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Nonlocal boundary value problems for integro-differential Langevin equation via the generalized Caputo proportional fractional derivative

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Abstract

Results reported in this paper study the existence and stability of a class of implicit generalized proportional fractional integro-differential Langevin equations with nonlocal fractional integral conditions. The main theorems are proved with the help of Banach's, Krasnoselskii's, and Schaefer's fixed point theorems and Ulam's approach. Finally, an example is given to demonstrate the applicability of our theoretical findings.

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Keywords: Fractional Langevin equation; Generalized proportional fractional derivative; Ulam stability; Existence and uniqueness; Nonlocal integral conditions

1 Introduction

Fractional calculus is a mathematical branch investigating the properties dealing with arbitrary order differential and integral operators. Fractional differential equation have been an excellent instrument in the mathematical modeling of dynamical systems and real world problems, such as physics, biological and chemical engineering, aerodynamics, earthquake vibrations, fractals and chaotic, nonlinear control theory, signal and image processing, artificial intelligence, *etc*. However, many researchers introduced various definitions of fractional derivative and integral operators of arbitrary order. For more details, we refer the reader to the books [1–5]. Some recent contributions to the theory of fractional differential equations and its applications can be seen in [6–9] and the references cited therein.

In this paper, we study the existence and stability of solutions for the following generalized proportional fractional (GPF) functional integro-differential Langevin equation with

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variable coefficient and nonlocal fractional integral conditions:

$$\begin{cases} {}^C_aD^{q_1,\rho}({}^C_aD^{q_2,\rho} + \lambda(t))x(t) = f(t, x(t), x(\theta(t)), (\mathcal{K}x)(t)), & t \in [a, T] \\ \sum_{i=1}^m \kappa_{i\,a}I^{\mu_i,\rho}x(\sigma_i) = \sum_{j=1}^n \alpha_{j\,a}I^{\beta_j,\rho}x(\eta_j), \\ \sum_{k=1}^p \omega_{k\,a}I^{\gamma_k,\rho}x(\psi_k) = \sum_{l=1}^r \nu_{l\,a}I^{\varphi_l,\rho}x(\xi_l), \end{cases} \tag{1.1}$$

where ${}^C_aD^{q,\rho}$ denotes the GPF derivative of order q, $q = \{q_1, q_2\}$ with $0 < q_1, q_2 \le 1$, $1 < q_1 + q_2 \le 2$, $\rho > 0$, in Caputo type, ${}_aI^{w,\rho}$ denotes the GPF integral of order $w > 0$, $w = \{\mu_i, \beta_j, \gamma_k, \varphi_l\}$, $\rho > 0$, κ_i, α_j, ω_k, $\nu_l \in \mathbb{R}$, σ_i, η_j, ψ_k, $\xi_l \in (a, T)$ for all $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $k = 1, 2, \dots, p$, $l = 1, 2, \dots, r$, $m, n, p, r \in \mathbb{N}$, $\lambda \in C([a, T], \mathbb{R})$, $f \in C([a, T] \times \mathbb{R}^3, \mathbb{R})$, $\theta \in C([a, T], [a, T])$, and

$$(\mathcal{K}x)(t) = \int_a^t \phi\bigl(t, s, x(s)\bigr)\,ds, \quad t \in [a, T],$$

where $\phi \in C([a, T]^2 \times \mathbb{R}, [a, \infty))$.

The Langevin equation has been used to describe the dynamics of physical phenomena in the fluctuating environment of mathematical physics [10, 11]. For a system in complex phenomena, it has been realized that the integer order of the Langevin equation does not provide the accurate representation of dynamical systems. Therefore, one of the best ways to overcome this disadvantage is to replace the integer order derivative by the fractional order derivative [12–15]. The popular research interest in fractional Langevin equations is focused on the investigation of existence and stability of solutions. In this context, the literature has witnessed the appearance of many results on Langevin equations within various types of fractional operators and using different techniques, we refer the reader to the papers [16–29] and the references therein. It is worth mentioning here that all of the above cited work has been conducted in the frame of the classical Riemann–Liouville, Caputo, and Hadamard fractional operators. Further, the problem of Langevin has been considered using some generalized fractional derivatives in which, for instance, Atangana–Baleanu and Hilfer fractional derivatives were employed [30, 31].

Inspired by the above work and with the hope of considering generalized fractional derivative that includes the classical derivatives as particular cases, we accommodate the newly defined GPF derivative to study the problem of Langevin equation. The new derivative $D^{p,q}x(t)$ involves two parameters and has the features that the semigroup property is preserved, nonlocal character is possessed, and upon limiting cases it converges to the original function and its derivative. The GPF derivative is well behaved and has a substantial advantage over the classical derivatives in the sense that it generalizes previously defined derivatives in the literature. We list here some recent results which have been elaborated in the frame of GPF derivative [32–37]. Exploring the literature and in view of equations considered in the aforesaid references, one can figure out that equation (1.1) is entirely different from the equations investigated earlier. The nonlinearity function incorporates an integral term, (1.1) includes variable coefficient, and the boundary conditions are formulated in general settings.

The manuscript is processed as follows. Sect. 2 is essential in its nature as it presents preliminary definitions and results. In Sect. 3, we establish some appropriate conditions for the existence and uniqueness of solutions of problem (1.1) via the technique of fixed point theorems. In Sect. 4, we set up applicable results under which the solution of problem (1.1) fulfills the conditions of different kinds of Ulam stability. The validity of discussed results is illustrated by a particular example in Sect. 5. We end the paper by a conclusion.

2 Preliminaries

This section presents some fundamental definitions and lemmas that will be used in this paper. For interpretations and proofs, the reader can consult the papers [38–40].

Throughout this paper, we define $\mathbb{E} = C([a,T],\mathbb{R})$ as the Banach space of all continuous functions from $[a,T]$ into $\mathbb{R}$ equipped with the norm $\|x\|_{\mathbb{E}} = \sup_{t\in[a,T]}\{|x(t)|\}$.

Definition 2.1 ([38]) For $\rho \in (0,1]$ and $\alpha \in \mathbb{R}^+$, the generalized proportional fractional (GPF) integral of function f of order α is defined by

$$\left({}_aI^{\alpha,\rho}f\right)(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}f(s)\,ds, \tag{2.1}$$

where $\Gamma(\cdot)$ represents the gamma function.

Definition 2.2 ([38]) For $\rho \in (0,1]$ and $\alpha \in \mathbb{R}^+$, the generalized proportional fractional (GPF) derivative of Caputo type of function f of order α is defined by

$$\left({}_a^CD^{\alpha,\rho}f\right)(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-\tau)}(t-\tau)^{n-\alpha-1}D^{n,\rho}f(\tau)\,d\tau, \tag{2.2}$$

where $n = [\alpha]+1$ with $[\alpha]$ represents the integer part of the real number α and $(D^{n,\rho}f)(t) = (D^\rho f(t))^n$ with $(D^\rho f)(t) = (1-\rho)f(t) - \rho f'(t)$. Note that $\lim_{\rho\to 0}(D^\rho f)(t) = f(t)$.

Lemma 2.3 ([38]) *For $\rho \in (0,1]$ and $n = [\alpha]+1$, we have $({}_a^CD^{\alpha,\rho}{}_aI^{\alpha,\rho}f)(t) = f(t)$, and*

$$\left({}_aI^{\alpha,\rho}{}_a^CD^{\alpha,\rho}f\right)(t) = f(t) - e^{\frac{\rho-1}{\rho}(t-a)}\sum_{k=0}^{n-1}\frac{D^{k,\rho}f(a)}{\rho^k k!}(t-a)^k. \tag{2.3}$$

Proposition 2.4 ([38]) *Let $\alpha \geq 0$ and $\beta > 0$. Then, for any $\rho \in (0,1]$ and $n = [\alpha]+1$, we have*

(i) $({}_aI^{\alpha,\rho}e^{\frac{\rho-1}{\rho}s}(s-a)^{\beta-1})(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\rho^\alpha}e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta+\alpha-1}$, $\alpha > 0$.

(ii) $({}_a^CD^{\alpha,\rho}e^{\frac{\rho-1}{\rho}s}(s-a)^{\beta-1})(t) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{\frac{\rho-1}{\rho}x}(t-a)^{\beta-\alpha-1}$, $\alpha > n$.

(iii) $({}_a^CD^{\alpha,\rho}e^{\frac{\rho-1}{\rho}s}(s-a)^k)(t) = 0$, $\alpha > n$, $k = 0,1,\ldots,n-1$.

Fixed point theorems play a major role in establishing the existence theory for problem (1.1). We collect here some well-known fixed point theorems for the sake of completeness.

Lemma 2.5 (Banach's fixed point theorem [40]) *Let D be a nonempty closed subset of a Banach space E. Then any contraction mapping T from D into itself has a unique fixed point.*

Lemma 2.6 (Krasnoselskii's fixed point theorem [41]) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Lemma 2.7 (Schaefer's fixed point theorem [40]) *Let $\mathbb{M}$ be a Banach space and $T : \mathbb{M} \to \mathbb{M}$ be a completely continuous operator, and let the set $D = \{x \in \mathbb{M} : x = \kappa Tx, 0 < \kappa \leq 1\}$ be bounded. Then T has a fixed point in $\mathbb{M}$.*

In order to transform the main problem into a fixed point problem, (1.1) must be converted to an equivalent Volterra integral equation. We provide the following lemma, which is important in our main results.

Lemma 2.8 *Let $h : [a,T] \to \mathbb{R}$ be a continuous function, $0 < q_1, q_2 \leq 1$, $1 < q_1 + q_2 \leq 2$ and ρ, μ_i, β_j, γ_k, φ_l, > 0, κ_i, α_j, ω_k, $\nu_l \in \mathbb{R}$ and σ_i, η_j, ψ_k, $\xi_l \in (a,T)$ for all $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, $k = 1,2,\dots,p$, $l = 1,2,\dots,r$, m, n, p, $r \in \mathbb{N}$. Then the function $x \in \mathbb{E}$ is the solution to the following linear GPF Langevin equation equipped with nonlocal fractional integral conditions*:

$$\begin{cases} {}_a^C D^{q_1,\rho}({}_a^C D^{q_2,\rho} + \lambda(t))x(t) = h(t), \quad t \in [a,T], \\ \sum_{i=1}^m \kappa_{i\,a} I^{\mu_i,\rho} x(\sigma_i) = \sum_{j=1}^n \alpha_{j\,a} I^{\beta_j,\rho} x(\eta_j), \\ \sum_{k=1}^p \omega_{k\,a} I^{\gamma_k,\rho} x(\psi_k) = \sum_{l=1}^r \nu_{l\,a} I^{\varphi_l,\rho} x(\xi_l), \end{cases} \tag{2.4}$$

if and only if x satisfies the following fractional integral equation:

$$\begin{aligned} x(t) &= {}_aI^{q_1+q_2,\rho}h(t) - {}_aI^{q_2,\rho}\lambda(t)x(t) \\ &\quad + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right)\Bigg(\sum_{j=1}^n \alpha_j\big[{}_aI^{q_1+q_2+\beta_j,\rho}h(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(\eta_j)x(\eta_j)\big] \\ &\quad - \sum_{i=1}^m \kappa_i\big[{}_aI^{q_1+q_2+\mu_i,\rho}h(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(\sigma_i)x(\sigma_i)\big]\Bigg) \\ &\quad + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\Bigg(\sum_{l=1}^r \nu_l\big[{}_aI^{q_1+q_2+\varphi_l,\rho}h(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(\xi_l)x(\xi_l)\big] \\ &\quad - \sum_{k=1}^p \omega_k\big[{}_aI^{q_1+q_2+\gamma_k,\rho}h(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(\psi_k)x(\psi_k)\big]\Bigg)\Bigg], \end{aligned} \tag{2.5}$$

where

$$\Omega_1 = \sum_{i=1}^m \frac{\kappa_i(\sigma_i-a)^{q_2+\mu_i}e^{\frac{\rho-1}{\rho}(\sigma_i-a)}}{\rho^{q_2+\mu_i}\Gamma(q_2+\mu_i+1)} - \sum_{j=1}^n \frac{\alpha_j(\eta_j-a)^{q_2+\beta_j}e^{\frac{\rho-1}{\rho}(\eta_j-a)}}{\rho^{q_2+\beta_j}\Gamma(q_2+\beta_j+1)}, \tag{2.6}$$

$$\Omega_2 = \sum_{i=1}^m \frac{\kappa_i(\sigma_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\sigma_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \sum_{j=1}^n \frac{\alpha_j(\eta_j-a)^{\beta_j}e^{\frac{\rho-1}{\rho}(\eta_j-a)}}{\rho^{\beta_j}\Gamma(\beta_j+1)}, \tag{2.7}$$

$$\Omega_3 = \sum_{k=1}^p \frac{\omega_k(\psi_k-a)^{q_2+\gamma_k}e^{\frac{\rho-1}{\rho}(\psi_k-a)}}{\rho^{q_2+\gamma_k}\Gamma(q_2+\gamma_k+1)} - \sum_{l=1}^r \frac{\nu_l(\xi_l-a)^{q_2+\varphi_l}e^{\frac{\rho-1}{\rho}(\xi_l-a)}}{\rho^{q_2+\varphi_l}\Gamma(q_2+\varphi_l+1)}, \tag{2.8}$$

$$\Omega_4 = \sum_{k=1}^p \frac{\omega_k(\psi_k-a)_k^{\gamma}e^{\frac{\rho-1}{\rho}(\psi_k-a)}}{\rho^{\gamma_k}\Gamma(\gamma_k+1)} - \sum_{l=1}^r \frac{\nu_l(\xi_l-a)^{\varphi_l}e^{\frac{\rho-1}{\rho}(\xi_l-a)}}{\rho^{\varphi_l}\Gamma(\varphi_l+1)}, \tag{2.9}$$

$$\Omega = \Omega_1\Omega_4 - \Omega_2\Omega_3 \neq 0. \tag{2.10}$$

Proof Let x be a solution of problem (2.4). By using Lemma 2.3 with Proposition 2.4 (i), the first equation of (2.4) can be written as an equivalent integral equation

$$x(t) = {}_aI^{q_1+q_2,\rho}h(t) - {}_aI^{q_2,\rho}\lambda(t)x(t) + c_1\frac{(t-a)^{q_2}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{q_2}\Gamma(q_2+1)} + c_2e^{\frac{\rho-1}{\rho}(t-a)}, \tag{2.11}$$

where arbitrary constants $c_1, c_2 \in \mathbb{R}$.

