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Multiple solutions for a class of nonlocal quasilinear elliptic systems in Orlicz–Sobolev spaces

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Abstract

In this paper, we study some results on the existence and multiplicity of solutions for a class of nonlocal quasilinear elliptic systems. In fact, we prove the existence of precise intervals of positive parameters such that the problem admits multiple solutions. Our approach is based on variational methods.

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1 Introduction

In this article, we are interested in establishing the existence of multiple solutions to the following Kirchhof-type systems in Orlicz–Sobolev spaces

$$\begin{cases} -M_i(\int_\Omega \Phi_i(|\nabla u_i|)\,dx)(\operatorname{div}(\alpha_i(|\nabla u_i|)\nabla u_i)) = \lambda F_{u_i}(x,u_1,\dots,u_n) & \text{in } \Omega, \\ u_i = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for $1 \le i \le n$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N \ge 3)$, with smooth boundary $\partial\Omega$ and λ is a positive parameter, $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function with respect to $x \in \Omega$ for every $(t_1,\dots,t_n) \in \mathbb{R}^n$ and is C^1 with respect to $(t_1,t_2,\dots,t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$; F_{t_i} denotes the partial derivative of F with respect to t_i. Also $M_i : \mathbb{R} \to \mathbb{R}$ $(i = 1,2,\dots,n)$, are continuous and increasing functions satisfying the following boundedness condition:

(**M**) There exist positive numbers m_i^0, M_i^0 such that

$$m_i^0 \le M_i(t) \le M_i^0, \quad \text{for all } t \ge 0 \ (i = 1,2,\dots,n).$$

Throughout this article we assume that for $i = 1,\dots,n$, the functions $\alpha_i : (0,+\infty) \to \mathbb{R}$ are such that the mappings $\varphi_i : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_i(t) = \begin{cases} \alpha_i(|t|)t & \text{for } t \ne 0, \\ 0 & \text{for } t = 0, \end{cases}$$

are odd, strictly increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. For the functions φ_i above, let us define $\Phi_i(t) = \int_0^t \varphi_i(s)\,ds$ for all $t \in \mathbb{R}$.

Notice that if $i = 1$, then problem (1.1) becomes

$$\begin{cases} -M(\int_\Omega \Phi(|\nabla u|)\,dx)(\operatorname{div}(\alpha(|\nabla u|)\nabla u)) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It should be mentioned that if $\varphi(t) = p|t|^{p-2}t$ for all $t \in \mathbb{R}$, $p > 1$ then problem (1.2) becomes the well-known p-Kirchhoff-type equation

$$\begin{cases} -M(\int_\Omega |\nabla u|^p\,dx)\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Problem (1.3) is related to the stationary problem

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, ρ_0, h, E, L are constants, for $0 < x < L$, $t \geq 0$, and where $u = u(x,t)$ is the lateral displacement at the space coordinate x and time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length, and ρ_0 the initial axial tension, proposed by Kirchhoff [17] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. This is an example of a nonlinear problem. One can refer to [3–5, 9, 13, 14, 20–23, 26–28, 30–32] for more relevant problems and techniques.

Now, we recall some basic facts about Orlicz and Orlicz–Sobolev spaces (see [2, 29] and the references therein). Let φ_i and Φ_i be as introduced at the beginning of the paper. Set

$$\Phi_i^*(t) = \int_0^t \varphi_i^{-1}(s)\,ds, \quad \text{for all } t \in \mathbb{R}.$$

We see that Φ_i, for $1 \leq i \leq n$, are Young functions, that is, $\Phi_i(0) = 0$, Φ_i are convex, and $\lim_{t\to\infty} \Phi_i(t) = +\infty$.

Also, since $\Phi_i(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi_i(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi_i(t)}{t} = +\infty,$$

then Φ_i are called N-functions. The functions Φ_i^*, for $1 \leq i \leq n$ are called the complementary functions of Φ_i and they satisfy

$$\Phi_i^*(t) = \sup\bigl\{st - \Phi_i(s); s \geq 0\bigr\}, \quad \text{for all } t \geq 0.$$

We observe that Φ_i^* are also N-functions and the following Young's inequality holds:

$$st \leq \Phi_i(s) + \Phi_i^*(t), \quad \text{for all } s, t \geq 0.$$

We define the numbers

$$(p_i)_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)}, \quad \text{and} \quad (p_i)^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}.$$

Throughout this paper, we assume the following condition:

$$N < (p_i)_0 \leq \frac{t\varphi_i(t)}{\Phi_i(t)} \leq (p_i)^0 < \infty, \quad \text{for all } t > 0. \tag{1.4}$$

The Orlicz spaces $L_{\Phi_i}(\Omega)$, for $1 \leq i \leq n$, defined by the N-functions Φ_i are the spaces of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_{\Phi_i}} := \sup\left\{\left|\int_\Omega u(x)v(x)\,dx\right| : \int_\Omega \Phi_i^*\bigl(|v(x)|\bigr)\,dx \leq 1\right\} < \infty.$$

Then $(L_{\Phi_i}(\Omega), \|\cdot\|_{L_{\Phi_i}})$ are Banach spaces whose norms are equivalent to the Luxemburg norm

$$\|u\|_{\Phi_i} := \inf\left\{k > 0; \int_\Omega \Phi_i\left(\frac{u(x)}{k}\right)dx \leq 1\right\}.$$

For Orlicz spaces, the Hölder's inequality takes the form

$$\int_\Omega uv\,dx \leq 2\|u\|_{L_{\Phi_i}}\|v\|_{L_{\Phi_i^*}} \quad \text{for all } u \in L_{\Phi_i}(\Omega) \text{ and } v \in L_{\Phi_i^*}(\Omega), 1 \leq i \leq n.$$

