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Hopf bifurcation in a delayed reaction–diffusion–advection equation with ideal free dispersal

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Abstract

In this paper, we investigate a delay reaction–diffusion–advection model with ideal free dispersal. The stability of positive steady-state solutions and the existence of the associated Hopf bifurcation are obtained by analyzing the principal eigenvalue of an elliptic operator. By the normal form theory and the center manifold reduction, the stability and bifurcation direction of Hopf bifurcating periodic solutions are obtained. Moreover, numerical simulations and a brief discussion are presented to illustrate our theoretical results.

Keywords: Hopf bifurcation; Reaction–diffusion; Delay; The ideal free distribution

1 Introduction

In recent years, biological mathematics has developed to be one of the most active research directions in the field of applied mathematics. The study of biological mathematics usually includes two aspects. One is to understand and predict the mechanism of biological processes by establishing and analyzing mathematical models. The other is to discover new mathematical problems, explore new mathematical research directions, and develop new mathematical methods via these models.

One important problem in spatial ecology is the effect of spatially inhomogeneous environment on the invasion of species. Spatially inhomogeneous environment refers to the heterogeneous distribution of various environmental conditions in space. For example, phytoplankton in an ocean or lake require light whose intensity in the vertical direction depends on depth. In heterogeneous environments, the movement of species also involves advection, besides random diffusion. Belgacem and Cosner [2] proposed a classical dispersal strategy by considering an advection term into a single population model with a spatially heterogeneous environment to describe that a population moves towards a more favorable environment,

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla[d\nabla u - \alpha u \nabla m] + u(m(x) - u), & (t,x) \in (0,+\infty) \times \Omega, \\ (d\nabla u - \alpha u\nabla m) \cdot \vec{n} = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where the flux of the population density $u(t,x)$ consists of two components: $d\nabla u$ and $\alpha u\nabla m$. The second one represents the movement upward along the gradient of the resource function, $\alpha > 0$ describes the advection speed/rate. $\vec{n}$ denotes the unit outward normal on the $\partial\Omega$. The result of Belgacem and Cosner [2] shows that, for an individual species, the movement upward along the gradient of the resource function will generally contribute to the survival of the species.

Since the movement of species may not perfectly track resource gradients in reality, Cantrell, Cosner and Lou [6] considered the biased movement strategy for the species

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla[d\nabla u - \alpha u\nabla P] + u(m(x) - u), & (t,x) \in (0,+\infty) \times \Omega, \\ (d\nabla u - \alpha u\nabla P) \cdot \vec{n} = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Here, the flux of the population density of species can be described by $-d\nabla u + \alpha u\nabla P$, where $P(x)$ describes the movement tendency of the species.

The ideal free distribution (IFD), introduced in [13], describes how populations distribute themselves if they move freely to optimize their fitness. From the viewpoint of the evolution of dispersal, an ideal free strategy refers to the idea that such a distribution would be expected if individuals have complete knowledge of their environment and are free to locate themselves wherever they want, specifically under the assumption that the presence of other individuals influences fitness; see [1, 4, 5, 7, 8, 12, 21] and the references therein. Motivated by [1], we say that P is an ideal free strategy if P can be found as follows.

To discuss the ideal free strategy, we require that $m(x) > 0$ in $\overline{\Omega}$, then (1.2) has a unique positive steady state $\widetilde{u}$ satisfying

$$\begin{cases} \nabla[d\nabla\widetilde{u} - \alpha\widetilde{u}\nabla P] + \widetilde{u}(m(x) - \widetilde{u}) = 0, & \text{in } \Omega, \\ (d\nabla\widetilde{u} - \alpha\widetilde{u}\nabla P) \cdot \vec{n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Integrating (1.3) and applying the divergence theorem, we obtain

$$\int_\Omega \nabla[d\nabla\widetilde{u} - \alpha\widetilde{u}\nabla P] + \widetilde{u}\big(m(x) - \widetilde{u}\big) = \int_\Omega \widetilde{u}\big(m(x) - \widetilde{u}\big) = 0.$$

Since $\widetilde{u}, m(x) > 0$ in Ω, either $\widetilde{u} \equiv m(x)$ or $\widetilde{u} - m(x)$ changes sign in Ω. Clearly, if $\widetilde{u} \equiv m(x)$, then we have

$$\nabla \cdot [d\nabla\widetilde{u} - \alpha\widetilde{u}\nabla P] = 0, \quad \text{in } \Omega,$$

which holds if

$$P = \ln m^{\frac{d}{\alpha}} + C \quad \text{for some constant } C. \tag{1.4}$$

The delay reaction–diffusion equation, which reflects the interaction between delay feedback of system and space migration impacts on the state of the system, is a kind of new and important mathematical model. During the past 30 years, it has appeared widely in many fields such as population biology, chemistry, physics, communication, and computer. In the real world, the phenomena of time delay and spatial diffusion are widespread.

For example, in a population model, time delay usually indicates resources regeneration time, mature period, or lactation time, etc., and in the infectious disease model, time delay usually indicates the incubation period, etc. Meanwhile, like cells, bacteria, chemicals, animals, each individual usually moves randomly, and their distribution is not uniform in space, which leads to the spread of the population in space.

Under homogeneous Neumann boundary conditions, the unique positive steady state is a constant and the stable bifurcating periodic orbit is spatially homogeneous. But for models with the homogeneous Dirichlet boundary conditions, the positive equilibrium is always spatially nonhomogeneous. Busenberg and Huang [3] first studied the Hopf bifurcation of the diffusive logistic equation with a delay effect and Dirichlet boundary condition,

$$\begin{cases} \frac{\partial u}{\partial t} = du_{xx} + ku(1-u(t-\tau)), & (t,x) \in (0,+\infty) \times (0,\pi), \\ u = 0, & t > 0, x = 0, \pi. \end{cases} \tag{1.5}$$

They have shown that:

1. If $k \le 1$, then system (1.5) does not have a positive equilibrium and the zero solution is a global attractor of all non-negative solutions.
2. If $k > 1$, then the zero solution of system (1.5) is unstable and there is a unique nonhomogeneous positive equilibrium U_k.
3. U_k is locally asymptotically stable if $(k\tau) \cdot \max_{x\in(0,\pi)}\{U_k(x)\} < \frac{\pi}{2}$.
4. One can give an estimate for $U_k(x)$ by using the implicit function theorem for $k \in [1, k^*]$.
5. There is a $\tau_k > 0$ such that the equilibrium $U_k(x)$ is locally stable if $0 \le \tau < \tau_k$, unstable if $\tau > \tau_k$.
6. There exists a sequence $\{\tau_{k_j}\}_{j=0}^{\infty}$ such that a Hopf bifurcation arising from $U_k(x)$ as the delay τ monotonically passes through each τ_{k_j}. Moreover, the periodic solution occurring from the Hopf bifurcation point τ_{k_0} is stable, and those occurring from the Hopf bifurcation points τ_{k_j}, $j > 0$, are unstable.

A population may tend to move up or down along the gradient of the habitats because of the heterogeneity of the environment. Chen, Lou, and Wei [9] considered the following model:

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla[d\nabla u - \alpha u \nabla m(x)] + u(m(x) - u(t-\tau, x)), & (t,x) \in (0,+\infty) \times \Omega, \\ u = 0, & t > 0, x \in \partial\Omega. \end{cases} \tag{1.6}$$

Their results imply that the increase of time delay can make the spatially nonhomogeneous positive steady state unstable for (1.6), and the model can exhibit an oscillatory pattern through Hopf bifurcation. They also considered the effect of advection on Hopf bifurcation values, and the Hopf bifurcation is more likely to occur with the increase of advection rate.

We also point out that there are several mathematical models formulated to describe the effect of time delay on Hopf bifurcation of the spatially nonhomogeneous positive equilibrium. These models include single population models [18–20, 23], competition diffusion systems [24, 26], predator–prey diffusion models [17, 25] and nonlocal delay models [10, 11, 14, 15].

In this paper, we introduce the notion of the IFD into the Hopf bifurcation problems to understand the Hopf bifurcation and bifurcation direction of the spatially nonhomogeneous positive steady state, and consider the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla[d\nabla u - \alpha u \nabla P] + u(m(x) - u(t-r,x)), & (t,x) \in (0,+\infty) \times \Omega, \\ (d\nabla u - \alpha u\nabla P) \cdot \vec{n} = 0, & t > 0, x \in \partial\Omega, \\ u(\theta,x) = \phi(\theta,x), & (\theta,x) \in [-\tau,0] \times \Omega, \end{cases} \tag{1.7}$$

where $u(t,x)$ represents the population density, $d > 0$ denotes the random diffusion coefficient, $P(x) \in C^2(\overline{\Omega})$ describes the movement tendency of the species, which is referred as the biased movement strategy for the species, $m(x) \in C^2(\overline{\Omega})$ is the intrinsic growth rate, $\alpha > 0$ measures the rate of population movement upward along the gradient of the function $P(x)$, and delay $r > 0$ denotes the maturation time. Throughout this paper, we assume that the function $P(x)$ and the resource function $m(x)$ have the relationship described as (1.4).

Letting $v(t,x) = e^{-\alpha/dP(x)}u(t,x)$, $\psi(\theta,x) = e^{-(\alpha/d)P(x)}\phi(\theta,x)$, $\widetilde{v}(\widetilde{t},x) = v(t,x)$, $t = \widetilde{t}/d$ and denoting $\lambda = 1/d$, $\tau = dr$, then dropping the tilde sign, system (1.7) can be transformed as follows:

$$\begin{cases} \frac{\partial v}{\partial t} = e^{-\frac{\alpha}{d}P(x)}\nabla[e^{\frac{\alpha}{d}P(x)}\nabla v] + \lambda v(m(x) - e^{\frac{\alpha}{d}P(x)}v(t-\tau,x)), & (t,x) \in (0,+\infty) \times \Omega, \\ \nabla v \cdot \vec{n} = 0, & t > 0, x \in \partial\Omega, \\ v(\theta,x) = \psi(\theta,x), & (\theta,x) \in [-\tau,0] \times \Omega, \end{cases} \tag{1.8}$$

or

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{m(x)}\nabla[m(x)\nabla v] + \lambda m(x)v(1 - e^{\frac{\alpha}{d}C}v(t-\tau,x)), & (t,x) \in (0,+\infty) \times \Omega, \\ \nabla v \cdot \vec{n} = 0, & t > 0, x \in \partial\Omega, \\ v(\theta,x) = \psi(\theta,x), & (\theta,x) \in [-\tau,0] \times \Omega. \end{cases} \tag{1.9}$$

For the convenience of calculation in the following sections, we only focus on (1.8). Moreover, it is easy to see that system (1.8) or (1.9) has a unique positive equilibrium $e^{-\frac{\alpha}{d}P(x)}m(x)$ or $e^{-\frac{\alpha}{d}C}$.

