bigl(u_1 - h(u)\bigr) \\ &\quad + \sum_{0<t_k<t} S_{\alpha,1}(t-t_k)I_k\bigl(u(t_k)\bigr) + \sum_{0<t_k<t} S_{\alpha,2}(t-t_k)J_k\bigl(u'(t_k)\bigr) \\ &\quad + \int_0^t S_{\alpha,\alpha}(t-s)\bigl[Bv(s) + f\bigl(s,u(s),Gu(s),Fu(s)\bigr)\bigr]\,ds, \quad t \in J. \end{aligned} \tag{2.9}$$

Proof The proof is similar to the proof in paper [12], here we omit it. □

Based on Lemma 2.11, we will give the definition of mild solutions for the problem (1.1).

Definition 2.6 A function $u : J \to E$ is called a mild solution of the problem (1.1) if $u(0) = (u_0 - g(u))$, $u'(0) = (u_1 - h(u))$, $\Delta u(t_k) = I_k(u(t_k))$, $\Delta u'(t_k) = J_k(u'(t_k))$, $u(\cdot)|_J \in \mathrm{PC}^1(J,E)$ and the following equation is satisfied:

$$\begin{aligned} u(t) &= S_{\alpha,1}(t)\bigl(u_0 - g(u)\bigr) + S_{\alpha,2}(t)\bigl(u_1 - h(u)\bigr) \\ &\quad + \sum_{0<t_k<t} S_{\alpha,1}(t-t_k)I_k\bigl(u(t_k)\bigr) + \sum_{0<t_k<t} S_{\alpha,2}(t-t_k)J_k\bigl(u'(t_k)\bigr) \\ &\quad + \int_0^t S_{\alpha,\alpha}(t-s)\bigl[Bv(s) + f\bigl(s,u(s),Gu(s),Fu(s)\bigr)\bigr]\,ds. \end{aligned}$$

Theorem 2.1 ([21]) *Let Ω be a closed convex subset of a Banach space E and $0 \in \Omega$. Assume that $Q : \Omega \to \Omega$ is a continuous map, which satisfies Mönch's condition, i.e., $D \subset \Omega$ is countable and $D \subset \overline{\mathrm{conv}}(\{0\} \cup Q(D)) \Rightarrow \overline{D}$ is compact. Then Q has at least one fixed point in Ω.*

3 Main results

In this section, we will establish the sufficient conditions for the controllability of the system (1.1). For arbitrary $u \in \mathrm{PC}^1(J,E)$, we denote the final stages of u by $u_b = u(b)$ and $u'_b = u'(b)$ at time b in the space E.

Definition 3.1 The system (1.1) is said to be controllable on J if for initial conditions $u_0 \in E$, $u_1 \in E$ and final stages u_b and u'_b in E, there exists a control $v \in L^2(J,U)$ such that the mild solution $u(t)$ of the system (1.1) corresponding to v satisfies $u(0) = (u_0 - g(u))$, $u'(0) = (u_1 - h(u))$, $\Delta u(t_k) = I_k(u(t_k))$, $\Delta u'(t_k) = J_k(u'(t_k))$, $k = 1,2,\ldots,m$, and $u(b) = u_b$, $u'(b) = u'_b$.

Let $B_{r_0} := \{u \in \mathrm{PC}^1(J,E) : \|u\|_{\mathrm{PC}^1} \le r_0$, where $r_0 = \max\{r_1, r_2\}$ such that $\|u\|_{\mathrm{PC}} \le r_1$, $\|u'\|_{\mathrm{PC}} \le r_2\}$. However, to achieve such a result, we assume certain conditions:

(H0) A generates a strongly continuous (α,β)-resolvent family $S_{\alpha,\beta}(t)$ $(t \ge 0)$ of the uniformly bounded linear operator on a Banach space E. That is, there exists a constants $M \ge 1$ such that $\|S_{\alpha,\beta}(t)\| \le M$ for all $t \ge 0$ and there exists a positive constant M_0 such that $\|AS_{\alpha,\alpha}(t)\|_{\mathcal{L}} \le M_0$ for all $t \ge 0$.

(H1) The function $f : J \times E \times E \times E \to E$ satisfies:

(i) for a.e. $t \in J$, the function $f(t,\cdot,\cdot,\cdot) : E \times E \times E \to E$ is continuous, and for each $(x,y) \in E \times E$, the function $f(\cdot,x,y,z) : J \to E$ is strongly measurable;

(ii) for any $r_0 > 0$, there exist a constant $q_2 \in (0,\alpha)$ and functions $m_{r_0} \in L^{\frac{1}{q_2}}(J,\mathbb{R}^+)$ such that

$$\sup\bigl\{\bigl\|f(t,x,y,z)\bigr\| : \|x\| \le r, \|y\| \le K^* r, \|z\| \le H^* r\bigr\} \le m_{r_0}(t), \quad t \in J,$$

where m_{r_0} satisfies $\lim_{r_0 \to +\infty} \inf \frac{1}{r_0} \|m_{r_0}\|_{L^{\frac{1}{q_2}}} \triangleq \gamma < \infty$;

(iii) there exist a constant $q_3 \in (0,\alpha)$ and functions $J_f \in L^{\frac{1}{q_3}}(J,\mathbb{R}^+)$ such that

$$\alpha\bigl(f(t,D_1,D_2,D_3)\bigr) \le J_f(t)\bigl(\alpha(D_1) + \alpha(D_2 + \alpha(D_3)\bigr), \quad t \in J,$$

for any countable subsets $D_1, D_2, D_3 \in \mathrm{PC}(J,E)$.

(H2) The function $I_k, J_k : \mathrm{PC}(J,E) \to E$, for $k = 1,2,\dots,m$, satisfies:

(i) There exists a nondecreasing function $L_k^j : \mathbb{R}^+ \to \mathbb{R}^+$ $(j = 1,2)$ such that

$$\bigl\|I_k(u)\bigr\| \le L_k^1\bigl(\|u\|_{\mathrm{PC}^1}\bigr), \qquad \bigl\|J_k(u)\bigr\| \le L_k^2\bigl(\|u\|_{\mathrm{PC}^1}\bigr) \quad \text{and}$$
$$\lim_{r_0 \to \infty} \frac{L_k^j(r_0)}{r_0} = \delta_k^j < \infty$$

for all $u \in E$ and $k = 1,2,\dots,m$.

(ii) There exist constants $M_j > 0$ such that, for any countable subsets $D_j \subset E$, and

$$\alpha\bigl(\bigl\{I_k(D_1)\bigr\}\bigr) \le M_k^1 \alpha(D_1), \qquad \alpha\bigl(\bigl\{J_k(D_2)\bigr\}\bigr) \le M_k^2 \alpha(D_2)$$

for all $j = 1,2$ and $i = 1,2,\dots,m$.