The Orlicz–Sobolev spaces $W^{1,\Phi_i}(\Omega)$, $1 \leq i \leq n$ are the spaces defined by

$$W^{1,\Phi_i}(\Omega) = \left\{u \in L_{\Phi_i}(\Omega), \frac{\partial u}{\partial x_j} \in L_{\Phi_i}(\Omega), j = 1, \ldots, N\right\}.$$

These are Banach spaces with respect to the norms:

$$\|u\|_{1,\Phi_i} := \|u\|_{\Phi_i} + \bigl\||\nabla u|\bigr\|_{\Phi_i}, \quad 1 \leq i \leq n.$$

Now, we introduce the Orlicz–Sobolev spaces $W_0^{1,\Phi_i}(\Omega)$, for $1 \leq i \leq n$, as the closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi_i}(\Omega)$ which can be renormed by equivalent norms:

$$\|u\|_i := \bigl\||\nabla u|\bigr\|_{\Phi_i}.$$

The relation (1.4) implies that Φ_i and Φ_i^*, for $1 \leq i \leq n$, both satisfy the Δ_2-condition [1, 12], i.e.,

$$\Phi_i(2t) \leq k\Phi_i(t) \quad \text{for all } t \geq 0,$$

where k is a positive constant. Furthermore, we assume that Φ_i satisfy in the following conditions:

$$\text{For each } x \in \bar{\Omega}, \text{ the functions } t \to \Phi_i(x, \sqrt{t}) \text{ are convex for all } t \in [0, \infty). \tag{1.5}$$

Condition Δ_2 for Φ_i assures that for each $i \in \{1, \ldots, n\}$ the Orlicz spaces $L_{\Phi_i}(\Omega)$ are separable. Also the Δ_2 condition and (1.5) assure that $L_{\Phi_i}(\Omega)$ are uniformly convex spaces, and thus reflexive Banach spaces (see [25, Proposition 2.2]), implying that Orlicz–Sobolev spaces $W_0^{1,\Phi_i}(\Omega)$, $i \in \{1, \ldots, n\}$ are reflexive Banach spaces also [16].

We define the space $X := \prod_{i=1}^{n} W_0^{1,\Phi_i}(\Omega)$ for problem (1.1) which is a reflexive Banach space with respect to the norm

$$\|u\| = \sum_{i=1}^{n} \|u_i\|_i, \quad u = (u_1, \dots, u_n) \in X.$$

Remark 1.1 In [12] we see that the Orlicz–Sobolev spaces $W_0^{1,\Phi_i}(\Omega)$, $i = 1, \dots, n$, are continuously embedded in $W_0^{1,(p_i)_0}(\Omega)$. On the other hand, since we assume that $(p_i)_0 > N$, we conclude that $W_0^{1,(p_i)_0}(\Omega)$ are compactly embedded in $C^0(\bar{\Omega})$, see [19]. Thus, we have that $W_0^{1,\Phi_i}(\Omega)$ are compactly embedded in $C^0(\bar{\Omega})$.

So, $X \hookrightarrow C^0(\bar{\Omega}) \times \cdots \times C^0(\bar{\Omega})$ is compact. We set a constant $C > 0$ such that

$$C := \max\left\{ \sup_{u_i \in W_0^{1,\Phi_i} \setminus \{0\}} \frac{\max_{x \in \bar{\Omega}} |u_i(x)|^{(p_i)^0}}{\|u_i\|_i^{(p_i)^0}} : \text{for } 1 \le i \le n \right\} < +\infty. \tag{1.6}$$

Proposition 1.1 ([24, Lemma 1]) *Let $u \in W_0^{1,\Phi_i}(\Omega)$, then the following relations hold*:

(I) $\|u\|_i^{(p_i)_0} \le \int_\Omega \Phi_i(|\nabla u(x)|)\,dx \le \|u\|_i^{(p_i)^0}$ *if* $\|u\|_i > 1$, $i = 1, \dots, n$,

(II) $\|u\|_i^{(p_i)^0} \le \int_\Omega \Phi_i(|\nabla u(x)|)\,dx \le \|u\|_i^{(p_i)_0}$ *if* $\|u\|_i < 1$, $i = 1, \dots, n$.

Proposition 1.2 ([21, Lemma 2.1]) *Let $u \in W_0^{1,\Phi_i}(\Omega)$ and*

$$\int_\Omega \Phi_i\big(|\nabla u(x)|\big)\,dx \le r$$

for some $0 < r < 1$. Then one has $\|u\|_i < 1$.

Proposition 1.3 ([7, Remark 2.1]) *Let $u \in W_0^{1,\Phi_i}(\Omega)$ be such that $\|u\|_i = 1$. Then*

$$\int_\Omega \Phi_i\big(|\nabla u(x)|\big)\,dx = 1.$$

Our aim is to prove the existence and multiplicity solutions for problem (1.1); so we study problem (1.1) by using the results as follows.

First, we recall the following three critical points theorem, obtained by G. Bonanno and S.A. Marano in [8].

Theorem 1.1 *Let X be a reflexive real Banach space, $J : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous and continuously Gâteaux differentiable functional that is bounded on bounded subsets of X and whose Gâteaux derivative admits a continuous inverse on X^*, and let $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact and satisfies $J(0) = I(0) = 0$. Assume that there exist $r > 0$ and $\bar{v} \in X$, with $r < J(\bar{v})$ such that*:

(a1) $\frac{\sup_{J^{-1}(-\infty,r]} I(u)}{r} < \frac{I(\bar{v})}{J(\bar{v})}$;

(a2) *for each $\lambda \in \Lambda_r := \left]\frac{J(\bar{v})}{I(\bar{v})}, \frac{r}{\sup_{J^{-1}(-\infty,r]} I(u)}\right[$ the functional $J - \lambda I$ is coercive.*

Then, for each compact interval $[\alpha,\beta]\subseteq\Lambda_r$, *there exists* $\rho>0$ *with the following property: for every* $\lambda\in[\alpha,\beta]$, *the equation*

$$J'(u)-\lambda I'(u)=0$$

has at least three solutions in X *whose norms are less than* ρ.