The organization of the paper is as follows. In the next section, we study the stability and Hopf bifurcation of the spatially nonhomogeneous positive steady state for system (1.8). In Sect. 3, we investigate the bifurcation direction of Hopf bifurcating period orbits by using the normal form theory and the center manifold reduction. Finally, we give some numerical simulations and a brief discussion in Sect. 4.

2 Stability and Hopf bifurcation

As in [9], throughout the paper, we denote the spaces $\mathbb{X} = H^2(\Omega) \cap H_0^1(\Omega)$, $\mathbb{Y} = L^2(\Omega)$. We also denote the complexification of the linear space $\mathbb{X}_c := \mathbb{X} \oplus i\mathbb{X} = \{a + ib \mid a, b \in \mathbb{X}\}$, the domain of a linear operator L by $\mathfrak{D}(L)$, the kernel of L by $\mathfrak{N}(L)$, and the range of L by $\mathfrak{R}(L)$. Moreover, we take $\langle u,v\rangle = \int_\Omega \bar{u}(x)v(x)\,dx$ as the inner product of Hilbert space $\mathbb{Y}_c$. The Banach space of continuous and differentiable mappings from $[-\tau,0]$ into $\mathbb{Y}$ is denoted by $C = C([-\tau,0],\mathbb{Y})$ and $C^1 = C^1([-\tau,0],\mathbb{Y})$, respectively.

For the following analysis, we decompose the spaces $\mathbb{X}$ and $\mathbb{Y}$ as follows:

$$\mathbb{X} = K \oplus \mathbb{X}_1, \qquad \mathbb{Y} = K \oplus \mathbb{Y}_1,$$

where

$$K = \operatorname{span}\{1\}, \qquad \mathbb{X}_1 = \left\{ y \in \mathbb{X} \,\middle|\, \int_\Omega y(x)\,dx = 0 \right\}, \qquad \mathbb{Y}_1 = \left\{ y \in \mathbb{Y} \,\middle|\, \int_\Omega y(x)\,dx = 0 \right\}.$$

The linearization of (1.8) at $e^{-\frac{\alpha}{d}P(x)}m(x)$ is given by

$$\begin{cases} \frac{\partial \omega}{\partial t} = e^{-\frac{\alpha}{d}P(x)} \nabla [e^{\frac{\alpha}{d}P(x)} \nabla \omega] - \lambda m(x)\omega(t-\tau), & (t,x) \in (0,+\infty) \times \Omega, \\ \nabla \omega \cdot \vec{n} = 0, & t > 0, x \in \partial\Omega. \end{cases} \tag{2.1}$$

It follows that the solution semigroup of the problem (2.1) has the infinitesimal generator satisfying

$$A_\tau(\lambda)\varphi = \dot{\varphi},$$

with

$$\mathfrak{D}\big(A_\tau(\lambda)\big) = \big\{\varphi \in C_c \cap C_c^1 : \dot{\varphi}(0) = e^{-\frac{\alpha}{d}P(x)} \nabla \big[e^{\frac{\alpha}{d}P(x)} \nabla \varphi(0)\big] - \lambda m(x)\varphi(-\tau)\big\},$$

where $C_c^1 = C^1([-\tau,0], \mathbb{Y}_c)$. Then the spectrum of $A_\tau(\lambda)$ is

$$\sigma\big(A_\tau(\lambda)\big) = \big\{\mu \in \mathcal{C}, \Lambda(\lambda,\mu,\tau)\varphi = 0, \text{ for some } \varphi \in \mathbb{X}_c \setminus \{0\}\big\}$$

with

$$\Lambda(\lambda,\mu,\tau)\varphi = e^{-\frac{\alpha}{d}P(x)} \nabla \big[e^{\frac{\alpha}{d}P(x)} \nabla \varphi\big] - \lambda m(x)\varphi e^{-\mu\tau} - \mu\varphi. \tag{2.2}$$

Lemma 2.1 *For any $\lambda > 0$, the steady state $e^{-\frac{\alpha}{d}P(x)}m(x)$ of (1.8) is locally asymptotically stable when $\tau = 0$. Moreover, 0 is not the spectrum of $A_\tau(\lambda)$, for any $\tau > 0$.*

Proof When $\tau = 0$, the spectrum of $A_\tau(\lambda)$ becomes

$$\Lambda(\lambda,\mu,0)\varphi = e^{-\frac{\alpha}{d}P(x)} \nabla \big[e^{\frac{\alpha}{d}P(x)} \nabla \varphi\big] - \lambda m(x)\varphi - \mu\varphi. \tag{2.3}$$

This, in turn, leads to the study of the linear eigenvalue problem

$$\begin{cases} e^{-\frac{\alpha}{d}P(x)} \nabla [e^{\frac{\alpha}{d}P(x)} \nabla \varphi] - \lambda m(x)\varphi = \mu\varphi, & x \in \Omega, \\ \nabla \varphi \cdot \vec{n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

Moreover, μ_1 has the following variational characterization:

$$\mu_1 = \max_{\varphi \in \mathbb{X}_c, \varphi \neq 0} \left[\frac{-\int_\Omega e^{\frac{\alpha}{d}P(x)} [(\nabla\varphi)^2 + \lambda m(x)\varphi^2]\,dx}{\int_\Omega e^{\frac{\alpha}{d}P(x)} \varphi^2\,dx} \right], \tag{2.5}$$

which yields $\mu_1 \le -\lambda \min_{x\in\Omega} m(x) < 0$. Thus we conclude that the steady state $e^{-\frac{\alpha}{d}P(x)}m(x)$ of (1.8) is locally asymptotically stable when $\tau = 0$.

Similarly, we can prove $0 \notin \sigma(A_\tau(\lambda))$ for any $\tau > 0$. □

Next, we will show that the eigenvalues of $A_\tau(\lambda)$ could pass through the imaginary axis for some $\tau > 0$. And, this is a necessary condition for hopf bifurcation to occur. Actually, $A_\tau(\lambda)$ has a purely imaginary eigenvalue $\mu = i\omega$ ($\omega > 0$) for some $\tau > 0$, if and only if

$$e^{-\frac{\alpha}{d}P(x)}\nabla\left[e^{\frac{\alpha}{d}P(x)}\nabla\varphi\right] - \lambda m(x)\varphi e^{-i\theta} - i\omega\varphi = 0 \tag{2.6}$$

is solvable for some value of $\omega > 0$, $\theta \in [0, 2\pi)$, $\omega\tau = \theta$, $\varphi \in \mathbb{X}_c$, $\varphi \neq 0$.

First, we give the following lemmas.

Lemma 2.2 *For some $\lambda^* > 0$, when $\lambda \in (0, \lambda^*)$, if there exist some $(\omega_\lambda, \theta_\lambda, \varphi_\lambda) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{X}_c\backslash\{0\}$ solving system (2.6), then $\frac{\omega_\lambda}{\lambda}$ is uniformly bounded.*

Proof Substituting $(\omega_\lambda, \theta_\lambda, \varphi_\lambda)$ into system (2.6) and multiplying $e^{\frac{\alpha}{d}P(x)}\overline{\varphi}_\lambda$, integrating the result over Ω, then we get

$$\left\langle\varphi_\lambda, \nabla\left[e^{\frac{\alpha}{d}P(x)}\nabla\varphi_\lambda\right]\right\rangle - \lambda\int_\Omega e^{\frac{\alpha}{d}P(x)}m(x)\varphi_\lambda^2 e^{-i\theta_\lambda} - i\omega_\lambda\int_\Omega e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2 = 0. \tag{2.7}$$

Separating the real and imaginary parts of the above equality, one can get

$$\lambda\sin\theta_\lambda\int_\Omega e^{\frac{\alpha}{d}P(x)}m(x)\varphi_\lambda^2 = \omega_\lambda\int_\Omega e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2.$$

Thus,

$$\frac{\omega_\lambda}{\lambda} = \frac{\sin\theta_\lambda\int_\Omega e^{\frac{\alpha}{d}P(x)}m(x)\varphi_\lambda^2}{\int_\Omega e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2} \le \max_\Omega m(x).$$ □

Lemma 2.3 *Let $\mathbf{L} := \nabla[e^{\frac{\alpha}{d}P(x)}\nabla]$. If $\nu \in \mathbb{X}_c$ and $\langle\nu, 1\rangle = 0$, then*

$$\left|\langle\mathbf{L}\nu, \nu\rangle\right| \ge \gamma_2\|\nu\|^2_{\mathbb{Y}_c},$$

where γ_2 is the second eigenvalue of operator $-\mathbf{L}$.