Taking the GPF integral operator ${}_aI^{w,\rho}$ into (2.11), we obtain

$$\begin{aligned} {}_aI^{w,\rho}x(t) &= {}_aI^{q_1+q_2+w,\rho}h(t) - {}_aI^{q_2+w,\rho}\lambda(t)x(t) \\ &\quad + c_1\frac{(t-a)^{q_2+w}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{q_2+w}\Gamma(q_2+w+1)} + c_2\frac{(t-a)^{w}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{w}\Gamma(w+1)}. \end{aligned} \tag{2.12}$$

Substituting $w = \{\mu_i, \beta_j, \gamma_k, \varphi_l\}$, $t = \{\sigma_i, \eta_j, \psi_k, \xi_l\}$ in (2.12), respectively, and applying the boundary conditions of problem (2.4), we have

$$\begin{aligned} \Omega_1c_1 + \Omega_2c_2 &= \sum_{j=1}^{n}\alpha_j\big[{}_aI^{q_1+q_2+\beta_j,\rho}h(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(\eta_j)x(\eta_j)\big] \\ &\quad - \sum_{i=1}^{m}\kappa_i\big[{}_aI^{q_1+q_2+\mu_i,\rho}h(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(\sigma_i)x(\sigma_i)\big], \\ \Omega_3c_1 + \Omega_4c_2 &= \sum_{l=1}^{r}\nu_l\big[{}_aI^{q_1+q_2+\varphi_l,\rho}h(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(\xi_l)x(\xi_l)\big] \\ &\quad - \sum_{k=1}^{p}\omega_k\big[{}_aI^{q_1+q_2+\gamma_k,\rho}h(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(\psi_k)x(\psi_k)\big]. \end{aligned}$$

Solving the above system for c_1 and c_2, we have

$$\begin{aligned} c_1 &= \frac{1}{\Omega}\Bigg[\Omega_4\Bigg(\sum_{j=1}^{n}\alpha_j\big[{}_aI^{q_1+q_2+\beta_j,\rho}h(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(\eta_j)x(\eta_j)\big] \\ &\quad - \sum_{i=1}^{m}\kappa_i\big[{}_aI^{q_1+q_2+\mu_i,\rho}h(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(\sigma_i)x(\sigma_i)\big]\Bigg) \\ &\quad - \Omega_2\Bigg(\sum_{l=1}^{r}\nu_l\big[{}_aI^{q_1+q_2+\varphi_l,\rho}h(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(\xi_l)x(\xi_l)\big] \\ &\quad - \sum_{k=1}^{p}\omega_k\big[{}_aI^{q_1+q_2+\gamma_k,\rho}h(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(\psi_k)x(\psi_k)\big]\Bigg)\Bigg], \\ c_2 &= \frac{1}{\Omega}\Bigg[\Omega_1\Bigg(\sum_{l=1}^{r}\nu_l\big[{}_aI^{q_1+q_2+\varphi_l,\rho}h(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(\xi_l)x(\xi_l)\big] \\ &\quad - \sum_{k=1}^{p}\omega_k\big[{}_aI^{q_1+q_2+\gamma_k,\rho}h(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(\psi_k)x(\psi_k)\big]\Bigg) \\ &\quad - \Omega_3\Bigg(\sum_{j=1}^{n}\alpha_j\big[{}_aI^{q_1+q_2+\beta_j,\rho}h(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(\eta_j)x(\eta_j)\big] \end{aligned}$$

$$- \sum_{i=1}^{m} \kappa_i \Big[{}_aI^{q_1+q_2+\mu_i,\rho} h(\sigma_i) - {}_aI^{q_2+\mu_i,\rho} \lambda(\sigma_i)x(\sigma_i) \Big] \Bigg) \Bigg],$$

where Ω_1, Ω_2, Ω_3, Ω_4, and Ω are defined by (2.6)–(2.10), respectively. Substituting the values c_1 and c_2 into (2.11), we get the fractional integral equation (2.5).

Conversely, it is easily to shown by direct calculation that the solution $x(t)$ given by (2.5) satisfies problem (2.4) under the given boundary conditions. The proof is completed. □

3 Main results

In this section, we establish the existence results of solutions for problem (1.1). Fixed point theorems are employed to prove the results.

Throughout this paper, the expression ${}_aI^{b,\rho} f(s, x(s), x(\theta(s)), (\mathcal{K}x)(s))(c)$ means that

$$\begin{aligned} &{}_aI^{b,\rho} f\big(s, x(s), x\big(\theta(s)\big), (\mathcal{K}x)(s)\big)(c) \\ &\quad = \frac{1}{\rho^b \Gamma(b)} \int_a^c e^{\frac{\rho-1}{\rho}(c-s)} (c-s)^{b-1} f\big(s, x(s), x\big(\theta(s)\big), (\mathcal{K}x)(s)\big)\, ds, \end{aligned}$$

where $b \in \{q_2, q_1+q_2, q_2+\mu_i, q_2+\beta_j, q_2+\gamma_k, q_2+\varphi_l, q_1+q_2+\mu_i, q_1+q_2+\beta_j, q_1+q_2+\gamma_k, q_1+q_2+\varphi_l\}$ and $c \in \{t, \sigma_i, \eta_j, \psi_k, \xi_l\}$. For simplicity, we set

$$F_x(t) = f\big(t, x(t), x\big(\theta(t)\big), (\mathcal{K}x)(t)\big).$$

In view of Lemma 2.8, an operator $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$ is defined by

$$\begin{aligned} (\mathcal{Q}x)(t) &= {}_aI^{q_1+q_2,\rho} F_x(s)(t) - {}_aI^{q_2,\rho} \lambda(s)x(s)(t) \\ &\quad + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega} \Bigg[\bigg(\frac{\Omega_4 (t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3 \bigg) \\ &\quad \times \Bigg(\sum_{j=1}^{n} \alpha_j \big[{}_aI^{q_1+q_2+\beta_j,\rho} F_x(s)(\eta_j) - {}_aI^{q_2+\beta_j,\rho} \lambda(s)x(s)(\eta_j) \big] \\ &\quad - \sum_{i=1}^{m} \kappa_i \big[{}_aI^{q_1+q_2+\mu_i,\rho} F_x(s)(\sigma_i) - {}_aI^{q_2+\mu_i,\rho} \lambda(s)x(s)(\sigma_i) \big] \Bigg) \\ &\quad + \bigg(\Omega_1 - \frac{\Omega_2 (t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} \bigg) \Bigg(\sum_{l=1}^{r} \nu_l \big[{}_aI^{q_1+q_2+\varphi_l,\rho} F_x(s)(\xi_l) - {}_aI^{q_2+\varphi_l,\rho} \lambda(s)x(s)(\xi_l) \big] \\ &\quad - \sum_{k=1}^{p} \omega_k \big[{}_aI^{q_1+q_2+\gamma_k,\rho} F_x(s)(\psi_k) - {}_aI^{q_2+\gamma_k,\rho} \lambda(s)x(s)(\psi_k) \big] \Bigg) \Bigg]. \end{aligned} \tag{3.1}$$

Then problem (1.1) has solutions if and only if the operator $\mathcal{Q}$ has fixed points.

To proceed further, we introduce the following hypotheses:

(H_1) Let $f : [a, T] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function.

(H_2) Let $\lambda : [a, T] \to \mathbb{R}$ be a continuous function.

(H_3) There exist positive constants L_1, L_2 such that

$$\big| f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \big| \le L_1 \big(|u_1 - u_2| + |v_1 - v_2| \big) + L_2 |w_1 - w_2|$$

for any $u_i, v_i, w_i \in \mathbb{R}$, $i = 1, 2$, and $t \in [a, T]$.

(H_4) Let $\phi : [a,T]^2 \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and there exists a positive constant ϕ_0 such that

$$\left|\phi(t,s,u) - \phi(t,s,v)\right| \le \phi_0|u-v|$$

for each $(t,s) \in [a,T]^2$ and $u, v \in \mathbb{R}$.

(H_5) $|f(t,u,v,w)| \le g(t)$, $\forall (t,u,v,w) \in [a,T] \times \mathbb{R}^3$ and $g \in C([a,T],\mathbb{R}^+)$.

(H_6) There exist nonnegative continuous functions $h_1, h_2, h_3, h_4 \in \mathbb{E}$ such that

$$\left|f(t,u,v,w)\right| \le h_1(t) + h_2(t)|u| + h_3(t)|v| + h_4(t)|w|, \quad u,v,w \in \mathbb{R}, \quad t \in [a,T],$$

with $h_1^* = \sup_{t\in[a,T]} h_1(t)$, $h_2^* = \sup_{t\in[a,T]} h_2(t)$, $h_3^* = \sup_{t\in[a,T]} h_3(t)$, $h_4^* = \sup_{t\in[a,T]} h_4(t)$.

For the sake of computational convenience, we make use of the following constants:

$$\Lambda_1 = \frac{1}{|\Omega|}\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_3|\right), \tag{3.2}$$

$$\Lambda_2 = \frac{1}{|\Omega|}\left(\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_1|\right), \tag{3.3}$$

$$\begin{aligned} \Lambda_3(u) ={}& \frac{(T-a)^u}{\rho^{q_1+q_2}\Gamma(u+1)} \\ &+ \Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{u+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(u+\beta_j+1)} + \sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{u+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(u+\mu_i+1)}\right) \\ &+ \Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{u+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(u+\varphi_l+1)} + \sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{u+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(u+\gamma_k+1)}\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \Lambda_4 ={}& {}_aI^{q_2,\rho}\left|\lambda(s)\right|(T) + \Lambda_1\left(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}\left|\lambda(s)\right|(\eta_j) + \sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}\left|\lambda(s)\right|(\sigma_i)\right) \\ &+ \Lambda_2\left(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}\left|\lambda(s)\right|(\xi_l) + \sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}\left|\lambda(s)\right|(\psi_k)\right), \end{aligned} \tag{3.5}$$

where $u = \{q_1+q_2, q_1+q_2+1\}$.

3.1 Existence and uniqueness result via Banach's contraction principle

The existence and uniqueness result of a solution for problem (1.1) will be proved by using Banach's fixed point theorem.

Theorem 3.1 *Suppose that hypotheses* (H_1), (H_2), (H_3), *and* (H_4) *are satisfied. If*

$$2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4 < 1 \tag{3.6}$$

and Λ_1, Λ_2, $\Lambda_3(u)$, $u = \{q_1+q_2, q_1+q_2+1\}$, *and* Λ_4 *are given by* (3.2), (3.3), (3.4), *and* (3.5), *respectively, then problem* (1.1) *has a unique solution in the space* $\mathbb{E}$.

Proof Firstly, we transform problem (1.1) into a fixed point problem $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined as in (3.1). Observe that the fixed points of the operator $\mathcal{Q}$ are solutions of problem (1.1). Applying the Banach contraction principle, we shall show that the operator $\mathcal{Q}$ has a fixed point which is the unique solution of problem (1.1).

Let $\sup_{t\in[a,T]}|f(t,0,0,0)| := M_1 < \infty$. Next, we set $B_{R_1} := \{x \in \mathbb{E} : \|x\|_{\mathbb{E}} \leq R_1\}$ with

$$R_1 \geq \frac{M_1\Lambda_3(q_1+q_2)}{1-[2L_1\Lambda_3(q_1+q_2)-L_2\phi_0\Lambda_3(q_1+q_2+1)-\Lambda_4]}, \tag{3.7}$$

where $\Lambda_3(u)$, $u = \{q_1+q_2, q_1+q_2+1\}$, and Λ_4 are given by (3.4) and (3.5), respectively. Note that B_{R_1} is a bounded, closed, and convex subset of $\mathbb{E}$. The proof is divided into two steps as follows.