Here, we recall a multiple critical points theorem of Bonanno et al. [6].

Theorem 1.2 *Let* X *be a reflexive real Banach space, let* $J,I:X\to\mathbb{R}$ *be two Gâteaux differentiable functionals such that* J *is strongly continuous, sequentially weakly lower semicontinuous and coercive, and* I *is sequentially weakly upper semicontinuous. For every* $r>\inf_X J$, *let*

$$\varphi(r):=\inf_{u\in J^{-1}(-\infty,r)}\frac{\sup_{v\in J^{-1}(-\infty,r)}I(v)-I(u)}{r-J(u)},$$

$$\gamma:=\liminf_{r\to+\infty}\varphi(r),\qquad \delta:=\liminf_{r\to(\inf_X J)^+}\varphi(r).$$

Then the following properties hold:

(a) *If* $\gamma<+\infty$, *then for each* $\lambda\in]0,\frac{1}{\gamma}[$, *either*

(a1) $h_\lambda:=J-\lambda I$ *possesses a global minimum, or*

(a2) *there is a sequence* $\{u_n\}$ *of critical points* (*local minima*) *of* h_λ *such that*

$$\lim_{n\to+\infty}J(u_n)=+\infty;$$

(b) *If* $\delta<+\infty$, *then for each* $\lambda\in]0,\frac{1}{\delta}[$, *either*

(b1) *there is a global minimum of* J *that is a local minimum of* h_λ, *or*

(b2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points* (*local minima*) *of* h_λ *that weakly converges to a global minimum of* J *with*

$$\lim_{n\to+\infty}J(u_n)=\inf_{u\in X}J(u).$$

2 Main results

Definition 2.1 We say that $u=(u_1,u_2,\dots,u_n)$ is a weak solution to the system (1.1) if $u=(u_1,u_2,\dots,u_n)\in X$ and

$$\begin{aligned}&\sum_{i=1}^{n}M_i\biggl(\int_\Omega\Phi_i\bigl(|\nabla u_i(x)|\bigr)\,dx\biggr)\int_\Omega\alpha_i\bigl(|\nabla u_i(x)|\bigr)\nabla u_i(x)\nabla v_i(x)\,dx\\&\quad-\lambda\int_\Omega\sum_{i=1}^{n}F_{u_i}\bigl(x,u_1(x),\dots,u_n(x)\bigr)v_i(x)\,dx=0,\end{aligned}$$

for every $v=(v_1,v_2,\dots,v_n)\in X$.

Set $\overline{p} := \max\{(p_i)^0 : i = 1,\dots,n\}$, $m_0 := \min\{m_i^0 : i = 1,\dots,n\}$ and $m_1 := \max\{M_i^0 : i = 1,\dots,n\}$. For all $\sigma > 0$, we define the set

$$Q(\sigma) := \left\{(t_1,\dots,t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |t_i| \le \sigma\right\}.$$

We need the following proposition in the proof of the main results.

Proposition 2.1 *Let $T : X \to X^*$ be the operator defined by*

$$T(u_1,\dots,u_n)(v_1,\dots,v_n) = \sum_{i=1}^{n} M_i\left(\int_\Omega \Phi_i\big(|\nabla u_i(x)|\big)\,dx\right) \times \int_\Omega \alpha_i\big(|\nabla u_i(x)|\big)\nabla u_i(x)\nabla v_i(x)\,dx,$$

for every $u = (u_1,\dots,u_n)$, $v = (v_1,\dots,v_n) \in X$. Then T admits a continuous inverse on X^*, where X^* denotes the dual of X.

Proof By applying the Minty–Browder theorem [33, Theorem 26.A(d)], it is sufficient to verify that T is coercive, hemicontinuous, and uniformly monotone. Since

$$(p_i)_0 \le \frac{t\varphi_i(t)}{\Phi_i(t)}, \quad \text{for all } t > 0,$$

by Proposition 1.1, for each $u \in X$ with $\|u_i\|_i > 1$, we have

$$\begin{aligned} &T(u_1,\dots,u_n)(u_1,\dots,u_n) \\ &\quad= \sum_{i=1}^{n} M_i\left(\int_\Omega \Phi_i\big(|\nabla u_i(x)|\big)\,dx\right)\int_\Omega \alpha_i\big(|\nabla u_i(x)|\big)|\nabla u_i(x)|^2\,dx \\ &\quad\ge \sum_{i=1}^{n} M_i\left(\int_\Omega \Phi_i\big(|\nabla u_i(x)|\big)\,dx\right)\int_\Omega \Phi_i\big(|\nabla u_i(x)|\big)\,dx \\ &\quad\ge m_0\sum_{i=1}^{n} \|u_i\|_i^{2(p_i)_0}, \end{aligned}$$

so if $(p_i)_0 > N$ then T is coercive. The fact that T is hemicontinuous can be verified using standard arguments. Similar to proof given in [18, Lemma 3.2], T is strictly monotone. Therefore, in view of Minty–Browder theorem, there exists $T^{-1} : X^* \to X$, and, by a similar method as that given in [10], one has that T^{-1} is continuous. □

Now, we define the energy functional of problem (1.1) by $h_\lambda : X \to \mathbb{R}$:

$$h_\lambda(u) = J(u) - \lambda I(u),$$

for all $u = (u_1,\dots,u_n) \in X$, where

$$J(u) = \sum_{i=1}^{n} \hat{M}_i\left(\int_\Omega \Phi_i\big(|\nabla u_i(x)|\big)\,dx\right), \qquad \hat{M}_i(t) = \int_0^t M_i(s)\,ds, \quad i = 1,2,\dots,n,$$

$$I(u) = \int_\Omega F\big(x, u_1(x), \dots, u_n(x)\big)\, dx.$$

Note that the weak solutions of (1.1) are exactly the critical points of h_λ. Similar arguments as in [25, Lemma 4.2] imply that J and I are continuously Gâteaux differentiable functionals and whose Gâteaux differentials at the point $u = (u_1, \dots, u_n) \in X$ are the functionals $J'(u)$ and $I'(u)$ given by

$$J'(u)(v) = \sum_{i=1}^n M_i\left(\int_\Omega \Phi_i\big(|\nabla u_i(x)|\big)\, dx\right) \int_\Omega \alpha_i\big(|\nabla u_i(x)|\big) \nabla u_i(x) \nabla v_i(x)\, dx,$$

$$I'(u)(v) = \int_\Omega \sum_{i=1}^n F_{u_i}\big(x, u_1(x), \dots, u_n\big) v_i(x)\, dx.$$

