Boulaaras et al. *Boundary Value Problems* (2020) 2020:172
https://doi.org/10.1186/s13661-020-01470-w

Boundary Value Problems
a SpringerOpen Journal

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General decay and blow-up of solutions for a nonlinear wave equation with memory and fractional boundary damping terms

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On the occasion of the 44th birthday of the first author's brother, Professor Djemai Mahmoud Mouha Boulaaras.

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Abstract
The paper studies the global existence and general decay of solutions using Lyapunov functional for a nonlinear wave equation, taking into account the fractional derivative boundary condition and memory term. In addition, we establish the blow-up of solutions with nonpositive initial energy.

MSC: General decay; Global existence; Fractional boundary dissipation; Blow-up; Memory term

Keywords: 35L35; 35L20

1 Introduction

Extraordinary differential equations, also known as fractional differential equations, are a generalization of differential equations through fractional calculus. Much attention has been accorded to fractional partial differential equations during the past two decades due to the many chemical engineering, biological, ecological, and electromagnetism phenomena that are modeled by initial boundary value problems with fractional boundary conditions. See Tarasov [16], Magin [15].

In this work we consider the nonlinear wave equation

$$\begin{cases} u_{tt} - \Delta u + au_t + \int_0^t g(t-s)\Delta u(s)\,ds = |u|^{p-2}u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = -b\partial_t^{\alpha,\eta} u, & x \in \Gamma_0, t > 0, \\ u = 0, & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$ of class C^2 and ν is the unit outward normal to $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed subsets of $\partial\Omega$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$.

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$a, b > 0$, $p > 2$, and $\partial_t^{\alpha,\eta}$ with $0 < \alpha < 1$ is the Caputo's generalized fractional derivative (see [11] and [7]) defined by

$$\partial_t^{\alpha,\eta} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} u_s(s)\, ds, \quad \eta \geq 0,$$

where Γ is the usual Euler gamma function. It can also be expressed by

$$\partial_t^{\alpha,\eta} u(t) = I^{1-\alpha,\eta} u'(t), \tag{1.2}$$

where $I^{\alpha,\eta}$ is the exponential fractional integro-differential operator given by

$$I^{\alpha,\eta} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} u(s)\, ds, \quad \eta \geq 0.$$

In the context of boundary dissipations of fractional order problems, the main research focus is on asymptotic stability of solutions starting by writing the equations as an augmented system. Then, various techniques are used such as LaSalle's invariance principle and the multiplier method mixed with frequency domain (see [1–16], and [18]).

Dai and Zhang [7] replaced $\int_0^t K(x, t-s) u_s(x,s)\, ds$ with $\partial_t^{\alpha} u(x,t)$ and $h(x,t)$ with $|u|^{m-1} u(x,t)$ and managed to prove exponential growth for the same problem.

Note that the nonlinear wave equation with boundary fractional damping case was first considered by authors in [4], where they used the augmented system to prove the exponential stability and blow-up of solutions in finite time.

Motivated by our recent work in [4] and based on the construction of a Lyapunov function, we prove in this paper under suitable conditions on the initial data the stability of a wave equation with fractional damping and memory term. This technique of proof was recently used by [9] and [4] to study the exponential decay of a system of nonlocal singular viscoelastic equations.

Here we also consider three different cases on the sign of the initial energy as recently examined by Zarai et al. [17], where they studied the blow-up of a system of nonlocal singular viscoelastic equations.

The organization of our paper is as follows. We start in Sect. 2 by giving some lemmas and notations in order to reformulate our problem (1.1) into an augmented system. In the following section, we use the potential well theory to prove the global existence result. Then, the general decay result is given in Sect. 4. In Sect. 5, following a direct approach, we prove blow-up of solutions.

2 Preliminaries

Let us introduce some notations, assumptions, and lemmas that are effective for proving our results.

Assume that the relaxation function g satisfies

(G_1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function with

$$g(0) > 0, \quad 1 - \int_0^{\infty} g(s)\, ds = l > 0; \tag{2.1}$$

(G_2) There exists a constant $\xi > 0$ such that

$$g'(t) \leq -\xi g(t), \quad \forall t > 0. \tag{2.2}$$

We denote

$$(g \circ u)(t) = \int_0^t g(t-s)\big\|u(t) - u(s)\big\|^2 \, ds \tag{2.3}$$

and

$$\aleph = \big\{w \in H_0^1 | I(w) > 0\big\} \cup \{0\},$$
$$H_{\Gamma_1}^1(\Omega) = \big\{u \in H^1(\Omega), u|_{\Gamma_1} = 0\big\}.$$

Lemma 1 (Sobolev–Poincaré inequality) *If either* $1 \leq q \leq \frac{N+2}{N-2}$ $(N \geq 3)$ *or* $1 \leq q \leq +\infty$ $(N = 2)$, *then there exists* $C_* > 0$ *such that*

$$\|u\|_{q+1} \leq C_* \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

Lemma 2 (Trace–Sobolev embedding) *For all p such that*

$$2 < p \leq \frac{2(n-1)}{n-2}, \tag{2.4}$$

we have

$$H_{\Gamma_1}^1(\Omega) \hookrightarrow L^p(\Gamma_0).$$

We denote by B_q *the embedding constant, i.e.,*

$$\|u\|_{p,\Gamma_0} \leq B_q \|u\|_2.$$

Lemma 3 ([17], p. 5, Lemma 2 or [3], p. 1406, Lemma 4.1) *Consider a nonnegative function* $B(t) \in C^2(0,\infty)$ *satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0, \tag{2.5}$$

where $\delta > 0$.

If

$$B'(0) > r_2 B(0) + l_0, \tag{2.6}$$

then

$$B'(t) \geq l_0, \quad \forall t > 0, \tag{2.7}$$

where $l_0 \in \mathbb{R}$, r_2 *represents the smallest root of the equation*

$$r^2 - 4(\delta+1)r + (\delta+1) = 0, \tag{2.8}$$

i.e., $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$.

Lemma 4 ([17], p. 5, Lemma 3 or [3], p. 1406, Lemma 4.2) *Let $J(t)$ be a nonincreasing function on $[t_0,\infty)$ verifying the differential inequality*

$$J'(t)^2 \geq \alpha + bJ(t)^{2+\frac{1}{\delta}}, \quad t \geq t_0 \geq 0, \tag{2.9}$$

where $\alpha > 0$, $b \in \mathbb{R}$, then there exists $T^* > 0$ such that

$$\lim_{t\to T^{*-}} J(t) = 0, \tag{2.10}$$

with the following upper bound cases for T^*:

(**i**) *When $b < 0$ and $J(t_0) < \min\{1, \sqrt{\alpha/(-b)}\}$,*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{\alpha}{-b}}}{\sqrt{\frac{\alpha}{-b}} - J(t_0)}. \tag{2.11}$$