Proof It is well known that the operator $-\mathbf{L}$ on the domain Ω with zero-Neumann boundary conditions has a sequence of eigenvalues $\{\gamma_n\}_{n=1}^\infty$ satisfying

$$0 = \gamma_1 < \gamma_2 \le \gamma_3 \le \cdots, \qquad \lim_{n\to\infty}\gamma_n = \infty$$

and the corresponding eigenfunctions $\{\phi_n\}_{n=1}^\infty$, construct an orthogonal basis of $\mathbb{Y}_c$, moreover $\phi_1 = 1$. In particular, for each $\nu \in \mathbb{X}_c$ satisfying that $\langle\nu, 1\rangle = 0$, there is a sequence of real numbers $\{c_n\}_{n=2}^\infty$ such that $\nu = \sum_{n=2}^\infty c_n\phi_n$ and therefore $\mathbf{L}\nu = \sum_{n=2}^\infty c_n\mathbf{L}\phi_n = \sum_{n=2}^\infty c_n\gamma_n\phi_n$. It follows from the above equality that

$$\left|\langle\mathbf{L}\nu, \nu\rangle\right| = \gamma_n\sum_{n=2}^\infty c_n^2\|\phi_n\|^2_{L^2} \ge \gamma_2\sum_{n=2}^\infty c_n^2\|\phi_n\|^2_{Y_\mathbb{C}} = \gamma_2\|\nu\|^2_{Y_\mathbb{C}}.$$ □

For $\lambda \in (0,\lambda^*)$, if (ω,θ,φ) satisfies system (2.6), then, ignoring a scalar factor, we have $\varphi = \beta + \lambda\nu$, with $\beta > 0$ and $\|\varphi\|_{\mathbb{Y}_c} = |\Omega|$. Letting $\omega = \lambda h$, and substituting these into Eq. (2.6), we obtain

$$\begin{cases} f_1(\nu,\beta,\theta,h,\lambda) := \nabla[e^{\frac{\alpha}{d}P(x)}\nabla\nu] - e^{\frac{\alpha}{d}P(x)}m(x)(\beta+\lambda\nu)e^{-i\theta} \\ \qquad\qquad - ihe^{\frac{\alpha}{d}P(x)}(\beta+\lambda\nu) = 0, \\ f_2(\nu,\beta,\theta,h,\lambda) := (\beta^2-1)|\Omega| + \lambda^2\|\nu\|_{\mathbb{Y}_c} = 0. \end{cases} \tag{2.8}$$

Define $F = (f_1,f_2)$ from $(\mathbb{X}_1)_c \times \mathbb{R}^4 \to \mathbb{Y}_c \times \mathbb{R}$. The following lemma confirms that $F(\nu,\beta,\theta,h,\lambda) = 0$ is uniquely solvable when $\lambda \to 0$.

Lemma 2.4 *Consider the following equation*:

$$\begin{cases} F(\nu,\beta,\theta,h,0) = 0, \\ \nu \in \mathbb{X}_c, \qquad \beta \geq 0, \qquad \theta \in [0,\frac{\pi}{2}], \qquad h \geq 0. \end{cases} \tag{2.9}$$

Equation (2.9) *has a unique solution* $(\nu_0,\beta_0,\theta_0,h_0)$ *as* $\lambda \to 0$. *Here* $\beta_0 = 1$, $\theta_0 = \frac{\pi}{2}$, $h_0 = \frac{\int_\Omega m^2(x)}{\int_\Omega m(x)}$, *and* ν_0 *satisfies the following equation*:

$$\nabla\big[e^{\frac{\alpha}{d}P(x)}\nabla\nu_0\big] + ie^{\frac{\alpha}{d}P(x)}m(x) - ie^{\frac{\alpha}{d}P(x)}h_0 = 0. \tag{2.10}$$

Proof From the second equation of (2.8), we see that $f_2(\nu,\beta,\theta,h,0)$ if and only if $\beta_0 = 1$. Substituting $\beta_0 = 1$ into the first equation of (2.8), we can see ν_0 satisfies

$$\nabla\big[e^{\frac{\alpha}{d}P(x)}\nabla\nu_0\big] - m(x)e^{\frac{\alpha}{d}P(x)}e^{-i\theta_0} - ie^{\frac{\alpha}{d}P(x)}h_0 = 0. \tag{2.11}$$

Next, integrating Eq. (2.11) over Ω, and separating the real and imaginary parts, we obtain

$$\begin{cases} \cos\theta_0\int_\Omega e^{\frac{\alpha}{d}P(x)}m(x) = \cos\theta_0 e^{\frac{\alpha C}{d}}\int_\Omega m^2(x) = 0, \\ \sin\theta_0 e^{\frac{\alpha C}{d}}\int_\Omega m^2(x) = h_0 e^{\frac{\alpha C}{d}}\int_\Omega m(x). \end{cases} \tag{2.12}$$

By the periodicity of θ_0, we can set $\theta_0 = \frac{\pi}{2}$. Then we have $h_0 = \frac{\int_\Omega m^2(x)}{\int_\Omega m(x)}$. □

We next prove that there exists a $\lambda_* > 0$ such that we can solve $F(\nu,\beta,\theta,h,\lambda) = 0$ for $\lambda \in (0,\lambda_*)$.

Lemma 2.5 *There exists a* $\lambda_* > 0$ *and a unique continuously differentiable mapping* $\lambda \to (\nu_\lambda,\beta_\lambda,\theta_\lambda,h_\lambda)$ *from* $(0,\lambda_*)$ *to* $(\mathbb{X}_1)_c \times \mathbb{R}^3$ *such that* $F(\nu_\lambda,\beta_\lambda,\theta_\lambda,h_\lambda,\lambda) = 0$.

Proof We define $T = (T_1,T_2)$ $(\mathbb{X}_1)_c \times \mathbb{R}^3 \to \mathbb{Y}_c \times \mathbb{R}$ be the Fréchet derivative of F with respect to (ν,β,θ,h) at the point $(\nu_0,\beta_0,\theta_0,h_0)$. Therefore, we have

$$\begin{cases} T_1(\nu_\epsilon,\beta_\epsilon,\theta_\epsilon,h_\epsilon) = \nabla[e^{\frac{\alpha}{d}P(x)}\nabla\nu_\epsilon] + ie^{\frac{\alpha}{d}P(x)}(m - \frac{\int_\Omega m(x)}{|\Omega|})\beta_\epsilon \\ \qquad\qquad + e^{\frac{\alpha}{d}P(x)}m(x)\theta_\epsilon - ie^{\frac{\alpha}{d}P(x)}h_\epsilon, \\ T_2(\nu_\epsilon,\beta_\epsilon,\theta_\epsilon,h_\epsilon) = 2\beta_\epsilon. \end{cases} \tag{2.13}$$

One can easily check that T is bijective from $(\mathbb{X}_1)_c \times \mathbb{R}^3$ to $\mathbb{Y}_c \times \mathbb{R}$. Thus by the implicit function theorem, there exists a $\lambda_* > 0$ and a continuously differentiable mapping $\lambda \to (\nu_\lambda, \beta_\lambda, \theta_\lambda, h_\lambda)$ from $(0, \lambda_*)$ to $(\mathbb{X}_1)_c \times \mathbb{R}^3$ such that $F(\nu_\lambda, \beta_\lambda, \theta_\lambda, h_\lambda, \lambda) = 0$.

We next claim that the uniqueness, by virtue of the uniqueness of the implicit function theorem, it is sufficient to show that, if $\nu^\lambda \in (\mathbb{X}_1)_c$, $\beta^\lambda, h^\lambda > 0$, $\theta^\lambda \in [0, 2\pi)$, and $F(\nu^\lambda, \beta^\lambda, \theta^\lambda, h^\lambda, \lambda) = 0$, then $(\nu^\lambda, \beta^\lambda, \theta^\lambda, h^\lambda) \to (\nu_0, \beta_0, \theta_0, h_0)$ as $\lambda \to 0$ in the norm of $(\mathbb{X}_1)_c \times \mathbb{R}^3$. First, the boundedness of the sequences $\{\beta^\lambda\}$, $\{\theta^\lambda\}$ and $\{h^\lambda\}$ can be easily obtained from the definition, hypothesis and Lemma 2.2, respectively. By Lemma 2.3, and the first equation of Eq. (2.8), we obtain

$$\|\nu^\lambda\|_{\mathbb{Y}_c}^2 \le \frac{1}{\gamma_2}\left|\langle \mathbf{L}\nu^\lambda, \nu^\lambda\rangle\right| = \frac{1}{\gamma_2}\left|\left\langle \left(m(x)e^{-i\theta^\lambda} + ih^\lambda\right)\left(\beta^\lambda + \lambda\nu^\lambda\right), \nu^\lambda\right\rangle\right|.$$

The boundedness of $m(x)$ and $\{h^\lambda\}$ implies that there exists a constant M_1 such that

$$\frac{1}{\gamma_2}\left\| m(x)e^{-i\theta^\lambda} + ih^\lambda\right\|_\infty \le M_1.$$

Hence, we obtain $\|\nu^\lambda\|_{\mathbb{Y}_c}^2 \le M_1|\beta^\lambda|\|\nu^\lambda\|_{\mathbb{Y}_c} + \lambda M_1\|\nu^\lambda\|_{\mathbb{Y}_c}^2$. Accordingly, if λ_* is sufficiently small, then we have $\lambda M_1 \le \frac{1}{2}$, and $\|\nu^\lambda\|_{\mathbb{Y}_c} \le 2M_1|\beta^\lambda|\|\nu^\lambda\|_{\mathbb{Y}_c}$. As a result, $\{\nu^\lambda\}$ is bounded in $\mathbb{Y}_c$ for $\lambda \in (0, \lambda_*)$. Since the operator $\mathbf{L}^{-1}$ is bounded, we see that $\{\nu^\lambda\}$ is bounded in $\mathbb{X}_c$, which implies that $(\nu^\lambda, \beta^\lambda, \theta^\lambda, h^\lambda)$ is precompact in $\mathbb{Y}_c \times \mathbb{R}^3$ for $\lambda \in (0, \lambda_*)$. Therefore, there is a subsequence $(\nu^{\lambda_i}, \beta^{\lambda_i}, \theta^{\lambda_i}, h^{\lambda_i})$ such that

$$\left(\nu^{\lambda_i}, \beta^{\lambda_i}, \theta^{\lambda_i}, h^{\lambda_i}\right) \to \left(\nu^0, \alpha^0, \theta^0, h^0\right) \quad \text{in } \mathbb{Y}_c \times \mathbb{R}^3\ \lambda_i \to 0, \text{ as } i \to \infty.$$

Taking the limit of the equation $\mathbf{L}^{-1}f_1(\nu^{\lambda_i}, \beta^{\lambda_i}, \theta^{\lambda_i}, h^{\lambda_i}, \lambda_i) = 0$ as $i \to \infty$, we see that

$$\left(\nu^{\lambda_i}, \beta^{\lambda_i}, \theta^{\lambda_i}, h^{\lambda_i}\right) \to \left(\nu^0, \alpha^0, \theta^0, h^0\right) \quad \text{in } \mathbb{X}_c \times \mathbb{R}^3\ \lambda_i \to 0, \text{ as } i \to \infty.$$

Moreover, $F(\nu, \beta, \theta, h, 0) = 0$ has a unique solution, which implies that $(\nu^0, \beta^0, \theta^0, h^0) = (\nu_0, \beta_0, \theta_0, h_0)$. This completes the proof. □

Remark 2.1 From Lemma 2.5, we derive that, for each $\lambda \in (0, \lambda_*)$, the eigenvalue problem $\Lambda(\lambda, i\omega, \tau)\varphi = 0$, $\omega > 0$, $\tau \ge 0$, $\varphi(\ne 0) \in \mathbb{X}_c$, has a solution (ω, τ, φ) if and only if

$$\omega = \omega_\lambda = \lambda h_\lambda, \qquad \varphi = c\varphi_\lambda, \quad \text{and} \quad \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{\omega_\lambda}, \quad n = 0, 1, 2, \ldots,$$

where $\varphi_\lambda = \beta_\lambda + \lambda\nu_\lambda$.

