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Positive ground states for nonlinearly coupled Choquard type equations with lower critical exponents

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Abstract

We study the coupled Choquard type system with lower critical exponents

$$\begin{cases} -\Delta u+\lambda_1(x)u=\mu_1(I_\alpha*|u|^{\frac{N+\alpha}{N}})|u|^{\frac{\alpha}{N}-1}u+\beta(I_\alpha*|v|^{\frac{N+\alpha}{N}})|u|^{\frac{\alpha}{N}-1}u, & x\in\mathbb{R}^N,\\ -\Delta v+\lambda_2(x)v=\mu_2(I_\alpha*|v|^{\frac{N+\alpha}{N}})|v|^{\frac{\alpha}{N}-1}v+\beta(I_\alpha*|u|^{\frac{N+\alpha}{N}})|v|^{\frac{\alpha}{N}-1}v, & x\in\mathbb{R}^N,\\ u,v\in H^1(\mathbb{R}^N), \end{cases}$$

where $N\geq 3$, $\mu_1,\mu_2,\beta>0$, and $\lambda_1(x)$, $\lambda_2(x)$ are nonnegative functions. The existence of at least one positive ground state of this system is proved under certain assumptions on λ_1, λ_2.

Keywords: Choquard system; Lower critical exponent; Positive ground state

1 Introduction

In this paper, we consider the following coupled nonlinear equations of Choquard type:

$$\begin{cases} -\Delta u+\lambda_1(x)u=\mu_1(I_\alpha*|u|^{\frac{N+\alpha}{N}})|u|^{\frac{\alpha}{N}-1}u+\beta(I_\alpha*|v|^{\frac{N+\alpha}{N}})|u|^{\frac{\alpha}{N}-1}u, & x\in\mathbb{R}^N,\\ -\Delta v+\lambda_2(x)v=\mu_2(I_\alpha*|v|^{\frac{N+\alpha}{N}})|v|^{\frac{\alpha}{N}-1}v+\beta(I_\alpha*|u|^{\frac{N+\alpha}{N}})|v|^{\frac{\alpha}{N}-1}v, & x\in\mathbb{R}^N,\\ u,v\in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N\geq 3$, $\alpha\in(0,N)$, $\mu_1,\mu_2,\beta>0$, $\frac{N+\alpha}{N}$ is the lower critical exponent due to the Hardy–Littlewood–Sobolev inequality (see [9, Theorem 3.1]), $I_\alpha:\mathbb{R}^N\setminus\{0\}\mapsto\mathbb{R}$ defined by

$$I_\alpha=\frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha\pi^{\frac{N}{2}}\Gamma(\frac{\alpha}{2})|x|^{N-\alpha}}$$

is the Riesz potential, and $\lambda_1(x)$ and $\lambda_2(x)$ are nonnegative functions. Elliptic equations of this type have wide application in physical problems, such as in Hartree–Fock theory [8, 10, 12] and in nonlinear optics [13, 14]. The readers can refer to [2, 18, 19] for more physical backgrounds.

Mathematically, Choquard type equations have received considerable attention in the past few years, see [1, 3–5, 7, 8, 11, 15–17] and the reference therein for scale equations. There are also some results concerned with solutions of a nonlinearly coupled Choquard system. In [21], Wang and Shi proved the existence of positive solutions of

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 (I_\alpha * |u|^2) u + \beta (I_\alpha * |v|^2) u, & x \in \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 (I_\alpha * |v|^2) v + \beta (I_\alpha * |u|^2) v, & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

for $\lambda_1, \lambda_2 > 0$ and $\beta \in (-\infty, \chi_0) \cup (\min\{\lambda^2 \mu, \lambda^{\frac{1}{2}} \nu\}, +\infty)$, where $\lambda = \lambda_2/\lambda_1$ and $\chi_0 > 0$ depends on μ_1, μ_2, λ. Particularly, when $\lambda_1 = \lambda_2 > 0$, they showed that system (1.2) has a positive ground state $(\sqrt{k_0} w_0, \sqrt{l_0} w_0)$, where (k_0, l_0) is the solution of

$$\begin{cases} \mu_1 k + \beta l = 1, \\ \mu_2 l + \beta k = 1, \end{cases} \tag{1.3}$$

and w_0 is a positive ground state of

$$-\Delta u + \lambda_1 u = \left(I_\alpha * |u|^2\right) u, \quad x \in \mathbb{R}^N, u \in H^1\left(\mathbb{R}^N\right). \tag{1.4}$$

In [22], Wang and Yang established the existence and nonexistence of normalized solutions of system (1.2) with trapping potentials. In [20], Wang obtained the multiplicity of nontrivial solutions of a nonlinearly coupled Choquard system with general subcritical exponents and perturbations.

For a Choquard system with upper critical exponents, You, Wang, and Zhao [25, 26] derived the existence of a positive ground state of the following system:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) u^{\frac{\alpha+2}{N-2}} + \beta (I_\alpha * |v|^{\frac{N+\alpha}{N-2}}) u^{\frac{\alpha+2}{N-2}}, & x \in \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 (I_\alpha * |v|^{\frac{N+\alpha}{N-2}}) v^{\frac{\alpha+2}{N-2}} + \beta (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) v^{\frac{\alpha+2}{N-2}}, & x \in \Omega, \\ u, v \in H_0^1(\Omega), \end{cases} \tag{1.5}$$

where $N \geq 5$, Ω is a bounded smooth domain in $\mathbb{R}^N$, $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$, and $\lambda_1(\Omega)$ represents the first eigenvalue of $-\Delta$ on Ω with the Dirichlet boundary condition. More precisely, they obtained that system (1.5) has a positive ground state if

$$\begin{cases} \beta \in (-\bar{\beta}, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, +\infty) & \text{for } \alpha = N-4, \\ \beta \in (-\infty, 0) \cup (\frac{\alpha+2}{N-2} \max\{\mu_1, \mu_2\}, +\infty) & \text{for } \alpha \in (0, N-4). \end{cases}$$

For the special case $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 < 0$, they proved that system (1.5) has a positive ground state $(\sqrt{\bar{k}} w^*, \sqrt{\bar{l}} w^*)$ if

$$\begin{cases} \beta \in (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, +\infty) & \text{for } \alpha = N-4, \\ \beta \in (\frac{\alpha+2}{N-2} \max\{\mu_1, \mu_2\}, +\infty) & \text{for } \alpha \in (0, N-4), \end{cases}$$

where w^* is a positive ground state of

$$-\Delta u + \lambda_1 u = \left(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\right)u^{\frac{\alpha+2}{N-2}}, \quad u \in H_0^1(\Omega), \tag{1.6}$$

and $\bar{k}$, $\bar{l}$ is a solution of

$$\begin{cases} \mu_1 k^{\frac{\alpha+2}{N-2}} + \beta k^{\frac{\alpha+4-N}{2(N-2)}} l^{\frac{N+\alpha}{2(N-2)}} = 1, \\ \mu_2 l^{\frac{\alpha+2}{N-2}} + \beta k^{\frac{N+\alpha}{2(N-2)}} l^{\frac{\alpha+4-N}{2(N-2)}} = 1, \\ k, l > 0, \end{cases} \tag{1.7}$$

satisfying

$$\bar{k} = \min\left\{k | (k,l) \text{ solves } (1.7)\right\}.$$

In the current paper, we study the nonlinearly coupled system (1.1) with lower critical exponents. Since system (1.1) with positive constant potentials has no nontrivial solution in $H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ by the Pohozaev identity, we assume that λ_1, λ_2 are functions dependent on $x \in \mathbb{R}^N$. We aim to prove the existence of positive ground states of system (1.1). Furthermore, for the case $\lambda_1(x) = \lambda_2(x) := \lambda(x)$, we will introduce an approach which is different with [21, 25, 26] to prove that system (1.1) has a positive ground state of the form (kw, lw), where w is a positive ground state of

$$-\Delta u + \lambda(x)u = \left(I_\alpha * |u|^{\frac{N+\alpha}{N}}\right)|u|^{\frac{\alpha}{N}-1}u, \quad x \in \mathbb{R}^N, u \in H^1\left(\mathbb{R}^N\right). \tag{1.8}$$

For this purpose, we assume that

$(C1)$ $\lambda_i(x) \geq 0$ for all $x \in \mathbb{R}^N$, $\lambda_i(x) \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} \lambda_i(x) = 1$, $i = 1, 2$;

$(C2)$ $\liminf_{|x|\to\infty}(1 - \lambda_i(x))|x|^2 \geq \frac{N^2(N-2)}{4(N+1)}$, $i = 1, 2$.