(H3) (i) The function $g, h : \mathrm{PC} \to E$ is Lipschitz continuous and bounded in E, that is, there exists aconstants $c_1, c_2 \ge 0$ and $c_3, c_4 \ge 0$ such that

$$\bigl\|g(u)\bigr\| \le c_1, \quad \bigl\|g(u) - g(v)\bigr\| \le c_2 \max_{t\in J} \|u - v\|_{\mathrm{PC}},$$
$$\bigl\|h(u)\bigr\| \le c_3, \quad \bigl\|h(u) - h(v)\bigr\| \le c_4 \max_{t\in J} \|u - v\|_{\mathrm{PC}},$$

for all $u, v \in \mathrm{PC}(J,E)$.

(ii) There exist constants $l_1, l_2 > 0$ such that, for any countable subsets $D_1, D_2 \subset E$, and

$$\alpha\bigl(g(D_1)\bigr) \le l_1 \alpha(D_1), \qquad \alpha\bigl(h(D_2)\bigr) \le l_2 \alpha(D_2).$$

(H4) Linear operator $W : L^2(J,U) \to E$ defined by

$$Wu = \begin{cases} \int_0^t S_{\alpha,\alpha}(t-s)Bv(s)\,ds, & v = v_1^u; \\ \int_0^t S_{\alpha,\alpha-1}(t-s)Bv(s)\,ds, & v = v_2^{u'}, \end{cases} \tag{3.1}$$

where v_1^u and $v_2^{u'}$ are defined in (3.6).

(i) W has an inverse operator W^{-1} which takes values in $L^2(J,U)\setminus \ker W$, and there exist two constants $M_1 > 0$, $M_2 > 0$ such that $\|B\| \le M_1$, $\|W^{-1}\| \le M_2$.

(ii) There exist a constant $q_1 \in (0,\alpha)$ and a function $K_w \in L^{\frac{1}{q_1}}(J,\mathbb{R}^+)$ such that

$$\alpha\bigl(W^{-1}(D_4)(t)\bigr) \le K_w(t)\alpha(D_4), \quad t \in J,$$

for any bounded subset $D_4 \subset E$.

For the sake of brevity, we introduce the notations

$$k_i = \frac{b^{q-q_i}}{(a_i+1)^{1-q_i}}, \qquad a_i = \frac{q-1}{1-q_i}, \quad i = 1,2,3;$$
$$h_i = \frac{b^{q-q_i-1}}{(b_i+1)^{1-q_i}}, \qquad b_i = \frac{q-2}{1-q_i}, \quad i = 1,2,3;$$
$$M_3 = k_1\|K_w\|_{L^{\frac{1}{q_1}}}; \qquad M_4 = k_3\|J_f\|_{L^{\frac{1}{q_3}}};$$
$$M_5 = k_2\|m_{r_0}\|_{L^{\frac{1}{q_2}}}; \qquad M_6 = h_2\|m_{r_0}\|_{L^{\frac{1}{q_2}}}, \qquad M_7 = h_3\|J_f\|_{L^{\frac{1}{q_3}}}.$$

Theorem 3.1 *Assume that the assumptions* (H0)–(H4) *are satisfied, then the system* (1.1) *is controllable on J provided that* $\max(\lambda_1,\lambda_2) < 1$, *where*

$$\begin{aligned} \lambda_1 = \max\Biggl\{&\Bigl[1 + \frac{MM_1M_2b^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\Bigr]\Bigl[\frac{MM_5}{\Gamma(\alpha)} + \sum_{k=1}^{m} M\bigl(\delta_k^1 + \delta_k^2\bigr)\Bigr], \\ &\Bigl[1 + \frac{MM_1M_2b^{\alpha-\frac{3}{2}}}{\Gamma(\alpha-1)}\Bigr]\Bigl[\frac{MM_6}{\Gamma(\alpha-1)} + \sum_{k=1}^{m}\bigl(M_0\delta_k^1 + M\delta_k^2\bigr)\Bigr]\Biggr\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \lambda_2 = \max\Biggl\{&\Bigl[M + \frac{2M^2M_1M_3}{\Gamma(\alpha)}\bigl\|K_w(s)\bigr\|_{L^{\frac{1}{q_1}}}\Bigr] \\ &\times\Bigl[\sum_{k=1}^{m} M_k^1 + b\sum_{k=1}^{m} M_k^2 + \frac{2M_4}{\Gamma(\alpha)}\|J_f\|_{L^{\frac{1}{q_3}}}\bigl(1+K^*\bigr)\Bigr], \\ &\Bigl[1 + \frac{2MM_1M_3}{\Gamma(\alpha)}\bigl\|K_w(s)\bigr\|_{L^{\frac{1}{q_1}}}\Bigr] \\ &\times\Bigl[M_0\sum_{k=1}^{m} M_k^1 + M\sum_{k=1}^{m} M_k^2 + \frac{2MM_7}{\Gamma(\alpha-1)}\|J_f\|_{L^{\frac{1}{q_3}}}\bigl(1+K^*\bigr)\Bigr]\Biggr\}. \end{aligned} \tag{3.3}$$

Proof Consider the operator $Q : \mathrm{PC}^1(J,E) \to \mathrm{PC}^1(J,E)$ defined by

$$\begin{aligned} (Qu)(t) = {} & S_{\alpha,1}(t)\bigl(u_0 - g(u)\bigr) + S_{\alpha,2}(t)\bigl(u_1 - h(u)\bigr) \\ & + \sum_{0<t_k<t} S_{\alpha,1}(t-t_k)I_k\bigl(u(t_k)\bigr) + \sum_{0<t_k<t} S_{\alpha,2}(t-t_k)J_k\bigl(u'(t_k)\bigr) \end{aligned}$$

$$+\int_0^t S_{\alpha,\alpha}(t-s)\big[Bv(s)+f\big(s,u(s),Gu(s),Fu(s)\big)\big]\,ds,\quad t\in J. \tag{3.4}$$

Furthermore, by Lemma 2.3, we get $(Qu)':\mathrm{PC}(J,E)\to\mathrm{PC}(J,E)$ such that

$$\begin{aligned}(Qu)'(t) &= AS_{\alpha,\alpha}(t)\big(u_0-g(u)\big)+S_{\alpha,1}(t)\big(u_1-h(u)\big)\\ &\quad+\sum_{0<t_k<t}AS_{\alpha,\alpha}(t-t_k)I_k\big(u(t_k)\big)+\sum_{0<t_k<t}S_{\alpha,1}(t-t_k)J_k\big(u'(t_k)\big)\\ &\quad+\int_0^t S_{\alpha,\alpha-1}(t-s)\big[Bv(s)+f\big(s,u(s),Gu(s),Fu(s)\big)\big]\,ds,\quad t\in J,\end{aligned} \tag{3.5}$$

where the control v is defined by

$$v(t)=\begin{cases}v_1^u(t), & u\in \mathrm{PC}^1(J,E),\\ v_2^{u'}(t), & u'\in \mathrm{PC}(J,E),\end{cases} \tag{3.6}$$