(**ii**) *When $b = 0$,*

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{\alpha}}. \tag{2.12}$$

(**iii**) *When $b > 0$,*

$$T^* \leq \frac{J(t_0)}{\sqrt{\alpha}} \tag{2.13}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha}} \left(1 - \left[1 + cJ(t_0)\right]^{\frac{1}{2\delta}}\right), \tag{2.14}$$

where

$$c = \left(\frac{b}{\alpha}\right)^{\delta/(2+\delta)}.$$

Definition 1 We say that u is a blow-up solution of (1.1) at finite time T^* if

$$\lim_{t\to T^{*-}} \frac{1}{(\|\nabla u\|_2)} = 0. \tag{2.15}$$

Theorem 1 ([12], Theorem 1) *Consider the constant*

$$\varrho = (\pi)^{-1} \sin(\alpha\pi)$$

and the function μ given by

$$\mu(\xi) = |\xi|^{\frac{(2\alpha-1)}{2}}, \quad 0 < \alpha < 1, \xi \in \mathbb{R}. \tag{2.16}$$

Then we can obtain

$$O = I^{1-\alpha,\eta} U, \tag{2.17}$$

which is a relation between U the "input" of the system

$$\partial_t\phi(\xi,t) + \big(\xi^2+\eta\big)\phi(\xi,t) - U(L,t)\mu(\xi) = 0, \quad t>0, \eta\geq 0, \xi\in\mathbb{R} \tag{2.18}$$

and the "output" O given by

$$O(t) = \varrho\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)\,d\xi, \quad \xi\in\mathbb{R}, t>0. \tag{2.19}$$

Now, using (1.2) and Theorem 1, the augmented system related to our system (1.1) may be given by

$$\begin{cases} u_{tt} - \Delta u + au_t + \int_0^t g(t-s)\Delta u(s)\,ds = |u|^{p-2}u, & x\in\Omega, t>0,\\ \partial_t\phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - u_t(x,t)\mu(\xi) = 0, & x\in\Gamma_0, \xi\in\mathbb{R}, t>0,\\ \frac{\partial u}{\partial\nu} = -b_1\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)\,d\xi, & x\in\Gamma_0, \xi\in\mathbb{R}, t>0,\\ u=0, & x\in\Gamma_1, t>0,\\ u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x), & x\in\Omega,\\ \phi(\xi,0)=0, & \xi\in\mathbb{R}, \end{cases} \tag{2.20}$$

where $b_1 = b\varrho$.

Lemma 5 ([2], p. 3, Lemma 2.1) *For all* $\lambda\in D_\eta = \{\lambda\in\mathbb{C} : \Im m\lambda\neq 0\}\cup\{\lambda\in\mathbb{C} : \Re e\lambda+\eta>0\}$, *we have*

$$A_\lambda = \int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\eta+\lambda+\xi^2}\,d\xi = \frac{\pi}{\sin(\alpha\pi)}(\eta+\lambda)^{\alpha-1}.$$

Theorem 2 (Local existence and uniqueness) *Assume that* (2.4) *holds. Then, for all* $(u_0,u_1,\phi_0)\in H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(-\infty,+\infty)$, *there exists some* T *small enough such that problem* (2.20) *admits a unique solution*

$$\begin{cases} u\in C([0,T), H^1_{\Gamma_0}(\Omega)),\\ u_t\in C([0,T), L^2(\Omega)),\\ \phi\in C([0,T), L^2(-\infty,+\infty)). \end{cases} \tag{2.21}$$

3 Global existence

Before proving the global existence for problem (2.20), let us introduce the functionals

$$I(t) = \left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)(t) - \|u\|_p^p$$

and

$$J(t) = \frac{1}{2}\left[\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\right] - \frac{1}{p}\|u\|_p^p.$$

The energy functional E associated with system (2.20) is given as follows:

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ &\quad - \frac{1}{p}\|u\|_p^p + \frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty} \left|\phi(\xi,t)\right|^2 d\xi\,d\rho. \end{aligned} \tag{3.1}$$

Lemma 6 *If* (u,ϕ) *is a regular solution to* (2.20), *then the energy functional given in* (3.1) *verifies*

$$\begin{aligned} \frac{d}{dt}E(t) &= -a\|u_t\|_2^2 - \frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}\left(g' \circ \nabla u\right)(t) \\ &\quad - b_1 \int_{\Gamma_0}\int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right)\left|\phi(\xi,t)\right|^2 d\xi\,d\rho \\ &\leq 0. \end{aligned} \tag{3.2}$$

Proof Multiplying by u_t in the first equation from (2.20), using integration by parts over Ω, we get

$$\begin{aligned} &\frac{1}{2}\|u_t\|_2^2 - \int_\Omega \Delta u u_t\,dx + a\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ &\quad = \int_\Omega |u|^{p-2}uu_t\,dx. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p\right] \\ &\quad + a\|u_t\|_2^2 + b_1\int_{\Gamma_0} u_t(x,t)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\,d\xi\,d\rho = 0. \end{aligned} \tag{3.3}$$

Multiplying by $b_1\phi$ in the second equation from (2.20) and integrating over $\Gamma_0 \times (-\infty,+\infty)$, we get

$$\begin{aligned} &\frac{b_1}{2}\frac{d}{dt}\int_{\Gamma_0}\int_{-\infty}^{+\infty} \left|\phi(\xi,t)\right|^2 d\xi\,d\rho + b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right)\left|\phi(\xi,t)\right|^2 d\xi\,d\rho \\ &\quad - b_1\int_{\Gamma_0} u_t(x,t)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\,d\xi\,d\rho = 0. \end{aligned} \tag{3.4}$$

From (3.1), (3.3), and (3.4) we obtain

$$\begin{aligned} \frac{d}{dt}E(t) &= -a\|u_t\|_2^2 - \frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}\left(g' \circ \nabla u\right)(t) \\ &\quad - b_1 \int_{\Gamma_0}\int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right)\left|\phi(\xi,t)\right|^2 d\xi\,d\rho \\ &\leq 0. \end{aligned}$$

□

Lemma 7 *Assuming that* (2.4) *holds and that for all* $(u_0, u_1, \phi_0) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(-\infty, +\infty)$ *verifying*

$$\begin{cases} \beta = C_*^p(\frac{2p}{p-2}E(0))^{\frac{p-2}{2}} < 1, \\ I(u_0) > 0. \end{cases} \tag{3.5}$$

Then $u(t) \in \aleph$, $\forall t \in [0, T]$.

Proof As $I(u_0) > 0$, there exists $T^* \leq T$ such that

$$I(u) \geq 0, \quad \forall t \in [0, T^*).$$

This leads to

$$\begin{aligned} \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g \circ \nabla u)(t) &\leq \frac{2p}{p-2}J(t), \quad \forall t \in [0, T^*) \\ &\leq \frac{2p}{p-2}E(0). \end{aligned} \tag{3.6}$$

Using the Poincare inequality, (3.1), (2.3), (3.5), and (3.6), we obtain

$$\begin{aligned} \|u\|_p^p &\leq C_*^p\|\nabla u\|_2^p \\ &\leq C_*^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}\|\nabla u\|_2^2. \end{aligned} \tag{3.7}$$

Thus

$$\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \|u\|_p^p > 0, \quad \forall t \in [0, T^*).$$

Consequently, $u \in H$, $\forall t \in [0, T^*)$.