Moreover, $I' : X \to X^*$ is a compact derivative. For this purpose, it is enough to show that I' is strongly continuous on X, so for a fixed $(u_1, u_2, \dots, u_n) \in X$, let $(u_{1k}, u_{2k}, \dots, u_{nk}) \rightharpoonup (u_1, u_2, \dots, u_n)$ weakly in X as $k \to +\infty$. Since X is compactly embedded in $C^0(\bar{\Omega}) \times \cdots \times C^0(\bar{\Omega})$, we have that $(u_{1k}, u_{2k}, \dots, u_{nk})$ converges uniformly to $(u_1, u_2, \dots, u_n)$ on Ω as $k \to +\infty$. Since $F(x, \cdot, \dots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$, and the partial derivatives of F are continuous in $\mathbb{R}^n$ for every $x \in \Omega$, $F_{u_i}(x, u_{1k}, \dots, u_{nk}) \to F_{u_i}(x, u_1, \dots, u_n)$ strongly as $k \to +\infty$, thus $I'(u_{1k}, \dots, u_{nk}) \to I'(u_1, \dots, u_n)$ strongly as $k \to +\infty$. So I' is strongly continuous on X, which implies that I' is a compact operator [33].

Lemma 2.1 *J is coercive and sequentially weakly lower semicontinuous.*

Proof For all $t \geq 0$, we have

$$J(u) \geq \sum_{i=1}^n m_i^0 \left(\int_\Omega \Phi_i\big(|\nabla u_i(x)|\big)\, dx\right), \quad i = 1, 2, \dots, n,$$

and, by Proposition 1.1, for all $u \in X$ with $\|u_i\|_i > 1$, we have

$$J(u) \geq \sum_{i=1}^n m_0 \|u_i\|_i^{(p_i)_0},$$

from which it follows that J is coercive. Moreover, since Φ_i for $1 \leq i \leq n$ are convex, J is a convex functional, and thus it is sequentially weakly lower semicontinuous. □

Three weak solutions

Theorem 2.1 *Assume that condition* (**M**) *holds and*

(h1) $F(x, 0, \dots, 0) = 0$, *for a.e.* $x \in \Omega$.

(h2) *There exist* $\alpha(x) \in L^1(\Omega)$ *and* n *positive constants* β_i, *with* $\beta_i < (p_i)_0$ *for* $1 \leq i \leq n$, *such that*

$$0 \leq F(x, t_1, \dots, t_n) \leq \alpha(x)\left(1 + \sum_{i=1}^n |t_i|^{\beta_i}\right),$$

for a.e. $x \in \Omega$, $(t_1, \dots, t_n) \in \mathbb{R}^n$.

(h3) *There exist* $x_0 \in \Omega$, $D > 0$, $\delta > 0$, $0 < b_i < (\frac{C}{m_0})^{\frac{1}{(p_i)^0}}$, *and*

$$m_0 \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\left(\frac{D}{2}\right)^N (2^N - 1) \sum_{i=1}^{n} \Phi_i\left(\frac{2\delta}{D}\right) > 1$$

such that

$$\int_\Omega \sup_{|t_1|<b_1,\dots,|t_n|<b_n} F(x,t_1,\dots,t_n)\,dx < \frac{\min\{\frac{m_0}{C}(b_i)^{(p_i)^0} : 1 \le i \le n\}}{m_1 \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}(\frac{D}{2})^N (2^N-1)\sum_{i=1}^n \Phi_i(\frac{2\delta}{D})} \times \int_{B(x_0,\frac{D}{2})} F(x,\delta,\dots,\delta)\,dx.$$

Furthermore, set

$$\underline{\lambda} := \frac{m_1 \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}(\frac{D}{2})^N (2^N-1)\sum_{i=1}^n \Phi_i(\frac{2\delta}{D})}{\int_{B(x_0,\frac{D}{2})} F(x,\delta,\dots,\delta)\,dx},$$
$$\overline{\lambda} := \frac{\min\{\frac{m_0}{C}(b_i)^{(p_i)^0} : 1 \le i \le n\}}{\int_\Omega \sup_{|t_1|<b_1,\dots,|t_n|<b_n} F(x,t_1,\dots,t_n)\,dx},$$

then, for each $\lambda \in \Lambda := (\underline{\lambda}, \overline{\lambda})$, *problem* (1.1) *possesses at least three distinct weak solutions in* X.

Proof Our aim is to apply Theorem 1.1 to our problem, so we check that the functionals J, I satisfy the conditions of Theorem 1.1. We set $u_0 = (0,\dots,0)$. Then by the definitions of I, J and from (h1), we have $J(u_0) = I(u_0) = 0$. Let $x_0 \in \Omega$, $D > 0$, and take

$$w(x) = \begin{cases} 0 & x \in \Omega \setminus B(x_0, D), \\ \delta, & x \in B(x_0, \frac{D}{2}), \\ \frac{2\delta}{D}(D - |x - x_0|) & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}). \end{cases}$$