and $v_1^u(t)$ and $v_2^{u'}(t)$ are given by

$$\begin{aligned}v_1^u(t) &= W^{-1}\bigg[u_b-S_{\alpha,1}(b)\big(u_0-g(u)\big)-S_{\alpha,2}(b)\big(u_1-h(u)\big)\sum_{0<t_k<b}S_{\alpha,1}(b-t_k)I_k\big(u(t_k)\big)\\ &\quad-\sum_{0<t_k<b}S_{\alpha,2}(b-t_k)J_k\big(u'(t_k)\big)-\int_0^b S_{\alpha,\alpha}(b-s))f\big(s,u(s),Gu(s),Fu(s)\big)\,ds\bigg](t),\end{aligned} \tag{3.7}$$

$$\begin{aligned}v_2^{u'}(t)) &= W^{-1}\bigg[u_b'-AS_{\alpha,\alpha}(b)\big(u_0-g(u)\big)-S_{\alpha,1}(b)\big(u_1-h(u)\big)\\ &\quad-\sum_{0<t_k<b}AS_{\alpha,\alpha}(b-t_k)I_k\big(u(t_k)\big)-\sum_{0<t_k<b}S_{\alpha,1}(b-t_k)J_k\big(u'(t_k)\big)\\ &\quad-\int_0^b S_{\alpha,\alpha-1}(b-s))f\big(s,u(s),Gu(s),Fu(s)\big)\,ds\bigg](t).\end{aligned} \tag{3.8}$$

Taking the control (3.7) and (3.8) in (3.4) and (3.5), respectively, we obtain $(Qu)(b)=u_b$ and $(Qu)'(b)=u_b'$, which means that the control $v(t)$ steers the system (1.1) from the initial conditions u_0 and u_1 to the final states u_b and u_b' in the time b, provided we can obtain a fixed point of the nonlinear operator Q.

Now, the objective is to prove that the operator Q has a fixed point. The proof will be carried out in three steps.

Step 1: $\exists r_0>0; Q(B_{r_0})\subset B_{r_0}$.

For this step, it will be carried out by contradiction. Suppose this is not true. Then, for each $r_0>0$, there exists $u_{r_0}(\cdot)\in B_{r_0}$ and, for some $t\in J$ such that $\|(Qu_{r_0})(t)\|>r_0$, we have

$$\begin{aligned}\big\|(Qu)(t)\big\| &= \big\|S_{\alpha,1}(t)\big(u_0-g(u)\big)\big\|+\big\|S_{\alpha,2}(t)\big(u_1-h(u)\big)\big\|+\sum_{k=1}^m\big\|S_{\alpha,1}(t-t_k)\big\|_{\mathcal{L}}\big\|I_k\big(u(t_k)\big)\big\|\\ &\quad+\sum_{k=1}^m\big\|S_{\alpha,2}(t-t_k)\big\|_{\mathcal{L}}\big\|J_k\big(u'(t_k)\big)\big\|+\int_0^t\big\|S_{\alpha,\alpha}(t-s)\big\|_{\mathcal{L}}\big\|Bv_1^u(s)\big\|\,ds\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \left\|S_{\alpha,\alpha}(t-s)\right\|_{\mathcal{L}} \left\|f\bigl(s,u(s),Gu(s),Fu(s)\bigr)\right\| ds \\
&\leq M\bigl(\|u_0\|+c_1\bigr)+Mb\bigl(\|u_1\|+c_3\bigr)+\sum_{k=1}^{m} ML_k^1\bigl(\|u\|_{\mathrm{PC}^1}\bigr)+\sum_{k=1}^{m} MbL_k^2\bigl(\|u'\|_{\mathrm{PC}^1}\bigr) \\
&+\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}M_1\int_0^t \left\|v_1^u(s)\right\| ds+\frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m_{r_0}(s)\, ds \\
&\leq M\bigl(\|u_0\|+c_1\bigr)+Mb\bigl(\|u_1\|+c_3\bigr)+\sum_{k=1}^{m} M\bigl[L_k^1\bigl(\|u\|_{\mathrm{PC}^1}\bigr)+bL_k^2\bigl(\|u'\|_{\mathrm{PC}^1}\bigr)\bigr] \\
&+\frac{MM_1 b^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\left\|v_1^u\right\|_{L^2}+\frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m_{r_0}(s)\, ds,
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{aligned}
\left\|v_1^u\right\|_{L^2} \leq M_2\Biggl[&\|u_b\|+M\bigl(\|u_0\|+c_1\bigr)+Mb\bigl(\|u_1\|+c_3\bigr)+\sum_{k=1}^{m} M\bigl[L_k^1\bigl(\|u\|_{\mathrm{PC}^1}\bigr) \\
&+bL_k^2\bigl(\|u'\|_{\mathrm{PC}^1}\bigr)\bigr]+\frac{M}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} m_{r_0}(s)\, ds\Biggr].
\end{aligned}
\tag{3.10}
$$

Now, from (3.9) and (3.10), we have

$$
\begin{aligned}
\left\|(Qu)(t)\right\|_{\mathrm{PC}} \leq \left[1+\frac{MM_1M_2 b^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\right]\Biggl[&M\bigl(\|u_0\|+c_1\bigr)+Mb\bigl(\|u_1\|+c_3\bigr) \\
&+\sum_{k=1}^{m} M\bigl[L_k^1\bigl(\|u\|_{\mathrm{PC}^1}\bigr)+bL_k^2\bigl(\|u'\|_{\mathrm{PC}^1}\bigr)\bigr]+\frac{M}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} m_{r_0}(s)\, ds\Biggr] \\
&+\frac{MM_1M_2 b^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\|u_b\|.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\left\|(Qu)'(t)\right\|_{\mathrm{PC}} \leq \left[1+\frac{MM_1M_2 b^{\alpha-\frac{3}{2}}}{\Gamma(\alpha-1)}\right]\Biggl[&M_0\bigl(\|u_0\|+c_1\bigr)+M\bigl(\|u_1\|+c_3\bigr) \\
&+\sum_{k=1}^{m}\bigl[M_0L_k^1\bigl(\|u\|_{\mathrm{PC}^1}\bigr)+ML_k^2\bigl(\|u'\|_{\mathrm{PC}^1}\bigr)\bigr] \\
&+\frac{M}{\Gamma(\alpha-1)}\int_0^b (b-s)^{\alpha-2} m_{r_0}(s)\, ds\Biggr]+\frac{MM_1M_2 b^{\alpha-\frac{3}{2}}}{\Gamma(\alpha-1)}\left\|u_b'\right\|.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
r_0 &\leq \left\|(Qu)\right\|_{\mathrm{PC}^1} \\
&=\max\Biggl\{\left[1+\frac{MM_1M_2 b^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\right]\Biggl[M\bigl(\|u_0\|+c_1\bigr)+Mb\bigl(\|u_1\|+c_3\bigr)+\sum_{k=1}^{m} M\bigl[L_k^1\bigl(\|u\|_{\mathrm{PC}^1}\bigr) \\
&+bL_k^2\bigl(\|u'\|_{\mathrm{PC}^1}\bigr)\bigr]+\frac{M}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} m_{r_0}(s)\, ds\Biggr]+\frac{MM_1M_2 b^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\|u_b\|,
\end{aligned}
$$