Repeating the procedure, T^* can be extended to T, and that makes the proof of our global existence result within reach. □

Theorem 3 *Assume that* (2.4) *holds. Then for all*

$$(u_0, u_1, \phi_0) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(-\infty, +\infty)$$

verifying (3.5), *the solution of system* (2.20) *is global and bounded.*

Proof From (3.2), we get

$$\begin{aligned} E(0) &\geq E(t) \\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p \\ &\quad + \frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi, t)\right|^2 d\xi\,d\rho \\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}I(t) + \frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi, t)\right|^2 d\xi\,d\rho. \end{aligned} \tag{3.8}$$

Or $I(t) > 0$, therefrom

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 + b_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} \left|\phi(\xi,t)\right|^2 d\xi\, d\rho \leq C_1 E(0),$$

where $C_1 = \max\{\frac{2}{b_1}, \frac{2p}{p-2}, 2\}$. □

4 Decay of solutions

To proceed for the energy decay result, we construct an appropriate Lyapunov functional as follows:

$$L(t) = \epsilon_1 E(t) + \epsilon_2 \psi_1(t) + \frac{\epsilon_2 b_1}{2} \psi_2(t), \tag{4.1}$$

where

$$\psi_1(t) = \int_\Omega u_t u\, dx,$$
$$\psi_2(t) = \int_{\Gamma_0} \int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right) \left(\int_0^t \phi(\xi,s)\, ds\right)^2 d\xi\, d\rho,$$

and ϵ_1, ϵ_2 are positive constants.

Lemma 8 *If (u,ϕ) is a regular solution of problem* (2.20), *then the following equality holds*:

$$\int_{\Gamma_0} \int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right) \phi(\xi,t) \int_0^t \phi(\xi,s)\, ds\, d\xi\, d\rho$$
$$= \int_{\Gamma_0} u(x,t) \int_{-\infty}^{+\infty} \phi(\xi,t)\mu(\xi)\, d\xi\, d\rho - \int_{\Gamma_0} \int_{-\infty}^{+\infty} \left|\phi(\xi,t)\right|^2 d\xi\, d\rho.$$

Proof From the second equation of (2.20), we have

$$\left(\xi^2 + \eta\right)\phi(\xi,t) = u_t(x,t)\mu(\xi) - \partial_t \phi(\xi,t), \quad \forall x \in \Gamma_0. \tag{4.2}$$

Integrating (4.2) over $[0,t]$ and using equations 3 and 6 from system (2.20), we get

$$\int_0^t \left(\xi^2 + \eta\right)\phi(\xi,s)\, ds = u(x,t)\mu(\xi) - \phi(\xi,t), \quad \forall x \in \Gamma_0, \tag{4.3}$$

hence,

$$\left(\xi^2 + \eta\right) \int_0^t \phi(\xi,s)\, ds = u(x,t)\mu(\xi) - \phi(\xi,t), \quad \forall x \in \Gamma_0. \tag{4.4}$$

Multiplying by ϕ followed by integration over $\Gamma_0 \times (-\infty,+\infty)$ leads to

$$\int_{\Gamma_0} \int_{-\infty}^{+\infty} \left(\xi^2 + \eta\right) \phi(\xi,t) \int_0^t \phi(\xi,s)\, ds\, d\xi\, d\rho$$
$$= \int_{\Gamma_0} u(x,t) \int_{-\infty}^{+\infty} \phi(\xi,t)\mu(\xi)\, d\xi\, d\rho - \int_{\Gamma_0} \int_{-\infty}^{+\infty} \left|\phi(\xi,t)\right|^2 d\xi\, d\rho.$$
□

Lemma 9 *For any (u,ϕ) solution of problem (2.20), we have*

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t), \tag{4.5}$$

where α_1, α_2 are positive constants.

Proof From (4.3), we get

$$\int_0^t \phi(\xi,s)\,ds = \frac{-\phi(\xi,t)}{\xi^2+\eta} + \frac{u(x,t)\mu(\xi)}{\xi^2+\eta}, \quad \forall x \in \Gamma_0. \tag{4.6}$$

Thus

$$\left(\int_0^t \phi(\xi,s)\,ds\right)^2 = \frac{|\phi(\xi,t)|^2}{(\xi^2+\eta)^2} + \frac{|u(x,t)|^2\mu^2(\xi)}{(\xi^2+\eta)^2} - 2\frac{\phi(\xi,t)u(x,t)\mu(\xi)}{(\xi^2+\eta)^2}. \tag{4.7}$$

Multiplying by $\xi^2+\eta$ in (4.7) followed by integration over $\Gamma_0 \times (-\infty,+\infty)$ leads to

$$\begin{aligned} \left|\psi_2(t)\right| \leq{}& \int_{\Gamma_0}\int_{-\infty}^{+\infty} \frac{|\phi(\xi,t)|^2}{\xi^2+\eta}\,d\xi\,d\rho + \int_{\Gamma_0}\left|u(x,t)\right|^2 \int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\,d\rho \\ &+ 2\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)u(x,t)\mu(\xi)|}{\xi^2+\eta}\,d\xi\,d\rho. \end{aligned} \tag{4.8}$$

Using Young's inequality in order to have an estimation of the last term in (4.8), we get for any $\delta > 0$

$$\begin{aligned} \int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)u(x,t)\mu(\xi)|}{\xi^2+\eta}\,d\xi\,d\rho &= \int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|}{(\xi^2+\eta)^{\frac{1}{2}}}\frac{|u(x,t)\mu(\xi)|}{(\xi^2+\eta)^{\frac{1}{2}}}\,d\xi\,d\rho \\ &\leq \frac{1}{4\delta}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|^2}{\xi^2+\eta}\,d\xi\,d\rho \\ &\quad + \delta\int_{\Gamma_0}\left|u(x,t)\right|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\,d\rho. \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9), we obtain

$$\begin{aligned} \left|\psi_2(t)\right| \leq{}& \left(\frac{2\delta+1}{2\delta}\right)\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|^2}{\xi^2+\eta}\,d\xi\,d\rho \\ &+ (2\delta+1)\int_{\Gamma_0}\left|u(x,t)\right|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\,d\rho. \end{aligned} \tag{4.10}$$

Since $\frac{1}{\xi^2+\eta} \leq \frac{1}{\eta}$, then

$$\begin{aligned} \left|\psi_2(t)\right| \leq{}& \left(\frac{2\delta+1}{2\delta\eta}\right)\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi,t)\right|^2\,d\xi\,d\rho \\ &+ (2\delta+1)\int_{\Gamma_0}\left|u(x,t)\right|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\,d\rho. \end{aligned} \tag{4.11}$$

Applying Lammas 2 and 5, we get

$$\left|\psi_2(t)\right| \le \left(\frac{2\delta+1}{2\delta\eta}\right)\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi,t)\right|^2 d\xi\, d\rho + A_0 B_q(2\delta+1)\|\nabla u\|_2^2. \tag{4.12}$$

By Poincare-type inequality and Young's inequality, we obtain

$$\left|\psi_1(t)\right| \le \frac{1}{2}\|u_t\|_2^2 + \frac{C_*}{2}\|\nabla u\|_2^2. \tag{4.13}$$