In the following section, we will always assume $\lambda \in (0, \lambda_*)$ for convenience. Actually, the interval of λ might be smaller, since further perturbation arguments are used.

Lemma 2.6 *Assume that* $0 < \lambda < \lambda_*$. *Then*

$$S_n(\lambda) = \int_\Omega e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2\left[\left(1 - \tau_n\lambda m(x)e^{-i\theta_\lambda}\right)\right] \ne 0.$$

Proof Since

$$\lim_{\lambda\to 0}\theta_\lambda = \frac{\pi}{2}, \qquad \lim_{\lambda\to 0}\varphi_\lambda = 1, \qquad \lim_{\lambda\to 0}\lambda\tau_n = \frac{(\frac{\pi}{2}+2n\pi)\int_\Omega m(x)}{\int_\Omega m^2(x)},$$

we have

$$\lim_{\lambda\to 0} S_n(\lambda) = e^{\frac{\alpha C}{d}}\left(1+i\left(\frac{\pi}{2}+2n\pi\right)\right)\int_\Omega m(x) \neq 0. \qquad \square$$

Theorem 2.1 *For $\lambda \in (0,\lambda_*)$, there is a neighborhood of $(\tau_n, i\omega_\lambda, \varphi_\lambda)$ such that $A_{\tau_n}(\lambda)$ has a simple eigenvalue $\mu(\tau_n) = a(\tau_n) + ib(\tau_n)$. Moreover, $a(\tau_n) = 0$, and $b(\tau_n) = \omega_\lambda$.*

Proof We know that $\mathcal{N}[A_{\tau_n}(\lambda) - i\omega_\lambda] = \mathrm{Span}[e^{i\omega_\lambda\theta}\varphi_\lambda]$, where $\theta \in [-\tau_n, 0]$. If $\phi_1 \in \mathcal{N}[A_{\tau_n}(\lambda) - i\omega_\lambda]^2$, then

$$\mathcal{N}\big[A_{\tau_n}(\lambda) - i\omega_\lambda\big]\phi_1 \in \mathcal{N}\big[A_{\tau_n}(\lambda) - i\omega_\lambda\big] = \mathrm{Span}\big[e^{i\omega_\lambda\theta}\varphi_\lambda\big].$$

Therefore, there exists a constant a such that

$$\mathcal{N}\big[A_{\tau_n}(\lambda) - i\omega_\lambda\big]\phi_1 = ae^{i\omega_\lambda\theta}\varphi_\lambda.$$

Hence,

$$\begin{cases} \dot{\phi}_1(\theta) = i\omega_\lambda\phi_1(\theta) + ae^{i\omega_\lambda\theta}\varphi_\lambda, \quad \theta\in[-\tau_n,0], \\ \dot{\phi}_1(0) = de^{-\frac{\alpha}{d}P(x)}\nabla[e^{\frac{\alpha}{d}P(x)}\nabla\phi_1(0)] - \lambda m(x)\phi_1(-\tau_n). \end{cases} \tag{2.14}$$

The first equation of Eq. (2.14) yields

$$\begin{cases} \phi_1(\theta) = \phi_1(0)e^{i\omega_\lambda\theta} + a\theta e^{i\omega_\lambda\theta}\varphi_\lambda, \\ \dot{\phi}_1(0) = i\omega_\lambda\phi_1(0) + a\varphi_\lambda. \end{cases} \tag{2.15}$$

From Eqs. (2.14) and (2.15), we have

$$\begin{aligned} e^{\frac{\alpha}{d}P(x)}\Lambda(\lambda, i\omega_\lambda, \tau_n)\phi_1(0) &= \nabla\big[e^{\frac{\alpha}{d}P(x)}\nabla\phi_1(0)\big] - \lambda e^{\frac{\alpha}{d}P(x)}m(x)\phi_1(0)e^{-i\theta_\lambda} \\ &\quad - ie^{\frac{\alpha}{d}P(x)}\omega_\lambda\phi_1(0) \\ &= ae^{\frac{\alpha}{d}P(x)}\big(\varphi_\lambda - \tau_n\lambda m(x)e^{-i\theta_\lambda}\varphi_\lambda\big). \end{aligned} \tag{2.16}$$

Hence

$$\begin{aligned} \int_\Omega \phi_1(0)\big[e^{\frac{\alpha}{d}P(x)}\Lambda(\lambda, i\omega_\lambda, \tau_n)\varphi_\lambda\big] &= \int_\Omega \varphi_\lambda\big[e^{\frac{\alpha}{d}P(x)}\Lambda(\lambda, i\omega_\lambda, \tau_n)\phi_1(0)\big] \\ &= a\int_\Omega e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2\big[\big(1-\tau_n\lambda m(x)e^{-i\theta_\lambda}\big)\big] = 0, \end{aligned} \tag{2.17}$$

which implies that $a = 0$. And it leads to that $\mu = i\omega_\lambda$ is a simple eigenvalue of $A_{\tau_n}(\lambda)$. It follows from the implicit function theorem that there is a neighborhood of $(\tau_n, i\omega_\lambda, \varphi_\lambda)$ such that $A_\tau(\lambda)$ has a simple eigenvalue $\mu(\tau) = a(\tau) + ib(\tau)$, for $\lambda \in (0,\lambda_*)$. $\square$

Theorem 2.2 *Assume that* $\lambda \in (0, \lambda_*)$, *we have* $\mathcal{R}e\frac{d\mu(\tau_n\lambda)}{d\tau} > 0$.

Proof Since

$$\begin{aligned} e^{\frac{\alpha}{d}P(x)}\Lambda\bigl(\lambda, \mu(\tau), \tau\bigr)\varphi(\tau) &= \nabla\bigl[e^{\frac{\alpha}{d}P(x)}\nabla\varphi(\tau)\bigr] - \lambda e^{\frac{\alpha}{d}P(x)}m(x)\varphi(\tau)e^{-\mu(\tau)\tau} \\ &\quad - \mu(\tau)e^{\frac{\alpha}{d}P(x)}\varphi(\tau) = 0. \end{aligned} \tag{2.18}$$

Differentiating above equation with respect to τ at $\tau = \tau_n$ yields

$$\begin{aligned} &e^{\frac{\alpha}{d}P(x)}\Lambda(\lambda, i\omega_\lambda, \tau_n)\frac{d\varphi(\tau_n)}{d\tau} + \frac{d\mu(\tau_n)}{d\tau}e^{\frac{\alpha}{d}P(x)}\bigl(\lambda\tau_n m(x)\varphi_\lambda e^{-i\theta_\lambda} - \varphi_\lambda\bigr) \\ &\quad + i\omega_\lambda\lambda m(x)e^{\frac{\alpha}{d}P(x)}\varphi_\lambda e^{-i\theta_\lambda} = 0. \end{aligned} \tag{2.19}$$

Then, multiplying the above equation by φ_λ and integrating the result over Ω, we have

$$\begin{aligned} \frac{d\mu(\tau_n)}{d\tau} &= \frac{\int_\Omega i\omega_\lambda\lambda m(x)e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2 e^{-i\theta_\lambda}}{\int_\Omega e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2[(1 - \tau_n\lambda m(x)e^{-i\theta_\lambda})]} \\ &= \frac{\int_\Omega i\omega_\lambda\lambda m(x)e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2 e^{-i\theta_\lambda}\int_\Omega e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2 - i\omega_\lambda\lambda^2\tau_n(\int_\Omega m(x)e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2)^2}{|S_n(\lambda)|^2}. \end{aligned} \tag{2.20}$$

Since $\lim_{\lambda\to 0}\sin\theta_\lambda = 1$,

$$\mathcal{R}e\frac{d\mu(\tau_n\lambda)}{d\tau} = \frac{\lambda\sin\theta_\lambda\omega_\lambda\int_\Omega m(x)e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2\int_\Omega e^{\frac{\alpha}{d}P(x)}\varphi_\lambda^2}{|S_n(\lambda)|^2} > 0 \quad \text{for } \lambda \in (0, \lambda_*). \tag{2.21}$$

□

From the above lemmas and theorems, we immediately have the following result.

Theorem 2.3 *For* $\lambda \in (0, \lambda_*)$, *the infinitesimal generator* $A_{\tau_n}(\lambda)$ *has exactly* $2(n + 1)$ *eigenvalues with positive real parts when* $\tau \in (\tau_n, \tau_{n+1}]$, *where* $n = 0, 1, 2, \ldots$.

Moreover, by virtue of [22], we have the local Hopf bifurcation theorem for partial functional differential equations as follows.