Step I. The operator $\mathcal{Q}$ defined by (3.1) satisfies the relation: $\mathcal{Q}B_{R_1} \subset B_{R_1}$.

For any $x \in B_{R_1}$, we have

$$\begin{aligned}
&\big|(\mathcal{Q}x)(t)\big| \\
&\leq {}_aI^{q_1+q_2,\rho}\big|F_x(s)\big|(t) + {}_aI^{q_2,\rho}\big|\lambda(s)\big|\big|x(s)\big|(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\Bigg[\bigg(\frac{|\Omega_4|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_3|\bigg) \\
&\quad\times\Bigg(\sum_{j=1}^{n}|\alpha_j|\big[{}_aI^{q_1+q_2+\beta_j,\rho}\big|F_x(s)\big|(\eta_j) + {}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\eta_j)\big] \\
&\quad+\sum_{i=1}^{m}|\kappa_i|\big[{}_aI^{q_1+q_2+\mu_i,\rho}\big|F_x(s)\big|(\sigma_i) + {}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\sigma_i)\big]\Bigg) \\
&\quad+\bigg(|\Omega_1| + \frac{|\Omega_2|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\bigg)\Bigg(\sum_{l=1}^{r}|\nu_l|\big[{}_aI^{q_1+q_2+\varphi_l,\rho}\big|F_x(s)\big|(\xi_l) + {}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\xi_l)\big] \\
&\quad+\sum_{k=1}^{p}|\omega_k|\big[{}_aI^{q_1+q_2+\gamma_k,\rho}\big|F_x(s)\big|(\psi_k) + {}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\psi_k)\big]\Bigg)\Bigg] \\
&\leq {}_aI^{q_1+q_2,\rho}\big(\big|F_x(s)-f(s,0,0,0)\big| + \big|f(s,0,0,0)\big|\big)(T) + {}_aI^{q_2,\rho}\big|\lambda(s)\big|\big|x(s)\big|(T) + \frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|} \\
&\quad\times\Bigg[\bigg(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_3|\bigg)\Bigg(\sum_{j=1}^{n}|\alpha_j|\big[{}_aI^{q_1+q_2+\beta_j,\rho}\big(\big|F_x(s)-f(s,0,0,0)\big| \\
&\quad+\big|f(s,0,0,0)\big|\big)(\eta_j) + {}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\eta_j)\big] \\
&\quad+\sum_{i=1}^{m}|\kappa_i|\big[{}_aI^{q_1+q_2+\mu_i,\rho}\big(\big|F_x(s)-f(s,0,0,0)\big| + \big|f(s,0,0,0)\big|\big)(\sigma_i) \\
&\quad+{}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\sigma_i)\big]\Bigg) + \bigg(|\Omega_1| + \frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\bigg) \\
&\quad\times\Bigg(\sum_{l=1}^{r}|\nu_l|\big[{}_aI^{q_1+q_2+\varphi_l,\rho}\big(\big|F_x(s)-f(s,0,0,0)\big| + \big|f(s,0,0,0)\big|\big)(\xi_l) \\
&\quad+{}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\xi_l)\big]
\end{aligned}$$

$$+\sum_{k=1}^{p}|\omega_k|\Big[{}_aI^{q_1+q_2+\gamma_k,\rho}\big(\big|F_x(s)-f(s,0,0,0)\big|+\big|f(s,0,0,0)\big|\big)(\psi_k)$$
$$+{}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\psi_k)\Big]\bigg)\bigg].$$

By using the property $e^{\frac{\rho-1}{\rho}(u-s)}\le 1$ for $a\le s<u<t\le T$ and (H_3)–(H_4), we obtain

$$\begin{aligned}
&\big|(\mathcal{Q}x)(t)\big|\\
&\le {}_aI^{q_1+q_2,\rho}\big(L_1\big(\big|x(s)\big|+\big|x\big(\theta(s)\big)\big|\big)+L_2|(\mathcal{K}x)(s)|+M_1\big)(T)+{}_aI^{q_2,\rho}\big|\lambda(s)\big|\big|x(s)\big|(T)\\
&\quad+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\bigg[\bigg(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\bigg)\\
&\quad\times\bigg(\sum_{j=1}^{n}|\alpha_j|\Big[{}_aI^{q_1+q_2+\beta_j,\rho}\big(L_1\big(\big|x(s)\big|+\big|x\big(\theta(s)\big)\big|\big)+L_2\big|(\mathcal{K}x)(s)\big|+M_1\big)(\eta_j)\\
&\quad+{}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\eta_j)\Big]+\sum_{i=1}^{m}|\kappa_i|\Big[{}_aI^{q_1+q_2+\mu_i,\rho}\big(L_1\big(\big|x(s)\big|+\big|x\big(\theta(s)\big)\big|\big)\\
&\quad+L_2\big|(\mathcal{K}x)(s)\big|+M_1\big)(\sigma_i)+{}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\sigma_i)\Big]\bigg)\\
&\quad+\bigg(|\Omega_1|+\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\bigg)\bigg(\sum_{l=1}^{r}|\nu_l|\Big[{}_aI^{q_1+q_2+\varphi_l,\rho}\big(L_1\big(\big|x(s)\big|+\big|x\big(\theta(s)\big)\big|\big)\\
&\quad+L_2\big|(\mathcal{K}x)(s)\big|+M_1\big)(\xi_l)+{}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\xi_l)\Big]\\
&\quad+\sum_{k=1}^{p}|\omega_k|\Big[{}_aI^{q_1+q_2+\gamma_k,\rho}\big(L_1\big(\big|x(s)\big|+\big|x\big(\theta(s)\big)\big|\big)\\
&\quad+L_2\big|(\mathcal{K}x)(s)\big|+M_1\big)(\psi_k)+{}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\psi_k)\Big]\bigg)\bigg]\\
&\le\frac{1}{\rho^{q_1+q_2}\Gamma(q_1+q_2)}\int_a^T(T-s)^{q_1+q_2-1}\big(2L_1R_1+L_2\phi_0(s-a)R_1+M_1\big)\,ds\\
&\quad+R_{1a}I^{q_2,\rho}\big|\lambda(s)\big|(T)+\frac{1}{|\Omega|}\bigg[\bigg(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\bigg)\\
&\quad\times\bigg(\sum_{j=1}^{n}|\alpha_j|\bigg[\frac{1}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j)}\int_a^{\eta_j}(\eta_j-s)^{q_1+q_2+\beta_j-1}\\
&\quad\times\big(2L_1R_1+L_2\phi_0(s-a)R_1+M_1\big)\,ds+R_{1a}I^{q_2+\beta_j,\rho}\big|\lambda(s)\big|(\eta_j)\bigg]\\
&\quad+\sum_{i=1}^{m}|\kappa_i|\bigg[\frac{1}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i)}\\
&\quad\times\int_a^{\sigma_i}(\sigma_i-s)^{q_1+q_2+\mu_i-1}\big(2L_1R_1+L_2\phi_0(s-a)R_1+M_1\big)\,ds+R_{1a}I^{q_2+\mu_i,\rho}\big|\lambda(s)\big|(\sigma_i)\bigg]\bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\left(|\Omega_1|+\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\left(\sum_{l=1}^{r}|\nu_l|\left[\frac{1}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l)}\int_a^{\xi_l}(\xi_l-s)^{q_1+q_2+\varphi_l-1}\right.\right.\\
&\times\bigl(2L_1R_1+L_2\phi_0(s-a)R_1+M_1\bigr)\,ds+R_{1a}I^{q_2+\varphi_l,\rho}\bigl|\lambda(s)\bigr|(\xi_l)\Bigr]\\
&+\sum_{k=1}^{p}|\omega_k|\left[\frac{1}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k)}\right.\\
&\times\int_a^{\psi_k}(\psi_k-s)^{q_1+q_2+\gamma_k-1}\bigl(2L_1R_1+L_2\phi_0(s-a)R_1+M_1\bigr)\,ds\\
&\left.\left.\left.+R_{1a}I^{q_2+\gamma_k,\rho}\bigl|\lambda(s)\bigr|(\psi_k)\right]\right)\right]\\
\le{}&(2L_1R_1+M_1)\left[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}+\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}\right.\right.\\
&\left.+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right)+\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}\right.\\
&\left.\left.+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\right]+L_2\phi_0R_1\left[\frac{(T-a)^{q_1+q_2+1}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+2)}\right.\\
&+\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\right)\\
&\left.+\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\right)\right]\\
&+R_1\left[{}_aI^{q_2,\rho}\bigl|\lambda(s)\bigr|(T)+\Lambda_1\left(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}\bigl|\lambda(s)\bigr|(\eta_j)+\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}\bigl|\lambda(s)\bigr|(\sigma_i)\right)\right.\\
&\left.+\Lambda_2\left(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}\bigl|\lambda(s)\bigr|(\xi_l)+\sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}\bigl|\lambda(s)\bigr|(\psi_k)\right)\right]\\
\le{}&(2L_1R_1+M_1)\Lambda_3(u)+L_2\phi_0R_1\Lambda_3(u+1)+R_1\Lambda_4\le R_1,
\end{aligned}
$$

which implies that $\|\mathcal{Q}x\|_{\mathbb{E}}\le R_1$. Therefore, $\mathcal{Q}$ maps bounded subsets of B_{R_1} into bounded subsets of B_{R_1}, that is, $\mathcal{Q}B_{R_1}\subset B_{R_1}$.

Step II. To show that an operator $\mathcal{Q}:\mathbb{E}\to\mathbb{E}$ is contraction.

For any $x\ y\in\mathbb{E}$ and for each $t\in[a,T]$, we have

$$
\begin{aligned}
&\bigl|(\mathcal{Q}x)(t)-(\mathcal{Q}y)(t)\bigr|\\
&\le{}_aI^{q_1+q_2,\rho}\bigl|F_x(s)-F_y(s)\bigr|(T)+{}_aI^{q_2,\rho}\bigl|\lambda(s)\bigr|\bigl|x(s)-y(s)\bigr|(T)\\
&+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\left[\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right)\right.\\
&\times\left(\sum_{j=1}^{n}|\alpha_j|\bigl[{}_aI^{q_1+q_2+\beta_j,\rho}\bigl|F_x(s)-F_y(s)\bigr|(\eta_j)+{}_aI^{q_2+\beta_j,\rho}\bigl|\lambda(s)\bigr|\bigl|x(s)-y(s)\bigr|(\eta_j)\bigr]\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}|\kappa_i|\bigl[{}_aI^{q_1+q_2+\mu_i,\rho}\bigl|F_x(s)-F_y(s)\bigr|(\sigma_i)+{}_aI^{q_2+\mu_i,\rho}\bigl|\lambda(s)\bigr|\bigl|x(s)-y(s)\bigr|(\sigma_i)\bigr]\Biggr)\\
&+\biggl(\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_1|\biggr)\Biggl(\sum_{l=1}^{r}|\nu_l|\bigl[{}_aI^{q_1+q_2+\varphi_l,\rho}\bigl|F_x(s)-F_y(s)\bigr|(\xi_l)\\
&+{}_aI^{q_2+\varphi_l,\rho}\bigl|\lambda(s)\bigr|\bigl|x(s)-y(s)\bigr|(\xi_l)\bigr]+\sum_{k=1}^{p}|\omega_k|\bigl[{}_aI^{q_1+q_2+\gamma_k,\rho}\bigl|F_x(s)-F_y(s)\bigr|(\psi_k)\\
&+{}_aI^{q_2+\gamma_k,\rho}\bigl|\lambda(s)\bigr|\bigl|x(s)-y(s)\bigr|(\psi_k)\bigr]\Biggr)\Biggr]\\
\le{}&{}_aI^{q_1+q_2,\rho}\bigl(2L_1+L_2\phi_0(s-a)\bigr)(T)\|x-y\|_{\mathbb{E}}+{}_aI^{q_2,\rho}\bigl|\lambda(s)\bigr|(T)\|x-y\|_{\mathbb{E}}\\
&+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\Biggl[\biggl(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\biggr)\\
&\times\Biggl(\sum_{j=1}^{n}|\alpha_j|\bigl[{}_aI^{q_1+q_2+\beta_j,\rho}\bigl(2L_1+L_2\phi_0(s-a)\bigr)(\eta_j)\|x-y\|_{\mathbb{E}}\\
&+{}_aI^{q_2+\beta_j,\rho}\bigl|\lambda(s)\bigr|(\eta_j)\|x-y\|_{\mathbb{E}}\bigr]\\
&+\sum_{i=1}^{m}|\kappa_i|\bigl[{}_aI^{q_1+q_2+\mu_i,\rho}\bigl(2L_1+L_2\phi_0(s-a)\bigr)(\sigma_i)\|x-y\|_{\mathbb{E}}\\
&+{}_aI^{q_2+\mu_i,\rho}\bigl|\lambda(s)\bigr|(\sigma_i)\|x-y\|_{\mathbb{E}}\bigr]\Biggr)+\biggl(\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_1|\biggr)\\
&\times\Biggl(\sum_{l=1}^{r}|\nu_l|\bigl[{}_aI^{q_1+q_2+\varphi_l,\rho}\bigl(2L_1+L_2\phi_0(s-a)\bigr)(\xi_l)\|x-y\|_{\mathbb{E}}\\
&+{}_aI^{q_2+\varphi_l,\rho}\bigl|\lambda(s)\bigr|(\xi_l)\|x-y\|_{\mathbb{E}}\bigr]\\
&+\sum_{k=1}^{p}|\omega_k|\bigl[{}_aI^{q_1+q_2+\gamma_k,\rho}\bigl(2L_1+L_2\phi_0(s-a)\bigr)(\psi_k)\|x-y\|_{\mathbb{E}}\\
&+{}_aI^{q_2+\gamma_k,\rho}\bigl|\lambda(s)\bigr|(\psi_k)\|x-y\|_{\mathbb{E}}\bigr]\Biggr)\Biggr]\\
\le{}&\Biggl[2L_1\Biggl[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}+\Lambda_1\Biggl(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}\\
&+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\Biggr)+\Lambda_2\Biggl(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}\\
&+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\Biggr)\Biggr]+L_2\phi_0\Biggl[\frac{(T-a)^{q_1+q_2+1}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+2)}\\
&+\Lambda_1\Biggl(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\Biggr)\\
&+\Lambda_2\Biggl(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\Biggr)\Biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+ {}_aI^{q_2,\rho}\big|\lambda(s)\big|(T) + \Lambda_1\left(\sum_{j=1}^{n}|\alpha_j|_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|(\eta_j) + \sum_{i=1}^{m}|\kappa_i|_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|(\sigma_i)\right)\\
&+ \Lambda_2\left(\sum_{l=1}^{r}|\nu_l|_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|(\xi_l) + \sum_{k=1}^{p}|\omega_k|_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|(\psi_k)\right)\Bigg]\|x-y\|_{\mathbb{E}}\\
\le{}& \big[2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4\big]\|x-y\|_{\mathbb{E}},
\end{aligned}
$$