Note that under assumptions $(C1)$ and $(C2)$, the scale equation

$$-\Delta u + \lambda_i(x)u = \mu_i\left(I_\alpha * |u|^{\frac{N+\alpha}{N}}\right)|u|^{\frac{\alpha}{N}-1}u, \quad x \in \mathbb{R}^N, u \in H^1\left(\mathbb{R}^N\right), i = 1, 2, \tag{1.9}$$

has a ground state w_i, $i = 1, 2$ (see [16, Theorem 3,Theorem 6]). Moreover, we may assume that w_i is positive since $|w_i|$ is also a ground state of (1.9). Clearly, system (1.1) has a trivial solution $(0,0)$ and two semi-trivial solutions $(w_1, 0)$ and $(0, w_2)$ for all $\beta \in \mathbb{R}$. Here we deal with the nontrivial solution, that is, a solution (u, v) of (1.1) with $u \not\equiv 0$ and $v \not\equiv 0$. Denote $\int_{\mathbb{R}^N} \cdot dx$ by $\int \cdot$ for simplicity, and define the functional $I : H \mapsto \mathbb{R}$ corresponding to system (1.1) by

$$\begin{aligned} I(u,v) = {} & \frac{1}{2}\int |\nabla u|^2 + \lambda_1(x)u^2 + |\nabla v|^2 + \lambda_2(x)v^2 \\ & - \frac{N}{2(N+\alpha)}\int \left(\mu_1\left(I_\alpha * |u|^{\frac{N+\alpha}{N}}\right)|u|^{\frac{N+\alpha}{N}} + \mu_2\left(I_\alpha * |v|^{\frac{N+\alpha}{N}}\right)|v|^{\frac{N+\alpha}{N}} \right. \\ & \left. + 2\beta\left(I_\alpha * |u|^{\frac{N+\alpha}{N}}\right)|v|^{\frac{N+\alpha}{N}}\right). \end{aligned}$$

Set

$$\mathcal{M} = \Big\{ (u,v) \in H, u, v \not\equiv 0, \\ \int |\nabla u|^2 + \lambda_1(x) u^2 = \int \mu_1 \big(I_\alpha * |u|^{\frac{N+\alpha}{N}}\big)|u|^{\frac{N+\alpha}{N}} + \beta \big(I_\alpha * |u|^{\frac{N+\alpha}{N}}\big)|v|^{\frac{N+\alpha}{N}}, \\ \int |\nabla v|^2 + \lambda_2(x) v^2 = \int \mu_2 \big(I_\alpha * |v|^{\frac{N+\alpha}{N}}\big)|v|^{\frac{N+\alpha}{N}} + \beta \big(I_\alpha * |u|^{\frac{N+\alpha}{N}}\big)|v|^{\frac{N+\alpha}{N}} \Big\}.$$

It is obvious that if (u,v) is a solution of system (1.1), then $(u,v) \in \mathcal{M}$. Define

$$\mathcal{B} = \inf_{\mathcal{M}} I(u,v).$$

A solution (u,v) of system (1.1) is called a positive solution if $u > 0$, $v > 0$ and a ground state if $I(u,v) = \mathcal{B}$. We first show that $\mathcal{B}$ is attained by some positive ground state of system (1.1) in the case when $\lambda_1(x) = \lambda_2(x) := \lambda(x)$.

Theorem 1.1 *Assume that* $(C1)$ *and* $(C2)$ *hold. If* $\lambda_1(x) = \lambda_2(x) := \lambda(x)$, *then* $(t_m s_m w, t_m w)$ *is a positive ground state of system* (1.1) *for all* $\beta > 0$, *where* w *is a positive ground state of* (1.8), $t_m = (\mu_2 + \beta s_m^{\frac{N+\alpha}{N}})^{-\frac{N}{2\alpha}}$, *and* $s_m > 0$ *is a minimum point of a function* $g(s) : \mathbb{R}^+ \mapsto \mathbb{R}$ *defined by*

$$g(s) = \frac{1+s^2}{(\mu_2 + \mu_1 s^{\frac{2(N+\alpha)}{N}} + 2\beta s^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}}. \tag{1.10}$$

Remark 1.2 If we apply a method as in the proof of [25, Theorem 1.3] and [26, Theorem 1.3] to our case, we can prove that system (1.1) has a ground state of the form (kw, lw) only if $\beta \geq \frac{\alpha}{N} \max\{\mu_1, \mu_2\}$. In the current paper, we use an alternative approach inspired by [24], which is based on studying the minimum point of $g(s)$, and we show that system (1.1) possesses a ground state of this form for all $\beta > 0$.

Remark 1.3 The method we adopted in the proof of Theorem 1.1 is also valid for the upper critical system (1.5). As we mentioned previously, system (1.5) has a ground state of the form (kw^*, lw^*) if $N \geq 5$, $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 < 0$, and

$$\begin{cases} \beta \in (\frac{\alpha+2}{N-2} \max\{\mu_1,\mu_2\}, +\infty) & \text{for } \alpha \in (0, N-4), \\ \beta \in (0, \min\{\mu_1,\mu_2\}) \cup (\max\{\mu_1,\mu_2\}, +\infty) & \text{for } \alpha = N-4, \end{cases}$$

(see [25, Theorem 1.3] and [26, Theorem 1.3]). However, we can prove that under the same assumptions on λ_1, λ_2, N, system (1.5) has a ground state in the same form if

$$\begin{cases} \beta \in (0, +\infty) & \text{for } \alpha \in (0, N-4), \\ \beta \in (\max\{\mu_1,\mu_2\}, +\infty) & \text{for } \alpha = N-4 \end{cases}$$

(see Theorem A.1 in Appendix). Although our approach can only deal with the case $\beta > \max\{\mu_1,\mu_2\}$ for $\alpha = N-4$, in the case $\alpha \in (0, N-4)$, the existence of a ground state of (kw^*, lw^*) type is obtained for all $\beta > 0$.

Next, for any $\lambda_1(x)$, $\lambda_2(x)$ satisfying $(C1)$ and $(C2)$, we have the following result.

Theorem 1.4 *Assume that* $(C1)$ *and* $(C2)$ *hold. Then system* (1.1) *has a positive ground state for all* $\beta > 0$.

In the proof of Theorem 1.4, we need to give an accurate estimate of the least energy so as to overcome the lack of compactness and show that both components of the solution we obtained are nontrivial. For this purpose, some results of equation (1.9) will be used. Denote the functional associated with (1.9) by

$$I_i(u) = \frac{1}{2}\int |\nabla u|^2 + \lambda_i(x)u^2 - \frac{N}{2(N+\alpha)}\mu_i \int \bigl(I_\alpha * |u|^{\frac{N+\alpha}{N}}\bigr)|u|^{\frac{N+\alpha}{N}},$$

and set

$$\mathcal{N}_i = \bigl\{u \in H^1\bigl(\mathbb{R}^N\bigr) \setminus \{0\} | \bigl\langle I_i'(u), u\bigr\rangle = 0\bigr\}, \qquad B_i = \inf_{\mathcal{N}_i} I_i(u), \quad i = 1, 2.$$

Then, from [16, Theorem 3,Theorem 6] and some calculation, we see that B_i is attained and

$$B_i \le \frac{\alpha}{2(N+\alpha)}\mu_i^{-\frac{N}{\alpha}} S_1^{\frac{N+\alpha}{\alpha}}, \tag{1.11}$$

where

$$S_1 = \inf_{(u,v)\in L^2(\mathbb{R}^N)\setminus\{0\}} \frac{\int u^2}{(\int (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}}. \tag{1.12}$$

By [9, Theorem 3.1], S_1 has a unique minimizer

$$U_*(x) := C\left(\frac{a}{a^2 + |x-b|^2}\right)^{\frac{N}{2}}. \tag{1.13}$$

We should also study the minimizing problem

$$\begin{aligned} S_0 = \inf_{\substack{(u,v)\in L\\ u\not\equiv 0, v\not\equiv 0}} &\biggl(\biggl(\int (u^2+v^2)\biggr) \\ &\bigg/\biggl(\biggl(\int \mu_1\bigl(I_\alpha * |u|^{\frac{N+\alpha}{N}}\bigr)|u|^{\frac{N+\alpha}{N}} + \mu_2\bigl(I_\alpha * |v|^{\frac{N+\alpha}{N}}\bigr)|v|^{\frac{N+\alpha}{N}} \\ &+ 2\beta\bigl(I_\alpha * |u|^{\frac{N+\alpha}{N}}\bigr)|v|^{\frac{N+\alpha}{N}}\biggr)^{\frac{N}{N+\alpha}}\biggr)\biggr), \end{aligned} \tag{1.14}$$

where $L = L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Problem (1.14) can be seen as an extension of the classical problem (1.12). By a similar approach as in the proof of Theorem 1.1, we obtain the following result.