Adding (4.13) to (4.12), we get

$$\begin{aligned}\left|\psi_1(t)+\frac{b_1}{2}\psi_2(t)\right| &\le \left|\psi_1(t)\right| + \frac{b_1}{2}\left|\psi_2(t)\right| \\ &\le \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\big[A_0B_qb_1(2\delta+1)+C_*\big]\|\nabla u\|_2^2 \\ &\quad + \frac{b_1}{2}\left[\frac{2\delta+1}{2\delta\eta}\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi,t)\right|^2 d\xi\, d\rho.\end{aligned} \tag{4.14}$$

Therefore, by the energy definition given in (3.1), for all $N>0$, we have

$$\begin{aligned}\left|\psi_1(t)+\frac{b_1}{2}\psi_2(t)\right| &\le NE(t) + \frac{1-N}{2}\|u_t\|_2^2 + \frac{N}{p}\|u_t\|_p^p \\ &\quad + \frac{1}{2}\big[A_0B_qb_1(2\delta+1)+C_*-N\big]\|\nabla u\|_2^2 \\ &\quad + \frac{b_1}{2}\left[\frac{2\delta+1}{2\delta\eta}-N\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi,t)\right|^2 d\xi\, d\rho.\end{aligned} \tag{4.15}$$

From (3.7) and (4.15), we finally get

$$\begin{aligned}\left|\psi_1(t)+\frac{b_1}{2}\psi_2(t)\right| &\le NE(t) + \frac{1-N}{2}\|u_t\|_2^2 \\ &\quad + \frac{1}{2}\left[A_0B_qb_1(2\delta+1)+C_*-\frac{p-2}{2p}N\right]\|\nabla u\|_2^2 \\ &\quad + \frac{b_1}{2}\left[\frac{2\delta+1}{2\delta\eta}-N\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi,t)\right|^2 d\xi\, d\rho,\end{aligned} \tag{4.16}$$

where N and ϵ_1 are chosen as follows:

$$N > \max\left\{\frac{2\delta+1}{2\delta\eta}, \frac{2p(A_0B_qb_1(2\delta+1)+C_*)}{p-2}, 1\right\},$$

$$\epsilon_1 \ge N\epsilon_2.$$

Then we conclude from (4.16)

$$\alpha_1 E(t) \le L(t) \le \alpha_2 E(t),$$

where

$$\alpha_1 = \epsilon_1 - N\epsilon_2$$

and

$$\alpha_2 = \epsilon_1 + N\epsilon_2. \qquad \square$$

Now, we prove the exponential decay of global solution.

Theorem 4 *If* (2.4) *and* (3.5) *hold, then there exist k and K, positive constants such that the global solution of* (2.20) *verifies*

$$E(t) \le Ke^{-kt}. \tag{4.17}$$

Proof By differentiation in (4.1), we get

$$\begin{aligned} L'(t) &= \epsilon_1 E'(t) + \epsilon_2 \|u_t\|_2^2 + \epsilon_2 \int_\Omega u_{tt} u \, dx \\ &\quad + \epsilon_2 b_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)\phi(\xi,t) \int_0^t \phi(\xi,s)\, ds\, d\xi\, d\rho. \end{aligned} \tag{4.18}$$

Combining with (2.20) to obtain

$$\begin{aligned} L'(t) &= \epsilon_1 E'(t) + \epsilon_2 \left[\|u_t\|_2^2 - \|\nabla u\|_2^2 + \|u\|_p^p - a\int_\Omega u u_t \, dx \right] \\ &\quad - b_1\epsilon_2 \int_{\Gamma_0} u(x,t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi\, d\rho \\ &\quad + b_1\epsilon_2 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)\phi(\xi,t) \int_0^t \phi(\xi,s)\, ds\, d\xi\, d\rho. \end{aligned} \tag{4.19}$$

An application of Lemma 8 leads to

$$\begin{aligned} L'(t) &= \epsilon_1 E'(t) + \epsilon_2 \|u_t\|_2^2 - \epsilon_2 \|\nabla u\|_2^2 + \epsilon_2 \|u\|_p^p \\ &\quad - b_1\epsilon_2 \int_{\Gamma_0} \int_{-\infty}^{+\infty} \left|\phi(\xi,t)\right|^2 d\xi\, d\rho - a\epsilon_2 \int_\Omega u u_t \, dx. \end{aligned} \tag{4.20}$$

Using Poincare-type inequality and Young's inequality on the last term of (4.20), we get, for all $\delta' > 0$,

$$\int_\Omega u u_t \, dx \le \frac{1}{4\delta'} \|u_t\|_2^2 + C_* \delta' \|\nabla u\|_2^2. \tag{4.21}$$

From (4.20), (4.21), and (3.2), we obtain

$$\begin{aligned} L'(t) &\le \left[-a\epsilon_1 + \epsilon_2 \left(1 + \frac{a}{4\delta'} \right) \right] \|u_t\|_2^2 + \epsilon_2 \left[-1 + \delta' C_* a \right] \|\nabla u\|_2^2 \\ &\quad + \epsilon_2 \|u\|_p^p - b_1\epsilon_2 \int_{\Gamma_0} \int_{-\infty}^{+\infty} \left|\phi(\xi,t)\right|^2 d\xi\, d\rho. \end{aligned} \tag{4.22}$$

We use (3.7) to get

$$\begin{aligned} L'(t) \le{}& \left[-a\epsilon_1 + \epsilon_2\left(1 + \frac{a}{4\delta'}\right)\right]\|u_t\|_2^2 + \epsilon_2\left[-1 + \delta' C_* a + C_*^p\left(\frac{2p}{p-2}\right)^{\frac{p-2}{2}}\right]\|\nabla u\|_2^2 \\ & - b_1\epsilon_2\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi,t)\right|^2 d\xi\, d\rho. \end{aligned} \tag{4.23}$$

On the other hand, from (3.5)

$$-1 + C_*^p\left(\frac{2p}{p-2}\right)^{\frac{p-2}{2}} < 0.$$

For a small enough δ', we may have

$$-1 + \delta' C_* a + C_*^p\left(\frac{2p}{p-2}\right)^{\frac{p-2}{2}} < 0.$$

Then choose $d > 0$ depending only on δ' such that

$$\begin{aligned} L'(t) \le{}& \left[-a\epsilon_1 + \epsilon_2\left(1 + \frac{a}{4\delta'}\right)\right]\|u_t\|_2^2 - \epsilon_2 d\|\nabla u\|_2^2 \\ & - b_1\epsilon_2\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi,t)\right|^2 d\xi\, d\rho. \end{aligned} \tag{4.24}$$

Equivalently, for all positive constant M, we have

$$\begin{aligned} L'(t) \le{}& \left[-a\epsilon_1 + \epsilon_2\left(1 + \frac{a}{4\delta'} + \frac{M}{2}\right)\right]\|u_t\|_2^2 + \epsilon_2\left[\frac{M}{2} - d\right]\|\nabla u\|_2^2 \\ & + b_1\epsilon_2\left[\frac{M}{2} - 1\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left|\phi(\xi,t)\right|^2 d\xi\, d\rho - \epsilon_2 M E(t). \end{aligned} \tag{4.25}$$

For ϵ_1 and $M < \min\{2, 2d\}$ chosen such that

$$\epsilon_1 > \frac{\epsilon_2(1 + \frac{a}{4\delta'} + \frac{M}{2})}{a}.$$

We obtain from (4.25)

$$L'(t) \le -M\epsilon_2 E(t) \le \frac{-\epsilon_2 M}{\alpha_2} L(t), \tag{4.26}$$

as a result of (4.5). Now, a simple integration of (4.26) yields

$$L(t) \le L(0)e^{-kt},$$

where $k = \frac{\epsilon_2 M}{\alpha_2}$. Another use of (4.5) provides (4.17). □

5 Blow-up

In the current section, we follow the same approach given in [11] to prove the blow-up of solution of problem (2.20).