which implies that $\|\mathcal{Q}x - \mathcal{Q}y\|_{\mathbb{E}} \le [2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4]\|x-y\|_{\mathbb{E}}$. As $2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4 < 1$, hence, by the Banach contraction principle (Lemma 2.5), the operator $\mathcal{Q}$ is a contraction, therefore, it has a unique fixed point that is the unique solution of problem (1.1) in $\mathbb{E}$. The proof is completed. □

3.2 Existence result via Krasnoselskii's fixed point theorem

By using Krasnoselskii's fixed point theorem, the existence theorem will be obtained.

Theorem 3.2 *Assume that* (H_1), (H_2), (H_4), *and* (H_5) *hold. Then problem* (1.1) *has at least one solution on* $[a,T]$ *provided* $\Lambda_4 < 1$, *where* Λ_4 *is defined by* (3.5).

Proof Let $\sup_{t\in[a,T]}|g(t)| = \|g\|_{\mathbb{E}}$. By choosing a suitable $B_{\bar{R}_2} = \{x\in\mathbb{E} : \|x\|_{\mathbb{E}} \le \bar{R}_2\}$, where

$$
\bar{R}_2 \ge \frac{\Lambda_3(q_1+q_2)\|g\|_{\mathbb{E}}}{1-\Lambda_4}, \tag{3.8}
$$

and $\Lambda_3(q_1+q_2)$ and Λ_4 are defined by (3.4) and (3.5), respectively. We define the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on $B_{\bar{R}_2}$ by

$$
\begin{aligned}
(\mathcal{Q}_1x)(t) ={}& {}_aI^{q_1+q_2,\rho}F_x(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right)\\
&\times\left(\sum_{j=1}^{n}\alpha_{ja}I^{q_1+q_2+\beta_j,\rho}F_x(s)(\eta_j) - \sum_{i=1}^{m}\kappa_{ia}I^{q_1+q_2+\mu_i,\rho}F_x(s)(\sigma_i)\right)\\
&+\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\\
&\times\left(\sum_{l=1}^{r}\nu_{la}I^{q_1+q_2+\varphi_l,\rho}F_x(s)(\xi_l) - \sum_{k=1}^{p}\omega_{ka}I^{q_1+q_2+\gamma_k,\rho}F_x(s)(\psi_k)\right)\Bigg], \quad t\in[a,T],\\
(\mathcal{Q}_2x)(t) ={}& \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right)\left(\sum_{i=1}^{m}\kappa_{ia}I^{q_2+\mu_i,\rho}\lambda(s)x(s)(\sigma_i)\right.\\
&\left.-\sum_{j=1}^{n}\alpha_{ja}I^{q_2+\beta_j,\rho}\lambda(s)x(s)(\eta_j)\right)\\
&+\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\left(\sum_{k=1}^{p}\omega_{ka}I^{q_2+\gamma_k,\rho}\lambda(s)x(s)(\psi_k)\right.\\
&\left.-\sum_{l=1}^{r}\nu_{la}I^{q_2+\varphi_l,\rho}\lambda(s)x(s)(\xi_l)\right)\Bigg] - {}_aI^{q_2,\rho}\lambda(s)x(s)(t), \quad t\in[a,T].
\end{aligned}
$$

Note that $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$. For any $x, y \in B_{\bar{R}_2}$, we have

$$
\begin{aligned}
&\|\mathcal{Q}_1 x + \mathcal{Q}_2 y\|_{\mathbb{E}} \\
&\leq \sup_{t\in[a,T]} \Biggl\{ {}_aI^{q_1+q_2,\rho}\bigl|F_x(s)\bigr|(t) + {}_aI^{q_2,\rho}\bigl|\lambda(s)\bigr|\bigl|y(s)\bigr|(t) \\
&\quad + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\Biggl[\biggl(\frac{|\Omega_4|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_3|\biggr) \\
&\quad \times \Biggl(\sum_{j=1}^{n}|\alpha_j|\bigl[{}_aI^{q_1+q_2+\beta_j,\rho}\bigl|F_x(s)\bigr|(\eta_j) + {}_aI^{q_2+\beta_j,\rho}\bigl|\lambda(s)\bigr|\bigl|y(s)\bigr|(\eta_j)\bigr] \\
&\quad + \sum_{i=1}^{m}|\kappa_i|\bigl[{}_aI^{q_1+q_2+\mu_i,\rho}\bigl|F_x(s)\bigr|(\sigma_i) + {}_aI^{q_2+\mu_i,\rho}\bigl|\lambda(s)\bigr|\bigl|y(s)\bigr|(\sigma_i)\bigr]\Biggr) \\
&\quad + \biggl(|\Omega_1| + \frac{|\Omega_2|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\biggr)\Biggl(\sum_{l=1}^{r}|\nu_l|\bigl[{}_aI^{q_1+q_2+\varphi_l,\rho}\bigl|F_x(s)\bigr|(\xi_l) \\
&\quad + {}_aI^{q_2+\varphi_l,\rho}\bigl|\lambda(s)\bigr|\bigl|y(s)\bigr|(\xi_l)\bigr] \\
&\quad + \sum_{k=1}^{p}|\omega_k|\bigl[{}_aI^{q_1+q_2+\gamma_k,\rho}\bigl|F_x(s)\bigr|(\psi_k) + {}_aI^{q_2+\gamma_k,\rho}\bigl|\lambda(s)\bigr|\bigl|y(s)\bigr|(\psi_k)\bigr]\Biggr)\Biggr]\Biggr\} \\
&\leq \|g\|_{\mathbb{E}}\Biggl[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)} + \Lambda_1\Biggl(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} \\
&\quad + \sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i)}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\Biggr) + \Lambda_2\Biggl(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)} \\
&\quad + \sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\Biggr)\Biggr] \\
&\quad + \bar{R}_2\Biggl[{}_aI^{q_2,\rho}\bigl|\lambda(s)\bigr|(T) + \Lambda_1\Biggl(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}\bigl|\lambda(s)\bigr|(\eta_j) + \sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}\bigl|\lambda(s)\bigr|(\sigma_i)\Biggr) \\
&\quad + \Lambda_2\Biggl(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}\bigl|\lambda(s)\bigr|(\xi_l) + \sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}\bigl|\lambda(s)\bigr|(\psi_k)\Biggr)\Biggr] \\
&= \Lambda_3(q_1+q_2)\|g\|_{\mathbb{E}} + \Lambda_4\bar{R}_2 \leq \bar{R}_2,
\end{aligned}
$$

which implies that $\|\mathcal{Q}_1 x + \mathcal{Q}_2 y\| \leq B_{\bar{R}_2}$. It follows that $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_{\bar{R}_2}$, which satisfies assumption (i) of Lemma 2.6.

To show that assumption (ii) of Lemma 2.6 is satisfied, let $\{x_n\}$ be a sequence such that $x_n \to x$ in $\mathbb{E}$. Then, for each $t \in [a,T]$, we take

$$
\begin{aligned}
&\bigl|(\mathcal{Q}_1 x_n)(t) - (\mathcal{Q}_1 x)(t)\bigr| \\
&\leq {}_aI^{q_1+q_2,\rho}\bigl|F_{x_n}(s) - F_x(s)\bigr|(T) + \frac{1}{|\Omega|}\Biggl[\biggl(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_3|\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\sum_{j=1}^{n}|\alpha_j|_aI^{q_1+q_2+\beta_j,\rho}\big|F_{x_n}(s)-F_x(s)\big|(\eta_j)+\sum_{i=1}^{m}|\kappa_i|_aI^{q_1+q_2+\mu_i,\rho}\big|F_{x_n}(s)-F_x(s)\big|(\sigma_i)\right)\\
&+\left(|\Omega_1|+\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\left(\sum_{l=1}^{r}|\nu_l|_aI^{q_1+q_2+\varphi_l,\rho}\big|F_{x_n}(s)-F_x(s)\big|(\xi_l)\right.\\
&\left.+\sum_{k=1}^{p}|\omega_k|_aI^{q_1+q_2+\gamma_k,\rho}\big|F_{x_n}(s)-F_x(s)\big|(\psi_k)\right)\Bigg]\\
\le{}&\left\{\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}\right.\\
&+\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right)\\
&\left.+\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\right\}\\
&\times\|F_{x_n}-F_x\|_{\mathbb{E}}\\
={}&\Lambda_3(q_1+q_2)\|F_{x_n}-F_x\|_{\mathbb{E}}.
\end{aligned}
$$

Since f and λ are continuous, by the Lebesgue dominated convergent theorem, we have

$$
\big|(\mathcal{Q}_1x_n)(t)-(\mathcal{Q}_1x)(t)\big|\to 0,\quad \text{as } n\to\infty.
$$

Hence,

$$
\|\mathcal{Q}_1x_n-\mathcal{Q}_1x\|_{\mathbb{E}}\to 0\quad \text{as } n\to\infty.
$$