Let $\bar{u} = (w(x),\dots,w(x))$ and $r = \min\{\frac{m_0}{C}(b_i)^{(p_i)^0} : 1 \le i \le n\}$. Clearly, $\bar{u} \in X$ and from (h3) we have

$$\begin{aligned} J(\bar{u}) &= \sum_{i=1}^n \hat{M}_i\left(\int_\Omega \Phi_i(|\nabla w(x)|)\,dx\right) \\ &\ge \sum_{i=1}^n m_i^0 \int_\Omega \Phi_i(|\nabla w(x)|)\,dx \\ &\ge \sum_{i=1}^n m_0 \Phi_i\left(\frac{2\delta}{D}\right)\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\left(\frac{D}{2}\right)^N (2^N - 1) \\ &> r. \end{aligned}$$

On the other way, when $J(u) \le r$ for $u = (u_1, \dots, u_n) \in X$,

$$\sum_{i=1}^{n} \hat{M}_i\left(\int_{\Omega} \Phi\left(\left|\nabla u_i(x)\right|\right) dx\right) \le r.$$

Hence, since $0 < b_i < (\frac{C}{m_0})^{\frac{1}{(p_i)^0}}$, using Propositions 1.1 and 1.2, we have

$$m_0 \|u_i\|_i^{(p_i)^0} < r,$$

and from (1.6) we obtain

$$\left|u_i(x)\right| < \left(\frac{Cr}{m_0}\right)^{\frac{1}{(p_i)^0}} = b_i, \quad \text{for } 1 \le i \le n.$$

Therefore, for every $u = (u_1, \dots, u_n) \in X$,

$$\begin{aligned} \sup_{u \in J^{-1}(-\infty, r)} I(u) &= \sup_{u \in J^{-1}(-\infty, r)} \int_{\Omega} F\left(x, u_1(x), \dots, u_n(x)\right) dx \\ &\le \int_{\Omega} \sup_{|t_1| \le b_1, \dots, |t_n| \le b_n} F(x, t_1, \dots, t_n)\, dx. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} J(\bar{u}) &= \sum_{i=1}^{n} \hat{M}_i\left(\int_{\Omega} \Phi_i\left(\left|\nabla w(x)\right|\right) dx\right) \\ &\le m_1 \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})}\left(\frac{D}{2}\right)^{N}\left(2^N - 1\right)\sum_{i=1}^{n} \Phi_i\left(\frac{2\delta}{D}\right) \end{aligned}$$

and

$$I(\bar{u}) > \int_{B(x_0, \frac{D}{2})} F(x, \delta, \dots, \delta)\, dx.$$

So, from (h3), we have

$$\frac{\sup_{u \in J^{-1}(-\infty, r)} I(u)}{r} \le \frac{\int_{\Omega} \sup_{|t_1| \le b_1, \dots, |t_n| \le b_n} F(x, t_1, \dots, t_n)\, dx}{\min\{\frac{m_0}{C}(b_i)^{(p_i)^0} : 1 \le i \le n\}} < \frac{I(\bar{u})}{J(\bar{u})}.$$

Hence, we observe that the condition (a1) of Theorem 1.1 is satisfied.

From (h2), it follows that the function $J - \lambda I$ is coercive for every positive parameter λ, in particular for every

$$\lambda \in \Lambda \subseteq \left(\frac{J(\bar{u})}{I(\bar{u})}, \frac{r}{\sup_{J(u) \le r} I(u)}\right),$$

so the condition (a2) of Theorem 1.1 holds. Then all the assumptions of Theorem 1.1 are fulfilled. By Theorem 1.1, we know that there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive constant ρ such that, for any $\lambda \in \Lambda$, problem (1.1) has at least three weak solutions whose norms are less than ρ. □

Theorem 2.2 *Assume that conditions* (**M**) *and* (h1) *hold and consider the following*:

(h4) $F(x,t_1,\dots,t_n) \geq 0$ *for every* $(x,t_1,\dots,t_n) \in \Omega \times \mathbb{R}_+^n$.

(h5) *There exist* $x_0 \in \Omega$ *and values* $D, \varrho > 0$ *such that* $\overline{B(x_0,D)} \subseteq \Omega$,

$$\lim_{t\to 0^+} \frac{\Phi_i(t)}{t^{(p_i)^0}} < \varrho, \tag{2.1}$$

and for

$$A := \liminf_{\sigma\to 0^+} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\sigma)} F(x,t_1,\dots,t_n)\,dx}{\sigma^{\overline{p}}},$$
$$B := \limsup_{(t_1,\dots,t_n)\to(0^+,\dots,0^+)} \frac{\int_{B(x_0,\frac{D}{2})} F(x,t_1,\dots,t_n)\,dx}{\sum_{i=1}^n t_i^{(p_i)^0}},$$

one has

$$A < LB,$$

where $L = \min\{L_{(p_i)^0}, i = 1,2,\dots,n\}$,

$$L_{(p_i)^0} = \frac{\Gamma(1+\frac{N}{2})}{(\sum_{i=1}^n (\frac{C}{m_0})^{\frac{1}{(p_i)^0}})^{\overline{p}} m_1 \varrho \pi^{\frac{N}{2}} (\frac{2}{D})^{(p_i)^0 - N}(2^N - 1)}. \tag{2.2}$$

Then for every

$$\lambda \in \Lambda := \frac{1}{(\sum_{i=1}^n (\frac{C}{m_0})^{\frac{1}{(p_i)^0}})^{\overline{p}}} \left(\frac{1}{LB}, \frac{1}{A}\right),$$

problem (1.1) *admits a sequence of pairwise distinct weak solutions which strongly converges to zero in* X.