Remark 1 By integration of (3.2) over $(0,t)$, we have

$$\begin{aligned} E(t) = E(0) - a\int_0^t \|u_s\|_2^2\,ds \\ + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) \\ - b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)\left|\phi(\xi,s)\right|^2 d\xi\,d\rho\,ds. \end{aligned} \tag{5.1}$$

Now, let us define $F(t)$:

$$\begin{aligned} F(t) = \|u\|_2^2 + a\int_0^t \|u\|_2^2\,ds \\ - \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t) + b_1H(t), \end{aligned} \tag{5.2}$$

where

$$H(t) = \int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)\left(\int_0^s \phi(\xi,z)\,dz\right)^2 d\xi\,d\rho\,ds. \tag{5.3}$$

Lemma 10 *Assume that* $\|\nabla u\|_2^2$ *is bounded on* $[0,T)$, *then*

$$H(t) \le C < +\infty. \tag{5.4}$$

More precisely

$$H(t) \le \frac{1}{2}C_1B_qe^{-\eta C_2}\left[C_2^{2\alpha-1}\alpha + C_2^{3-2\alpha}\eta\right]\Gamma(\alpha)T^4,$$

where

$$C_1 = \sup_{t\in[0,T)}\left\{\|\nabla u\|_2^2, 1\right\}.$$

Proof Using (2.18), we obtain

$$\phi(\xi,t) = \int_0^t \mu(\xi)e^{-(\xi^2+\eta)(t-s)}u(x,s)\,ds, \quad \forall x\in\Gamma_0. \tag{5.5}$$

Hölder's inequality yields

$$\phi(\xi,t) \le \left(\int_0^t \mu^2(\xi)e^{-2(\xi^2+\eta)(t-s)}\,ds\right)^{\frac{1}{2}}\left(\int_0^t \left|u(x,s)\right|^2 ds\right)^{\frac{1}{2}}, \quad \forall x\in\Gamma_0. \tag{5.6}$$

On the other hand,

$$\left(\int_0^t \phi(\xi,s)\,ds\right)^2 \le T\int_0^t \left|\phi(\xi,s)\right|^2 ds. \tag{5.7}$$

From (5.6) in (5.7), we obtain

$$\left(\int_0^t \phi(\xi,s)\,ds\right)^2 \le T\int_0^t \left[\int_0^s \mu^2(\xi)e^{-2(\xi^2+\eta)(s-z)}\,dz \int_0^s \left|u(x,z)\right|^2 dz\right] ds. \tag{5.8}$$

Applying Lemma 2 leads to

$$\int_{\Gamma_0}\left(\int_0^t \phi(\xi,s)\,ds\right)^2 d\rho \le B_q C_1 T\int_0^t \left[\int_0^s \mu^2(\xi)e^{-2(\xi^2+\eta)(s-z)}\,dz\right] ds. \tag{5.9}$$

Since $z \in (0,s)$, we choose $\exists C_2 \ge 0$ such that $s-z \ge \frac{C_2}{2}$ to term (5.9) into

$$\int_{\Gamma_0}\left(\int_0^t \phi(\xi,s)\,ds\right)^2 d\rho \le \frac{1}{2}B_q C_1 T^3 \mu^2(\xi)e^{-C_2(\xi^2+\eta)}. \tag{5.10}$$

Multiplication by $\xi^2+\eta$ followed by integration over $(0,t)\times(-\infty,+\infty)$ yields

$$\begin{aligned} H(t) &\le C_1 B_q e^{-\eta C_2} T^3 \int_0^t \left[\int_0^{+\infty} \xi^{2\alpha+1} e^{-C_2\xi^2}\,d\xi\right] ds \\ &\quad + C_1 B_q e^{-\eta C_2}\eta T^3 \int_0^t \left[\int_0^{+\infty} \xi^{2\alpha-1} e^{-C_2\xi^2}\,d\xi\right] ds. \end{aligned} \tag{5.11}$$

Then

$$\begin{aligned} H(t) &\le \frac{1}{2}C_1 B_q e^{-\eta C_2} C_2^{2\alpha-1} T^3 \int_0^t \left[\int_0^{+\infty} y^{\alpha} e^{-y}\,dy\right] ds \\ &\quad + \frac{1}{2}C_1 B_q e^{-\eta C_2} C_2^{3-2\alpha}\eta T^3 \int_0^t \left[\int_0^{+\infty} y^{\alpha-1} e^{-y}\,dy\right] ds. \end{aligned} \tag{5.12}$$

Applying a special integral (Euler gamma function), we obtain

$$H(t) \le \frac{1}{2}C_1 B_q e^{-\eta C_2}\left[C_2^{2\alpha-1}\alpha + C_2^{3-2\alpha}\eta\right]\Gamma(\alpha)T^4. \tag{5.13}$$

Lemma 11 *Suppose $p>2$, then*

$$\begin{aligned} F''(t) &\ge (p+2)\|u_t\|_2^2 \\ &\quad + 2p\left\{-E(0) + a\int_0^t \|u_s\|_2^2\,ds - \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)\right. \\ &\quad \left. + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left|\phi(\xi,s)\right|^2 d\xi\,d\rho\,ds\right\}. \end{aligned} \tag{5.14}$$

Proof We differentiate with respect to t in (5.2), then we get

$$\begin{aligned} F'(t) &= 2\int_\Omega u u_t\,dx + a\|u\|_2^2 \\ &\quad + \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\bigl(g' \circ \nabla u\bigr)(t) \\ &\quad + 2b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} \bigl(\xi^2+\eta\bigr)\phi(\xi,s)\int_0^s \phi(\xi,z)\,dz\,d\xi\,d\rho\,ds. \end{aligned} \tag{5.15}$$

Using divergence theorem and (2.20), we obtain

$$\begin{aligned} F''(t) &= 2\|u_t\|_2^2 - 2\int_\Omega \nabla u \int_0^t g(t-s)\nabla u(s)\,ds\,dx \\ &\quad + 2\|u\|_p^p + 2b_1 \int_{\Gamma_0} u(x,t)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\,d\xi\,d\rho \\ &\quad + 2b_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} \bigl(\xi^2+\eta\bigr)\phi(\xi,t)\int_0^t \phi(\xi,s)\,ds\,d\xi\,d\rho. \end{aligned} \tag{5.16}$$