Theorem 1.5 *If $\beta > 0$, then $S_0 = g(s_m)S_1$, and $(s_m U_*, U_*)$ is a solution of (1.14), where $g(s)$ is defined in (1.10) and s_m is a minimum point of $g(s)$. If $\beta < 0$, then*

$$S_0 = \left(\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}}\right)^{\frac{\alpha}{N+\alpha}} S_1$$

and S_0 is not attained.

Theorem 1.5 not only plays an important role in the proof of Theorem 1.4, but also extends the classical results of [9, Theorem 3.1].

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we study the minimizing problem

$$\begin{aligned} A = \inf_{\substack{(u,v)\in H \\ u\not\equiv 0, v\not\equiv 0}} &\left(\left(\int |\nabla u|^2 + \lambda(x)u^2 + |\nabla v|^2 + \lambda(x)v^2\right)\right. \\ &\Big/\left(\left(\int \mu_1\left(I_\alpha * |u|^{\frac{N+\alpha}{N}}\right)|u|^{\frac{N+\alpha}{N}} + \mu_2\left(I_\alpha * |v|^{\frac{N+\alpha}{N}}\right)|v|^{\frac{N+\alpha}{N}}\right.\right. \\ &\left.\left.\left. + 2\beta\left(I_\alpha * |u|^{\frac{N+\alpha}{N}}\right)|v|^{\frac{N+\alpha}{N}}\right)^{\frac{N}{N+\alpha}}\right)\right). \end{aligned}$$

Up to multiplication by a scalar, we know that a minimizer of A is a ground state of system (1.1) for $\lambda_1(x) = \lambda_2(x) := \lambda(x)$. Set

$$A_1 = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int |\nabla u|^2 + \lambda(x)u^2}{\left(\int (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}}\right)^{\frac{N}{N+\alpha}}}. \tag{2.1}$$

Letting w be a solution of (1.8), we know that A_1 is attained by w. By studying a function $g : \mathbb{R}^+ \mapsto \mathbb{R}$ defined by

$$g(s) = \frac{1+s^2}{\left(\mu_2 + \mu_1 s^{\frac{2(N+\alpha)}{N}} + 2\beta s^{\frac{N+\alpha}{N}}\right)^{\frac{N}{N+\alpha}}},$$

we are able to obtain the relationship between A and A_1 and show that A is attained.

Lemma 2.1 *If $\beta > 0$, then there is $s_m > 0$ such that $g(s_m) = \min_{s\ge 0} g(s)$.*

Proof By simple calculation, we have

$$g'(s) = \frac{2s\left(\mu_2 - \mu_1 s^{\frac{2\alpha}{N}} - \beta s^{\frac{\alpha-N}{N}} + \beta s^{\frac{N+\alpha}{N}}\right)}{\left(\mu_2 + \mu_1 s^{\frac{2(N+\alpha)}{N}} + 2\beta s^{\frac{N+\alpha}{N}}\right)^{\frac{2N+\alpha}{N+\alpha}}}.$$

Let $h(s) = \mu_2 - \mu_1 s^{\frac{2\alpha}{N}} - \beta s^{\frac{\alpha-N}{N}} + \beta s^{\frac{N+\alpha}{N}}$. If $\beta > 0$, then $h(s) \to -\infty$ as $s \to 0$, and $h(s) \to +\infty$ as $s \to +\infty$. Thus, there exists $s_m > 0$ such that $h(s_m) = 0$ and $g(s_m) = \min_{s\ge 0} g(s)$. □

Lemma 2.2 *Assume that (C1) and (C2) hold. If $\beta > 0$, then $A = g(s_m)A_1$.*

Proof We follow a similar approach as in [6, Theorem 1.1] and [24, Lemma 2.1] to prove this Lemma. For any $z \in H^1(\mathbb{R}^N) \setminus \{0\}$, we set $(u, v) := (s_m z, z)$. Then it follows that

$$A \leq \frac{(1 + s_m^2) \int |\nabla z|^2 + \lambda(x) z^2}{((\mu_2 + \mu_1 s_m^{\frac{2(N+\alpha)}{N}} + 2\beta s_m^{\frac{N+\alpha}{N}}) \int (I_\alpha * |z|^{\frac{N+\alpha}{N}}) z^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}}, \tag{2.2}$$

which indicates

$$A \leq g(s_m) A_1. \tag{2.3}$$

Let $(u_n, v_n) \in H$ be a minimizing sequence of A, and set $\xi_n = \tau_n u_n$, where

$$\tau_n = \left(\frac{\int (I_\alpha * |v_n|^{\frac{N+\alpha}{N}}) |v_n|^{\frac{N+\alpha}{N}}}{\int (I_\alpha * |u_n|^{\frac{N+\alpha}{N}}) |u_n|^{\frac{N+\alpha}{N}}} \right)^{\frac{N}{2(N+\alpha)}}.$$

Then we have

$$\int \left(I_\alpha * |\xi_n|^{\frac{N+\alpha}{N}}\right) |\xi_n|^{\frac{N+\alpha}{N}} = \int \left(I_\alpha * |v_n|^{\frac{N+\alpha}{N}}\right) |v_n|^{\frac{N+\alpha}{N}}. \tag{2.4}$$

From (2.4) and the property of the Riesz potential that $I_\alpha = I_{\frac{\alpha}{2}} * I_{\frac{\alpha}{2}}$, we obtain

$$\begin{aligned}
&\int \left(I_\alpha * |\xi_n|^{\frac{N+\alpha}{N}}\right) |v_n|^{\frac{N+\alpha}{N}} \\
&\quad = \int \left(I_{\frac{\alpha}{2}} * |\xi_n|^{\frac{N+\alpha}{N}}\right)\left(I_{\frac{\alpha}{2}} * |v_n|^{\frac{N+\alpha}{N}}\right) \\
&\quad \leq \left(\int \left| I_{\frac{\alpha}{2}} * |\xi_n|^{\frac{N+\alpha}{N}} \right|^2 \right)^{\frac{1}{2}} \left(\int \left| I_{\frac{\alpha}{2}} * |v_n|^{\frac{N+\alpha}{N}} \right|^2 \right)^{\frac{1}{2}} \\
&\quad = \left(\int \left(I_\alpha * |\xi_n|^{\frac{N+\alpha}{N}}\right) |\xi_n|^{\frac{N+\alpha}{N}} \right)^{\frac{1}{2}} \left(\int \left(I_\alpha * |v_n|^{\frac{N+\alpha}{N}}\right) |v_n|^{\frac{N+\alpha}{N}} \right)^{\frac{1}{2}} \\
&\quad = \int \left(I_\alpha * |v_n|^{\frac{N+\alpha}{N}}\right) |v_n|^{\frac{N+\alpha}{N}}.
\end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we have

$$\begin{aligned}
&A + o(1) \\
&\quad = \left(\int |\nabla u_n|^2 + \lambda(x) u_n^2 + |\nabla v_n|^2 + \lambda(x) v_n^2 \right) \\
&\qquad \Big/ \left(\left(\int \mu_1 \left(I_\alpha * |u_n|^{\frac{N+\alpha}{N}}\right) |u_n|^{\frac{N+\alpha}{N}} + \mu_2 \left(I_\alpha * |v_n|^{\frac{N+\alpha}{N}}\right) |v_n|^{\frac{N+\alpha}{N}} \right.\right. \\
&\qquad \left.\left. + 2\beta \left(I_\alpha * |u_n|^{\frac{N+\alpha}{N}}\right) |v_n|^{\frac{N+\alpha}{N}} \right)^{\frac{N}{N+\alpha}} \right) \\
&\quad \geq \frac{\tau_n^{-2} \int |\nabla \xi_n|^2 + \lambda(x) \xi_n^2 + \int |\nabla v_n|^2 + \lambda(x) v_n^2}{((\mu_2 + \mu_1 \tau_n^{-\frac{2(N+\alpha)}{N}} + 2\beta \tau_n^{-\frac{N+\alpha}{N}}) \int (I_\alpha * |\xi_n|^{\frac{N+\alpha}{N}}) |\xi_n|^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}} \\
&\quad = g\left(\tau_n^{-1}\right) A_1 \geq g(s_m) A_1.
\end{aligned} \tag{2.6}$$