Theorem 2.4 *For each fixed* $\lambda \in (0, \lambda_*)$, *the positive steady state* $e^{-\frac{\alpha}{d}P(x)}m(x)$ *of* (1.8) *is locally asymptotically stable when* $\tau \in [0, \tau_0)$, *and is unstable when* $\tau \in [\tau_0, \infty)$. *Furthermore, in system* (1.8) *there occurs a Hopf bifurcation at the positive steady state* $e^{-\frac{\alpha}{d}P(x)}m(x)$, *when* $\tau = \tau_n$ $(n = 0, 1, 2, \ldots)$.

3 The direction of the Hopf bifurcation

Taking advantage of the previous section, we find that a periodic solution bifurcates from the spatially nonhomogeneous steady-state solution $e^{-\frac{\alpha}{d}P(x)}m(x)$ as the delay τ passes through the critical value τ_n $(n = 0, 1, 2, \ldots)$.

In this section, by applying the normal form theory and the center manifold reduction we analyze the direction of Hopf bifurcation occurring around the positive steady-state solution with τ as a bifurcation parameter.

We first transform the steady state $e^{-\frac{\alpha}{d}P(x)}m(x)$ of system (1.8) and the critical value τ_n to the origin via the translations

$$V(t) = \upsilon(t,x) - e^{-\frac{\alpha}{d}P(x)}m(x), \quad t = \tau\widehat{t}, \tau = \tau_n + \varrho,$$

then, dropping the tilde sign, system (1.8) can be transformed as follows:

$$\frac{dV(t)}{dt} = \tau_n e^{-\frac{\alpha}{d}P(x)}\nabla\big[e^{\frac{\alpha}{d}P(x)}\nabla V(t)\big] - \tau_n\lambda m(x)V(t-1) + F(V_t,\varrho), \tag{3.1}$$

where

$$F(V_t,\varrho) = \varrho e^{-\frac{\alpha}{d}P(x)}\nabla\big[e^{\frac{\alpha}{d}P(x)}\nabla V(t)\big] - \varrho\lambda m(x)V(t-1) - (\tau_n+\varrho)\lambda e^{\frac{\alpha}{d}P(x)}V(t)V(t-1).$$

Similar to Sect. 2, we define $\mathbb{A}_{\tau_n}(\lambda)$ to be the infinitesimal generator of the linearized equation (3.1), then

$$\mathbb{A}_{\tau_n}(\lambda)\varphi = \dot{\varphi},$$

with

$$\mathfrak{D}\big(\mathbb{A}_{\tau_n}(\lambda)\big) = \big\{\varphi \in C_c \cap C_c^1 : \dot{\varphi}(0) = \tau_n e^{-\frac{\alpha}{d}P(x)}\nabla\big[e^{\frac{\alpha}{d}P(x)}\nabla\varphi(0)\big] - \lambda\tau_n m(x)\varphi(-1)\big\},$$

where $C_c^1 = C^1([-1,0],\mathbb{Y}_c)$.

Define

$$\mathbb{F}(V_t,\varrho)(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ F(V_t,\varrho), & \theta = 0, \end{cases}$$

then (3.1) can be rewritten as

$$\frac{dV_t}{dt} = \mathbb{A}_{\tau_n}(\lambda)V_t + \mathbb{F}(V_t,\varrho)(\theta). \tag{3.2}$$

It follows from the previous section that $\mathbb{A}_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\omega_\lambda\tau_n$, which are simple. The eigenfunction associated with $i\omega_\lambda\tau_n$ (respectively, $-i\omega_\lambda\tau_n$) is $\gamma(\theta) = \varphi_\lambda e^{i\omega_\lambda\tau_n\theta}$ (respectively, $\bar{\gamma}(\theta) = \bar{\varphi}_\lambda e^{-i\omega_\lambda\tau_n\theta}$) for $\theta \in [-1,0]$, where φ_λ is defined as in Remark 2.1.

Following [9], we introduce a bilinear inner product as follows:

$$\langle\!\langle \Phi, \widehat{\Phi} \rangle\!\rangle = \big\langle \Phi(0), \widehat{\Phi}(0)\big\rangle_1 - \lambda\tau_n \int_{-1}^{0} \big\langle \Phi(s+1), m(x)\widehat{\Phi}(s)\big\rangle_1 \, ds$$

for $\Phi \in C_c$ and $\widehat{\Phi} \in C_c^*$.

Here $\langle u,v\rangle_1 = \int_\Omega e^{\frac{\alpha}{d}P(x)}\bar{u}(x)v(x)\,dx$.

As in [9], we have the following lemma to compute the formal adjoint operator of $\mathbb{A}_{\tau_n}$ satisfy the above bilinear inner product.

Lemma 3.1 *Let*

$$\mathbb{A}^*_{\tau_n}\widehat{\Phi}(s) = \begin{cases} \dot{\widehat{\Phi}}(s), & s \in (0,1], \\ \tau_n e^{-\frac{\alpha}{d}P(x)}\nabla[e^{\frac{\alpha}{d}P(x)}\nabla\widehat{\Phi}(0)] - \lambda\tau_n m(x)\widehat{\Phi}(1), & s = 0, \end{cases}$$

for $\widehat{\Phi} \in C_c^* \cap C_c^{1*}$, *where* $C_c^* = C([0,1], \mathbb{Y}_c)$. *Then* $\mathbb{A}_{\tau_n}$ *and* $\mathbb{A}^*_{\tau_n}$ *satisfy*

$$\langle\!\langle \widehat{\Phi}, \mathbb{A}_{\tau_n}\Phi \rangle\!\rangle = \langle\!\langle \mathbb{A}^*_{\tau_n}\widehat{\Phi}, \Phi \rangle\!\rangle.$$

Proof For $\Phi \in \mathfrak{D}(\mathbb{A}_{\tau_n})$ and $\widehat{\Phi} \in \mathfrak{D}(\mathbb{A}^*_{\tau_n})$, we have

$$\begin{aligned} \langle\!\langle \widehat{\Phi}, \mathbb{A}_{\tau_n}\Phi \rangle\!\rangle &= \langle \widehat{\Phi}(0), \mathbb{A}_{\tau_n}\Phi(0) \rangle - \lambda\tau_n \int_{-1}^0 \langle \widehat{\Phi}(s+1), m(x)\mathbb{A}_{\tau_n}\Phi(s) \rangle\, ds \\ &= \langle \widehat{\Phi}(0), \tau_n e^{-\frac{\alpha}{d}P(x)}\nabla\big[e^{\frac{\alpha}{d}P(x)}\nabla\Phi(0)\big] - \lambda\tau_n m(x)\Phi(-1) \rangle \\ &\quad - \lambda\tau_n \int_{-1}^0 \langle \widehat{\Phi}(s+1), m(x)\dot{\Phi}(s) \rangle\, ds \\ &= \langle \widehat{\Phi}(0), \tau_n e^{-\frac{\alpha}{d}P(x)}\nabla\big[e^{\frac{\alpha}{d}P(x)}\nabla\Phi(0)\big] \rangle - \lambda\tau_n\big[\langle \widehat{\Phi}(s+1), m(x)\Phi(s) \rangle\big]_{-1}^0 \\ &\quad + \langle \widehat{\Phi}(0), -\lambda\tau_n m(x)\Phi(-1) \rangle + \lambda\tau_n \int_{-1}^0 \langle \dot{\widehat{\Phi}}(s+1), m(x)\Phi(s) \rangle\, ds \\ &= \langle (\mathbb{A}^*_{\tau_n}\widehat{\Phi})(0), \Phi(0) \rangle - \lambda\tau_n \int_{-1}^0 \langle -\dot{\widehat{\Phi}}(s+1), m(x)\Phi(s) \rangle\, ds \\ &= \langle\!\langle \mathbb{A}^*_{\tau_n}\widehat{\Phi}, \Phi \rangle\!\rangle. \end{aligned}$$

□

Similarly, we know that $\mathbb{A}^*_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\omega_\lambda\tau_n$, which are simple. The eigenfunction associated with $i\omega_\lambda\tau_n$ (respectively, $-i\omega_\lambda\tau_n$) is $\widehat{\gamma}(s) = \varphi_\lambda e^{-i\omega_\lambda\tau_n s}$ (respectively, $\bar{\widehat{\gamma}}(s) = \bar{\varphi}_\lambda e^{i\omega_\lambda\tau_n s}$) for $s \in [0,1]$, where φ_λ is defined as in Remark 2.1. The center subspace of (3.2) is $P = \operatorname{span}\{\gamma(\theta), \bar{\gamma}(\theta)\}$. Moreover, the basis of eigenfunction space of the adjoint operator $\mathbb{A}^*_{\tau_n}$ associated with the eigenvalues $\pm i\omega_\lambda\tau_n$ is $P^* = \operatorname{span}\{\widehat{\gamma}(s), \bar{\widehat{\gamma}}(s)\}$. And the formal adjoint subspace of P is P^*. As usual, C_c can be decomposed as $C_c = P \oplus Q$, where $Q = \{\psi \in C_c \mid \langle\!\langle \widehat{\psi}, \psi \rangle\!\rangle = 0, \text{ for all } \widehat{\psi} \in P^*\}$. Let

$$\Phi_\gamma = \big(\gamma(\theta), \bar{\gamma}(\theta)\big), \quad \text{for } \theta \in [-1,0],$$

$$\Psi_{\widehat{\gamma}} = \left(\frac{\widehat{\gamma}(s)}{\bar{S}_n(\lambda)}, \frac{\bar{\widehat{\gamma}}(s)}{S_n(\lambda)}\right) \quad \text{for } s \in [0,1],$$

and one can easily check that $\langle\!\langle \Phi_\gamma, \Psi_{\widehat{\gamma}} \rangle\!\rangle = I$, where I is the identity matrix in $\mathbb{R}^{2\times 2}$. Since the formulas to be developed for the bifurcation direction and stability are all relative to $\varrho = 0$ only, we set $\varrho = 0$ in system (3.2) and define

$$z(t) = \frac{1}{S_n(\lambda)}\langle\!\langle \widehat{\gamma}, V_t \rangle\!\rangle. \tag{3.3}$$