By definition of energy functional in (3.1) and relation (5.1), we give the following evaluation of the third term of (5.16):

$$\begin{aligned} 2\|u\|_p^p &= p\|u_t\|_2^2 + p\|\nabla u\|_2^2 + pb_1 \int_{\Gamma_0}\int_{-\infty}^{+\infty} \bigl|\phi(\xi,t)\bigr|^2\,d\xi\,d\rho - 2pE(0) \\ &\quad + 2p\biggl[a\int_0^t \|u_s\|_2^2\,ds - \frac{1}{2}\biggl(1-\int_0^t g(s)\,ds\biggr)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t) \\ &\quad + b_1 \int_0^t \int_{\Gamma_0}\int_{-\infty}^{+\infty} \bigl(\xi^2+\eta\bigr)\bigl|\phi(\xi,s)\bigr|^2\,d\xi\,d\rho\,ds\biggr]. \end{aligned} \tag{5.17}$$

We can also estimate the last term of (5.16) using Lemma 8:

$$\begin{aligned} &\int_{\Gamma_0}\int_{-\infty}^{+\infty} \bigl(\xi^2+\eta\bigr)\phi(\xi,t)\int_0^t \phi(\xi,s)\,ds\,d\xi\,d\rho \\ &\quad = \int_{\Gamma_0} u(x,t)\int_{-\infty}^{+\infty} \phi(\xi,t)\mu(\xi)\,d\xi\,d\rho - \int_{\Gamma_0}\int_{-\infty}^{+\infty} \bigl|\phi(\xi,t)\bigr|^2\,d\xi\,d\rho. \end{aligned} \tag{5.18}$$

From (5.17), (5.18), and (5.16), we get

$$\begin{aligned} F''(t) &\geq (p+2)\|u_t\|_2^2 + (p-2)\|\nabla u\|_2^2 + b_1(p-2)\int_{\Gamma_0}\int_{-\infty}^{+\infty} \bigl|\phi(\xi,t)\bigr|^2\,d\xi\,d\rho \\ &\quad + 2p\biggl[-E(0) + a\int_0^t \|u_s\|_2^2\,ds - \frac{1}{2}\biggl(1-\int_0^t g(s)\,ds\biggr)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t) \\ &\quad + b_1 \int_0^t \int_{\Gamma_0}\int_{-\infty}^{+\infty} \bigl(\xi^2+\eta\bigr)\bigl|\phi(\xi,s)\bigr|^2\,d\xi\,d\rho\,ds\biggr]. \end{aligned} \tag{5.19}$$

Taking $p > 2$, we obtain the needed estimation

$$\begin{aligned} F''(t) &\geq (p+2)\|u_t\|_2^2 \\ &\quad + 2p\biggl\{-E(0) + a\int_0^t \|u_s\|_2^2\,ds - \frac{1}{2}\biggl(1-\int_0^t g(s)\,ds\biggr)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t) \\ &\quad + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}\bigl(\xi^2+\eta\bigr)\bigl|\phi(\xi,s)\bigr|^2\,d\xi\,d\rho\,ds\biggr\}. \end{aligned}$$

□

Lemma 12 *Suppose that $p > 2$ and that either one of the next assumptions is verified*:

(*i*) $E(0) < 0$;

(*ii*) $E(0) = 0$, *and*

$$F'(0) > a\|u_0\|_2^2; \tag{5.20}$$

(*iii*) $E(0) > 0$, *and*

$$F'(0) > \bigl[F(0) + l_0\bigr] + a\|u_0\|_2^2, \tag{5.21}$$

where

$$r = p - 2\sqrt{p^2 - p}$$

and

$$l_0 = a\|u_0\|_2^2 - 2E(0). \tag{5.22}$$

Then $F'(t) > a\|u_0\|_2^2$ for $t > t_0$, where

$$t^* > \max\biggl\{0, \frac{F'(0) - a\|u_0\|_2^2]}{2pE(0)}\biggr\}, \tag{5.23}$$

where $t_0 = t^*$ in case (*i*), *and $t_0 = 0$ in cases* (*ii*) *and* (*iii*).

Proof (i) Case of $E(0) < 0$.

From (5.14), we have

$$F''(t) \geq -2pE(0),$$

which clearly leads to

$$F'(t) \geq F'(0) - 2pE(0)t.$$

Then

$$F'(t) > a\|u_0\|_2^2, \quad \forall t \geq t^*,$$

where t^* as given in (5.23).

(ii) Case $E(0) = 0$.

Using (5.14) we get

$$F''(t) \geq 0, \quad \forall t \geq 0.$$

Thus

$$F'(t) \geq F'(0), \quad \forall t \geq 0.$$

Then, by (5.20),

$$F'(t) > a\|u_0\|_2^2, \quad \forall t \geq 0.$$

(iii) Case $E(0) > 0$.

The proof of this case consists of getting to a differential inequality: $B''(t) - pB'(t) + pB(t) \geq 0$ pursued by a use of Lemma 3. Indeed, from (5.15) we have

$$\begin{aligned} F'(t) = {} & 2\int_\Omega uu_t\,dx + a\|u\|_2^2 \\ & + \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\big(g' \circ \nabla u\big)(t) \\ & + 2b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} \big(\xi^2 + \eta\big)\phi(\xi,s) \int_0^s \phi(\xi,z)\,dz\,d\xi\,d\rho\,ds. \end{aligned} \tag{5.24}$$

Or, the last term in (5.24) can be estimated using Young's inequality

$$\begin{aligned} & \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} \big(\xi^2 + \eta\big)\phi(\xi,s) \int_0^s \phi(\xi,z)\,dz\,d\xi\,d\rho\,ds \\ & \quad \leq \frac{1}{2} \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} \big(\xi^2 + \eta\big)\big|\phi(\xi,s)\big|^2\,d\xi\,d\rho\,ds \\ & \qquad + \frac{1}{2} \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} \big(\xi^2 + \eta\big)\bigg(\int_0^s \phi(\xi,z)\,dz\bigg)^2\,d\xi\,d\rho\,ds. \end{aligned} \tag{5.25}$$

On the other hand,

$$2\int_0^t \int_\Omega u_s u\,dx\,ds = \int_0^t \frac{d}{ds}\|u_s\|_2^2\,ds = \|u\|_2^2 - \|u_0\|_2^2. \tag{5.26}$$

By Young's inequality, we get

$$\|u\|_2^2 \leq \int_0^t \|u_s\|_2^2\,ds + \int_0^t \|u\|_2^2\,ds + \|u_0\|_2^2. \tag{5.27}$$

Now, we remake (5.24) using (5.25) and (5.27):

$$\begin{aligned} F'(t) \leq{}& \|u\|_2^2 + \|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2\,ds + a\int_0^t \|u\|_2^2\,ds + a\|u_0\|_2^2 \\ &- \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t) \\ &+ b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)\left|\phi(\xi,s)\right|^2\,d\xi\,d\rho\,ds \\ &+ b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)\left(\int_0^s \phi(\xi,z)\,dz\right)^2\,d\xi\,d\rho\,ds. \end{aligned} \tag{5.28}$$

