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On a new structure of the pantograph inclusion problem in the Caputo conformable setting

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Abstract

In this work, we reformulate and investigate the well-known pantograph differential equation by applying newly-defined conformable operators in both Caputo and Riemann–Liouville settings simultaneously for the first time. In fact, we derive the required existence criteria of solutions corresponding to the inclusion version of the three-point Caputo conformable pantograph BVP subject to Riemann–Liouville conformable integral conditions. To achieve this aim, we establish our main results in some cases including the lower semi-continuous, the upper semi-continuous and the Lipschitz set-valued maps. Eventually, the last part of the present research is devoted to proposing two numerical simulative examples to confirm the consistency of our findings.

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1 Introduction

Over the years, human beings have needed to be acquainted with various natural phenomena more and more. One possible way to achieve this aim is to utilize the techniques and tools available in mathematics and particularly the mathematical operators in modeling of different processes. Numerous fractional operators have been introduced during years and their applicability is becoming increasingly apparent to researchers every day that passes. In this direction, it is better that we formulate and investigate various complicated modelings of processes from all aspects by applying the fractional operators in boundary problems.

In much of the literature we can see various complicated fractional modelings in which one of the well-known fractional Caputo or the Riemann–Liouville operators has been utilized (see for example, [1–13]). Also, some generalizations of these operators such as the Hadamard, Caputo–Hadamard and Hilfer fractional operators were utilized by other researchers in the next period and different modelings are investigated using these new operators (see, for instance, [14–30]). Five years ago, a novel derivative in the fractional

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frame was formulated by Fabrizio and Caputo [31] in which the kernel has no singularity in any point. This new operator is called the fractional Caputo–Fabrizio operator. Immediately after this work, Nieto and Losada [32] turned to several important computational aspects of this newly-defined operator. Some useful aspects of mentioned nonsingular operator led to publishing of numerous research articles on the fractional modelings in this context (see [33–41]).

More recently, Abdeljawad [42] extended some notions presented in [43] and studied some applied specifications of the well-behaved conformable derivatives of arbitrary order. Next, Jarad et al. [44] proceeded to answer this key problem if we can generalize the usual fractional Riemann–Liouville integral provided that we obtain a unification to remaining useful operators such as Caputo, Riemann–Liouville, Hadamard, and Caputo–Hadamard derivatives [45]. To achieve this purpose, they tried to derive two corresponding integration and differentiation operators of arbitrary order based on the existing conformable operators. In this way, the authors first designed functional spaces and then verified some fundamental applied specifications of both newly-defined combined operators.

Until now, there have been published a limited number of papers based on these novel operators. For example, the authors employed new Riemann–Liouville and Caputo conformable operators in the following BVP for the first time. Indeed, Aphithana, Ntouyas and Tariboon [46] regraded a modern BVP including the Caputo conformable differential equation along with integral conditions:

$$\begin{cases} {}^{CC}\mathcal{D}_c^{\zeta, \nu^*} \phi(s) = \hat{f}^*(s, \phi(s)), & (s \in [c, M]) \\ \phi(c) = \vartheta_1 \phi(\xi) + \vartheta_2, & \phi(M) = \vartheta_3 {}^{\mathcal{R}C}\mathcal{J}_c^{\zeta, p^*} \phi(\sigma), \end{cases}$$

so that ${}^{CC}\mathcal{D}_c^{\zeta, \nu^*}$ indicates the conformable derivative in the Caputo frame of order $\nu^* \in (1, 2)$ along with $\zeta \in (0, 1]$. Also, ${}^{\mathcal{R}C}\mathcal{J}_c^{\zeta, p^*}$ stands for the conformable integral in the Riemann–Liouville frame of order $p^* > 0$. The authors utilized several techniques to establish desired theorems. Furthermore, different types of Ulam stability of the proposed problem were studied by authors [46]. Recently, Baleanu, Etemad and Rezapour [47] turned to the differential inclusion in the Caputo fractional conformable frame illustrated by

$$\begin{cases} {}^{CC}\mathcal{D}_c^{\zeta, \nu^*} \phi(s) \in \check{\mathfrak{R}}(s, \phi(s)) & (s \in [c, M], c \geq 0), \\ \phi(c) = {}^{\mathcal{R}C}\mathcal{D}_c^{\zeta, p^*} \phi(\xi) + \mu_1, & \phi(M) = {}^{\mathcal{R}C}\mathcal{J}_c^{\zeta, q^*} \phi(\sigma) + \mu_2, \end{cases}$$

so that ${}^{CC}\mathcal{D}_c^{\zeta, \nu^*}$, ${}^{\mathcal{R}C}\mathcal{D}_c^{\zeta, p^*}$ and ${}^{\mathcal{R}C}\mathcal{J}_c^{\zeta, q^*}$ stand for the Caputo- and the Riemann–Liouville conformable derivatives and the Riemann–Liouville conformable integral of order $q^* > 0$, respectively. The main aim of the authors in that manuscript is to discuss the existence aspects for mentioned BVP by employing several methods based on the α - ψ -contractives and operators involving approximate endpoint specification [47].