Let

$$W(z,\bar{z}) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \mathbf{O}\big(|z|^3\big)$$

be the center manifold with the range in Q, and then the flow of Eq. (3.2) on the center manifold can be written as

$$V_t = \Phi_\gamma \cdot \big(z(t), \bar{z}(t)\big)^T + W\big(z(t), \bar{z}(t)\big).$$

Since $\varrho = 0$, we have

$$\dot{z}(t) = \frac{1}{S_n(\lambda)} \frac{d\langle\!\langle \widehat{\gamma}, V_t \rangle\!\rangle}{dt} = i\omega_\lambda \tau_n z(t) + g\big(z(t), \bar{z}(t)\big),$$

where

$$\begin{aligned} g\big(z(t), \bar{z}(t)\big) &= \frac{1}{S_n(\lambda)} \big\langle \gamma(0), F(V_t, 0) \big\rangle_1 \\ &= \frac{1}{S_n(\lambda)} \big\langle \gamma(0), F\big(\Phi_\gamma \cdot \big(z(t), \bar{z}(t)\big)^T + W\big(z(t), \bar{z}(t)\big), 0\big) \big\rangle_1 \\ &= g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots, \end{aligned}$$

and obviously an easy calculation implies that

$$\begin{aligned} g_{20} &= -\frac{2\lambda\tau_n}{S_n(\lambda)} e^{-i\omega_\lambda\tau_n} \int_\Omega e^{\frac{2\alpha}{d}P(x)} \varphi_\lambda^3 \, dx, \\ g_{11} &= -\left[\frac{\lambda\tau_n}{S_n(\lambda)}\big(e^{i\omega_\lambda\tau_n} + e^{-i\omega_\lambda\tau_n}\big)\right] \int_\Omega e^{\frac{2\alpha}{d}P(x)} \varphi_\lambda |\varphi_\lambda|^2 \, dx, \\ g_{02} &= -\frac{2\lambda\tau_n}{S_n(\lambda)} e^{i\omega_\lambda\tau_n} \int_\Omega e^{\frac{2\alpha}{d}P(x)} \varphi_\lambda \bar{\varphi}_\lambda^2 \, dx, \\ g_{21} &= -\frac{2\lambda\tau_n}{S_n(\lambda)} \int_\Omega e^{\frac{2\alpha}{d}P(x)} \varphi_\lambda^2 W_{11}(-1) \, dx - \frac{\lambda\tau_n}{S_n(\lambda)} \int_\Omega e^{\frac{2\alpha}{d}P(x)} |\varphi_\lambda|^2 W_{20}(-1) \, dx \\ &\quad - \frac{2\lambda\tau_n}{S_n(\lambda)} e^{-i\omega_\lambda\tau_n} \int_\Omega e^{\frac{2\alpha}{d}P(x)} \varphi_\lambda^2 W_{11}(0) \, dx - \frac{\lambda\tau_n}{S_n(\lambda)} e^{i\omega_\lambda\tau_n} \int_\Omega e^{\frac{2\alpha}{d}P(x)} |\varphi_\lambda|^2 W_{20}(0) \, dx. \end{aligned} \tag{3.4}$$

To determine the bifurcation direction and stability of bifurcating periodic orbits, we need to compute the following quantities:

$$\begin{aligned} \mathcal{C}_1(0) &= \frac{i}{2\omega_\lambda\tau_n}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathcal{R}e\{\mathcal{C}_1(0)\}}{\mathcal{R}e\{\mu'(\tau_n)\}}, \\ \beta_2 &= 2\mathcal{R}e\big\{\mathcal{C}_1(0)\big\}, \\ T_2 &= -\frac{\mathcal{I}m\{\mathcal{C}_1(0)\} + \mu_2 \mathcal{I}m\{\mu'(\tau_n)\}}{\tau_n}. \end{aligned}$$

Inspired by [16] and [22], we have the following results.

Lemma 3.2 *For $\lambda \in (0, \lambda_*)$, in Eq. (1.8) there occurs a Hopf bifurcation at the positive steady state $e^{-\frac{\alpha}{d}P(x)} m(x)$, moreover,*

(i) μ_2 *determines the direction of the Hopf bifurcation: if* $\mu_2 > 0$ $(\mu_2 < 0)$, *then the bifurcating periodic solutions exist for* $\tau > \tau_n$ $(\tau < \tau_n)$, *and the bifurcation is called forward (backward);*
(ii) β_2 *determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions are orbitally asymptotically stable (unstable) if* $\beta_2 < 0$ $(\beta_2 > 0)$;
(iii) T_2 *determines the period of the bifurcating periodic solutions: the period increases (decreases) if* $T_2 > 0$ $(T_2 < 0)$.

To compute g_{21}, we need to figure out $W_{11}(\theta)$ and $W_{20}(\theta)$. Note that $W(z,\bar{z})$ satisfies

$$\dot{W} = \mathbb{A}_{\tau_n} W + H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots, \tag{3.5}$$

and $W_{11}(\theta)$, $W_{20}(\theta)$, $H_{11}(\theta)$ and $H_{20}(\theta)$ satisfy

$$\begin{cases} -\mathbb{A}_{\tau_n} W_{11}(\theta) = H_{11}(\theta), \\ (2i\omega_\lambda\tau_n - \mathbb{A}_{\tau_n})W_{20}(\theta) = H_{20}(\theta), \\ H_{11}(\theta) = -(g_{11}\gamma(\theta) + \bar{g}_{11}\bar{\gamma}(\theta)), & \theta \in [-1,0), \\ H_{20}(\theta) = -(g_{20}\gamma(\theta) + \bar{g}_{02}\bar{\gamma}(\theta)), & \theta \in [-1,0), \\ H_{11}(0) = -(g_{11}\gamma(0) + \bar{g}_{11}\bar{\gamma}(0)) - \lambda\tau_n e^{\frac{\alpha}{d}P(x)}(e^{i\omega_\lambda\tau_n} + e^{-i\omega_\lambda\tau_n})|\varphi|^2, \\ H_{20}(0) = -(g_{20}\gamma(0) + \bar{g}_{02}\bar{\gamma}(0)) - 2\lambda\tau_n e^{\frac{\alpha}{d}P(x)}e^{-i\omega_\lambda\tau_n}\varphi^2. \end{cases} \tag{3.6}$$

Then from Eq. (3.6), we have

$$\begin{cases} W_{20}(\theta) = \frac{ig_{20}}{\omega_\lambda\tau_n}\gamma(\theta) + \frac{i\bar{g}_{02}}{3\omega_\lambda\tau_n}\bar{\gamma}(\theta) + Ee^{2i\omega_\lambda\tau_n\theta}, \\ W_{11}(\theta) = \frac{ig_{11}}{\omega_\lambda\tau_n}\gamma(\theta) + \frac{i\bar{g}_{11}}{\omega_\lambda\tau_n}\bar{\gamma}(\theta) + F. \end{cases} \tag{3.7}$$

In what follows, we shall solve E and F. From Eqs. (3.6) and (3.7), the definition of $\mathbb{A}_{\tau_n}$, we see that E and F satisfy

$$\begin{cases} \Lambda(\lambda, 2i\omega_\lambda, \tau_n)E = 2\lambda e^{\frac{\alpha}{d}P(x)}e^{-i\omega_\lambda\tau_n}\varphi^2, \\ \Lambda(\lambda, 0, \tau_n)F = \lambda e^{\frac{\alpha}{d}P(x)}(e^{i\omega_\lambda\tau_n} + e^{-i\omega_\lambda\tau_n})|\varphi|^2. \end{cases} \tag{3.8}$$

Note that $2i\omega_\lambda$ is not the eigenvalue of $\mathbb{A}_{\tau_n}$ for $\lambda \in (0, \lambda^*)$, and hence

$$E = 2\lambda\Lambda(\lambda, 2i\omega_\lambda, \tau_n)^{-1}e^{\frac{\alpha}{d}P(x)}e^{-i\omega_\lambda\tau_n}\varphi^2.$$

Similarly, we obtain

$$F = \lambda\Lambda(\lambda, 0, \tau_n)^{-1}e^{\frac{\alpha}{d}P(x)}\left(e^{i\omega_\lambda\tau_n} + e^{-i\omega_\lambda\tau_n}\right)|\varphi|^2.$$

Lemma 3.3 *Assume that E and F satisfy system* (3.8). *Then*

$$E = c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda, \qquad F = \widetilde{\vartheta}_\lambda,$$

where ϑ_λ *and* $\widetilde{\vartheta}_\lambda$ *satisfy*

$$\left\langle e^{-\frac{\alpha}{d}P(x)}m(x), \vartheta_\lambda\right\rangle = 0, \qquad \lim_{\lambda\to 0}\|\vartheta_\lambda\|_{\mathbb{Y}_c} = 0, \qquad \lim_{\lambda\to 0}\|\widetilde{\vartheta}_\lambda\|_{\mathbb{Y}_c} = 0.$$

Moreover, the constant c_λ *satisfies* $\lim_{\lambda\to 0} c_\lambda = \frac{2i}{e^{-\frac{2C\alpha}{d}}(2i-1)}$.

Proof We just give the estimate for E, and that for F can be obtained similarly.