Combining (2.3) with (2.6), we conclude that $A = g(s_m) A_1$. □

Proof of Theorem 1.1 From the proof of Lemma 2.1, we see that there exists $s_m > 0$ such that $h(s_m) = 0$, that is,

$$\mu_2 - \mu_1 s_m^{\frac{2\alpha}{N}} - \beta s_m^{\frac{\alpha-N}{N}} + \beta s_m^{\frac{N+\alpha}{N}} = 0. \tag{2.7}$$

From (2.7), we get

$$\mu_1 s_m^{\frac{2(N+\alpha)}{N}} + \beta s_m^{\frac{N+\alpha}{N}} = s^2\bigl(\mu_1 s_m^{\frac{2\alpha}{N}} + \beta s_m^{\frac{\alpha-N}{N}}\bigr) = s^2\bigl(\mu_2 + \beta s_m^{\frac{N+\alpha}{N}}\bigr).$$

Then it follows

$$\mu_2 + \mu_1 s_m^{\frac{2(N+\alpha)}{N}} + 2\beta s_m^{\frac{N+\alpha}{N}} = \mu_2 + \beta s_m^{\frac{N+\alpha}{N}} + \bigl(\mu_1 s_m^{\frac{2(N+\alpha)}{N}} + \beta s_m^{\frac{N+\alpha}{N}}\bigr) = \bigl(1+s^2\bigr)\bigl(\mu_2 + \beta s_m^{\frac{N+\alpha}{N}}\bigr),$$

which yields

$$g(s_m) = \frac{1+s_m^2}{(\mu_2 + \mu_1 s_m^{\frac{2(N+\alpha)}{N}} + 2\beta s_m^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}} = \frac{(1+s_m^2)^{\frac{\alpha}{N+\alpha}}}{(\mu_2 + \beta s_m^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}}. \tag{2.8}$$

Let $t_m = (1 + \beta s_m^{\frac{N+\alpha}{N}})^{-\frac{N}{2\alpha}}$, then $t_m(s_m w, w)$ is a positive solution of system (1.1). Moreover, by (2.7), (2.8), and Lemma 2.2, we have

$$\begin{aligned}
\mathcal{B} \le I\bigl(t_m(s_m w, w)\bigr) &= \frac{\alpha}{2(N+\alpha)} t_m^2\bigl(1+s_m^2\bigr)\int |\nabla w|^2 + \lambda(x) w^2 \\
&= \frac{\alpha}{2(N+\alpha)}\bigl(1+s_m^2\bigr)\bigl(\mu_2 + \beta s_m^{\frac{N+\alpha}{N}}\bigr)^{-\frac{N}{\alpha}} A_1^{\frac{N+\alpha}{\alpha}} \\
&= \frac{\alpha}{2(N+\alpha)}\bigl(g(s_m)A_1\bigr)^{\frac{N+\alpha}{\alpha}} \\
&= \frac{\alpha}{2(N+\alpha)} A^{\frac{N+\alpha}{\alpha}}.
\end{aligned}$$

On the other hand, $\forall (u,v) \in \mathcal{M}$, we have

$$I(u,v) = \frac{\alpha}{2(N+\alpha)}\int |\nabla u|^2 + \lambda(x)u^2 + |\nabla v|^2 + \lambda(x)v^2 \ge \frac{\alpha}{2(N+\alpha)} A^{\frac{N+\alpha}{\alpha}},$$

which indicates that $\mathcal{B} \ge \frac{\alpha}{2(N+\alpha)} A^{\frac{N+\alpha}{\alpha}}$. Thus, $\mathcal{B} = \frac{\alpha}{2(N+\alpha)} A^{\frac{N+\alpha}{\alpha}} = I(t_m s_m w, t_m w)$, that is, $(t_m s_m w, t_m w)$ is a positive ground state of system (1.1). □

3 Proof of Theorem 1.5

In this section, we prove Theorem 1.5, which is essential in the proof of Theorem 1.4. Recalling the definition of U_*, we have the following lemma.

Lemma 3.1 *If* $\beta > 0$, *then* $S_0 = g(s_m)S_1$, *and* S_0 *is attained by* $(s_m U_*, U_*)$.

Proof By a similar approach as that in Lemma 2.2, we see that $S_0 = g(s_m)S_1$. Then the conclusion follows from

$$\frac{(1+s_m^2)\int |U_*|^2}{((\mu_2 + \mu_1 s_m^{\frac{2(N+\alpha)}{N}} + 2\beta s_m^{\frac{N+\alpha}{N}})\int (I_\alpha * |U_*|^{\frac{N+\alpha}{N}})|U_*|^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}} = g(s_m)S_1.$$

□

Lemma 3.2 *If $\beta < 0$, then $S_0 = (\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}})^{\frac{\alpha}{N+\alpha}} S_1$, and S_0 is not attained.*

Proof Denote $(u_0, v_y) := (\mu_1^{-\frac{N}{2\alpha}} U_*(x), \mu_2^{-\frac{N}{2\alpha}} U_*(x + e_1 y))$, where $e = (1, 0, \dots, 0) \in \mathbb{R}^N$. Then $v_y^{\frac{N+\alpha}{N}} \rightharpoonup 0$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ as $y \to +\infty$. Taking account of the fact that $I_\alpha * |u_0|^{\frac{N+\alpha}{N}} \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, we have

$$\lim_{y \to +\infty} \int (I_\alpha * |u_0|^{\frac{N+\alpha}{N}}) |v_y|^{\frac{N+\alpha}{N}} = 0.$$

Then, for $|y|$ sufficiently large,

$$\begin{aligned} S_0 &\le \left(\int u_0^2 + v_y^2\right) \\ &\quad \Big/ \left(\left(\int \mu_1 \big(I_\alpha * |u_0|^{\frac{N+\alpha}{N}}\big)|u_0|^{\frac{N+\alpha}{N}} + \mu_2 \big(I_\alpha * |v_y|^{\frac{N+\alpha}{N}}\big)|v_y|^{\frac{N+\alpha}{N}} \right.\right. \\ &\quad \left.\left. + 2\beta \big(I_\alpha * |u_0|^{\frac{N+\alpha}{N}}\big)|v_y|^{\frac{N+\alpha}{N}} \right)^{\frac{N}{N+\alpha}}\right) \\ &= \frac{(\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}}) \int U_*^2}{((\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}}) \int (I_\alpha * |U_*|^{\frac{N+\alpha}{N}})|U_*|^{\frac{N+\alpha}{N}} + o(1))^{\frac{N}{N+\alpha}}}. \end{aligned}$$

By letting $y \to +\infty$, we get

$$S_0 \le \big(\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}}\big)^{\frac{\alpha}{N+\alpha}} S_1.$$

On the other hand, since $\beta < 0$, we know that

$$\begin{aligned} S_0 &\ge \inf_{\substack{(u,v)\in L \\ u\not\equiv 0, v\not\equiv 0}} \frac{\int u^2 + v^2}{(\int \mu_1 (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} + \mu_2 (I_\alpha * |v|^{\frac{N+\alpha}{N}})|v|^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}} \\ &\ge \frac{(\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}}) \int {U_*}^2}{((\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}}) \int (I_\alpha * |U_*|^{\frac{N+\alpha}{N}})|U_*|^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}} \\ &= \big(\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}}\big)^{\frac{\alpha}{N+\alpha}} S_1. \end{aligned}$$

Therefore,

$$S_0 = \big(\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}}\big)^{\frac{\alpha}{N+\alpha}} S_1. \tag{3.1}$$