$$
\begin{aligned}
&\left[1+\frac{MM_1M_2b^{\alpha-\frac{3}{2}}}{\Gamma(\alpha-1)}\right]\\
&\times\left[M_0\bigl(\|u_0\|+c_1\bigr)+M\bigl(\|u_1\|+c_3\bigr)+\sum_{k=1}^{m}\bigl[M_0L_k^1\bigl(\|u\|_{\mathrm{PC}^1}\bigr)+ML_k^2\bigl(\|u'\|_{\mathrm{PC}^1}\bigr)\bigr]\right.\\
&\left.+\frac{M}{\Gamma(\alpha-1)}\int_0^b(b-s)^{\alpha-2}m_{r_0}(s)\,ds\right]+\frac{MM_1M_2b^{\alpha-\frac{3}{2}}}{\Gamma(\alpha-1)}\|u_b\|\biggr\}. \qquad (3.11)
\end{aligned}
$$

Dividing both sides of (3.12) by r_0, and taking $r_0\to\infty$, we get

$$
\begin{aligned}
1\le \max&\left\{\left[1+\frac{MM_1M_2b^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\right]\left[\frac{MM_5}{\Gamma(\alpha)}+\sum_{k=1}^{m}M\bigl(\delta_k^1+\delta_k^2\bigr)\right],\right.\\
&\left.\left[1+\frac{MM_1M_2b^{\alpha-\frac{3}{2}}}{\Gamma(\alpha-1)}\right]\left[\frac{MM_6}{\Gamma(\alpha-1)}+\sum_{k=1}^{m}\bigl(M_0\delta_k^1+M\delta_k^2\bigr)\right]\right\}. \qquad (3.12)
\end{aligned}
$$

This contradicts (3.2). Therefore for some $r_0>0$, $Q(B_{r_0})\subset B_{r_0}$, which means that $Q(B_{r_0})\subset B_{r_0}$.

Step 2: We show that Q is continuous on B_{r_0}. To show this, let $\{u_n\}_{n=1}^{\infty},\{u_n'\}_{n=1}^{\infty}\subset B_{r_0}$ be a sequence such that $u_n\to u$, $u_n'\to u'$ in B_{r_0}. Then there exists a number $r_0>0$ such that $\|u^n\|_{\mathrm{PC}}\le r_0$, $\|u'^n\|_{\mathrm{PC}}\le r_0$ and $\|u\|_{\mathrm{PC}}\le r_0$ and $\|u'\|_{\mathrm{PC}}\le r_0$ for all $n\ge 1$, and we define

$$
\widetilde{F}_n(s)=f\bigl(s,u^n(s),G\bigl(u^n(s)\bigr),F\bigl(u^n(s)\bigr)\bigr)\quad\text{and}\quad \widetilde{F}(s)=f\bigl(s,u(s),G\bigl(u(s)\bigr),F\bigl(u(s)\bigr)\bigr).
$$

Then we obtain

$$
\begin{aligned}
&\bigl\|\bigl(Qu^n\bigr)(t)-(Qu)(t)\bigr\|\\
&\quad\le\bigl\|S_{\alpha,1}(t)\bigr\|_{\mathcal{L}}\bigl\|\bigl(g(u^n)-g(u)\bigr)\bigr\|+\bigl\|S_{\alpha,2}(t)\bigr\|_{\mathcal{L}}\bigl\|\bigl(h(u^n)-h(u)\bigr)\bigr\|\\
&\qquad+\sum_{k=1}^{m}\bigl\|S_{\alpha,1}(t-t_k)\bigr\|_{\mathcal{L}}\bigl\|I_k\bigl(u^n(t_k)\bigr)-I_k\bigl(u(t_k)\bigr)\bigr\|\\
&\qquad+\sum_{k=1}^{m}\bigl\|S_{\alpha,2}(t-t_k)\bigr\|_{\mathcal{L}}\bigl\|J_k\bigl(u'^n(t_k)\bigr)-J_k\bigl(u'(t_k)\bigr)\bigr\|\\
&\qquad+\int_0^t\bigl\|S_{\alpha,\alpha}(t-s)\bigr\|_{\mathcal{L}}\bigl\|Bv_1^{u^n}(s)-Bv_1^{u}(s)\bigr\|\,ds+\int_0^t\bigl\|S_{\alpha,\alpha}(t-s)\bigr\|_{\mathcal{L}}\bigl\|\widetilde{F}_n(s)-\widetilde{F}(s)\bigr\|\,ds\\
&\quad\le Mc_2\max_{t\in J}\bigl\|u^n-u\bigr\|_{\mathrm{PC}}+Mbc_4\max_{t\in J}\bigl\|u^n-u\bigr\|_{\mathrm{PC}}\\
&\qquad+\sum_{k=1}^{m}M\bigl\|I_k\bigl(u^n(t_k)\bigr)-I_k\bigl(u(t_k)\bigr)\bigr\|+\sum_{k=1}^{m}Mb\bigl\|J_k\bigl(u'^n(t_k)\bigr)-J_k\bigl(u'(t_k)\bigr)\bigr\|\\
&\qquad+\frac{MM_1b^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\bigl\|v_1^{u^n}(s)-v_1^{u}(s)\bigr\|_{L^2}+\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^t\bigl\|\widetilde{F}_n(s)-\widetilde{F}(s)\bigr\|\,ds, \qquad (3.13)
\end{aligned}
$$

where

$$\begin{aligned}
&\left\| v_1^{u^n}(s) - v_1^u(s) \right\|_{L^2} \\
&= M_2 \Bigg[Mc_2 \max_{t\in J} \left\| u^n - u \right\|_{\mathrm{PC}} + Mbc_4 \max_{t\in J} \left\| u^n - u \right\|_{\mathrm{PC}} + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \left\| \widetilde{F}_n(s) - \widetilde{F}(s) \right\| ds \\
&\quad + \sum_{k=1}^{m} M \left\| I_k\big(u^n(t_k)\big) - I_k\big(u(t_k)\big) \right\| + \sum_{k=1}^{m} Mb \left\| J_k\big(u'^n(t_k)\big) - J_k\big(u'(t_k)\big) \right\| \Bigg]. \qquad (3.14)
\end{aligned}$$

By continuity of f, I_k, J_k, and the Lebesgue dominated convergence theorem combined with (3.13), (3.14), we get $\|Qu^n - Qu\|_{\mathrm{PC}} \to 0$, as $n \to \infty$.