Substituting $E = c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda$ into the first equation of system (3.8), one can easily have

$$\begin{aligned} &e^{-\frac{\alpha}{d}P(x)}\nabla\left[e^{\frac{\alpha}{d}P(x)}\nabla\left(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda\right)\right] - \lambda m(x)\left(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda\right)e^{-2i\omega_\lambda\tau} \\ &\quad - 2i\omega_\lambda\left(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda\right) = 2\lambda e^{\frac{\alpha}{d}P(x)}e^{-i\omega_\lambda\tau_n}\varphi^2. \end{aligned}$$

Since $e^{-\frac{\alpha}{d}P(x)}m(x)$ is a positive steady state of system (1.8) which satisfies

$$\begin{cases} e^{-\frac{\alpha}{d}P(x)}\nabla[e^{\frac{\alpha}{d}P(x)}\nabla v] + \lambda v(m(x) - e^{\frac{\alpha}{d}P(x)}v) = 0, & x \in \Omega, \\ \nabla v\cdot\vec{n} = 0, & x \in \partial\Omega, \end{cases} \tag{3.9}$$

we obtain

$$\begin{aligned} &\nabla\left[e^{\frac{\alpha}{d}P(x)}\nabla(\vartheta_\lambda)\right] - \lambda e^{\frac{\alpha}{d}P(x)}m(x)\left(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda\right)e^{-2i\omega_\lambda\tau_n} \\ &\quad - 2ie^{\frac{\alpha}{d}P(x)}\omega_\lambda\left(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda\right) = 2\lambda e^{\frac{2\alpha}{d}P(x)}e^{-i\omega_\lambda\tau_n}\varphi^2. \end{aligned} \tag{3.10}$$

Multiplying the above equation by $e^{-\frac{\alpha}{d}P(x)}m(x)$, and integrating the result over Ω, we obtain

$$\begin{aligned} &\lambda c_\lambda e^{-2i\omega_\lambda\tau_n}\int_\Omega e^{-\frac{\alpha}{d}P(x)}m^3(x)\,dx + 2i\omega_\lambda c_\lambda\int_\Omega e^{-\frac{\alpha}{d}P(x)}m^2(x)\,dx \\ &\quad = -\lambda e^{-2i\omega_\lambda\tau_n}\int_\Omega e^{-\frac{\alpha}{d}P(x)}m^2(x)\vartheta_\lambda\,dx - 2i\omega_\lambda\int_\Omega e^{-\frac{\alpha}{d}P(x)}m(x)\vartheta_\lambda\,dx \\ &\qquad - 2\lambda e^{-i\omega_\lambda\tau_n}\int_\Omega \varphi^2 e^{\frac{\alpha}{d}P(x)}m(x)\,dx. \end{aligned} \tag{3.11}$$

Multiplying (3.10) by $\bar{\vartheta}_\lambda$ and integrating the result over Ω, we have

$$\begin{aligned} &\left\langle\vartheta_\lambda, d\nabla\left[e^{\frac{\alpha}{d}P(x)}\nabla\vartheta_\lambda\right]\right\rangle - \lambda c_\lambda e^{-2i\omega_\lambda\tau_n}\int_\Omega m^2(x)\bar{\vartheta}_\lambda\,dx - 2i\omega_\lambda c_\lambda\int_\Omega m(x)\bar{\vartheta}_\lambda\,dx \\ &\quad = \lambda e^{-2i\omega_\lambda\tau_n}\int_\Omega e^{\frac{\alpha}{d}P(x)}m(x)|\vartheta_\lambda|^2\,dx + 2i\omega_\lambda\int_\Omega e^{-\frac{\alpha}{d}P(x)}|\vartheta_\lambda|^2\,dx \\ &\qquad + 2\lambda e^{-i\omega_\lambda\tau_n}\int_\Omega e^{\frac{2\alpha}{d}P(x)}\varphi^2\bar{\vartheta}_\lambda\,dx. \end{aligned} \tag{3.12}$$

Combining the above lemma leads easily to

$$\varphi \to 1, \qquad \frac{\omega_\lambda}{\lambda} \to h_0, \qquad \omega_\lambda\tau_n \to \frac{\pi}{2} + 2n\pi,$$

as $\lambda \to 0$.

Hence, it follows from Eq. (3.11) that there exist constants $\widehat{\lambda} > 0$ and $M_0, M_1 > 0$ such that, for any $\lambda \in (0, \widehat{\lambda})$,

$$|c_\lambda| \leq M_0 \|\vartheta_\lambda\|_{\mathbb{Y}_c} + M_1. \tag{3.13}$$

Furthermore, from Eqs. (3.12) and (3.13) and combining with Lemma 2.3, we find that there exist constants $M_2, M_3 > 0$ such that, for any $\lambda \in (0, \widehat{\lambda})$,

$$\gamma_2 \cdot \|\vartheta_\lambda\|^2_{\mathbb{Y}_c} \leq \lambda M_2 \|\vartheta_\lambda\|^2_{\mathbb{Y}_c} + \lambda M_3 \|\vartheta_\lambda\|_{\mathbb{Y}_c}. \tag{3.14}$$

So, we have $\lim_{\lambda \to 0} \|\vartheta_\lambda\|_{\mathbb{Y}_c} = 0$. This, together with Eq. (3.11), implies

$$\lim_{\lambda \to 0} c_\lambda = \frac{2i}{e^{-\frac{2C\alpha}{d}}(2i-1)},$$

where C is a constant determined by $P(x) = \ln m^{\frac{d}{\alpha}} + C$. □

By similar arguments to [10] and [9], one can easily have the following result.

Theorem 3.1 *For $\lambda \in (0, \lambda_*)$, in Eq. (1.8) there occurs a Hopf bifurcation at the positive steady state $e^{-\frac{\alpha}{d}P(x)} m(x)$, near τ_n ($n \in \mathbb{N} \cup \{0\}$). Moreover, the direction of the Hopf bifurcation at $\tau = \tau_n$ is forward and the bifurcating periodic solution at $\tau = \tau_0$ is orbitally asymptotically stable.*

Proof Since

$$\lim_{\lambda \to 0} S_n(\lambda) = e^{\frac{\alpha C}{d}} \left(1 + \left(\frac{\pi}{2} + 2n\pi\right) i\right) \int_\Omega m(x), \qquad \lim_{\lambda \to 0} \lambda \tau_n = \frac{(\frac{\pi}{2} + 2n\pi) \int_\Omega m(x)}{\int_\Omega m^2(x)},$$

we have

$$\lim_{\lambda \to 0} g_{20} = -\frac{2\lambda\tau_n}{S_n(\lambda)} e^{-i\omega_\lambda \tau_n} \int_\Omega e^{\frac{2\alpha}{d}P(x)} \varphi^3 \, dx = \frac{i(\pi + 4n\pi)}{e^{-\frac{\alpha C}{d}}(1 + i(\frac{\pi}{2} + 2n\pi))}, \tag{3.15}$$

$$\lim_{\lambda \to 0} g_{11} = -\left[\frac{\lambda\tau_n}{S_n(\lambda)} \left(e^{i\omega_\lambda \tau_n} + e^{-i\omega_\lambda \tau_n}\right)\right] \int_\Omega e^{\frac{\alpha}{d}P(x)} \varphi |\varphi|^2 \, dx = 0, \tag{3.16}$$

$$\lim_{\lambda \to 0} g_{02} = -\frac{2\lambda\tau_n}{S_n(\lambda)} e^{i\omega_\lambda \tau_n} \int_\Omega e^{\frac{\alpha}{d}P(x)} |\varphi|^2 \varphi \, dx = \frac{-i(\pi + 4n\pi)}{e^{-\frac{\alpha C}{d}}(1 + i(\frac{\pi}{2} + 2n\pi))}. \tag{3.17}$$

The conclusions in front yield

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega_\lambda \tau_n} \gamma(\theta) + \frac{i\bar{g}_{02}}{3\omega_\lambda \tau_n} \bar{\gamma}(\theta) + E e^{2i\omega_\lambda \tau_n \theta}, \\ W_{11}(\theta) &= \frac{ig_{11}}{\omega_\lambda \tau_n} \gamma(\theta) + \frac{i\bar{g}_{11}}{\omega_\lambda \tau_n} \bar{\gamma}(\theta) + F, \end{aligned} \tag{3.18}$$

where

$$\gamma(\theta) = \varphi e^{i\omega_\lambda \tau_n \theta}, \qquad \bar{\gamma}(\theta) = \bar{\varphi} e^{-i\omega_\lambda \tau_n \theta}.$$

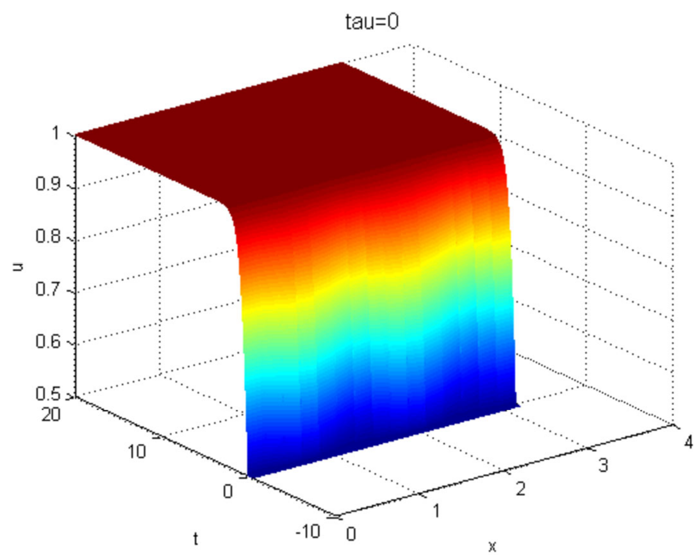

Figure 1 Numerical solutions of system (1.8). Parameters were chosen as: $m(x) = \sin(x) + 2$, $\alpha = 1$, $d = 2$, $\tau = 0 < \tau_0 \approx 0.7$, $C = 0$