If S_0 is attained by (u, v) with $u \not\equiv 0$, $v \not\equiv 0$, then

$$\begin{aligned} S_0 &= \frac{\int u^2 + v^2}{(\int \mu_1 (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} + \mu_2 (I_\alpha * |v|^{\frac{N+\alpha}{N}})|v|^{\frac{N+\alpha}{N}} + 2\beta (I_\alpha * |u|^{\frac{N+\alpha}{N}})|v|^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}} \\ &> \frac{\int u^2 + v^2}{(\int \mu_1 (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} + \mu_2 (I_\alpha * |v|^{\frac{N+\alpha}{N}})|v|^{\frac{N+\alpha}{N}})^{\frac{N}{N+\alpha}}} \\ &\ge \big(\mu_1^{-\frac{N}{\alpha}} + \mu_2^{-\frac{N}{\alpha}}\big)^{\frac{\alpha}{N+\alpha}} S_1, \end{aligned}$$

which contradicts (3.1). Thus, the conclusion holds. □

Proof of Theorem 1.5 By Lemmas 3.1 and 3.2, we see that Theorem 1.5 holds. □

4 Proof of Theorem 1.4

In this section, we define

$$B=\inf_{\eta\in\Gamma}\max_{t\in[0,1]}I\bigl(\eta(t)\bigr),$$

where

$$\Gamma=\bigl\{\eta\in\mathcal{C}\bigl([0,1],H\bigr)|\eta(0)=(0,0),I\bigl(\eta(1)\bigr)<0\bigr\}.$$

Set

$$\mathcal{N}=\bigl\{(u,v)\in H\setminus\bigl\{(0,0)\bigr\}|\bigl\langle I'(u,v),(u,v)\bigr\rangle=0\bigr\}.$$

By simple calculation and analysis, we see that for any $(u,v)\neq(0,0)$, there exists $t_0>0$ such that $t_0(u,v)\in\mathcal{N}$ and $I(t_0u,t_0v)=\max_{t\geq 0}I(tu,tv)$. Then, as in the proof of [23, Theorem 4.2], we know that

$$B=\inf_{(u,v)\in H\setminus(0,0)}\max_{t\geq 0}I(tu,tv)=\inf_{\mathcal{N}}I(u,v).$$

Moreover, since $\mathcal{M}\subset\mathcal{N}$, we have $B\leq\mathcal{B}$. We will show that B is attained by some positive solution (u,v) of system (1.1). To begin with, we give an estimate of the upper bound of B, which is important in recovering the compactness of the Palais–Smale sequence.

Lemma 4.1 *Assume that* (C1) *and* (C2) *hold. If* $\beta>0$, *then*

$$B<\min\biggl\{B_1,B_2,\frac{\alpha}{2(N+\alpha)}S_0^{\frac{N+\alpha}{\alpha}}\biggr\}.$$

Proof We first show that

$$B<\frac{\alpha}{2(N+\alpha)}S_0^{\frac{N+\alpha}{\alpha}}. \tag{4.1}$$

Recall (s_mU_*,U_*) defined in Theorem 1.5, and let $t>0$ be the constant such that $t(s_mU_*,U_*)\in\mathcal{N}$. Then, by Theorem 1.5 and direct calculation, we see that

$$\begin{aligned}B&\leq I\bigl(t(s_mU_*,U_*)\bigr)\\&=\frac{1}{2}t^2\int\bigl(1+s_m^2\bigr)|\nabla U_*|^2+\bigl(\lambda_1(x)s_m^2+\lambda_2(x)\bigr)U_*^2\\&\quad-\frac{N}{2(N+\alpha)}t^{\frac{2(N+\alpha)}{N}}\bigl(\mu_2+\mu_1s_m^{\frac{2(N+\alpha)}{N}}+2\beta s_m^{\frac{N+\alpha}{N}}\bigr)\int\bigl(I_\alpha*|U_*|^{\frac{N+\alpha}{N}}\bigr)|U_*|^{\frac{N+\alpha}{N}}\\&=\frac{1}{2}t^2\int\bigl(1+s_m^2\bigr)U_*^2\\&\quad-\frac{N}{2(N+\alpha)}t^{\frac{2(N+\alpha)}{N}}\bigl(\mu_2+\mu_1s_m^{\frac{2(N+\alpha)}{N}}+2\beta s_m^{\frac{N+\alpha}{N}}\bigr)\int\bigl(I_\alpha*|U_*|^{\frac{N+\alpha}{N}}\bigr)|U_*|^{\frac{N+\alpha}{N}}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}t^2\int s_m^2|\nabla U_*|^2+s_m^2\big(\lambda_1(x)-1\big)U_*^2+|\nabla U_*|^2+\big(\lambda_2(x)-1\big)U_*^2\\
&\le\frac{\alpha}{2(N+\alpha)}\big(g(s_m)S_1\big)^{\frac{N+\alpha}{\alpha}}\\
&+\frac{1}{2}t^2\int\big(1+s_m^2\big)|\nabla U_*|^2+s_m^2\big(\lambda_1(x)-1\big)U_*^2+\big(\lambda_2(x)-1\big)U_*^2\\
&=\frac{\alpha}{2(N+\alpha)}S_0^{\frac{N+\alpha}{\alpha}}+\frac{1}{2}t^2\int\big(1+s_m^2\big)|\nabla U_*|^2+s_m^2\big(\lambda_1(x)-1\big)U_*^2+\big(\lambda_2(x)-1\big)U_*^2.
\end{aligned}
$$

Denote $\phi_i(u)=\frac{1}{2}\int|\nabla u|^2+(\lambda_i(x)-1)u^2$, $i=1,2$. To get (4.1), it suffices to show

$$
\phi_i(U_*)<0,\quad i=1,2, \tag{4.2}
$$

for some $b\in\mathbb{R}^N$. By the fact that

$$
\int\frac{|x|^2}{(1+|x|^2)^{N+2}}=\frac{N-2}{4(N+1)}\int\frac{1}{x^2(1+x^2)^N},
$$

we obtain

$$
\int|\nabla U_*|^2=\frac{N^2(N-2)}{4(N+1)}\int\frac{|U_*|^2}{|x|^2}.
$$

After a transformation $x=b+ay$, we have

$$
a^2\phi_i(U_*)=\int\left(\frac{N^2(N-2)}{4(N+1)|y|^2}-a^2\big(1-\lambda_i(b+ay)\big)\right)\frac{C^2}{(1+|y|^2)^N}\,dy.
$$

Then from $(C2)$ we see that (4.2) holds for $b=0$, and (4.1) follows.

Next, we show $B<B_i$, $i=1,2$. Let w_i be a positive solution of (1.9) for $i=1,2$ and $t(\tau)>0$ such that $(\sqrt{t(\tau)}w_1,\sqrt{t(\tau)}\tau w_1)\in\mathcal{N}$. Then

$$
t(\tau)^{\frac{\alpha}{N}}=\frac{\int|\nabla w_1|^2+\lambda_1(x)w_1^2+\tau^2(|\nabla w_1|^2+\lambda_2(x)w_1^2)}{(\mu_1+2\beta\tau^{\frac{N+\alpha}{N}}+\mu_2\tau^{\frac{2(N+\alpha)}{N}})\int(I_\alpha*|w_1|^{\frac{N+\alpha}{N}})|w_1|^{\frac{N+\alpha}{N}}}.
$$

By simple calculation, we get

$$
\lim_{\tau\to0^+}\frac{t'(\tau)}{|\tau|^{\frac{\alpha}{N}-1}\tau}=-\frac{2(N+\alpha)}{\alpha\mu_1}\beta.
$$

It follows that

$$
t(\tau)=1-\frac{2N}{\alpha\mu_1}\beta\tau^{\frac{N+\alpha}{N}}\big(1+o(1)\big),\quad\text{as }\tau\to0,
$$

and

$$
t(\tau)^{\frac{N+\alpha}{N}}=1-\frac{2(N+\alpha)}{\alpha\mu_1}\beta\tau^{\frac{N+\alpha}{N}}\big(1+o(1)\big),\quad\text{as }\tau\to0.
$$

Therefore,

$$\begin{aligned}
B &\le I\big(\sqrt{t(\tau)}w_1, \sqrt{t(\tau)}\tau w_1\big)\\
&= \frac{\alpha}{2(N+\alpha)} t(\tau)^{\frac{N+\alpha}{N}} \big(\mu_1 + 2\beta\tau^{\frac{N+\alpha}{N}} + \mu_2 \tau^{\frac{2(N+\alpha)}{N}}\big) \int \big(I_\alpha * |w_1|^{\frac{N+\alpha}{N}}\big)|w_1|^{\frac{N+\alpha}{N}}\\
&= \frac{\alpha}{2(N+\alpha)} \int \mu_1 \big(I_\alpha * |w_1|^{\frac{N+\alpha}{N}}\big)|w_1|^{\frac{N+\alpha}{N}}\\
&\quad - \frac{N}{N+\alpha}\beta\tau^{\frac{N+\alpha}{N}} \int \big(I_\alpha * |w_1|^{\frac{N+\alpha}{N}}\big)|w_1|^{\frac{N+\alpha}{N}} + o\big(\tau^{\frac{N+\alpha}{N}}\big)\\
&< B_1 \quad \text{for } \tau > 0 \text{ small enough.}
\end{aligned}$$

Similarly, we have $B < B_2$. □

Next, we prove a Brezis–Lieb type lemma.