Therefore, the operator $\mathcal{Q}_1$ is continuous. Also, the set $\mathcal{Q}_1B_{\bar{R}_2}$ is uniformly bounded as

$$
\|\mathcal{Q}_1x\|_{\mathbb{E}}\le\Lambda_3(q_1+q_2)\|g\|_{\mathbb{E}}.
$$

Next, we prove the compactness of the operator $\mathcal{Q}_1$. Set $\sup_{(t,z_1,z_2,z_3)\in[a,T]\times B^3_{\bar{R}_2}}|f(t,z_1,z_2,z_3)|=f^*<\infty$, then for each $t_1, t_2\in[a,T]$ with $a\le t_1\le t_2\le T$, we have

$$
\begin{aligned}
&\big|(\mathcal{Q}_1x)(t_2)-(\mathcal{Q}_1x)(t_1)\big|\\
&\le\big|{}_aI^{q_1+q_2,\rho}F_x(s)(t_2)-{}_aI^{q_1+q_2,\rho}F_x(s)(t_1)\big|+\frac{1}{|\Omega|}\big|e^{\frac{\rho-1}{\rho}(t_2-a)}-e^{\frac{\rho-1}{\rho}(t_1-a)}\big|\\
&\times\left[|\Omega_4|\left(\frac{(t_2-a)^{q_2}-(t_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\right.\\
&\times\left(\sum_{j=1}^{n}|\alpha_j|_aI^{q_1+q_2+\beta_j,\rho}F_x(s)(\eta_j)+\sum_{i=1}^{m}|\kappa_i|_aI^{q_1+q_2+\mu_i,\rho}F_x(s)(\sigma_i)\right)\\
&+|\Omega_2|\left(\frac{(t_2-a)^{q_2}-(t_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\\
&\left.\times\left(\sum_{l=1}^{r}|\nu_l|_aI^{q_1+q_2+\varphi_l,\rho}F_x(s)(\xi_l)+\sum_{k=1}^{p}|\omega_k|_aI^{q_1+q_2+\gamma_k,\rho}F_x(s)(\psi_k)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq f^*\Bigg\{\frac{1}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}\big(\big|(t_2-a)^{q_1+q_2}-(t_1-a)^{q_1+q_2}-(t_2-t_1)^{q_1+q_2}\big| \\
&\quad+(t_2-t_1)^{q_1+q_2}\big)+\frac{1}{|\Omega|}\big|e^{\frac{\rho-1}{\rho}(t_2-a)} \\
&\quad-e^{\frac{\rho-1}{\rho}(t_1-a)}\big|\Bigg[|\Omega_4|\bigg(\frac{(t_2-a)^{q_2}-(t_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\bigg)\bigg(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} \\
&\quad+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\bigg)+|\Omega_2|\bigg(\frac{(t_2-a)^{q_2}-(t_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\bigg) \\
&\quad\times\bigg(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\bigg)\Bigg]\Bigg\},
\end{aligned}
$$

which is independent of x and $|(\mathcal{Q}_1x)(t_2)-(\mathcal{Q}_1x)(t_1)|\to 0$ as $t_2\to t_1$. Therefore, the set $\mathcal{Q}_1B_{\bar{R}_2}$ is equicontinuous, the operator $\mathcal{Q}_1$ maps bounded subsets into relatively compact subsets, it follows that the set $\mathcal{Q}_1B_{\bar{R}_2}$ is relatively compact. Then, by the Arzelá–Ascoli theorem, the operator $\mathcal{Q}_1$ is compact on $B_{\bar{R}_2}$. It is easy to see that using $\Lambda_4<1$ leads to the operator $\mathcal{Q}_2$ is a contraction mapping and also assumption (iii) of Lemma 2.6 holds. Thus, all the assumptions of Lemma 2.6 are satisfied. Hence, the conclusion of Theorem 3.2 implies that problem (1.1) has at least one solution on $[a,T]$. This completes the proof. □

3.3 Existence result via Schaefer's fixed point theorem

The last existence result is based on Schaefer's fixed point theorem.

Theorem 3.3 *Assume that* (H_1), (H_2), (H_4), *and* (H_6) *hold. Then problem* (1.1) *has at least one solution on* $[a,T]$.

Proof To show that the operator $\mathcal{Q}$ has at least a fixed point in $\mathbb{E}$, the proof is divided into a sequence of four steps.

Step I The operator $\mathcal{Q}$ is continuous.

Let x_n be a sequence such that $x_n\to x$ in $\mathbb{E}$. Then, for each $t\in[a,T]$, we obtain

$$
\begin{aligned}
&\big|(\mathcal{Q}x_n)(t)-(\mathcal{Q}x)(t)\big| \\
&\quad\leq {}_aI^{q_1+q_2,\rho}\big|F_{x_n}(s)-F_x(s)\big|(t)+{}_aI^{q_2,\rho}\big|\lambda(s)\big|\big|x_n(s)-x(s)\big|(t) \\
&\qquad+\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\Bigg[\bigg(\frac{|\Omega_4|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\bigg)\bigg(\sum_{j=1}^{n}|\alpha_j|\big[{}_aI^{q_1+q_2+\beta_j,\rho}\big|F_{x_n}(s)-F_x(s)\big|(\eta_j) \\
&\qquad+{}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|\big|x_n(s)-x(s)\big|(\eta_j)\big]+\sum_{i=1}^{m}|\kappa_i|\big[{}_aI^{q_1+q_2+\mu_i,\rho}\big|F_{x_n}(s)-F_x(s)\big|(\sigma_i) \\
&\qquad+{}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|\big|x_n(s)-x(s)\big|(\sigma_i)\big]\bigg)+\bigg(|\Omega_1|+\frac{|\Omega_2|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\bigg) \\
&\qquad\times\bigg(\sum_{l=1}^{r}|\nu_l|\big[{}_aI^{q_1+q_2+\varphi_l,\rho}\big|F_{x_n}(s)-F_x(s)\big|(\xi_l)+{}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|\big|x_n(s)-x(s)\big|(\xi_l)\big] \\
&\qquad+\sum_{k=1}^{p}|\omega_k|\big[{}_aI^{q_1+q_2+\gamma_k,\rho}\big|F_{x_n}(s)-F_x(s)\big|(\psi_k)+{}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|\big|x_n(s)-x(s)\big|(\psi_k)\big]\bigg)\Bigg]
\end{aligned}
$$

$$\begin{aligned}
&\leq \Bigg[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)} \\
&\quad + \Lambda_1 \Bigg(\sum_{j=1}^{n} \frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} + \sum_{i=1}^{m} \frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)} \Bigg) \\
&\quad + \Lambda_2 \Bigg(\sum_{l=1}^{r} \frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)} + \sum_{k=1}^{p} \frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)} \Bigg) \Bigg] \\
&\quad \times \|F_{x_n}-F_x\|_{\mathbb{E}} \\
&\quad + \Bigg[{}_aI^{q_2,\rho}\big|\lambda(s)\big|(T) + \Lambda_1 \Bigg(\sum_{j=1}^{n} |\alpha_j| {}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|(\eta_j) + \sum_{i=1}^{m} |\kappa_i| {}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|(\sigma_i) \Bigg) \\
&\quad + \Lambda_2 \Bigg(\sum_{l=1}^{r} |\nu_l| {}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|(\xi_l) + \sum_{k=1}^{p} |\omega_k| {}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|(\psi_k) \Bigg) \Bigg] \|x_n-x\|_{\mathbb{E}} \\
&= \Lambda_3(q_1+q_2)\|F_{x_n}-F_x\|_{\mathbb{E}} + \Lambda_4\|x_n-x\|_{\mathbb{E}}.
\end{aligned}$$

Since f and λ are continuous, this implies that the operator $\mathcal{Q}$ is also continuous. Hence, we obtain

$$\|F_{x_n}-F_x\|_{\mathbb{E}} \to 0 \quad \text{and} \quad \|x_n-x\|_{\mathbb{E}} \to 0, \quad \text{as } n \to \infty.$$

Step II The operator $\mathcal{Q}$ maps a bounded set into a bounded set in $\mathbb{E}$.

For $R_3 > 0$, there exists a constant $M_3 > 0$ such that, for each $x \in \bar{B}_{R_3} = \{x \in \mathbb{E} : \|x\|_{\mathbb{E}} \leq R_3\}$, then $\|\mathcal{Q}x\|_{\mathbb{E}} \leq M_3$. Then, for any $t \in [a,T]$ and $x \in \bar{B}_{R_3}$, we have

$$\begin{aligned}
&\big|(\mathcal{Q}x)(t)\big| \\
&\quad \leq {}_aI^{q_1+q_2,\rho}\big|F_x(s)\big|(t) + {}_aI^{q_2,\rho}\big|\lambda(s)\big|\big|x(s)\big|(t) \\
&\quad + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|} \Bigg[\Bigg(\frac{|\Omega_4|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_3| \Bigg) \\
&\quad \times \Bigg(\sum_{j=1}^{n} |\alpha_j| \big[{}_aI^{q_1+q_2+\beta_j,\rho}\big|F_x(s)\big|(\eta_j) + {}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\eta_j) \big] \\
&\quad + \sum_{i=1}^{m} |\kappa_i| \big[{}_aI^{q_1+q_2+\mu_i,\rho}\big|F_x(s)\big|(\sigma_i) + {}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\sigma_i) \big] \Bigg) \\
&\quad + \Bigg(|\Omega_1| + \frac{|\Omega_2|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} \Bigg) \\
&\quad \times \Bigg(\sum_{l=1}^{r} |\nu_l| \big[{}_aI^{q_1+q_2+\varphi_l,\rho}\big|F_x(s)\big|(\xi_l) + {}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\xi_l) \big] \\
&\quad + \sum_{k=1}^{p} |\omega_k| \big[{}_aI^{q_1+q_2+\gamma_k,\rho}\big|F_x(s)\big|(\psi_k) + {}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\psi_k) \big] \Bigg) \Bigg].
\end{aligned}$$

It follows from hypotheses (H_4) and (H_6) that

$$\begin{aligned}
{}_aI^{u,\rho}\big|F_x(s)\big|(z) &\le {}_aI^{u,\rho}\big(h_1(s)+h_2(s)\big|x(s)\big|+h_3(s)|x(\epsilon s)+h_4(s)\big|(\mathcal{K}x)(s)\big|\big)(z)\\
&\le \frac{1}{\rho^u\Gamma(u)}\int_a^z (z-s)^{u-1}\big(h_1^*+h_2^*R_3+h_3^*R_3+h_4^*R_3\phi_0(s-a)\big)\,ds\\
&\le \big(h_1^*+h_2^*R_3+h_3^*R_3\big)\frac{(z-a)^u}{\rho^u\Gamma(u+1)}+h_4^*R_3\phi_0\frac{(z-a)^{u+1}}{\rho^u\Gamma(u+2)},
\end{aligned} \tag{3.9}$$

where $u=\{q_1+q_2, q_1+q_2+\mu_i, q_1+q_2+\beta_j, q_1+q_2+\varphi_l, q_1+q_2+\gamma_k\}$ and $z=\{t,T,\sigma_i,\eta_j,\xi_l,\psi_k\}$.
This implies that

$$\begin{aligned}
&\big|(\mathcal{Q}x)(t)\big|\\
&\le \Bigg[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}\\
&\quad+\Lambda_1\Bigg(\sum_{j=1}^n\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}+\sum_{i=1}^m\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\Bigg)\\
&\quad+\Lambda_2\Bigg(\sum_{l=1}^r\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^p\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\Bigg)\Bigg]\\
&\quad\times\big(h_1^*+h_2^*R_3+h_3^*R_3\big)\\
&\quad+\Bigg[\frac{(T-a)^{q_1+q_2+1}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+2)}\\
&\quad+\Lambda_1\Bigg(\sum_{j=1}^n\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}+\sum_{i=1}^m\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\Bigg)\\
&\quad+\Lambda_2\Bigg(\sum_{l=1}^r\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)}+\sum_{k=1}^p\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\Bigg)\Bigg]\\
&\quad\times h_4^*R_3\phi_0\\
&\quad+\Bigg[{}_aI^{q_2,\rho}\big|\lambda(s)\big|(T)+\Lambda_1\Bigg(\sum_{j=1}^n|\alpha_j|{}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|(\eta_j)+\sum_{i=1}^m|\kappa_i|{}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|(\sigma_i)\Bigg)\\
&\quad+\Lambda_2\Bigg(\sum_{l=1}^r|\nu_l|{}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|(\xi_l)+\sum_{k=1}^p|\omega_k|{}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|(\psi_k)\Bigg)\Bigg]R_3\\
&\le \Lambda_3(q_1+q_2)\big(h_1^*+h_2^*R_3+h_3^*R_3\big)+\Lambda_3(q_1+q_2+1)h_4^*R_3\phi_0+\Lambda_4R_3:=M_3,
\end{aligned}$$

we estimate

$$\|\mathcal{Q}x\|_{\mathbb{E}}\le \Lambda_3(q_1+q_2)\big(h_1^*+h_2^*R_3+h_3^*R_3\big)+\Lambda_3(q_1+q_2+1)h_4^*R_3\phi_0+\Lambda_4R_3:=M_3,$$

where Λ_1, Λ_2, $\Lambda_3(u)$, $u=\{q_1+q_2,q_1+q_2+1\}$, and Λ_4 are given by (3.2), (3.3), (3.4), and (3.5), respectively.

Step III The operator $\mathcal{Q}$ maps a bounded set into an equicontinuous set of $\mathbb{E}$.