Hence

$$\begin{aligned}
\lim_{\lambda\to 0} W_{20}(-1) &= \frac{ig_{20}}{\omega_\lambda\tau_n}\varphi e^{-i\omega_\lambda\tau_n} + \frac{i\bar{g}_{02}}{3\omega_\lambda\tau_n}\bar{\varphi}e^{i\omega_\lambda\tau_n} + \bigl(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda\bigr)e^{-2i\omega_\lambda\tau_n},\\
\lim_{\lambda\to 0} e^{i\omega_\lambda\tau_n} W_{20}(0) &= \frac{ig_{20}}{\omega_\lambda\tau_n}\varphi e^{i\omega_\lambda\tau_n} + \frac{i\bar{g}_{02}}{3\omega_\lambda\tau_n}\bar{\varphi}e^{i\omega_\lambda\tau_n} + \bigl(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda\bigr)e^{i\omega_\lambda\tau_n},
\end{aligned} \tag{3.19}$$

where

$$\begin{cases}
\lim_{\lambda\to 0}\frac{ig_{20}}{\omega_\lambda\tau_n}\varphi e^{-i\omega_\lambda\tau_n} = -\lim_{\lambda\to 0}\frac{ig_{20}}{\omega_\lambda\tau_n}\varphi e^{i\omega_\lambda\tau_n} = \frac{(\pi+4n\pi)+i2}{e^{-\frac{\alpha C}{d}}(1+\frac{\pi}{2}+2n\pi)},\\
\lim_{\lambda\to 0}\frac{i\bar{g}_{02}}{3\omega_\lambda\tau_n}\bar{\varphi}e^{i\omega_\lambda\tau_n} = \frac{(\pi+4n\pi)-i2}{3e^{-\frac{\alpha C}{d}}(1+\frac{\pi}{2}+2n\pi)},\\
\lim_{\lambda\to 0}(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda)e^{-2i\omega_\lambda\tau_n} = \frac{-4e^{\frac{C\alpha}{d}}+2e^{\frac{C\alpha}{d}}i}{5},\\
\lim_{\lambda\to 0}(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda)e^{i\omega_\lambda\tau_n} = \frac{i4e^{\frac{C\alpha}{d}}+2e^{\frac{C\alpha}{d}}}{5}.
\end{cases} \tag{3.20}$$

Then

$$\begin{aligned}
&\lim_{\lambda\to 0} g_{21}\\
&= \lim_{\lambda\to 0}\biggl(-\frac{2\lambda\tau_n}{S_n(\lambda)}\int_\Omega e^{\frac{2\alpha}{d}P(x)}\varphi^2 W_{11}(-1)\,dx - \frac{\lambda\tau_n}{S_n(\lambda)}\int_\Omega e^{\frac{2\alpha}{d}P(x)}|\varphi|^2 W_{20}(-1)\,dx\\
&\quad - \frac{2\lambda\tau_n}{S_n(\lambda)}e^{-i\omega_\lambda\tau_n}\int_\Omega e^{\frac{2\alpha}{d}P(x)}\varphi^2 W_{11}(0)\,dx - \frac{\lambda\tau_n}{S_n(\lambda)}e^{i\omega_\lambda\tau_n}\int_\Omega e^{\frac{2\alpha}{d}P(x)}|\varphi|^2 W_{20}(0)\,dx\biggr)\\
&= \lim_{\lambda\to 0}\biggl(-\frac{\lambda\tau_n}{S_n(\lambda)}\int_\Omega e^{\frac{2\alpha}{d}P(x)}|\varphi|^2 W_{20}(-1)\,dx - \frac{\lambda\tau_n}{S_n(\lambda)}e^{i\omega_\lambda\tau_n}\int_\Omega e^{\frac{2\alpha}{d}P(x)}|\varphi|^2 W_{20}(0)\,dx\biggr)\\
&= \lim_{\lambda\to 0}\biggl[-\frac{\lambda\tau_n}{S_n(\lambda)}e^{\frac{2C\alpha}{d}}\int_\Omega m^2(x)\biggl(\frac{i2\bar{g}_{02}}{3\omega_\lambda\tau_n}\bar{\varphi}e^{i\omega_\lambda\tau_n} + \bigl(c_\lambda e^{-\frac{\alpha}{d}P(x)}m(x) + \vartheta_\lambda\bigr)e^{-2i\omega_\lambda\tau_n}
\end{aligned} \tag{3.21}$$

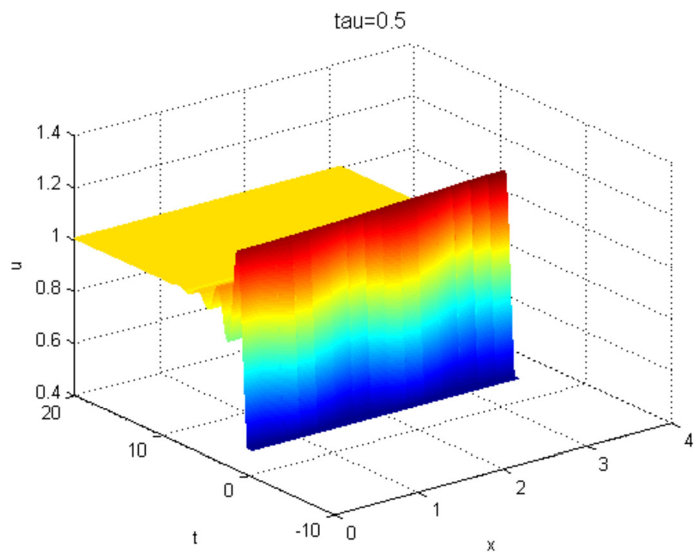

Figure 2 Numerical solutions of system (1.8). Parameters were chosen as: $m(x) = \sin(x) + 2$, $\alpha = 1$, $d = 2$, $\tau = 0.5 < \tau_0 \approx 0.7$, $C = 0$

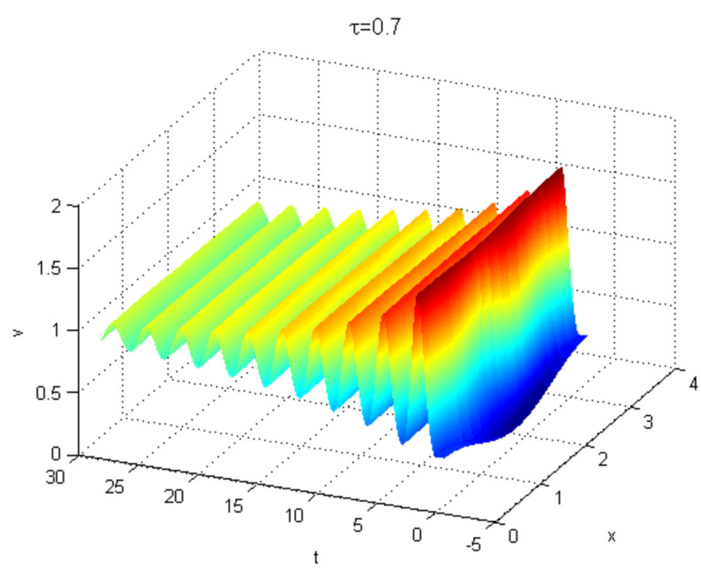

Figure 3 Numerical solutions of system (1.8). Parameters were chosen as: $m(x) = \sin(x) + 2$, $\alpha = 1$, $d = 2$, $\tau = 0.7 < \tau_0 \approx 0.8$, $C = 0$

$$
\begin{aligned}
&+ \left(c_\lambda e^{-\frac{\alpha}{d}P(x)} m(x) + \vartheta_\lambda\right) e^{-i\omega_\lambda \tau_n}\Bigg)\Bigg] \\
&= \Bigg[-\frac{\lambda \tau_n}{S_n(\lambda)} e^{\frac{2C\alpha}{d}} \int_\Omega m^2(x)\Bigg(\frac{(2\pi + 8n\pi)e^{\frac{C\alpha}{d}} - i4e^{\frac{C\alpha}{d}}}{3(1 + \frac{\pi}{2} + 2n\pi)} + \frac{-4e^{\frac{C\alpha}{d}} + 2e^{\frac{C\alpha}{d}} i}{5} \\
&\quad + \frac{i4e^{\frac{C\alpha}{d}} + 2e^{\frac{C\alpha}{d}}}{5}\Bigg)\Bigg] \\
&= -\frac{\frac{\pi}{2} + 2n\pi}{1 + i(\frac{\pi}{2} + 2n\pi)} e^{\frac{2C\alpha}{d}} \left(\frac{(2\pi + 8n\pi) - i4}{3(1 + \frac{\pi}{2} + 2n\pi)} + \frac{i6 - 2}{5}\right).
\end{aligned}
$$

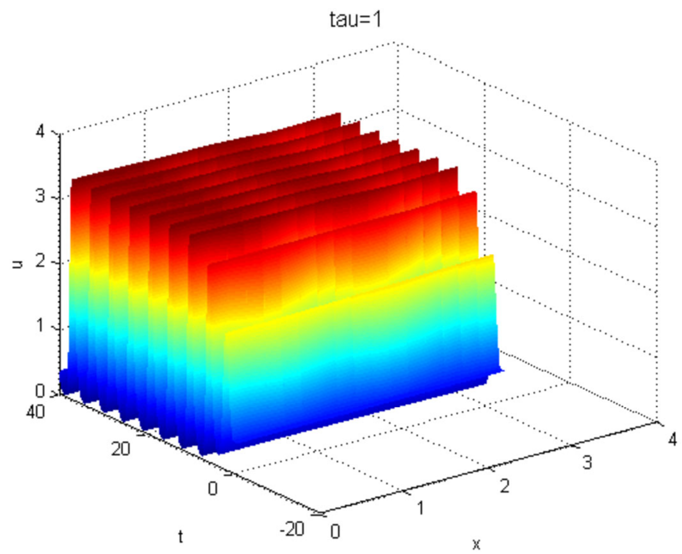

Figure 4 Numerical solutions of system (1.8). Parameters were chosen as: $m(x) = \sin(x) + 2$, $\alpha = 1$, $d = 2$, $\tau = 1 > \tau_0 \approx 0.7$, $C = 0$

Then we can compute that $\lim_{\lambda\to 0} g_{21} < 0$, which implies that $\lim_{\lambda\to 0} \mathcal{C}_1(0) < 0$. From Lemma 3.2, the proof of this theorem is finished. □

4 The numerical simulations and conclusions

This article is concerned with a delayed reaction–diffusion equation with ideal free dispersal. The local asymptotic stability of positive equilibrium solutions $e^{-\frac{\alpha}{d}P(x)}m(x)$ is studied by analyzing the associated eigenvalue problem. Moreover, it is demonstrated that the positive equilibrium solutions $e^{-\frac{\alpha}{d}P(x)}m(x)$ is asymptotically stable when there is no delay (see Fig. 1) or the delay is less than a certain critical value τ_0 (see Figs. 2 and 3), and unstable when the delay is greater than this critical value τ_0 (see Fig. 4). Besides, it is also found that the system under consideration can undergo a Hopf bifurcation when the delay crosses through a sequence of critical values τ_n.

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