Lemma 4.2 *Let $\{(u_n, v_n)\}$ be a bounded sequence in H, and $(u_n, v_n) \to (u, v)$ a.e on $\mathbb{R}^N$ as $n \to \infty$. Then*

$$\int \big(I_\alpha * |u_n|^{\frac{N+\alpha}{N}}\big)|v_n|^{\frac{N+\alpha}{N}} - \int \big(I_\alpha * |u_n - u|^{\frac{N+\alpha}{N}}\big)|v_n - v|^{\frac{N+\alpha}{N}} \to \int \big(I_\alpha * |u|^{\frac{N+\alpha}{N}}\big)|v|^{\frac{N+\alpha}{N}}$$

as $n \to \infty$.

Proof From the Brezis–Lieb lemma [23], we know that

$$\begin{aligned}
&|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}} \to |u|^{\frac{N+\alpha}{N}}, \quad \text{in } L^{\frac{2N}{N+\alpha}}\big(\mathbb{R}^N\big),\\
&|v_n|^{\frac{N+\alpha}{N}} - |v_n - v|^{\frac{N+\alpha}{N}} \to |v|^{\frac{N+\alpha}{N}}, \quad \text{in } L^{\frac{2N}{N+\alpha}}\big(\mathbb{R}^N\big),
\end{aligned}$$

as $n \to \infty$. Then, according to the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned}
&I_\alpha * \big(|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}}\big) \to I_\alpha * |u|^{\frac{N+\alpha}{N}} \quad \text{in } L^{\frac{2N}{N-\alpha}}\big(\mathbb{R}^N\big),\\
&I_\alpha * \big(|v_n|^{\frac{N+\alpha}{N}} - |v_n - v|^{\frac{N+\alpha}{N}}\big) \to I_\alpha * |v|^{\frac{N+\alpha}{N}} \quad \text{in } L^{\frac{2N}{N-\alpha}}\big(\mathbb{R}^N\big),
\end{aligned}$$

as $n \to \infty$. Observing that

$$\begin{aligned}
&\int \big(I_\alpha * |u_n|^{\frac{N+\alpha}{N}}\big)|v_n|^{\frac{N+\alpha}{N}} - \int \big(I_\alpha * |u_n - u|^{\frac{N+\alpha}{N}}\big)|v_n - v|^{\frac{N+\alpha}{N}}\\
&\quad = \int I_\alpha * \big(|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}}\big)\big(|v_n|^{\frac{N+\alpha}{N}} - |v_n - v|^{\frac{N+\alpha}{N}}\big)\\
&\qquad + \int I_\alpha * \big(|v_n|^{\frac{N+\alpha}{N}} - |v_n - v|^{\frac{N+\alpha}{N}}\big)|u_n - u|^{\frac{N+\alpha}{N}}\\
&\qquad + \int I_\alpha * \big(|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}}\big)|v_n - v|^{\frac{N+\alpha}{N}},
\end{aligned} \tag{4.3}$$

and

$$|u_n - u|^{\frac{N+\alpha}{N}} \rightharpoonup 0, \qquad |v_n - v|^{\frac{N+\alpha}{N}} \rightharpoonup 0 \quad \text{in } L^{\frac{2N}{N+\alpha}}\big(\mathbb{R}^N\big),$$

we see that the conclusion holds. □

Proof of Theorem 1.4 According to the mountain pass theorem [23], we obtain that there is $\{(u_n, v_n)\} \subset \mathcal{N}$ satisfying

$$I(u_n, v_n) \to B, I'(u_n, v_n) \to 0 \quad \text{in } H^{-1}.$$

It follows that

$$\begin{aligned} B + o(1) &\geq I(u_n, v_n) - \frac{N}{2(N+\alpha)}\langle I'(u_n, v_n), (u_n, v_n)\rangle \\ &= \frac{\alpha}{2(N+\alpha)} \int |\nabla u_n|^2 + \lambda_1(x)u_n^2 + |\nabla v_n|^2 + \lambda_2(x)v_n^2 \end{aligned}$$

for n large enough, which combined with assumption $(C1)$ implies that $\{(u_n, v_n)\}$ is bounded in H. Then we may assume that

$$\begin{aligned} &(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } H, \\ &(u_n, v_n) \to (u, v) \quad \text{in } L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N), \\ &(u_n, v_n) \to (u, v) \quad \text{a.e on } \mathbb{R}^N. \end{aligned}$$

Since $|u_n|^{\frac{N+\alpha}{N}}$ and $|v_n|^{\frac{N+\alpha}{N}}$ are bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, we have

$$|u_n|^{\frac{N+\alpha}{N}} \rightharpoonup |u|^{\frac{N+\alpha}{N}}, \qquad |v_n|^{\frac{N+\alpha}{N}} \rightharpoonup |v|^{\frac{N+\alpha}{N}} \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Using the Hardy–Littlewood–Sobolev inequality, we obtain

$$I_\alpha * |u_n|^{\frac{N+\alpha}{N}} \rightharpoonup I_\alpha * |u|^{\frac{N+\alpha}{N}}, \qquad I_\alpha * |v_n|^{\frac{N+\alpha}{N}} \rightharpoonup I_\alpha * |v|^{\frac{N+\alpha}{N}} \quad \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).$$

Observing that

$$|u_n|^{\frac{\alpha}{N}-1}u_n \to |u|^{\frac{\alpha}{N}-1}u, \qquad |v_n|^{\frac{\alpha}{N}-1}v_n \to |v|^{\frac{\alpha}{N}-1}v \quad \text{in } L^{\frac{2N}{\alpha}}_{loc}(\mathbb{R}^N),$$

we have, for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} &\int \big(I_\alpha * |u_n|^{\frac{N+\alpha}{N}}\big)|u_n|^{\frac{\alpha}{N}-1}u_n\phi \to \int \big(I_\alpha * |u|^{\frac{N+\alpha}{N}}\big)|u|^{\frac{\alpha}{N}-1}u\phi, \\ &\int \big(I_\alpha * |v_n|^{\frac{N+\alpha}{N}}\big)|v_n|^{\frac{\alpha}{N}-1}v_n\phi \to \int \big(I_\alpha * |v|^{\frac{N+\alpha}{N}}\big)|v|^{\frac{\alpha}{N}-1}v\phi, \\ &\int \big(I_\alpha * |u_n|^{\frac{N+\alpha}{N}}\big)|v_n|^{\frac{\alpha}{N}-1}v_n\phi \to \int \big(I_\alpha * |u|^{\frac{N+\alpha}{N}}\big)|v|^{\frac{\alpha}{N}-1}v\phi, \\ &\int \big(I_\alpha * |v_n|^{\frac{N+\alpha}{N}}\big)|u_n|^{\frac{\alpha}{N}-1}u_n\phi \to \int \big(I_\alpha * |v|^{\frac{N+\alpha}{N}}\big)|u|^{\frac{\alpha}{N}-1}u\phi, \end{aligned} \tag{4.4}$$

as $n \to \infty$. Taking account of $I'(u_n, v_n) \to 0$, (4.4), and the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, we have $I'(u, v) = 0$. Denote $z_n = u_n - u$, $\omega_n = v_n - v$, then $(z_n, \omega_n) \rightharpoonup (0, 0)$ in H, $(z_n, \omega_n) \to (0, 0)$ in $L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N)$, and $(z_n, \omega_n) \to (0, 0)$ a.e on $\mathbb{R}^N$. By $(C1)$, there exists