For $a \le t_1 < t_2 \le T$ and $x \in \bar{B}_{R_3}$ where $\bar{B}_{R_3}$ is as defined in Step II, by using the property f is bounded on the compact set $[a, T] \times \bar{B}_{R_3}$, we have

$$
\begin{aligned}
&\bigl|(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)\bigr| \\
&\quad\le \bigl|{}_aI^{q_1+q_2,\rho}F_x(s)(t_2) - {}_aI^{q_1+q_2,\rho}F_x(s)(t_1)\bigr| + \bigl|{}_aI^{q_2,\rho}\lambda(s)x(s)(t_2) - {}_aI^{q_2,\rho}\lambda(s)x(s)(t_1)\bigr| \\
&\qquad + \frac{1}{|\Omega|}\bigl|e^{\frac{\rho-1}{\rho}(t_2-a)} - e^{\frac{\rho-1}{\rho}(t_1-a)}\bigr| \\
&\qquad \times \Biggl[|\Omega_4|\biggl(\frac{(t_2-a)^{q_2} - (t_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\biggr)\Biggl(\sum_{j=1}^{n}|\alpha_j|\bigl[{}_aI^{q_1+q_2+\beta_j,\rho}\bigl|F_x(s)\bigr|(\eta_j) \\
&\qquad + {}_aI^{q_2+\beta_j,\rho}\bigl|\lambda(s)\bigr|\bigl|x(s)\bigr|(\eta_j)\bigr] \\
&\qquad + \sum_{i=1}^{m}|\kappa_i|\bigl[{}_aI^{q_1+q_2+\mu_i,\rho}\bigl|F_x(s)\bigr|(\sigma_i) + {}_aI^{q_2+\mu_i,\rho}\bigl|\lambda(s)\bigr|\bigl|x(s)\bigr|(\sigma_i)\bigr]\Biggr) \\
&\qquad + |\Omega_2|\biggl(\frac{(t_2-a)^{q_2} - (t_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\biggr) \\
&\qquad \times \Biggl(\sum_{l=1}^{r}|\nu_l|\bigl[{}_aI^{q_1+q_2+\varphi_l,\rho}F_x(s)(\xi_l) + {}_aI^{q_2+\varphi_l,\rho}\bigl|\lambda(s)\bigr|\bigl|x(s)\bigr|(\xi_l)\bigr] \\
&\qquad + \sum_{k=1}^{p}|\omega_k|\bigl[{}_aI^{q_1+q_2+\gamma_k,\rho}F_x(s)(\psi_k) + {}_aI^{q_2+\gamma_k,\rho}\bigl|\lambda(s)\bigr|\bigl|x(s)\bigr|(\psi_k)\bigr]\Biggr)\Biggr] \\
&\quad\le \frac{1}{\rho^{q_1+q_2}\Gamma(q_1+q_2)}\int_a^{t_1}\bigl|e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{q_1+q_2-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{q_1+q_2-1}\bigr| \\
&\qquad \times \bigl(h_1^* + h_2^*R_3 + h_3^*R_3 + h_4^*R_3\phi_0(s-a)\bigr)\,ds \\
&\qquad + \frac{1}{\rho^{q_1+q_2}\Gamma(q_1+q_2)}\int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{q_1+q_2-1} \\
&\qquad \times \bigl(h_1^* + h_2^*R_3 + h_3^*R_3 + h_4^*R_3\phi_0(s-a)\bigr)\,ds \\
&\qquad + \frac{R_3}{\rho^{q_2}\Gamma(q_2)}\int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{q_1+q_2-1}\bigl|\lambda(s)\bigr|\,ds \\
&\qquad + \frac{R_3}{\rho^{q_2}\Gamma(q_2)}\int_a^{t_1}\bigl|e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{q_1+q_2-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{q_1+q_2-1}\bigr|\bigl|\lambda(s)\bigr|\,ds \\
&\qquad + \frac{h_1^* + h_2^*R_3 + h_3^*R_3 + h_4^*R_3\phi_0}{|\Omega|}\bigl|e^{\frac{\rho-1}{\rho}(t_2-a)} - e^{\frac{\rho-1}{\rho}(t_1-a)}\bigr| \\
&\qquad \times \Biggl[|\Omega_4|\biggl(\frac{(t_2-a)^{q_2} - (t_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\biggr) \\
&\qquad \times \Biggl(\sum_{j=1}^{n}|\alpha_j|\biggl(\frac{(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} + \frac{(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}\biggr) \\
&\qquad + \sum_{i=1}^{m}|\kappa_i|\biggl(\frac{(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)} + \frac{(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\biggr)\Biggr) \\
&\qquad + \Biggl(\sum_{l=1}^{r}|\nu_l|\biggl(\frac{(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)} + \frac{(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)}\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{p}|\omega_k|\left(\frac{(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}+\frac{(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\right)\Bigg)\\
&\times|\Omega_2|\left(\frac{(t_2-a)^{q_2}-(t_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\Bigg]+R_3\Bigg[\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|(\eta_j)\\
&+\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|(\sigma_i)+\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|(\xi_l)\\
&+\sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|(\psi_k)\Bigg].
\end{aligned}
$$

The R.H.S of the above inequality tends to zero as $t_2 \to t_1$ implies that $\|(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)\|_{\mathbb{E}} \to 0$ as $t_2 \to t_1$. Thus, by Steps I to III, together with the Arzelá–Ascoli theorem, we conclude that the operator $\mathcal{Q}$ is completely continuous.

Step IV The set $\mathbb{D} = \{x \in \mathbb{E} : x = \varrho\mathcal{Q}x, 0 \le \varrho \le 1\}$ is bounded (a priori bounds).

Let $x \in \mathbb{D}$, then $x = \varrho\mathcal{Q}x$ for some $0 < \varrho < 1$. From (H_4)–(H_5), for each $t \in [a, T]$, one can get the estimates

$$
\begin{aligned}
&\big|(\mathcal{Q}x)(t)\big|\\
&\quad=\big|\varrho(\mathcal{Q}x)(t)\big|\\
&\quad\le {}_aI^{q_1+q_2,\rho}\big|F_x(s)\big|(T)+{}_aI^{q_2,\rho}\big|\lambda(s)\big|\big|x(s)\big|(T)+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\Bigg[\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right)\\
&\qquad\times\Bigg(\sum_{j=1}^{n}|\alpha_j|\big[{}_aI^{q_1+q_2+\beta_j,\rho}\big|F_x(s)\big|(\eta_j)+{}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\eta_j)\big]\\
&\qquad+\sum_{i=1}^{m}|\kappa_i|\big[{}_aI^{q_1+q_2+\mu_i,\rho}\big|F_x(s)\big|(\sigma_i)+{}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\sigma_i)\big]\Bigg)\\
&\qquad+\left(|\Omega_1|+\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\Bigg(\sum_{l=1}^{r}|\nu_l|\big[{}_aI^{q_1+q_2+\varphi_l,\rho}\big|F_x(s)\big|(\xi_l)\\
&\qquad+{}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\xi_l)\big]\\
&\qquad+\sum_{k=1}^{p}|\omega_k|\big[{}_aI^{q_1+q_2+\gamma_k,\rho}\big|F_x(s)\big|(\psi_k)+{}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|\big|x(s)\big|(\psi_k)\big]\Bigg)\Bigg]\\
&\quad\le\Bigg[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}\\
&\qquad+\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right)\\
&\qquad+\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\Bigg]\\
&\qquad\times\big(h_1^*+h_2^*R_3+h_3^*R_3\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[\frac{(T-a)^{q_1+q_2+1}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+2)}\\
&+\Lambda_1\Bigg(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\Bigg)\\
&+\Lambda_2\Bigg(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\Bigg)\Bigg]\\
&\times h_4^*R_3\phi_0\\
&+\Bigg[{}_aI^{q_2,\rho}\big|\lambda(s)\big|(T)+\Lambda_1\Bigg(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}\big|\lambda(s)\big|(\eta_j)+\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}\big|\lambda(s)\big|(\sigma_i)\Bigg)\\
&+\Lambda_2\Bigg(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}\big|\lambda(s)\big|(\xi_l)+\sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}\big|\lambda(s)\big|(\psi_k)\Bigg)\Bigg]R_3\\
=&\ \Lambda_3(q_1+q_2)\big(h_1^*+h_2^*R_3+h_3^*R_3\big)+\Lambda_3(q_1+q_2+1)h_4^*R_3\phi_0+\Lambda_4R_3.
\end{aligned}
$$

Thus, $\|\mathcal{Q}x\|_{\mathbb{E}}\le\Lambda_3(q_1+q_2)(h_1^*+h_2^*R_3+h_3^*R_3)+\Lambda_3(q_1+q_2+1)h_4^*R_3\phi_0+\Lambda_4R_3:=N<\infty$. This implies that the set $\mathbb{D}$ is bounded.

By all the hypotheses of Theorem 3.3, we conclude that there exists a positive constant N such that $\|x\|_{\mathbb{E}}\le N<\infty$. By applying Schaefer's fixed point theorem (Lemma 2.7), the operator $\mathcal{Q}$ has at least one fixed point which is a solution of problem (1.1). This completes the proof. □

4 Ulam–Hyers stability results

In this section, we investigate the Ulam stability of problem (1.1), namely Ulam–Hyers stable, generalized Ulam–Hyers stable, Ulam–Hyers–Rassias stable, and generalized Ulam–Hyers–Rassias stable.

Definition 4.1 ([39]) Problem (1.1) is said to be Ulam–Hyers stable if there exists $\Phi\in\mathbb{R}^+\setminus\{0\}$ such that, for each $\varrho>0$ and solution $z\in\mathbb{E}^1=C^1([a,T],\mathbb{R})$ of the inequality

$$\big|{}^C_aD^{q_1,\rho}\big({}^C_aD^{q_2,\rho}+\lambda(t)\big)z(t)-f\big(t,z(t),z\big(\theta(t)\big),(\mathcal{K}z)(t)\big)\big|\le\varrho,\quad t\in[a,T],\tag{4.1}$$

there exists a solution $x\in\mathbb{E}^1$ of problem (1.1) such that

$$\big|z(t)-x(t)\big|\le\Phi\varrho,\quad t\in[a,T].\tag{4.2}$$

Definition 4.2 ([39]) Problem (1.1) is said to be generalized Ulam–Hyers stable if there exists $\Phi_f\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\Phi_f(0)=0$ such that, for each solution $z\in\mathbb{E}^1$ of inequality (4.1), there exists a solution $x\in\mathbb{E}^1$ of problem (1.1) such that

$$\big|z(t)-x(t)\big|\le\Phi_f\varrho,\quad t\in[a,T].\tag{4.3}$$

Definition 4.3 ([39]) Problem (1.1) is said to be Ulam–Hyers–Rassias stable with respect to $\Phi_f\in C([a,T],\mathbb{R}^+)$ if there exists a real number $C_{f,\Phi}>0$ such that, for $\varrho>0$ and for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left|{}_a^C D^{q_1,\rho}\left({}_a^C D^{q_2,\rho} + \lambda(t)\right)z(t) - f\left(t, z(t), z\left(\theta(t)\right), (\mathcal{K}z)(t)\right)\right| \le \varrho\Phi_f(t), \quad t \in [a,T], \tag{4.4}$$

there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) such that

$$\left|z(t) - x(t)\right| \le C_{f,\Phi}\varrho\Phi_f(t), \quad t \in [a,T]. \tag{4.5}$$

Definition 4.4 ([39]) Problem (1.1) is said to be generalized Ulam–Hyers–Rassias stable with respect to $\Phi_f \in C([a,T],\mathbb{R}^+)$ if there exists a real number $C_{f,\Phi} > 0$ such that, for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left|{}_a^C D^{q_1,\rho}\left({}_a^C D^{q_2,\rho} + \lambda(t)\right)z(t) - f\left(t, z(t), z\left(\theta(t)\right), (\mathcal{K}z)(t)\right)\right| \le \Phi_f(t), \quad t \in [a,T], \tag{4.6}$$

there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) such that

$$\left|z(t) - x(t)\right| \le C_{f,\Phi}\Phi_f(t), \quad t \in [a,T]. \tag{4.7}$$

Remark 4.5 It is clear that
(i) Definition 4.1 $\Rightarrow$ Definition 4.2;
(ii) Definition 4.3 $\Rightarrow$ Definition 4.4;
(iii) Definition 4.3 for $\Phi_f(\cdot) = 1 \Rightarrow$ Definition 4.1.

Remark 4.6 A function $z \in \mathbb{E}^1$ is a solution of inequality (4.1) if and only if there exists a function $\Psi \in C([a,T],\mathbb{R})$ (dependent on z) such that
(i) $|\Psi(t)| \le \varrho$, $\forall t \in [a,T]$;
(ii) ${}_a^C D^{q_1,\rho}({}_a^C D^{q_2,\rho} + \lambda(t))z(t) = f(t, z(t), z(\theta(t)), (\mathcal{K}z)(t)) + \Psi(t)$, $t \in [a,T]$.