Then, in a similar manner to above, we get

$$\begin{aligned}
&\left\| \big(Qu^n\big)'(t) - (Qu)'(t) \right\| \\
&\le \left\| AS_{\alpha,\alpha}(t) \right\|_{\mathcal{L}} \cdot \left\| (g(u^n) - g(u)) \right\| + \left\| AS_{\alpha,1}(t) \right\|_{\mathcal{L}} \cdot \left\| (h(u^n) - h(u)) \right\| \\
&\quad + \sum_{k=1}^{m} \left\| AS_{\alpha,\alpha}(t-t_k) \right\|_{\mathcal{L}} \left\| I_k\big(u^n(t_k)\big) - I_k\big(u(t_k)\big) \right\| \\
&\quad + \sum_{k=1}^{m} \left\| S_{\alpha,1}(t-t_k) \right\|_{\mathcal{L}} \left\| J_k\big(u'^n(t_k)\big) - J_k\big(u'(t_k)\big) \right\| \\
&\quad + \int_0^t \left\| S_{\alpha,\alpha-1}(t-s) \right\|_{\mathcal{L}} \left\| Bv_1^{u'^n}(s) - Bv_1^{u'(s)} \right\| ds \\
&\quad + \int_0^t \left\| S_{\alpha,\alpha-1}(t-s) \right\|_{\mathcal{L}} \left\| \widetilde{F}_n(s) - \widetilde{F}(s) \right\| ds \\
&\le M_0 c_2 \max_{t\in J} \left\| u^n - u \right\|_{\mathrm{PC}} + Mc_4 \max_{t\in J} \left\| u^n - u \right\|_{\mathrm{PC}} \\
&\quad + \sum_{k=1}^{m} M_0 \left\| I_k\big(u^n(t_k)\big) - I_k\big(u(t_k)\big) \right\| + \sum_{k=1}^{m} M \left\| J_k\big(u'^n(t_k)\big) - J_k\big(u'(t_k)\big) \right\| \\
&\quad + \frac{MM_1 b^{\alpha-\frac{3}{2}}}{\Gamma(\alpha-1)} \left\| v_1^{u'^n}(s) - v_1^{u'(s)} \right\|_{L^2} + \frac{Mb^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^t \left\| \widetilde{F}_n(s) - \widetilde{F}(s) \right\| ds, \qquad (3.15)
\end{aligned}$$

where

$$\begin{aligned}
&\left\| v_2^{u'^n}(s) - v_2^{u'(s)} \right\|_{L^2} \\
&= M_2 \Bigg[M_0 c_2 \max_{t\in J} \left\| u^n - u \right\|_{\mathrm{PC}} + Mc_4 \max_{t\in J} \left\| u^n - u \right\|_{\mathrm{PC}} \\
&\quad + \frac{Mb^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^t \left\| \widetilde{F}_n(s) - \widetilde{F}(s) \right\| ds \\
&\quad + \sum_{k=1}^{m} M_0 \left\| I_k\big(u^n(t_k)\big) - I_k\big(u(t_k)\big) \right\| \\
&\quad + \sum_{k=1}^{m} M \left\| J_k\big(u'^n(t_k)\big) - J_k\big(u'(t_k)\big) \right\| \Bigg]. \qquad (3.16)
\end{aligned}$$

By continuity of f, I_k, J_k, and the Lebesgue dominated convergence theorem combined with (3.15), (3.16), we get $\|(Qu^n)' - (Qu)'\|_{\mathrm{PC}} \to 0$, as $n \to \infty$.

Hence, we have

$$\|Qu^n - Qu\|_{\mathrm{PC}^1} \to 0, \quad \text{as } n \to \infty.$$

Step 3: We will prove that Q satisfies Mönch's condition. To this end, let us assume that D and D' are countable subsets of B_{r_0} and $D \subset \overline{\mathrm{conv}}(\{0\} \cup Q(D))$ and $D' \subset \overline{\mathrm{conv}}(\{0\} \cup (Q(D))')$. Then we show that $\alpha_{\mathrm{PC}^1}(D) = 0$.

First, without loss of generality, we consider that $D = \{u^n\}_{n=1}^{\infty}$ and $D' = \{u'^n\}_{n=1}^{\infty}$. If we are able to show that $\{(Qz^n)'\}_{n=1}^{\infty}$ is equicontinuous on J_k, $k = 0, 1, 2, \dots, m$, then $D \subset \overline{\mathrm{conv}}(\{0\} \cup Q(D))$ and $D' \subset \overline{\mathrm{conv}}(\{0\} \cup (Q(D))')$ are also equicontinuous on J_k, $k = 0, 1, 2, \dots, m$. For this fact, let $l_1, l_2 \in J_p$ be such that $t_p \le l_1 \le l_2 \le t_{p+1}$ for some $p \in \{0, 1, 2, \dots, m\}$, and we get

$$\begin{aligned}
\|(Qu^n)'(l_2) - (Qu^n)'(l_1)\| &\le \|AS_{\alpha,\alpha}(l_2) - AS_{\alpha,\alpha}(l_1)\|_{\mathcal{L}} \|(u_0 + g(u))\| \\
&\quad + \|S_{\alpha,1}(l_2) - S_{\alpha,1}(l_1)\|_{\mathcal{L}} \|(u_1 + h(u))\| \\
&\quad + \sum_{k=1}^{p} \|AS_{\alpha,\alpha}(l_2 - t_k) - AS_{\alpha,\alpha}(l_1 - t_k)\|_{\mathcal{L}} \|I_k(u^n(t_k))\| \\
&\quad + \sum_{k=1}^{p} \|S_{\alpha,1}(l_2 - t_k) - S_{\alpha,1}(l_1 - t_k)\|_{\mathcal{L}} \|J_k(u'^n(t_k))\| \\
&\quad + \int_0^{l_1} \|S_{\alpha,\alpha-1}(l_2 - s) - S_{\alpha,\alpha-1}(l_1 - s)\|_{\mathcal{L}} \big[Bv_2^{u'^n}(s) + \widetilde{F}_n(s)\big]\,ds \\
&\quad + \int_{l_1}^{l_2} \|S_{\alpha,\alpha-1}(l_1 - s)\|_{\mathcal{L}} \big[Bv_2^{u'^n}(s) + \widetilde{F}_n(s)\big]\,ds.
\end{aligned}$$

By equicontinuity of $S_{\alpha,\beta}(t)$ and absolute continuity of the Lebesgue integral, we conclude that the right side of the above inequality tends to zero as $l_2 \to l_1$ independently of u. Thus, $Q(D)$ shows equicontinuity on J_k for all $k = 0, 1, 2, \dots, m$.