$R > 0$ sufficiently large such that

$$\begin{aligned} \int \lambda_1(x)z_n^2 + \lambda_2(x)\omega_n^2 &= \int_{\mathbb{R}^N \setminus B(0,R)} z_n^2 + \omega_n^2 + \int_{B(0,R)} \lambda_1(x)z_n^2 + \lambda_2(x)\omega_n^2 + o(1) \\ &= \int z_n^2 + \omega_n^2 + o(1). \end{aligned} \tag{4.5}$$

Denote

$$\begin{aligned} J(u,v) &= \frac{1}{2}\int |\nabla u|^2 + u^2 + |\nabla v|^2 + v^2 \\ &\quad - \frac{N}{2(N+\alpha)}\int \bigl(\mu_1\bigl(I_\alpha * |u|^{\frac{N+\alpha}{N}}\bigr)|u|^{\frac{N+\alpha}{N}} + \mu_2\bigl(I_\alpha * |v|^{\frac{N+\alpha}{N}}\bigr)|v|^{\frac{N+\alpha}{N}} \\ &\quad + 2\beta\bigl(I_\alpha * |u|^{\frac{N+\alpha}{N}}\bigr)|v|^{\frac{N+\alpha}{N}}\bigr). \end{aligned}$$

Combining (4.5) with Lemma 4.2, we have, for n large enough,

$$\bigl\langle J'(z_n,w_n),(z_n,\omega_n)\bigr\rangle = \bigl\langle I'(u_n,v_n),(u_n,v_n)\bigr\rangle - \bigl\langle I'(u,v),(u,v)\bigr\rangle = o(1) \tag{4.6}$$

and

$$B + o(1) = I(u_n,v_n) = I(u,v) + J(z_n,\omega_n) + o(1). \tag{4.7}$$

Set

$$C_n = \int |\nabla z_n|^2 + z_n^2, \qquad D_n = \int |\nabla \omega_n|^2 + \omega_n^2.$$

Then it follows

$$B = I(u,v) + \frac{\alpha}{2(N+\alpha)}(C_n + D_n) + o(1). \tag{4.8}$$

We will show that $u \not\equiv 0$, $v \not\equiv 0$ by excluding the following three cases:

(i) $(u,v) \equiv (0,0)$. By (4.8), we know that

$$C_n + D_n > 0.$$

Denote

$$E_n = \int \mu_1\bigl(I_\alpha * |z_n|^{\frac{N+\alpha}{N}}\bigr)|z_n|^{\frac{N+\alpha}{N}}, \qquad F_n = \int \mu_2\bigl(I_\alpha * |w_n|^{\frac{N+\alpha}{N}}\bigr)|w_n|^{\frac{N+\alpha}{N}}.$$

If $E_n \to 0$, then $\int (I_\alpha * |w_n|^{\frac{N+\alpha}{N}})|z_n|^{\frac{N+\alpha}{N}} \to 0$. So we have

$$\begin{aligned} \int |\nabla z_n|^2 + z_n^2 + |\nabla w_n|^2 + w_n^2 &= \int \mu_1\bigl(I_\alpha * |w_n|^{\frac{N+\alpha}{N}}\bigr)|w_n|^{\frac{N+\alpha}{N}} + o(1). \\ &\le \mu_1 S_1^{-\frac{N+\alpha}{N}} \left(\int |w_n|^2\right)^{\frac{N+\alpha}{N}}. \end{aligned}$$

$$\leq \mu_1 S_1^{-\frac{N+\alpha}{N}} \left(\int |\nabla z_n|^2 + z_n^2 + |\nabla w_n|^2 + w_n^2 \right)^{\frac{N+\alpha}{N}},$$

which implies

$$\int |\nabla z_n|^2 + z_n^2 + |\nabla w_n|^2 + w_n^2 \geq \mu_1^{-\frac{N}{\alpha}} S_1^{\frac{N+\alpha}{\alpha}}.$$

Then, by (4.8) and (1.11), we obtain

$$B = I(u,v) + \frac{\alpha}{2(N+\alpha)}(C_n + D_n) + o(1) \geq \frac{\alpha}{2(N+\alpha)} \mu_1^{-\frac{N}{\alpha}} S_1^{\frac{N+\alpha}{\alpha}} > B_1,$$

which contradicts Lemma 4.1. Similarly, $F_n \to 0$ also leads to a contradiction. Thus, $E_n \geq \delta$ and $F_n \geq \delta$ for some $\delta > 0$ and n large enough. Then there exists $t_n > 0$ such that

$$\langle J'(t_n z_n, t_n \omega_n), (t_n z_n, t_n \omega_n) \rangle = 0$$

and

$$\begin{aligned} & J(t_n z_n, t_n \omega_n) \\ &\quad = \max_{s_n \geq 0} J(s_n z_n, s_n \omega_n) \\ &\quad \geq \max_{s_n \geq 0} \frac{1}{2} s_n^2 \int |z_n|^2 + \omega_n^2 \\ &\qquad - \frac{N s_n^{\frac{2(N+\alpha)}{N}}}{2(N+\alpha)} \int \Bigl(\mu_1 \bigl(I_\alpha * |z_n|^{\frac{N+\alpha}{N}} \bigr) |z_n|^{\frac{N+\alpha}{N}} + \mu_2 \bigl(I_\alpha * |\omega_n|^{\frac{N+\alpha}{N}} \bigr) |\omega_n|^{\frac{N+\alpha}{N}} \\ &\qquad + 2\beta \bigl(I_\alpha * |z_n|^{\frac{N+\alpha}{N}} \bigr) |\omega_n|^{\frac{N+\alpha}{N}} \Bigr) \\ &\quad \geq \frac{\alpha}{2(N+\alpha)} S_0^{\frac{N+\alpha}{\alpha}}, \end{aligned} \tag{4.9}$$

where the last inequality follows by Theorem 1.5. Moreover, by (4.6), we have $t_n \to 1$. Then we have

$$B = I(u,v) + J(z_n, \omega_n) = J(t_n z_n, t_n \omega_n) \geq \frac{\alpha}{2(N+\alpha)} S_0^{\frac{N+\alpha}{\alpha}},$$

which also contradicts Lemma 4.1.

(ii) $u \equiv 0$, $v \not\equiv 0$. In this case, it is clear that v is a solution of (1.9) for $i = 2$. Then, by (4.7), we have $B \geq I(0,v) \geq B_2$, which contradicts Lemma 4.1.

(iii) $v \equiv 0$, $u \not\equiv 0$. By similar arguments as in case (ii), we see that $B \geq B_1$, which also contradicts Lemma 4.1.

Thus, we have proved that $u \not\equiv 0$, $v \not\equiv 0$, and $I'(u,v) = 0$. Then $I(u,v) \geq B$, which combining with (4.7), (4.8) indicates $I(u,v) = B$. Hence, (u,v) is a ground state of system (1.1). Moreover, since $I(|u|,|v|) = B$ and $(|u|,|v|) \in \mathcal{N}$, we know that $(|u|,|v|)$ is also a ground state of (1.1). By the strong maximum principle, we have $|u| > 0$, $|v| > 0$. Thus, system (1.1) has a positive ground state $(|u|,|v|)$. □

Remark 4.3 Let (u,v) be a solution obtained in Theorem 1.4. Then it is obvious that $(u,v) \in \mathcal{M}$. Moreover, since $\mathcal{M} \in \mathcal{N}$, we have

$$B = I(u,v) \le \mathcal{B} \le I(u,v),$$

which implies that $B = \mathcal{B}$.