By Remark 4.6, the solution of the problem

$${}_a^C D^{q_1,\rho}\left({}_a^C D^{q_2,\rho} + \lambda(t)\right)z(t) = f\left(t, z(t), z\left(\theta(t)\right), (\mathcal{K}z)(t)\right) + \Psi(t), \quad t \in [a,T],$$

can be written as follows:

$$\begin{aligned} z(t) &= {}_aI^{q_1+q_2,\rho}F_z(s)(t) - {}_aI^{q_2,\rho}\lambda(s)x(s)(t) \\ &\quad + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\ &\quad \times \left(\sum_{j=1}^{n}\alpha_j\left[{}_aI^{q_1+q_2+\beta_j,\rho}F_x(s)(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(s)x(s)(\eta_j)\right]\right. \\ &\quad \left. - \sum_{i=1}^{m}\kappa_i\left[{}_aI^{q_1+q_2+\mu_i,\rho}F_x(s)(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(s)x(s)(\sigma_i)\right]\right) \\ &\quad + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\left(\sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}F_x(s)(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(s)x(s)(\xi_l)\right]\right. \end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=1}^{p}\omega_k\big[{}_aI^{q_1+q_2+\gamma_k,\rho}F_x(s)(\psi_k)-{}_aI^{q_2+\gamma_k,\rho}\lambda(s)x(s)(\psi_k)\big]\Bigg)\Bigg]+{}_aI^{q_1+q_2,\rho}\Psi(s)(t)\\
&+\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\bigg(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}-\Omega_3\bigg)\\
&\times\Bigg(\sum_{j=1}^{n}\alpha_{ja}I^{q_1+q_2+\beta_j,\rho}\Psi(s)(\eta_j)-\sum_{i=1}^{m}\kappa_{ia}I^{q_1+q_2+\mu_i,\rho}\Psi(s)(\sigma_i)\Bigg)\\
&+\bigg(\Omega_1-\frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\bigg)\\
&\times\Bigg(\sum_{l=1}^{r}\nu_{la}I^{q_1+q_2+\varphi_l,\rho}\Psi(s)(\xi_l)-\sum_{k=1}^{p}\omega_{ka}I^{q_1+q_2+\gamma_k,\rho}\Psi(s)(\psi_k)\Bigg)\Bigg].
\end{aligned}
\tag{4.8}
$$

Firstly, we present an important lemma that will be used in the proofs of the first stability theorem.

Lemma 4.7 *If $z\in\mathbb{E}^1$ satisfies inequality (4.1), then the function z is a solution of the following inequality*:

$$
\big|z(t)-(\mathcal{Q}z)(t)\big|\le\Lambda_3(q_1+q_2)\varrho,\quad 0<\varrho\le 1,
\tag{4.9}
$$

where $\Lambda_3(q_1+q_2)$ is given by (3.4).

Proof From Remark 4.6 with (4.8), we obtain

$$
\begin{aligned}
&\big|z(t)-(\mathcal{Q}z)(t)\big|\\
&=\Bigg|{}_aI^{q_1+q_2,\rho}\Psi(s)(t)+\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\bigg(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}-\Omega_3\bigg)\Bigg(\sum_{j=1}^{n}\alpha_{ja}I^{q_1+q_2+\beta_j,\rho}\Psi(s)(\eta_j)\\
&\quad-\sum_{i=1}^{m}\kappa_{ia}I^{q_1+q_2+\mu_i,\rho}\Psi(s)(\sigma_i)\Bigg)+\bigg(\Omega_1-\frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\bigg)\Bigg(\sum_{l=1}^{r}\nu_{la}I^{q_1+q_2+\varphi_l,\rho}\Psi(s)(\xi_l)\\
&\quad-\sum_{k=1}^{p}\omega_{ka}I^{q_1+q_2+\gamma_k,\rho}\Psi(s)(\psi_k)\Bigg)\Bigg]\Bigg|\\
&\le\Bigg\{\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}\\
&\quad+\Lambda_1\Bigg(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\Bigg)\\
&\quad+\Lambda_2\Bigg(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\Bigg)\Bigg\}\varrho\\
&=\Lambda_3(q_1+q_2)\varrho,
\end{aligned}
$$

where $\Lambda_3(q_1+q_2)$ is given by (3.4), from which inequality (4.9) is obtained. □

Now, we present the Ulam–Hyers stability and generalized Ulam–Hyers stability results.

Theorem 4.8 *Assume that* (H_1), (H_2), (H_3), (H_4) *are satisfied with*

$$2L_1\Lambda_3(q_1+q_2)+L_2\phi_0\Lambda_3(q_1+q_2+1)+\Lambda_4<1,$$

where $\Lambda_3(u)$, $u=\{q_1+q_2,q_1+q_2+1\}$, Λ_4 *are defined by* (3.4) *and* (3.5), *respectively. Then problem* (1.1) *is both Ulam–Hyers stable and generalized Ulam–Hyers stable on* $[a,T]$.

Proof Let $z\in\mathbb{E}^1$ be a solution of inequality (4.1), and let x be the unique solution of problem (1.1),

$$\begin{cases} {}^C_aD^{q_1,\rho}({}^C_aD^{q_2,\rho}+\lambda(t))x(t)=f(t,x(t),x(\theta(t)),(\mathcal{K}x)(t)), \quad t\in[a,T] \\ \sum_{i=1}^m\kappa_{ia}I^{\mu_i,\rho}x(\sigma_i)=\sum_{j=1}^n\alpha_{ja}I^{\beta_j,\rho}x(\eta_j), \qquad \sum_{k=1}^p\omega_{ka}I^{\gamma_k,\rho}x(\psi_k)=\sum_{l=1}^r\nu_{la}I^{\varphi_l,\rho}x(\xi_l). \end{cases}$$

By applying the triangle inequality, $|u-v|\leq|u|+|v|$, and Lemma 4.7, we have

$$\begin{aligned} &\bigl|z(t)-x(t)\bigr| \\ &\quad=\Biggl|z(t)-\Biggl\{{}_aI^{q_1+q_2,\rho}F_x(s)(t)-{}_aI^{q_2,\rho}\lambda(s)x(s)(t) \\ &\qquad+\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Biggl[\biggl(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}-\Omega_3\biggr) \\ &\qquad\times\Biggl(\sum_{j=1}^n\alpha_j\bigl[{}_aI^{q_1+q_2+\beta_j,\rho}F_x(s)(\eta_j)-{}_aI^{q_2+\beta_j,\rho}\lambda(s)x(s)(\eta_j)\bigr] \\ &\qquad-\sum_{i=1}^m\kappa_i\bigl[{}_aI^{q_1+q_2+\mu_i,\rho}F_x(s)(\sigma_i)-{}_aI^{q_2+\mu_i,\rho}\lambda(s)x(s)(\sigma_i)\bigr]\Biggr) \\ &\qquad+\biggl(\Omega_1-\frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\biggr)\Biggl(\sum_{l=1}^r\nu_l\bigl[{}_aI^{q_1+q_2+\varphi_l,\rho}F_x(s)(\xi_l)-{}_aI^{q_2+\varphi_l,\rho}\lambda(s)x(s)(\xi_l)\bigr] \\ &\qquad-\sum_{k=1}^p\omega_k\bigl[{}_aI^{q_1+q_2+\gamma_k,\rho}F_x(s)(\psi_k)-{}_aI^{q_2+\gamma_k,\rho}\lambda(s)x(s)(\psi_k)\bigr]\Biggr)\Biggr]\Biggr\}\Biggr| \\ &\quad=\bigl|z(t)-(\mathcal{Q}z)(t)+(\mathcal{Q}z)(t)-(\mathcal{Q}x)(t)\bigr| \\ &\quad\leq\bigl|z(t)-(\mathcal{Q}z)(t)\bigr|+\bigl|(\mathcal{Q}z)(t)-(\mathcal{Q}x)(t)\bigr| \\ &\quad\leq\Lambda_3(q_1+q_2)\varrho+\bigl[2L_1\Lambda_3(q_1+q_2)+L_2\phi_0\Lambda_3(q_1+q_2+1)+\Lambda_4\bigr]\bigl|z(t)-x(t)\bigr|, \end{aligned}$$

where $\Lambda_3(u)$, $u=\{q_1+q_2,q_1+q_2+1\}$, and Λ_4 are defined by (3.4) and (3.5), respectively. This yields that

$$\bigl|z(t)-x(t)\bigr|\leq\frac{\Lambda_3(q_1+q_2)\varrho}{1-[2L_1\Lambda_3(q_1+q_2)+L_2\phi_0\Lambda_3(q_1+q_2+1)+\Lambda_4]}.$$

By setting

$$\Phi:=\frac{\Lambda_3(q_1+q_2)}{1-[2L_1\Lambda_3(q_1+q_2)+L_2\phi_0\Lambda_3(q_1+q_2+1)+\Lambda_4]}, \tag{4.10}$$

we end up with

$$\left|z(t)-x(t)\right| \le \Phi\varrho.$$

Hence, problem (1.1) is Ulam–Hyers stable. Moreover, if we set $\Phi_f(\varrho) = \Phi\varrho$ such that $\Phi_f(0) = 0$, then problem (1.1) is generalized Ulam–Hyers stable. The proof is completed. □

Remark 4.9 A function $z \in \mathbb{E}^1$ is a solution of inequality (4.4) if and only if there exists a function $\Theta \in C([a,T],\mathbb{R})$ (dependent on z) such that

(i) $|\Theta(t)| \le \varrho\Psi_\Theta(t)$, $\forall t \in [a,T]$;

(ii) ${}_a^C D^{q_1,\rho}({}_a^C D^{q_2,\rho} + \lambda(t))z(t) = f(t,z(t),z(\theta(t)),(\mathcal{K}z)(t)) + \Theta(t)$, $t \in [a,T]$.

By Remark 4.9, the solution of the problem

$${}_a^C D^{\beta,\rho}\left({}_a^C D^{\alpha,\rho} + \lambda(t)\right)z(t) = f\left(t,z(t),z\left(\theta(t)\right),(\mathcal{K}z)(t)\right) + \Theta(t), \quad t \in [a,T],$$

can be written as follows:

$$\begin{aligned}
z(t) &= {}_aI^{q_1+q_2,\rho}F_z(s)(t) - {}_aI^{q_2,\rho}\lambda(s)x(s)(t) \\
&\quad + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\
&\quad \times \Bigg(\sum_{j=1}^{n}\alpha_j\left[{}_aI^{q_1+q_2+\beta_j,\rho}F_x(s)(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(s)x(s)(\eta_j)\right] \\
&\quad - \sum_{i=1}^{m}\kappa_i\left[{}_aI^{q_1+q_2+\mu_i,\rho}F_x(s)(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(s)x(s)(\sigma_i)\right]\Bigg) \\
&\quad + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\Bigg(\sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}F_x(s)(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(s)x(s)(\xi_l)\right] \\
&\quad - \sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}F_x(s)(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(s)x(s)(\psi_k)\right]\Bigg)\Bigg] + {}_aI^{q_1+q_2,\rho}\Theta(s)(t) \\
&\quad + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\
&\quad \times \left(\sum_{j=1}^{n}\alpha_{j\,a}I^{q_1+q_2+\beta_j,\rho}\Theta(s)(\eta_j) - \sum_{i=1}^{m}\kappa_{i\,a}I^{q_1+q_2+\mu_i,\rho}\Theta(s)(\sigma_i)\right) \\
&\quad + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
&\quad \times \left(\sum_{l=1}^{r}\nu_{l\,a}I^{q_1+q_2+\varphi_l,\rho}\Theta(s)(\xi_l) - \sum_{k=1}^{p}\omega_{k\,a}I^{q_1+q_2+\gamma_k,\rho}\Theta(s)(\psi_k)\right)\Bigg].
\end{aligned} \tag{4.11}$$

Lemma 4.10 *Let $z \in \mathbb{E}^1$ be a solution of inequality (4.4). Then the function z satisfies the inequality*

$$\left|z(t) - (\mathcal{Q}z)(t)\right| \leq \Lambda_3(q_1 + q_2)\Psi_\Theta(t)\varrho, \quad 0 < \varrho \leq 1, \tag{4.12}$$

where $\Lambda_3(q_1 + q_2)$ is given by (3.4).

Proof From Remark 4.9, we obtain the inequality

$$\begin{aligned}
&\left|z(t) - (\mathcal{Q}z)(t)\right| \\
&= \Bigg| {}_aI^{q_1+q_2,\rho}\Psi_\Theta(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right)\Bigg(\sum_{j=1}^{n}\alpha_{j\,a}I^{q_1+q_2+\beta_j,\rho}\Psi_\Theta(s)(\eta_j) \\
&\quad - \sum_{i=1}^{m}\kappa_{i\,a}I^{q_1+q_2+\mu_i,\rho}\Psi_\Theta(s)(\sigma_i)\Bigg) \\
&\quad + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\Bigg(\sum_{l=1}^{r}\nu_{l\,a}I^{q_1+q_2+\varphi_l,\rho}\Psi_\Theta(s)(\xi_l) \\
&\quad - \sum_{k=1}^{p}\omega_{k\,a}I^{q_1+q_2+\gamma_k,\rho}\Psi_\Theta(s)(\psi_k)\Bigg)\Bigg]\Bigg| \\
&\leq \Bigg\{\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)} \\
&\quad + \Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} + \sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right) \\
&\quad + \Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)} + \sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\Bigg\} \\
&\quad \times \Psi_\Theta(t)\varrho \\
&= \Lambda_3(q_1+q_2)\Psi_\Theta(t)\varrho,
\end{aligned}$$

where Λ_1, Λ_2, $\Lambda_3(q_1 + q_2)$ are given by (3.2), (3.3), and (3.4), respectively, which leads to the inequality in (4.9). □

Next, we are ready to prove the Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stability results.