Now, by Lemma 2.9 and (H1)(iii), (H2)(iii) and (H3)(ii), we have

$$\begin{aligned}
&\alpha\big(\{v_1^{u^n}(\xi)\}_{n=1}^{\infty}\big) \\
&\quad \le K_w(\xi)\Bigg[\alpha\Bigg(\Big\{S_{\alpha,1}(b)g(u^n)\Big\}_{n=1}^{\infty}\Bigg) + \alpha\Bigg(\Big\{S_{\alpha,2}(b)h(u^n)\Big\}_{n=1}^{\infty}\Bigg) \\
&\qquad + \alpha\Bigg(\Bigg\{\sum_{k=1}^{m} S_{\alpha,1}(b - t_k)I_k(u^n(t_k))\Bigg\}_{n=1}^{\infty}\Bigg) \\
&\qquad + \alpha\Bigg(\Bigg\{\sum_{k=1}^{m} S_{\alpha,2}(b - t_k)J_k(u'^n(t_k))\Bigg\}_{n=1}^{\infty}\Bigg) \\
&\qquad + \alpha\Bigg(\Bigg\{\int_0^b S_{\alpha,\alpha}(b - s)f\big(s, u^n(s), (Gu^n(s)), (Fu^n(s))\big)\,ds\Bigg\}_{n=1}^{\infty}\Bigg)\Bigg] \\
&\quad \le K_w(\xi)\Bigg[Ml_1 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) + Mbl_2 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big)
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{k=1}^{m} MM_k^1 \sup_{t_k\in J} \alpha\big(\{u^n(t_k)\}_{n=1}^{\infty}\big) + \sum_{k=1}^{m} MbM_k^2 \sup_{t_k\in J} \alpha\big(\{u'^n(t_k)\}_{n=1}^{\infty}\big) \\
&+ \frac{2M}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} J_f(s) \Big(\sup_{s\in J} \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big) + K^* \sup_{s\in J} \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\Big)\,ds \\
&+ H^* \sup_{s\in J} \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\Big)\,ds\Big] \\
&\le K_w(\xi) M\Big[l_1 \sup_{t\in J} \alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) + bl_2 \sup_{t\in J} \alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) \\
&+ \sum_{k=1}^{m} M_k^1 \sup_{t_k\in J} \alpha\big(\{u^n(t_k)\}_{n=1}^{\infty}\big) + b\sum_{k=1}^{m} M_k^2 \sup_{t_k\in J} \alpha\big(\{u'^n(t_k)\}_{n=1}^{\infty}\big) \\
&+ \frac{2}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} J_f(s) \big(1+K^*+H^*\big) \sup_{s\in J} \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\,ds\Big].
\end{aligned} \tag{3.17}$$

Similarly, we obtain

$$\begin{aligned}
&\alpha\big(\{v_2^{u'^n}(\xi)\}_{n=1}^{\infty}\big) \\
&\le K_w(\xi)\Bigg[\alpha\Bigg(\Big\{AS_{\alpha,\alpha}(b)g(u^n)\Big\}_{n=1}^{\infty} + \alpha\Bigg(\Big\{S_{\alpha,1}(b)h(u^n)\Big\}_{n=1}^{\infty}\Bigg) \\
&+ \alpha\Bigg(\Bigg\{\sum_{k=1}^{m} AS_{\alpha,\alpha}(b-t_k) I_k\big(u^n(t_k)\big)\Bigg\}_{n=1}^{\infty}\Bigg) \\
&+ \alpha\Bigg(\Bigg\{\sum_{k=1}^{m} S_{\alpha,1}(b-t_k) J_k\big(u'^n(t_k)\big)\Bigg\}_{n=1}^{\infty}\Bigg) \\
&+ \alpha\Bigg(\Bigg\{\int_0^b S_{\alpha,\alpha-1}(b-s) f\big(s,u^n(s),\big(Gu^n(s)\big),\big(Fu^n(s)\big)\big)\,ds\Bigg\}_{n=1}^{\infty}\Bigg)\Bigg] \\
&\le K_w(\xi)\Big[M_0 l_1 \sup_{t\in J} \alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) + Ml_2 \sup_{t\in J} \alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) \\
&+ \sum_{k=1}^{m} M_0 M_k^1 \sup_{t_k\in J} \alpha\big(\{u^n(t_k)\}_{n=1}^{\infty}\big) + \sum_{k=1}^{m} MM_k^2 \sup_{t_k\in J} \alpha\big(\{u'^n(t_k)\}_{n=1}^{\infty}\big) \\
&+ \frac{2M}{\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} J_f(s) \Big(\sup_{s\in J} \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big) + K^* \sup_{s\in J} \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big) \\
&+ H^* \sup_{s\in J} \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\Big)\,ds\Big] \\
&\le K_w(\xi)\Big[M_0 l_1 \sup_{t\in J} \alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) + Ml_2 \sup_{t\in J} \alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) \\
&+ M_0 \sum_{k=1}^{m} M_k^1 \sup_{t_k\in J} \alpha\big(\{u^n(t_k)\}_{n=1}^{\infty}\big) + M\sum_{k=1}^{m} M_k^2 \sup_{t_k\in J} \alpha\big(\{u'^n(t_k)\}_{n=1}^{\infty}\big) \\
&+ \frac{2M\big(1+K^*+H^*\big)}{\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} J_f(s) \sup_{s\in J} \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\,ds\Big].
\end{aligned} \tag{3.18}$$

Furthermore, by Lemma 2.9, we get

$$\begin{aligned}
\alpha\big((Qu^n)(t)\big) &\le \alpha\left(\left\{S_{\alpha,1}(t)g(u^n)\right\}_{n=1}^{\infty}\right) + \alpha\left(\left\{S_{\alpha,2}(t)h(u^n)\right\}_{n=1}^{\infty}\right) \\
&\quad + \alpha\left(\left\{\sum_{0<t_k<t} S_{\alpha,1}(t-t_k)I_k\big(u^n(t_k)\big)\right\}_{n=1}^{\infty}\right) \\
&\quad + \alpha\left(\left\{\sum_{0<t_k<t} S_{\alpha,2}(t-t_k)J_k\big(u'(t_k)\big)\right\}_{n=1}^{\infty}\right) \\
&\quad + \alpha\left(\left\{\int_0^t S_{\alpha,\alpha}(t-s)Bv_1^{u^n}(s)\,ds\right\}_{n=1}^{\infty}\right) \\
&\quad + \alpha\left(\left\{\int_0^t S_{\alpha,\alpha}(t-s)f\big(s,u^n(s),\big(Gu^n(s)\big),\big(Fu^n(s)\big)\big)\,ds\right\}_{n=1}^{\infty}\right) \\
&\le Ml_1 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) + Mbl_2 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) \\
&\quad + \sum_{k=1}^{m} M_k^1 \sup_{t_k\in J}\alpha\big(\{u^n(t_k)\}_{n=1}^{\infty}\big) \\
&\quad + Mb\sum_{k=1}^{m} M_k^2 \sup_{t_k\in J}\alpha\big(\{u'^n(t_k)\}_{n=1}^{\infty}\big) + \frac{2MM_1b^{\alpha-1}}{\Gamma(\alpha)}\int_0^b \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\,ds \\
&\quad + \frac{2(1+K^*+H^*)}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}J_f(s)\sup_{s\in J}\alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\,ds.
\end{aligned} \tag{3.19}$$