Appendix

Theorem A.1 *Assume that $N \ge 5$ and $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 < 0$. If*

$$\begin{cases} \beta > 0, & \alpha \in (0, N-4), \\ \beta > \max\{\mu_1, \mu_2\}, & \alpha = N-4. \end{cases}$$

Then system (1.5) has a positive ground state $\zeta_m(s_m^* w^*, w^*)$, where $\zeta_m = (\mu_2 + \beta s_m^{*\frac{N+\alpha}{N-2}})^{-\frac{N-2}{2(\alpha+2)}}$, s_m^* is a minimum point of a function $l(s) : \mathbb{R}^+ \mapsto \mathbb{R}$ defined by

$$l(s) = \frac{1+s^2}{(\mu_1 s^{\frac{2(N+\alpha)}{N-2}} + \mu_2 + 2\beta s^{\frac{N+\alpha}{N-2}})^{\frac{N-2}{N+\alpha}}},$$

and w^* is a positive ground state of (1.6).

In order to prove Theorem A.1, we define the functional associated with (1.5) by

$$\begin{aligned} E(u,v) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2 \\ &\quad - \frac{N-2}{2(N+\alpha)}\int_\Omega \Big(\mu_1\big(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\big)|u|^{\frac{N+\alpha}{N-2}} + \mu_2\big(I_\alpha * |v|^{\frac{N+\alpha}{N-2}}\big)|v|^{\frac{N+\alpha}{N-2}} \\ &\quad + 2\beta\big(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\big)|v|^{\frac{N+\alpha}{N-2}}\Big). \end{aligned}$$

Set $H^* = H_0^1(\Omega) \times H_0^1(\Omega)$ and

$$\begin{aligned} \mathcal{M}^* = \bigg\{ &(u,v) \in H^*, u,v \not\equiv 0, \\ &\int_\Omega |\nabla u|^2 + \lambda_1 u^2 = \int_\Omega \mu_1\big(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\big)|u|^{\frac{N+\alpha}{N-2}} + \beta\big(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\big)|v|^{\frac{N+\alpha}{N-2}}, \\ &\int_\Omega |\nabla v|^2 + \lambda_2 v^2 = \int_\Omega \mu_2\big(I_\alpha * |v|^{\frac{N+\alpha}{N-2}}\big)|v|^{\frac{N+\alpha}{N-2}} + \beta\big(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\big)|v|^{\frac{N+\alpha}{N-2}} \bigg\}, \end{aligned}$$

and $\mathcal{B}^* = \inf_{\mathcal{M}^*} E(u,v)$. Set

$$\begin{aligned} A_0^* = \inf_{\substack{(u,v)\in H^* \\ u\not\equiv 0, v\not\equiv 0}} &\bigg(\bigg(\int_\Omega |\nabla u|^2 + \lambda u^2 + |\nabla v|^2 + \lambda v^2\bigg) \\ &\Big/ \bigg(\bigg(\int_\Omega \mu_1\big(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\big)|u|^{\frac{N+\alpha}{N-2}} + \mu_2\big(I_\alpha * |v|^{\frac{N+\alpha}{N-2}}\big)|v|^{\frac{N+\alpha}{N-2}} \\ &+ 2\beta\big(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\big)|v|^{\frac{N+\alpha}{N-2}}\bigg)^{\frac{N-2}{N+\alpha}}\bigg)\bigg) \end{aligned}$$

and

$$A_1^* = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 + \lambda_1 u^2}{(\int_\Omega (I_\alpha * |u|^{\frac{N+\alpha}{N-2}})|u|^{\frac{N+\alpha}{N-2}})^{\frac{N-2}{N+\alpha}}}.$$

By studying the minimum point of $l(s)$ and analyzing as in the proof of Lemma 2.2, we have the following.

Lemma A.2 *Assume that* $N \geq 5$ *and*

$$\begin{cases} \beta > 0 & \text{for } \alpha \in (0, N-4), \\ \beta > \max\{\mu_1, \mu_2\} & \text{for } \alpha = N-4. \end{cases}$$

Then $A_0^* = l(s_m^*)A_1^*$, *where* s_m^* *is a minimum point of* $l(s)$.

Proof By some calculation, we have

$$l'(s) = \frac{2s(\mu_2 + \beta s^{\frac{N+\alpha}{N-2}} - \mu_1 s^{\frac{2\alpha+4}{N-2}} - \beta s^{\frac{\alpha-N+4}{N-2}})}{(\mu_1 s^{\frac{2(N+\alpha)}{N-2}} + \mu_2 + 2\beta s^{\frac{N+\alpha}{N-2}})^{\frac{2(N-2)}{N+\alpha}}}.$$

Denote

$$p(s) = \begin{cases} \mu_2 + \beta s^{\frac{N+\alpha}{N-2}} - \mu_1 s^{\frac{2\alpha+4}{N-2}} - \beta s^{\frac{\alpha-N+4}{N-2}} & \text{for } \alpha \in (0, \alpha-4), \\ \mu_2 - \beta - (\mu_1 - \beta)s^2 & \text{for } \alpha = N-4. \end{cases}$$

If $N \geq 5$, $\alpha \in (0, N-4)$, then $p(s) \to -\infty$ as $s \to 0$, and $p(s) \to +\infty$ as $s \to +\infty$. So there exists $s_{min}^* > 0$ such that $p(s_{min}^*) = 0$ and $l(s_{min}^*) = \min_{s\geq 0} l(s)$. If $N \geq 5$, $\alpha = N-4$, and $\beta > \max\{\mu_1, \mu_2\}$, it is clear that $p(s)$ has a zero point $s_{min}^* > 0$ such that $l(s_{min}^*) = \min_{s\geq 0} l(s)$. Then, by a similar argument as in the proof of Lemma 2.2, we see that

$$A_0^* = l(s_m^*)A_1^*.$$ □

Proof of Theorem A.1 From Lemma A.2, we know that $p(s_m^*) = 0$. Then it follows that

$$\mu_1 {s_m^*}^{\frac{2(N+\alpha)}{N-2}} + \mu_2 + 2\beta {s_m^*}^{\frac{(N+\alpha)}{N-2}} = (1 + {s_m^*}^2)(\mu_2 + \beta {s_m^*}^{\frac{(N+\alpha)}{N-2}}).$$

By the definition of $l(s)$, we have

$$l(s_m^*) = \frac{(1 + {s_m^*}^2)^{\frac{\alpha+2}{N+\alpha}}}{(\mu_2 + \beta {s_m^*}^{\frac{(N+\alpha)}{N-2}})^{\frac{N-2}{N+\alpha}}}.$$

Let $\zeta_m = (\mu_2 + \beta s_m^{*\frac{N+\alpha}{N-2}})^{-\frac{N-2}{2(\alpha+2)}}$, then $\zeta_m(s_m^* w^*, w^*)$ is a positive solution of system (1.5). Moreover, by Lemma A.2 and direct calculation, we have

$$\begin{aligned}\mathcal{B}^* \le E\big(\zeta_m\big(s_m^* w^*, w^*\big)\big) &= \frac{\alpha+2}{2(N+\alpha)}\big(1+s_m^{*2}\big)\int_\Omega |\nabla w^*|^2 + \lambda|w^*|^2 \\ &= \frac{\alpha+2}{2(N+\alpha)}\big(1+s_m^{*2}\big)\big(\mu_2+\beta s_m^{*\frac{N+\alpha}{N-2}}\big)^{-\frac{N-2}{\alpha+2}} A_1^{*\frac{N+\alpha}{\alpha+2}} \\ &= \frac{\alpha+2}{2(N+\alpha)} A^{*\frac{N+\alpha}{\alpha+2}}.\end{aligned}$$

On the other hand, for any $(u,v) \in \mathcal{M}^*$, by Lemma A.2 again, we have

$$E(u,v) \ge \mathcal{B}^* = \frac{\alpha+2}{2(N+\alpha)} A^{*\frac{N+\alpha}{\alpha+2}}.$$

Thus, $\zeta_m(s_m^* w^*, w^*)$ is a positive ground state of (1.5). □

Remark A.3 For the case $N \ge 5$, $\alpha = N-4$, and $0 < \beta < \min\{\mu_1, \mu_2\}$, we see from the proof of Lemma A.2 that there exists s_0 such that $p(s_0) = 0$. Then, arguing as in the proof of Theorem A.1, we see that (1.5) has a positive solution $\zeta_0(s_0 w^*, w^*)$, where $\zeta_0 = (\mu_2 + \beta s_0^{\frac{N+\alpha}{N-2}})^{-\frac{N-2}{2(\alpha+2)}}$. However, by our method, we do not know whether this solution is a ground state or not.

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Authors' contributions
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