From the definition of F in (5.2), inequality (5.28) also becomes

$$\begin{aligned} F'(t) \leq{}& F(t) + \|u_t\|_2^2 + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)\left|\phi(\xi,s)\right|^2\,d\xi\,d\rho\,ds \\ &+ a\int_0^t \|u_s\|_2^2\,ds + a\|u_0\|_2^2. \end{aligned} \tag{5.29}$$

Thus, by (5.14), we get

$$\begin{aligned} F''(t) - p\left\{F'(t) - F(t)\right\} \geq{}& 2\|u_t\|_2^2 + ap\int_0^t \|u_s\|_2^2\,ds - pa\|u_0\|_2^2 - 2pE(0) \\ &+ pb_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)\left|\phi(\xi,s)\right|^2\,d\xi\,d\rho\,ds. \end{aligned} \tag{5.30}$$

Hence

$$F''(t) - pF'(t) + pF(t) + pl_0 \geq 0, \tag{5.31}$$

where

$$l_0 = a\|u_0\|_2^2 - 2E(0).$$

Posing

$$B(t) = F(t) + l_0$$

leads to

$$B''(t) - pB'(t) + pB(t) \geq 0. \tag{5.32}$$

By Lemma 3 and for $p = \delta + 1$, we conclude that if

$$B'(t) > \left(p - 2\sqrt{p^2 - p}\right)B(0) + a\|u_0\|_2^2, \tag{5.33}$$

then

$$F'(t) = B'(t) > a\|u_0\|_2^2 \quad \forall t \geq 0.$$

□

Theorem 5 *Suppose that* $p > 2$ *and that either one of the next assumptions is verified*:

(*i*) $E(0) < 0$;

(*ii*) $E(0) = 0$ *and* (5.20) *holds*;

(*iii*) $0 < E(0) < \frac{(2p-4)(F'(t_0)-a\|u_0\|_2^2)^2 J(t_0)^{\frac{1}{\gamma_1}}}{16p}$ *and* (5.21) *holds*.

Then, in the sense of Definition 1, *the solution* (u, ϕ) *blows up at finite time* T^*.

For case (*i*):

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}. \tag{5.34}$$

Moreover, if $J(t_0) < \min\{1, \sqrt{\frac{\sigma}{-b}}\}$, *we get*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{\sigma}{-b}}}{\sqrt{\frac{\sigma}{-b}} - J(t_0)}. \tag{5.35}$$

For case (*ii*), *we get either*

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)} \tag{5.36}$$

or

$$T^* \leq t_0 + \frac{J(t_0)}{J'(t_0)}. \tag{5.37}$$

For case (*iii*):

$$T^* \leq \frac{J(t_0)}{\sqrt{\sigma}}, \tag{5.38}$$

or else

$$T^* \leq t_0 + 2^{\frac{3\gamma_1+1}{2\gamma_1}} \frac{\gamma_1 c}{\sqrt{\sigma}} \left\{1 - \left[1 - cJ(t_0)\right]^{\frac{1}{2\gamma_1}}\right\}, \tag{5.39}$$

where $\gamma_1 = \frac{p-4}{4}$, $c = (\frac{b}{\sigma})^{\frac{\gamma_1}{2+\gamma_1}}$, $J(t)$, b *and* σ *are as in* (5.40) *and* (5.54) *respectively*.

Note that $t_0 = 0$ in cases (ii) and (iii). For case (i), we have as in (5.23): $t_0 = t^*$.

Proof Consider

$$J(t) = \left[F(t) + a(T-t)\|u_0\|_2^2\right]^{-\gamma_1}, \quad t \in [t_0, T]. \tag{5.40}$$

We differentiate on $J(t)$ to get

$$J'(t) = -\gamma_1 J(t)^{1+\frac{1}{\gamma_1}} \left[F'(t) - a\|u_0\|_2^2\right] \tag{5.41}$$

and again

$$J''(t) = -\gamma_1 J(t)^{1+\frac{2}{\gamma_1}} G(t), \tag{5.42}$$

where

$$G(t) = F''(t)\big[F(t) + a(T-t)\|u_0\|_2^2\big] - (1+\gamma_1)\big\{F'(t) - a\|u_0\|_2^2\big\}^2. \tag{5.43}$$

Using (5.14), we obtain

$$\begin{aligned} F''(t) &\geq (p+2)\|u_t\|_2^2 \\ &\quad + 2p\left\{-E(0) + a\int_0^t \|u_s\|_2^2\,ds - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)\right. \\ &\quad \left. + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\big|\phi(\xi,s)\big|^2\,d\xi\,d\rho\,ds\right\}. \end{aligned}$$

Consequently,

$$\begin{aligned} F''(t) &\geq -2pE(0) \\ &\quad \times p\left\{\|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2\,ds - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)\right. \\ &\quad \left. + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\big|\phi(\xi,s)\big|^2\,d\xi\,d\rho\,ds\right\}. \end{aligned} \tag{5.44}$$

Or, from (5.15) and the fact that $\|u\|_2^2 - \|u_0\|_2^2 = 2\int_0^t\int_\Omega u_s u\,dx\,ds$, we attain

$$\begin{aligned} F'(t) - a\|u_0\|_2^2 &= 2\int_\Omega u u_t\,dx + 2a\int_0^t\int_\Omega u_s u\,dx\,ds \\ &\quad + 2b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z)\,dz\,d\xi\,d\rho\,ds. \end{aligned} \tag{5.45}$$

Going back to (5.43) with (5.44) and (5.45) in hand, we get

$$\begin{aligned} G(t) &\geq -2pE(0)J(t)^{\frac{-1}{\gamma_1}} \\ &\quad + p\left\{\|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2\,ds - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)\right. \\ &\quad \left. + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\big|\phi(\xi,s)\big|^2\,d\xi\,d\rho\,ds\right\} \\ &\quad \times\left[\|u\|_2^2 + a\int_0^t \|u\|_2^2\,ds - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)\right. \\ &\quad \left. + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s \phi(\xi,z)\,dz\right)^2 d\xi\,d\rho\,ds\right] \\ &\quad - 4(1+\gamma_1)\left\{\int_\Omega u u_t\,dx + a\int_0^t\int_\Omega u_s u\,dx\,ds + \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\big(g'\circ\nabla u\big)(t)\right. \\ &\quad \left. + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z)\,dz\,d\xi\,d\rho\,ds\right\}^2. \end{aligned} \tag{5.46}$$

For the sake of simplicity, we introduce the following notations:

$$\begin{aligned}
\mathbf{A} &= \|u\|_2^2 + a\int_0^t \|u\|_2^2\,ds - \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t) \\
&\quad + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s \phi(\xi,z)\,dz\right)^2 d\xi\,d\rho\,ds, \\
\mathbf{B} &= \int_\Omega u u_t\,dx + a\int_0^t\int_\Omega u_s u\,dx\,ds + \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\big(g'\circ\nabla u\big)(t) \\
&\quad + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z)\,dz\,d\xi\,d\rho\,ds, \\
\mathbf{C} &= \|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2\,ds - \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t) \\
&\quad + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\big|\phi(\xi,s)\big|^2\,d\xi\,d\rho\,ds.
\end{aligned}$$