Similarly, by Lemma 2.9, we have

$$\begin{aligned}
\alpha\big((Qu^n)'(t)\big) &\le M_0l_1 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) + Ml_2 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) \\
&\quad + \frac{2MM_1b^{\alpha-1}}{\Gamma(\alpha)}\int_0^b \alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\,ds \\
&\quad + \frac{2(1+K^*+H^*)}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}J_f(s)\sup_{s\in J}\alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\,ds \\
&\quad + M_0\sum_{k=1}^{m} M_k^1 \sup_{t_k\in J}\alpha\big(\{u^n(t_k)\}_{n=1}^{\infty}\big) \\
&\quad + M\sum_{k=1}^{m} M_k^2 \sup_{t_k\in J}\alpha\big(\{u'^n(t_k)\}_{n=1}^{\infty}\big).
\end{aligned} \tag{3.20}$$

By (3.17) and (3.19), we obtain

$$\begin{aligned}
\alpha\big((Qu^n)(t)\big) &\le Ml_1 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) + Mbl_2 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) \\
&\quad + M\sum_{k=1}^{m} M_k^1 \sup_{t_k\in J}\alpha\big(\{u^n(t_k)\}_{n=1}^{\infty}\big) + Mb\sum_{k=1}^{m} M_k^2 \sup_{t_k\in J}\alpha\big(\{u'^n(t_k)\}_{n=1}^{\infty}\big) \\
&\quad + \frac{2MM_1}{\Gamma(\alpha)}\left(\int_0^b (b-s)^{\alpha-1}K_w(s)\,ds\right)
\end{aligned}$$

$$
\begin{aligned}
&\times M\Bigg[l_1 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) + bl_2 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) \\
&\quad + \sum_{k=1}^{m} M_k^1 \sup_{t_k\in J}\alpha\big(\{u^n(t_k)\}_{n=1}^{\infty}\big) + b\sum_{k=1}^{m} M_k^2 \sup_{t_k\in J}\alpha\big(\{u'^n(t_k)\}_{n=1}^{\infty}\big) \\
&\quad + \frac{2(1+K^*+H^*)}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} J_f(s)\sup_{s\in J}\alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\,ds\Bigg] \\
&\quad + \frac{2(1+K^*+H^*)}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} J_f(s)\sup_{s\in J}\alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\,ds \\
&\le \bigg[M + \frac{2M^2M_1M_3}{\Gamma(\alpha)}\big\|K_w(s)\big\|_{L^{\frac{1}{q_1}}}\bigg] \\
&\quad \times \Bigg[l_1 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) + bl_2 \sup_{t\in J}\alpha\big(\{u^n(t)\}_{n=1}^{\infty}\big) \\
&\quad + \sum_{k=1}^{m} M_k^1 \sup_{t_k\in J}\alpha\big(\{u^n(t_k)\}_{n=1}^{\infty}\big) + b\sum_{k=1}^{m} M_k^2 \sup_{t_k\in J}\alpha\big(\{u'^n(t_k)\}_{n=1}^{\infty}\big) \\
&\quad + \frac{2(1+K^*+H^*)}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} J_f(s)\sup_{s\in J}\alpha\big(\{u^n(s)\}_{n=1}^{\infty}\big)\,ds\Bigg] \\
&\le \bigg[M + \frac{2M^2M_1M_3}{\Gamma(\alpha)}\big\|K_w(s)\big\|_{L^{\frac{1}{q_1}}}\bigg] \\
&\quad \times \Bigg[l_1 + bl_2 + \sum_{k=1}^{m} M_k^1 + b\sum_{k=1}^{m} M_k^2 + \frac{2M_4(1+K^*+H^*)}{\Gamma(\alpha)}\|J_f\|_{L^{\frac{1}{q_3}}}\Bigg]\alpha_{PC^1}(D).
\end{aligned}
\tag{3.21}
$$

Similarly, by (3.18) and (3.20), we get

$$
\begin{aligned}
\alpha\big((Qu^n)'(t)\big) &\le \bigg[1 + \frac{2MM_1M_3}{\Gamma(\alpha)}\big\|K_w(s)\big\|_{L^{\frac{1}{q_1}}}\bigg]\Bigg[M_0l_1 + Ml_2 + M_0\sum_{k=1}^{m} M_k^1 + M\sum_{k=1}^{m} M_k^2 \\
&\quad + \frac{2MM_7}{\Gamma(\alpha-1)}\|J_f\|_{L^{\frac{1}{q_3}}}\big(1+K^*+H^*\big)\Bigg]\alpha_{PC^1}(D).
\end{aligned}
\tag{3.22}
$$

Now, by Lemma 2.10, we have

$$
\begin{aligned}
\alpha_{PC^1}(QD) &= \max\Big\{\sup_{t\in J}\alpha\big((Qu^n)(t)\big), \sup_{t\in J}\alpha\big((Qu^n)'(t)\big)\Big\} \\
&\le \max\Bigg\{\bigg[M + \frac{2M^2M_1M_3}{\Gamma(\alpha)}\big\|K_w(s)\big\|_{L^{\frac{1}{q_1}}}\bigg]\Bigg[l_1 + bl_2 + \sum_{k=1}^{m} M_k^1 + b\sum_{k=1}^{m} M_k^2 \\
&\quad + \frac{2M_3}{\Gamma(\alpha)}\|J_f\|_{L^{\frac{1}{q_1}}}\big(1+K^*+H^*\big)\Bigg], \\
&\quad \bigg[1 + \frac{2MM_1b^{\alpha-1}}{\Gamma(\alpha)}\big\|K_w(s)\big\|_{L^1}\bigg]\Bigg[M_0l_1 + Ml_2 + M_0\sum_{k=1}^{m} M_k^1 + M\sum_{k=1}^{m} M_k^2 \\
&\quad + \frac{2MM_7}{\Gamma(\alpha-1)}\|J_f\|_{L^{\frac{1}{q_3}}}\big(1+K^*+H^*\big)\Bigg]\Bigg\}\alpha_{PC^1}(D).
\end{aligned}
$$

This implies that $\alpha_{\mathrm{PC}^1}(QD) \le \lambda_2 \alpha_{\mathrm{PC}^1}(D)$. Therefore, we get

$$\alpha_{\mathrm{PC}^1}(D) \le \alpha_{\mathrm{PC}^1}\big(\overline{\mathrm{conv}}\big(\{0\} \cup Q(D)\big)\big) = \alpha_{\mathrm{PC}^1}(QD) \le \lambda_2 \alpha_{\mathrm{PC}^1}(D).$$

Since $\lambda_2 < 1$, we obtain $\alpha_{\mathrm{PC}^1}(D) = 0$. That is, D is relatively compact. Hence by Lemma 2.11, Q has at least one fixed point $u \in B_{r_0}$, which is a mild solution of the system (1.1) and it satisfies $u(b) = u_b$ and $u'(b) = u'_b$. Therefore, the system (1.1) is controllable on J. This completes the proof. □

4 Application

In this section, we give an example to demonstrate the feasibility of our results.