Proof We apply the part (b) of Theorem 1.2 and show that $\delta < \infty$. Let $\{\sigma_k\}$ be a sequence of positive numbers such that $\lim_{k\to+\infty} \sigma_k = 0$ then

$$\begin{aligned} &\lim_{k\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\sigma_k)} F(x,t_1,\dots,t_n)\,dx}{\sigma_k^{\overline{p}}} \\ &= \liminf_{\sigma\to 0^+} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\sigma)} F(x,t_1,\dots,t_n)\,dx}{\sigma^{\overline{p}}} \\ &= A < +\infty. \end{aligned} \tag{2.3}$$

Putting

$$r_k = \frac{\sigma_k^{\overline{p}}}{(\sum_{i=1}^n (\frac{C}{m_0})^{\frac{1}{(p_i)^0}})^{\overline{p}}} \quad \text{for all } k \in \mathbb{N},$$

$$\begin{aligned} J^{-1}\big(]-\infty,r_k[\big) &:= \big\{u=(u_1,u_2,\dots,u_n)\in X : J(u)<r_k\big\} \\ &\subseteq \left\{u\in X : \sum_{i=1}^{n}\hat{M}_i\left(\int_\Omega \Phi_i\big(|\nabla u_i|\big)\,dx\right)\le r_k\right\}, \end{aligned}$$

by Propositions 1.1 and 1.2, for k large enough ($0<r_k<1$),

$$m_0\|u_i\|_i^{(p_i)^0}<r_k,$$

and from (1.6) we have $\max_{x\in\bar\Omega}|u_i(x)|^{(p_i)^0}\le C\|u_i\|_i^{(p_i)^0}$. Then we obtain for all $x\in\Omega$,

$$\big|u_i(x)\big|\le\left(\frac{Cr_k}{m_0}\right)^{\frac{1}{(p_i)^0}}.$$

Thus

$$\sum_{i=1}^{n}\big|u_i(x)\big|\le\sum_{i=1}^{n}\left(\frac{Cr_k}{m_0}\right)^{\frac{1}{(p_i)^0}}\le r_k^{\frac{1}{\overline{p}}}\sum_{i=1}^{n}\left(\frac{C}{m_0}\right)^{\frac{1}{(p_i)^0}}\le\sigma_k.$$

Then we have

$$J^{-1}(-\infty,r_k)\subseteq\left\{u\in X : \sum_{i=1}^{n}\big|u_i(x)\big|\le\sigma_k\right\}.$$

From condition (h1), we have $\min_X J=J(0,\dots,0)=I(0,\dots,0)=0$.

$$\begin{aligned} \varphi(r_k) &= \inf_{u\in J^{-1}(]-\infty,r_k[)}\frac{\sup_{v\in J^{-1}(]-\infty,r_k[)}I(v)-I(u)}{r_k-J(u)} \\ &\le \frac{\sup_{v\in J^{-1}(]-\infty,r_k[)}I(v)}{r_k} \\ &\le \left(\sum_{i=1}^{n}\left(\frac{C}{m_0}\right)^{\frac{1}{(p_i)^0}}\right)^{\overline{p}}\frac{\int_\Omega\sup_{(t_1,t_2,\dots,t_n)\in Q(\sigma_k)}F(x,t_1,\dots,t_n)\,dx}{\sigma_k^{\overline{p}}}. \end{aligned}\tag{2.4}$$

Let $\delta:=\liminf_{r\to0^+}\varphi(r)$. It follows from (2.3) and (2.4) that

$$\begin{aligned} \delta &\le \liminf_{k\to+\infty}\varphi(r_k) \\ &\le \left(\sum_{i=1}^{n}\left(\frac{C}{m_0}\right)^{\frac{1}{(p_i)^0}}\right)^{\overline{p}}\lim_{k\to+\infty}\frac{\int_\Omega\sup_{(t_1,t_2,\dots,t_n)\in Q(\sigma_k)}F(x,t_1,\dots,t_n)\,dx}{\sigma_k^{\overline{p}}} \\ &\le \left(\sum_{i=1}^{n}\left(\frac{C}{m_0}\right)^{\frac{1}{(p_i)^0}}\right)^{\overline{p}}A<+\infty. \end{aligned}$$

So $\Lambda\subseteq]0,\frac{1}{\delta}[$. For a fixed $\lambda\in\Lambda$, we claim that the functional h_λ is unbounded from below. Indeed, since

$$\frac{1}{\lambda}<\left(\sum_{i=1}^{n}\left(\frac{C}{m_0}\right)^{\frac{1}{(p_i)^0}}\right)^{\overline{p}}LB,$$

we can consider n positive real sequences $\{d_{i,k}\}_{i=1}^{n}$ and $\eta > 0$ such that $\sqrt{\sum_{i=1}^{n} d_{i,k}^2} \to 0$ as $k \to +\infty$ and

$$\frac{1}{\lambda} < \eta < L\left(\sum_{i=1}^{n}\left(\frac{C}{m_0}\right)^{\frac{1}{(p_i)^0}}\right)^{\overline{p}} \frac{\int_{B(x_0,\frac{D}{2})} F(x, d_{1,k}, \dots, d_{n,k})\, dx}{\sum_{i=1}^{n} d_{i,k}^{(p_i)^0}}. \tag{2.5}$$

Let $\{u_k(x) = (u_{1k}, u_{2k}, \dots, u_{nk})\} \subseteq X$ be a sequence defined by

$$u_{ik}(x) = \begin{cases} 0, & x \in \bar{\Omega} \setminus B(x_0, D), \\ \frac{2d_{i,k}}{D}(D - |x - x_0|), & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}), \\ d_{i,k}, & x \in B(x_0, \frac{D}{2}), \end{cases}$$

for $1 \le i \le n$. Then

$$\begin{aligned} J(u_k) &= \sum_{i=1}^{n} \hat{M}_i\left(\int_{\Omega} \Phi_i\big(|\nabla u_{ik}|\big)\, dx\right) \\ &< m_1 \int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} \Phi_i\left(\frac{2d_{i,k}}{D}\right) dx. \end{aligned}$$