Theorem 4.11 *Assume that (H_1), (H_2), (H_3), (H_4) are satisfied with*

$$2L_1\Lambda_3(q_1 + q_2) + L_2\phi_0\Lambda_3(q_1 + q_2 + 1) + \Lambda_4 < 1.$$

Then problem (1.1) is both Ulam–Hyers–Rassias stable and generalized Ulam–Hyers Rassias stable on $[a, T]$.

Proof Let $z \in \mathbb{E}^1$ be a solution of inequality (4.4), and let x be the unique solution of problem (1.1). By applying the triangle inequality and Lemma 4.7 with (4.11), we get

$$\begin{aligned}
&\left|z(t) - x(t)\right| \\
&= \Bigg| z(t) - \Bigg\{ {}_aI^{q_1+q_2,\rho} F_z(s)(t) - {}_aI^{q_2,\rho}\lambda(s)x(s)(t) \\
&\quad + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\
&\quad \times \Bigg(\sum_{j=1}^{n}\alpha_j\big[{}_aI^{q_1+q_2+\beta_j,\rho}F_x(s)(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(s)x(s)(\eta_j)\big] \\
&\quad - \sum_{i=1}^{m}\kappa_i\big[{}_aI^{q_1+q_2+\mu_i,\rho}F_x(s)(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(s)x(s)(\sigma_i)\big]\Bigg) \\
&\quad + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\Bigg(\sum_{l=1}^{r}\nu_l\big[{}_aI^{q_1+q_2+\varphi_l,\rho}F_x(s)(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(s)x(s)(\xi_l)\big] \\
&\quad - \sum_{k=1}^{p}\omega_k\big[{}_aI^{q_1+q_2+\gamma_k,\rho}F_x(s)(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(s)x(s)(\psi_k)\big]\Bigg)\Bigg] + {}_aI^{q_1+q_2,\rho}\Theta(s)(t) \\
&\quad + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\
&\quad \times \left(\sum_{j=1}^{n}\alpha_{j\,a}I^{q_1+q_2+\beta_j,\rho}\Theta(s)(\eta_j) - \sum_{i=1}^{m}\kappa_{i\,a}I^{q_1+q_2+\mu_i,\rho}\Theta(s)(\sigma_i)\right) \\
&\quad + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
&\quad \times \left(\sum_{l=1}^{r}\nu_{l\,a}I^{q_1+q_2+\varphi_l,\rho}\Theta(s)(\xi_l) - \sum_{k=1}^{p}\omega_{k\,a}I^{q_1+q_2+\gamma_k,\rho}\Theta(s)(\psi_k)\right)\Bigg]\Bigg\}\Bigg| \\
&= \left|z(t) - (\mathcal{Q}z)(t) + (\mathcal{Q}z)(t) - (\mathcal{Q}x)(t)\right| \\
&\le \left|z(t) - (\mathcal{Q}z)(t)\right| + \left|(\mathcal{Q}z)(t) - (\mathcal{Q}x)(t)\right| \\
&\le \Lambda_3(q_1+q_2)\Psi_\Theta(t)\varrho + \big[2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4\big]\left|z(t) - x(t)\right|,
\end{aligned}$$

where $\Lambda_3(u)$, $u = \{q_1 + q_2, q_1 + q_2 + 1\}$, and Λ_4 are defined by (3.4) and (3.5), respectively, which implies that

$$\left|z(t) - x(t)\right| \le \frac{\Lambda_3(q_1+q_2)\Psi_\Theta(t)\varrho}{1 - [2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4]}.$$

By setting

$$C_{f,\Phi} := \frac{\Lambda_3(q_1+q_2)}{1 - [2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4]},$$

we get the following inequality:

$$\left|z(t) - x(t)\right| \leq C_{f,\Phi}\varrho\Psi_{\Theta}(t).$$

Hence, problem (1.1) is Ulam–Hyers–Rassias stable. Moreover, if we set $\Phi_f(t) = \varrho\Psi_{\Theta}(t)$, with $\Phi_f(0) = 0$, then problem (1.1) is generalized Ulam–Hyers–Rassias stable. The proof is completed. □

5 Example

In this section, we present an example which illustrates the validity and applicability of the main results.

Example 5.1 Consider the following nonlinear GPF functional integro-differential Langevin equation involving nonlocal integral conditions:

$$\begin{cases} {}^{C}_{a}D^{\frac{3}{4},\frac{1}{3}}({}^{C}_{a}D^{\frac{1}{2},\frac{1}{3}} + \frac{1}{25}(t-a)^2 e^{\frac{\rho-1}{\rho}(t-a)})x(t) = f(t,x(t),x(\theta(t)),(\mathcal{K}x)(t)), \quad t \in (0,2], \\ \sum_{i=1}^{3}(\frac{i}{3})_{a}I^{\frac{i}{i+1},\frac{1}{3}}x(\frac{i}{2(i+1)}) = \sum_{j=1}^{2}(\frac{j}{4})_{a}I^{\frac{j+1}{j+2},\frac{1}{3}}x(\frac{j}{j+6}), \\ \sum_{k=1}^{2}(\frac{k}{5})_{a}I^{\frac{k+2}{k+3},\frac{1}{3}}x(\frac{\sqrt{k}}{k^2+2}) = \sum_{l=1}^{3}(\frac{l}{6})_{a}I^{\frac{l+3}{l+4},\frac{1}{3}}x(\frac{2l}{3l+2}). \end{cases} \tag{5.1}$$

Here, $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{2}$, $\rho = \frac{1}{3}$, $a = 0$, $T = 2$, $m = 3$, $n = 2$, $p = 2$, $r = 3$, $\kappa_i = \frac{i}{3}$, $\sigma_i = \frac{i}{2(i+1)}$, $\mu_i = \frac{i}{i+1}$, $i = 1,2,3$, $\alpha_j = \frac{j}{4}$, $\eta_j = \frac{j}{j+6}$, $\beta_j = \frac{j+1}{j+2}$, $j = 1,2$, $\omega_k = \frac{k}{5}$, $\psi_j = \frac{\sqrt{k}}{k^2+2}$, $\gamma_k = \frac{k+2}{k+3}$, $k = 1,2$, $\nu_l = \frac{l}{6}$, $\xi_l = \frac{2l}{3l+2}$, $\varphi_l = \frac{l+3}{l+4}$, $l = 1,2,3$, $\theta(t) = \frac{t}{2}$, and

$$\lambda(t) = \frac{1}{25}(t-a)^2 e^{\frac{\rho-1}{\rho}(t-a)}.$$

Obviously, the function λ satisfies (H_2) for all $t \in [a,T]$. From the given data, we obtain that $\Omega_1 \approx 0.6995071719$, $\Omega_2 \approx 0.7639237899$, $\Omega_3 \approx -0.3023660189$, $\Omega_4 \approx -0.2312067168$, $\Omega \approx 0.0662301783 \neq 0$. Furthermore, we assume the nonlinearity as follows:

(i) Let $f : [a,T] \times \mathbb{R}^3 \to \mathbb{R}$ be a function, which is given by

$$\begin{aligned} &f\bigl(t,x(t),x\bigl(\theta(t)\bigr),(\mathcal{K}x)(t)\bigr) \\ &\quad = \frac{1}{2} + \frac{t^2}{3} + \frac{2\cos^2(\pi t)}{(t+16)^2}\frac{|x|}{1+|x|} - \frac{x(0.5t)}{(t+16)^2} + \frac{(t+2)^3}{(8e^t+1)^2}\int_a^t \frac{\cos^2(\pi t)}{(e^{s^2}+1)^2}x(s)\,ds. \end{aligned}$$

For $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in [a,T]$, we have

$$\begin{aligned} &\bigl|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\bigr| \leq \frac{1}{(t+16)^2}\bigl(|x_1-y_1| + |x_2-y_2|\bigr) + \frac{(t+2)^3}{(8e^t+1)^2}|z_1-z_2|, \\ &\bigl|\phi(t,s,x_1) - \phi(t,s,y_1)\bigr| \leq \frac{1}{4}|x_1-y_1|. \end{aligned}$$

Hypotheses (H_1)–(H_4) are satisfied with $L_1 = \frac{1}{256}$, $L_2 = \frac{1}{81}$, and $\phi_0 = \frac{1}{4}$. Hence

$$2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4 \approx 0.7833485782 < 1.$$

Since all the hypotheses of Theorem 3.1 are satisfied, problem (5.1) has a unique solution on $[0, 2]$. Moreover, we can also compute that

$$\Phi := \frac{\Lambda_3(q_1 + q_2)}{1 - [2L_1\Lambda_3(q_1 + q_2) + L_2\phi_0\Lambda_3(q_1 + q_2 + 1) + \Lambda_4]} \approx 375.8602857 > 0.$$

Hence, by Theorem 4.8, problem (5.1) is both Ulam–Hyers and also generalized Ulam–Hyers stable.

(ii) Let $f : [a, T] \times \mathbb{R}^3 \to \mathbb{R}$ be a function which is given by

$$\begin{aligned} f\bigl(t, x(t), x\bigl(\theta(t)\bigr), (\mathcal{K}x)(t)\bigr) &= \frac{2e^t}{(t+1)^2} + \frac{e^{-t}}{2(t+9)^2} \cdot \frac{|x|}{2+|x|} \\ &\quad + \frac{1}{(t+9)^2} \cdot \frac{|x(0.75t)|}{|x(0.75t)|+4} + \frac{\sin^2(\pi t)}{e^t+1} \int_a^t \frac{\cos^2(t-s)}{(e^{t-s}+1)^2} x(s)\,ds. \end{aligned}$$

It is easy to see that, for all $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in [a, T]$, we get

$$\begin{aligned} \bigl|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\bigr| &\leq \frac{1}{4(t+9)^2}\bigl(|x_1 - y_1| + |x_2 - y_2|\bigr) \\ &\quad + \frac{1}{(e^t+1)^3}|z_1 - z_2|, \\ \bigl|\phi(t, s, x_1) - \phi(t, s, y_1)\bigr| &\leq \frac{1}{16}|x_1 - y_1|. \end{aligned}$$

Hypotheses (H_1)–(H_4) are satisfied with $L_1 = \frac{1}{324}$, $L_2 = \frac{1}{8}$, and $\phi_0 = \frac{1}{16}$. Hence

$$2L_1\Lambda_3(q_1 + q_2) + L_2\phi_0\Lambda_3(q_1 + q_2 + 1) + \Lambda_4 \approx 0.7348101092 < 1.$$

Furthermore, for $x, y, z \in \mathbb{R}$ and $t \in [a, T]$, it follows that

$$\bigl|f(t, x, y, z)\bigr| \leq \frac{2e^t}{(t+1)^2} + \frac{e^{-t}}{4(t+9)^2}|x| + \frac{1}{4(t+9)^2}|y| + \frac{1}{(e^t+1)^3}|z|.$$

Hypothesis (H_5) is also valid with $h_1(t) = \frac{2e^t}{(t+1)^2}$, $h_2(t) = \frac{e^{-t}}{4(t+9)^2}$, $h_3(t) = \frac{1}{4(t+9)^2}$, $h_4(t) = \frac{1}{(e^t+1)^3}$, and $h_1^* = 2$, $h_2^* = h_3^* = \frac{1}{324}$, $h_4^* = \frac{1}{8}$. Therefore, all the hypotheses of Theorem 3.3 are fulfilled, which concludes that problem (5.1) has at least one solution on $[0, 2]$. Moreover, we obtain

$$C_{f,\Phi} := \frac{\Lambda_3(q_1 + q_2)}{1 - [2L_1\Lambda_3(q_1 + q_2) + L_2\phi_0\Lambda_3(q_1 + q_2 + 1) + \Lambda_4]} \approx 307.0654958 > 0.$$

Hence, by Theorem 4.11, problem (5.1) is both Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stable.

6 Conclusion

Langevin equation is an important equation of mathematical physics that is used in modeling the phenomena occurring in fluctuating environment such as Brownian motion. In the literature, it is also referred to as a stochastic differential equation as it governs the fast motion of microscopic variables of the dynamical systems. It has been realized, however, that the integer order Langevin equation cannot provide elaborate description to the complex systems that involve disordered or fractal medium. Therefore, attention toward considering noninteger order Langevin equation becomes urgent and compulsory. Thus, the boundary value problems defined by fractional Langevin equation have been extensively studied in recent years. Based on their interests and demands, the authors have considered Langevin equation within different types of fractional derivatives and boundary conditions. The fractional derivatives have been often utilized in the frame of Caputo, Riemann–Liouville, or Hadamard settings, whereas the supplemented boundary conditions have been of nonlocal, anti periodic, or mixed types.

In this paper, we study new Langevin equation within the so-called GPF derivative. Fixed point theorems and Ulam's approach are employed to investigate the existence, uniqueness, and different types of stability. The results of this paper not only generalize previous results but also provide a totally different approach in the sense that different fractional derivative is accommodated, different boundary conditions are associated, different fixed point theorems are used, and Ulam stability within GPF derivative is discussed. We believe that the results of this paper will provide considerable potential to interested researchers to produce relevant results concerning qualitative properties of nonlinear GPF differential equations.

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