Example 4.1 We consider the impulsive fractional parabolic partial differential equation

$$\begin{cases} \frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}} u(x,t) = \frac{\partial^2}{\partial x^2} u(t,x) + Bv(t,x) + \frac{e^{-2t}}{1+e^t}[u(t,y) \\ \qquad + \int_0^t \sin(t-s)u(s,x)\,ds + \int_0^1 \cos(t-s)u(s,x)\,ds], \quad x \in [0,\pi], t \in J, \\ u(t,0) = u(t,\pi) = 0, \quad t \in J, \\ \Delta u(t,x)|_{t=t_k} = \int_0^{t_k} a_k(t_k - s)u(s,x)\,ds, \quad k = 1,2,\ldots,m, \\ \Delta \frac{\partial u(t,x)}{\partial t}|_{t=t_k} = \int_0^{t_k} \widetilde{a}_k(t_k - s)\frac{\partial u(s,x)}{\partial s}\,ds, \quad k = 1,2,\ldots,m, \\ u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = u_1(x), \end{cases} \tag{4.1}$$

where $\frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}}$ is the Caputo fractional partial derivative of order $1 < \alpha < 2$, $J = [0,1]$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, $\Delta u(t,x)|_{t=t_k} = u(t^+,x) - u(t^-,x)$, $\Delta \frac{\partial u(t,x)}{\partial t}|_{t=t_k} = \Delta \frac{\partial u(t,x)}{\partial t}|_{t=t_k^+} - \Delta \frac{\partial u(t,x)}{\partial t}|_{t=t_k^-}$. $a_k, \widetilde{a}_k \in C(\mathbb{R},\mathbb{R})$. We choose $E = U = L^2([0,\pi])$ to be endowed with the norm $\|\cdot\|_{L^2}$. The function $v: J \times [0,\pi] \to [0,\pi]$ is a control function and $B: U \to E$ is a bounded linear operator.

Define $u(t)(x) = u(t,x)$, $v(t)(x) = v(t,x)$, and ${}^cD_{0+}^{\frac{3}{2}} u(t)(x) = \frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}} u(x,t)$, and

$$\begin{aligned} &f\big(t,u(t),Gu(t),Fu(t)\big)(x) \\ &\quad = \frac{e^{-2t}}{1+e^t}\left[u(t,y) + \int_0^t \sin(t-s)u(s,x)\,ds + \int_0^1 \cos(t-s)u(s,x)\,ds\right], \\ &I_k(u_{t_k}) = \int_0^{t_k} a_k(t_k - s)u(s,x)\,ds, \quad k = 1,2,\ldots,m, \\ &J_k(u_{t_k}) = \int_0^{t_k} \widetilde{a}_k(t_k - s)\frac{\partial u(s,x)}{\partial s}\,ds, \quad k = 1,2,\ldots,m. \end{aligned}$$

We define $A: D(A) \subset E \to E$ by $Au = \frac{\partial^2}{\partial x^2} u$ with each domain $D(A)$ given by

$$D(A) := \big\{u \in E : u, u_x \text{ are absolute continuous}, u_{xx} \in E, u(t)(0) = u(t)(\pi) = 0\big\}.$$

Then the operator A is given by

$$Au = \sum_{n=1}^{\infty} -n^2(u,u_n)u_n, \quad u \in D(A),$$

where $u_n(t) = \sqrt{\frac{2}{\pi}}\sin nt$, $n = 1, 2, \ldots$, is the orthogonal set of eigenfunctions corresponding to the eigenvalues $\lambda_n = -n^2$ of A. Then A will be a generator of the (α,β)-resolvent family such that

$$S_{\alpha,\beta}(t)u = \sum_{n=1}^{\infty} \cos\frac{n^2 t}{1+n^2}(u,u_n)u_n.$$

Moreover, we have $\|S_{\alpha,\beta}(t)\|_{\mathcal{L}} \le M = 1$. Then the system (4.1) is the abstract form of the system (1.1). Obviously, f satisfies (H1)(i) and (ii). Thus, for $(t,u) \in J \times \mathrm{PC}(J,E)$, we have

$$\begin{aligned} &f\bigl(t,u(t),Gu(t),Fu(t)\bigr) \\ &\quad\le \frac{e^{-2t}}{1+e^t}\left[\int_0^{\pi}\left(u(t,y) + \int_0^t \sin(t-s)u(s,x)\,ds + \int_0^1 \cos(t-s)u(s,x)\,ds\right)^2\right]^{\frac{1}{2}} \\ &\quad\le \frac{e^{-2t}}{1+e^t}\sqrt{\pi}\bigl(\|u\|_{\mathrm{PC}} + k^0\|u\|_{\mathrm{PC}} + h^0\|u\|_{\mathrm{PC}}\bigr), \end{aligned}$$

where $k^0 = \sup_{t\in J}\int_0^t \|\sin(t-s)\|\,ds \le 1$, $h^0 = \sup_{t\in J}\int_0^1 \|\cos(t-s)\|\,ds \le 1$. Furthermore, for $D_1, D_2, D_3 \subset \mathrm{PC}(J,E)$, we have

$$\alpha\bigl(f(t,D_1,D_2,D_3)\bigr) \le J_f(t)\Bigl[\sup_{t\in J}\alpha\bigl(D_1(t)\bigr) + k^0\sup_{t\in J}\alpha\bigl(D_2(t)\bigr) + h^0\sup_{t\in J}\alpha\bigl(D_2(t)\bigr)\Bigr],$$

where $J_f(t) = \frac{e^{-2t}}{1+e^t}\sqrt{\pi}$. Similarly, we can show that the condition (H2) is satisfied with $L_k^1 = M_k^1 = (\int_0^{t_k}|a_k(t_k-s)|^2\,ds)^{\frac{1}{2}}$ and $L_k^2 = M_k^2 = (\int_0^{t_k}|\widetilde{a}_k(t_k-s)|^2\,ds)^{\frac{1}{2}}$.

When $B = I$, and using the above defined linear operator, we conclude that the operator $W : L^2(J,U) \to E$, defined as in [26, (4.6)],

$$Wu(s) = \begin{cases} \sum_{n=1}^{\infty}\int_0^1 \frac{1}{n^2}\sin[(\frac{n^2}{1+n^2})(1-s)](u(s),u_n)u_n\,ds, & u \in \mathrm{PC}^1; \\ \sum_{n=1}^{\infty}\int_0^1 \frac{1}{1+n^2}\cos[(\frac{n^2}{1+n^2})(1-s)](u(s),u_n)u_n\,ds, & u' \in \mathrm{PC}, \end{cases} \tag{4.2}$$

has a bounded inverse operator and satisfies the condition (H3). Thus the conditions (H0)–(H3) are satisfied, and, by Theorem 3.1, the system (4.1) is controllable on J.

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