Moreover, from (2.1) and since $\lim_{k\to\infty} \frac{2d_{i,k}}{D} = 0$, there exist $\zeta > 0$ and $n_i \in \mathbb{N}$ $i = 1, \dots, n$ such that $\frac{2d_{i,k}}{D} \in (0, \zeta)$, and

$$\begin{aligned} &\Phi_i\left(\frac{2d_{i,k}}{D}\right) < \varrho\left(\frac{2}{D}\right)^{(p_i)^0} d_{i,k}^{(p_i)^0} \quad \text{for all } n \ge n_i\ (i = 1, \dots, n), \\ &\int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} \Phi_i\left(\frac{2d_{i,k}}{D}\right) dx < \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})}\varrho\left(\frac{2}{D}\right)^{(p_i)^0 - N} d_{i,k}^{(p_i)^0}\left(2^N - 1\right). \end{aligned}$$

From (2.2), for all $n \ge \max\{n_1, \dots, n_2\}$, we have

$$J(u_k) \le \frac{1}{(\sum_{i=1}^{n}(\frac{C}{m_0})^{\frac{1}{(p_i)^0}})^{\overline{p}}} \sum_{i=1}^{n} \frac{d_{i,k}^{(p_i)^0}}{L_{(p_i)^0}}. \tag{2.6}$$

By (h4), we have

$$I(u_k) = \int_{\Omega} F(x, u_{1k}, \dots, u_{nk})\, dx \ge \int_{B(x_0,\frac{D}{2})} F(x, d_{1,k}, \dots, d_{n,k})\, dx. \tag{2.7}$$

By (2.5), (2.6), and (2.7), we have

$$\begin{aligned} h_\lambda(u_k) &= J(u_k) - \lambda I(u_k) \\ &\le \frac{1}{(\sum_{i=1}^{n}(\frac{C}{m_0})^{\frac{1}{(p_i)^0}})^{\overline{p}}} \sum_{i=1}^{n} \frac{d_{i,k}^{(p_i)^0}}{L_{(p_i)^0}} - \lambda \int_{B(x_0,\frac{D}{2})} F(x, d_{1,k}, \dots, d_{i,k})\, dx \end{aligned}$$

$$< \frac{1-\lambda\eta}{L(\sum_{i=1}^{n}(\frac{C}{m_0})^{\frac{1}{(p_i)^0}})^{\overline{p}}}\sum_{i=1}^{n} d_{i,k}^{(p_i)^0}$$

$$< 0 = h_\lambda(0,\ldots,0),$$

for every $n \in \mathbb{N}$ large enough. Then $(0,\ldots,0)$ is not a local minimum of h_λ. Thus, owing to the fact that $(0,\ldots,0)$ is the unique global minimum of J, there exists a sequence $\{u_k = (u_{1k},\ldots,u_{nk})\}$ of pairwise distinct critical points of h_λ such that $\lim_{k\to+\infty}\|u_k\| = 0$, and this completes the proof. □

We illustrate this abstract existence result with the following example.

Example 2.1 Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $|\Omega| = 1$ and assume $i = 2$. Similar to [15, Remark 3.6], we have

$$\varphi_1(t) = \begin{cases} \frac{|t|^4 t}{\log(1+|t|)} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

By [11, Example 3], one has $(p_1)_0 = 5 < (p_1)^0 = 6$. Thus the condition (1.4) is satisfied. Moreover, owing to

$$\lim_{t\to 0^+} \frac{1}{t^5}\int_0^t \frac{|s|^4 s}{\log(1+|s|)}\,ds = \frac{1}{5},$$

the condition (2.1) is also fulfilled (for example, take $\varrho = \frac{1}{N} = \frac{1}{3}$). Now let

$$\varphi_2(t) = \log\bigl(1+|t|^2\bigr)|t|^2 t, \quad t \in \mathbb{R}.$$

Then by [11, Example 2], one has $(p_2)_0 = 4 < (p_2)^0 = 6$. So the condition (1.4) is satisfied. Moreover, owing to

$$\lim_{t\to 0^+} \frac{1}{t^4}\int_0^t \log\bigl(1+|s|^2\bigr)|s|^2 s\,ds = 0,$$

the condition (2.1) is also fulfilled (here we take $\varrho = \frac{1}{N} = \frac{1}{3}$, again). So we see that with the above choices, φ_1 and φ_2 satisfy the assumptions of Theorem 2.2. Let $F : \mathbb{R}^2 \to [0,\infty)$ be a continuous function defined by

$$F(s,t) = \begin{cases} s^6(1+\sin(\ln(1+|t|))), & (s,t) \neq (0,0), \\ 0, & (s,t) = (0,0), \end{cases}$$

$$A = \liminf_{\sigma\to 0^+} \frac{\int_\Omega \max_{|s|+|t|\le\sigma} F(s,t)\,dx}{\sigma^6} = |\Omega| \liminf_{\sigma\to 0^+} \frac{\max_{|s|+|t|\le\sigma} F(s,t)}{\sigma^6} = 2,$$

$$B = \limsup_{s,t\to 0^+} \frac{\int_{B(x_0,\frac{D}{2})} F(s,t)\,dx}{s^6+t^6} = \left|B\left(x_0,\frac{D}{2}\right)\right| \limsup_{s,t\to 0^+} \frac{F(s,t)}{s^6+t^6} = \left|B\left(x_0,\frac{D}{2}\right)\right|.$$

Then

$$\lambda_1 = \frac{7m_1}{3}\left(\frac{2}{D}\right)^6 > 0 \quad \text{and} \quad \lambda_2 = \frac{m_0}{2^7 C} > 0,$$

with this condition $2^{13} < \frac{3m_0 D^6}{7m_1 C}$. Then for $\lambda \in]\lambda_1, \lambda_2[$, the following system:

$$\begin{cases} -M_1(\int_\Omega \Phi_1(|\nabla u|)\,dx)\operatorname{div}(\frac{|\nabla u|^4}{\log(1+|\nabla u|)}\nabla u) = \lambda F_u(x,u,v) \quad \text{in } \Omega, \\ -M_2(\int_\Omega \Phi_2(|\nabla v|)\,dx)\operatorname{div}(\log(1+|\nabla v|^2)|\nabla v|^2\nabla v) = \lambda F_v(x,u,v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$.