Therefore

$$Q(t) \geq -2pE(0)J(t)^{\frac{-1}{\gamma_1}} + p\big\{\mathbf{AC}-\mathbf{B}^2\big\}. \tag{5.47}$$

Note that, $\forall w \in R$ and $\forall t > 0$,

$$\begin{aligned}
\mathbf{A}w^2 + 2\mathbf{B}w + \mathbf{C} &= \left[w^2\|u\|_2^2 + 2w\int_\Omega u u_t\,dx + \|u_t\|_2^2\right] \\
&\quad + a\int_0^t\left[w^2\|u\|_2^2 + 2w\int_\Omega u u_s\,dx + \|u_s\|_2^2\right] ds \\
&\quad + \big(w^2+1\big)\left(-\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)\right) \\
&\quad + w\left(\frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\big(g'\circ\nabla u\big)(t)\right) \\
&\quad + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left[w^2\left(\int_0^s \phi(\xi,z)\,dz\right)^2\right. \\
&\quad \left. + 2w\phi(\xi,s)\int_0^s \phi(\xi,z)\,dz + \big|\phi(\xi,s)\big|^2\right] d\xi\,d\rho\,ds.
\end{aligned} \tag{5.48}$$

Hence

$$\begin{aligned}
&\mathbf{A}w^2 + 2\mathbf{B}w + \mathbf{C} \\
&\quad = \big[w\|u\|_2 + \|u_t\|_2\big]^2 + a\int_0^t \big[w\|u\|_2 + \|u_s\|_2\big]^2\,ds \\
&\quad + \big(w^2+1\big)\left(-\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)\right) \\
&\quad + w\left(\frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\big(g'\circ\nabla u\big)(t)\right) \\
&\quad + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left[w\int_0^s \phi(\xi,z)\,dz + \big|\phi(\xi,s)\big|\right]^2 d\xi\,d\rho\,ds.
\end{aligned} \tag{5.49}$$

It is clear that

$$\mathbf{A}w^2 + 2\mathbf{B} + \mathbf{C} \geq 0$$

and

$$\mathbf{B}^2 - \mathbf{AC} \leq 0. \tag{5.50}$$

Then, from (5.47) and (5.50), we obtain

$$G(t) \geq -2pE(0)J(t)^{\frac{-1}{\gamma_1}}, \quad t \geq t_0. \tag{5.51}$$

Hence, by (5.42) and (5.51),

$$J''(t) \leq \frac{p^2 - 4p}{2} E(0)J(t)^{1+\frac{1}{\gamma_1}}, \quad t \geq t_0. \tag{5.52}$$

Or, by Lemma [6], $J'(t) < 0$, where $t \geq t_0$.

Multiplication by $J'(t)$ in (5.52), followed by integration from t_0 to t, leads to

$$J'(t)^2 \geq \sigma + bJ(t)^{2+\frac{1}{\gamma_1}}, \tag{5.53}$$

where

$$\begin{cases} \sigma = [\frac{(p-4)^2}{16}(F'(t_0) - \|u_0\|_2^2)^2 - \frac{p(p-4)^2}{2p-4}E(0)J(t_0)^{\frac{-1}{\gamma_1}}]J(t_0)^{2+\frac{2}{\gamma_1}}, \\ b = \frac{p(p-4)^2}{2p-4}E(0). \end{cases} \tag{5.54}$$

Note that $\sigma > 0$ is equivalent to $E(0) < \frac{(2p-4)(F'(t_0)-a\|u_0\|_2^2)^2 J(t_0)^{\frac{1}{\gamma_1}}}{16p}$, which by Lemma 4 ensures the existence of a finite time $T^* > 0$ such that

$$\lim_{t \to T^{*-}} J(t) = 0.$$

That involves

$$\begin{aligned} \lim_{t \to T^{*-}} \biggl[& \|u\|_2^2 + a \int_0^t \|u\|_2^2 \, ds - \frac{1}{2}\biggl(1 - \int_0^t g(s)\, ds\biggr) \|\nabla u\|_2^2 \\ & - \frac{1}{2}(g \circ \nabla u)(t) + b_1 H(t) \biggr]^{-1} = 0, \end{aligned} \tag{5.55}$$

i.e.,

$$\begin{aligned} \lim_{t \to T^{*-}} \biggl[& \|u\|_2^2 + a \int_0^t \|u\|_2^2 \, ds - \frac{1}{2}\biggl(1 - \int_0^t g(s)\, ds\biggr) \|\nabla u\|_2^2 \\ & - \frac{1}{2}(g \circ \nabla u)(t) + b_1 H(t) \biggr] = +\infty. \end{aligned} \tag{5.56}$$

So, there exists T such that $t_0 < T \leq T^*$ and $\|\nabla u\|_2^2 \longrightarrow +\infty$ as $t \longrightarrow T^-$.

Indeed, if it is not the case, then $\|\nabla u\|_2^2$ remained bounded on $[t_0, T^*)$, which by Lemma 10 leads to

$$\lim_{t\to T^{*-}}\left[\|u\|_2^2 + b_1 H(t)\right] = C < +\infty,$$

contradicting (5.56). □

6 Conclusion

Much attention has been accorded to fractional partial differential equations during the past two decades due to the many chemical engineering, biological, ecological, and electromagnetism phenomena that are modeled by initial boundary value problems with fractional boundary conditions. In the context of boundary dissipations of fractional order problems, the main research focus is on asymptotic stability of solutions starting by writing the equations as an augmented system. Then, various techniques are used such as LaSalle's invariance principle and the multiplier method mixed with frequency domain. We prove in this paper under suitable conditions on the initial data the stability of a wave equation with fractional damping and memory term. This technique of proof was recently used by [4] to study the exponential decay of a system of nonlocal singular viscoelastic equations. Here we also considered three different cases on the sign of the initial energy as recently examined by Zarai et al. [17], where they studied the blow-up of a system of nonlocal singular viscoelastic equations.

In the next work, we will try to extend the same study of this paper to a general source term case.

Acknowledgements

The authors are grateful to the anonymous referees for the careful reading and their important observations/suggestions for the sake of improving this paper. The first author (Pr. Salah Boulaaras) would like to thank all the professors of the mathematics department at the University of Annaba in Algeria, especially his Professors/Scientists Pr. Mohamed Haiour, Pr. Ahmed-Salah Chibi, and Pr. Azzedine Benchettah for the important content of masters and PhD courses in pure and applied mathematics which he received during his studies. Moreover, he thanks them for the additional help they provided to him during office hours in their office about the few concepts/difficulties he had encountered, and he appreciates their talent and dedication for their postgraduate students currently and previously.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Ethics approval and consent to participate

Not applicable.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

Consent for publication

Not applicable.

Authors' contributions

The authors contributed equally in this article. They have all read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 22 September 2020 Accepted: 20 November 2020 Published online: 30 November 2020

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