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References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Adams, R.A., Fournier, J.: Sobolev Spaces. Academic Press, London (2003)
3. Allegue, O., Bezzarga, M.: Three solutions for a class of quasilinear elliptic systems in Orlicz–Sobolev spaces. Complex Var. Elliptic Equ. **58**(9), 1215–1227 (2013)
4. Alves, C.O., Santos, J.A.: Multivalued elliptic equation with exponential critical growth in $\mathbb{R}^2$. J. Differ. Equ. **261**(9), 4758–4788 (2016)
5. Behboudi, F., Razani, A., Oveisiha, M.: Existence of a mountain pass solution for a nonlocal fractional (p,q)-Laplacians problem. Bound. Value Probl. **2020**, 149 (2020)
6. Bonanno, G., Bisci, G.M.: Infinitely many solutions for a boundary value problem with discontinuous nonlinearities. Bound. Value Probl. **2009**, 670675 (2009)
7. Bonanno, G., Bisci, G.M., Rădulescu, V.D.: Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz–Sobolev spaces. Nonlinear Anal. **75**(12), 4441–4456 (2012)
8. Bonanno, G., Marano, S.A.: On the structure of the critical set of non-differentiable functions with a weak compactness condition. Appl. Anal. **89**, 1–10 (2010)
9. Cheng, B., Wu, X., Liu, J.: Multiplicity of solutions for nonlocal elliptic system of (p,q)-Kirchhoff type. Abstr. Appl. Anal. **2011**, 526026 (2011)
10. Chung, N.T.: Three solutions for a class of nonlocal problems in Orlicz–Sobolev spaces. J. Korean Math. Soc. **250**(6), 1257–1269 (2013)
11. Clément, P., de Pagter, B., Sweers, G., de Thélin, F.: Existence of solutions to a semilinear elliptic system through Orlicz–Sobolev spaces. Mediterr. J. Math. **1**(3), 241–267 (2004)
12. Clément, P., García Huidobro, M., Manásevich, R., Schmitt, K.: Mountain pass type solutions for quasilinear elliptic equations. Calc. Var. Partial Differ. Equ. **11**, 33–62 (2000)
13. Cowan, C., Razani, A.: Singular solutions of a p-Laplace equation involving the gradient. J. Differ. Equ. **269**, 3914–3942 (2020)
14. Figueiredo, G.M., Santos, J.A.: Existence of least energy nodal solution with two nodal domains for a generalized Kirchhoff problem in an Orlicz–Sobolev space. Math. Nachr. **290**(4), 583–603 (2017)
15. Heidarkhani, S., Caristi, G., Ferrara, M.: Perturbed Kirchhoff-type Neumann problems in Orlicz–Sobolev spaces. Comput. Math. Appl. **71**(10), 2008–2019 (2016)
16. Hudzik, H.: The problem of separability, duality, reflexivity and of comparison for generalized Orlicz–Sobolev spaces $(W_M^k(\Omega))$. Comment. Math. **21**(2), 315–324 (1979)
17. Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)
18. Kristály, A., Mihăilescu, M., Rădulescu, V.: Two non-trivial solutions for a non-homogeneous Neumann problem: an Orlicz–Sobolev space setting. Proc. R. Soc. Edinb., Sect. A **139**(2), 367–379 (2009)
19. Kurdila, A.J., Zabarankin, M.: Convex Functional Analysis, Systems Control: Foundations Applications. Birkhäuser, Basel (2005)

20. Li, Q., Yang, Z.: Existence of positive solutions for a quasilinear elliptic systems of p-Kirchhoff type. Differ. Equ. Appl. **6**(1), 73–80 (2014)
21. Makvand Chaharlang, M., Razani, A.: Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition. Commun. Korean Math. Soc. **34**(1), 155–167 (2019)
22. Makvand Chaharlang, M., Razani, A.: A fourth order singular elliptic problem involving p-biharmonic operator. Taiwan. J. Math. **23**(3), 589–599 (2019)
23. Makvand Chaharlang, M., Razani, A.: Two weak solutions for some Kirchhoff-type problem with Neumann boundary condition. Georgian Math. J. (2020). https://doi.org/10.1515/gmj-2019-2077
24. Mihăilescu, M., Rădulescu, V.: Eigenvalue problems associated to nonhomogeneous differential operators in Orlicz–Sobolev spaces. Anal. Appl. **6**(1), 1–16 (2008)
25. Mihăilescu, M., Rădulescu, V.: Neumann problems associated to nonhomogeneous differential operators in Orlicz–Sobolev spaces. Ann. Inst. Fourier **58**(6), 2087–2111 (2008)
26. Mihăilescu, M., Repovs, D.: Multiple solutions for a nonlinear and non-homogeneous problem in Orlicz–Sobolev spaces. Appl. Math. Comput. **217**, 6624–6632 (2011)
27. Ragusa, M.A.: Elliptic boundary value problem in vanishing mean oscillation hypothesis. Comment. Math. Univ. Carol. **40**(4), 651–663 (1999)
28. Ragusa, M.A., Tachikawa, A.: Boundary regularity of minimizers of $p(x)$-energy functionals. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **33**(2), 451–476 (2016)
29. Rao, M.M., Ren, Z.D.: Applications of Orlicz Spaces. Monographs and Textbooks in Pure and Applied Mathematics, vol. 250. Dekker, New York (2002)
30. Rasouli, S.H.: Existence of solutions for singular (p,q)-Kirchhoff type systems with multiple parameters. Electron. J. Differ. Equ. **2016**, 69 (2016)
31. Razani, A.: Subsonic detonation waves in porous media. Phys. Scr. **94**, 085209 (2019)
32. Safari, F., Razani, A.: Existence of radially positive solutions for Neumann problem on the Heisenberg group. Bound. Value Probl. **2020**, 88 (2020)
33. Zeidler, E.: Nonlinear Functional Analysis and Applications. Nonlinear Monotone Operators, vol. II/B. Springer, New York (1990)