One of the most famous categories of differential equations is related to the pantograph equation. This kind of equation is considered as proportional delay functional differential equations and they have many applications in applied and pure mathematics. In other words, pantograph equations arise in rather various contexts: control systems, quantum mechanics, electrodynamics, probability, etc. For the first time, Balachandran et al. [48] formulated a pantograph equation of fractional order and derived existence and

also uniqueness criteria for the proposed problem. After that, different researchers studied fractional pantograph equations with the help of various numerical methods such as the operational method, the spectral-collocation method, and the Hermite wavelet method [49–51]. Recently, other researchers investigated various versions of fractional pantograph equations relying on analytical methods (see [52–54]). By taking into account the aforementioned new operators introduced by Jarad et al. [44] and inspired by some existing ideas in the above articles, in the current manuscript, for the first time, we formulate an inclusion version of the pantograph boundary problem in the fractional Caputo conformable settings subject to three-point Riemann–Liouville conformable integral conditions as follows:

$$\begin{cases} {}^{CC}\mathcal{D}_c^{\zeta, \nu^*} \phi(s) \in \tilde{\mathcal{D}}(s, \phi(s), \phi(\lambda^*s)) & (s \in [c, M], c \geq 0), \\ \phi(c) = 0, \quad \mu_1^* \phi(M) + \mu_2^* \mathcal{R}^C \mathcal{J}_c^{\zeta, \theta^*} \phi(\sigma) = \xi^* \end{cases} \tag{1}$$

so that ${}^{CC}\mathcal{D}_c^{\zeta, \nu^*}$ indicates the derivative in the Caputo conformable settings of order $\nu^* \in (1, 2)$ along with $\zeta \in (0, 1]$ and $\mathcal{R}^C \mathcal{J}_c^{\zeta, \theta^*}$ stands for the integral in the Riemann–Liouville conformable frame of order $\theta^* > 0$. Furthermore, $\sigma \in (c, M)$, $\mu_1^*, \mu_2^*, \xi^* \in \mathbb{R}$, $\lambda^* \in (0, 1)$ and $\tilde{\mathcal{D}} : [c, M] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multifunction furnished with several necessary specifications which are indicated in the rest of the manuscript. It is important that the reader pays attention to the fact that this structure of a pantograph inclusion problem in the Caputo conformable operators is novel and such a kind of construction has not been discussed in any literature so far. In fact, we reformulate the well-known pantograph differential equation by applying newly-defined conformable operators in both Caputo and Riemann–Liouville settings simultaneously for the first time. We demonstrate the contents of the current research manuscript as follows. In Sect. 2, we briefly review fundamental and auxiliary concepts and notions. In Sect. 3, we employ some well-known analytical techniques to establish existence criteria corresponding to the given pantograph inclusion BVP (1). In this way, we deduce key results in three cases including the lower semi-continuous, the upper semi-continuous and the Lipschitz set-valued maps. In fact, we derive desired existence results for three different structures considered on the set-valued maps and this cover a vast range of multifunctions satisfying our given conditions. the last part of the present research is devoted to proposing two numerical simulative examples to demonstrate the consistency of the analytical findings.

2 Auxiliary notions

Now, we review some fundamental and auxiliary notions and some specifications of the fractional Riemann–Liouville and Caputo conformable operators. As we see in much of the literature, the concept of the Riemann–Liouville integral of order $\nu^* > 0$ for a real function $\phi : [0, +\infty) \rightarrow \mathbb{R}$ is illustrated by $\mathcal{R}^C \mathcal{J}_0^{\nu^*} \phi(s) = \int_0^s \frac{(s-q)^{\nu^*-1}}{\Gamma(\nu^*)} \phi(q) dq$ such that the RHS integral possesses finite values [55, 56]. In the current position, we assume that $\nu^* \in (k - 1, k)$ so that $k = [\nu^*] + 1$. For a given function $\phi \in \mathcal{AC}_{\mathbb{R}}^{(k)}([0, +\infty))$, the fractional derivative in the Caputo settings is defined by

$${}^C \mathcal{D}_0^{\nu^*} \phi(s) = \int_0^s \frac{(s-q)^{k-\nu^*-1}}{\Gamma(k-\nu^*)} \phi^{(k)}(q) dq$$

so that the existing R.H.S integral involves the finite values [55, 56]. Subsequently, the left conformable derivative at $s_0 = c$ for $\phi : [c, \infty) \rightarrow \mathbb{R}$ along with $\zeta \in (0, 1]$ was introduced as

$$\mathfrak{D}_c^\zeta \phi(s) = \lim_{\lambda \rightarrow 0} \frac{\phi(s + \lambda(s - c)^{1-\zeta}) - \phi(s)}{\lambda}$$

provided that the limit exists [43]. Notice that, if $\mathfrak{D}_c^\zeta \phi(s)$ exists on (c, d) , in this case we have $\mathfrak{D}_c^\zeta \phi(c) = \lim_{s \rightarrow c^+} \mathfrak{D}_c^\zeta \phi(s)$. Also, if we assume that the given function ϕ is differentiable, then it is clear that $\mathfrak{D}_c^\zeta \phi(s) = (s - c)^{1-\zeta} \phi'(s)$. The left conformable integral of ϕ along with $\zeta \in (0, 1]$ is defined in the form $\mathfrak{J}_c^\zeta \phi(s) = \int_c^s \phi(q) \frac{dq}{(q-c)^{1-\zeta}}$ whenever the RHS integral is finite-valued [43]. Jarad et al. [44] presented a new formulation of integro-derivative operators which generalize conformable operators to fractional orders in both Riemann–Liouville and Caputo settings. To see this, let $\nu^* \in \mathbb{C}$ with $\text{Re}(\nu^*) \geq 0$. In this phase, the Riemann–Liouville conformable integral for ϕ of order ν^* along with $\zeta \in (0, 1]$ is introduced as follows:

$$\mathcal{R}C \mathfrak{J}_c^{\zeta, \nu^*} \phi(s) = \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \phi(q) \frac{dq}{(q-c)^{1-\zeta}}$$

so that the RHS integral is finite [44]. One can simply deduce that, if $c = 0$ and $\zeta = 1$, then $\mathcal{R}C \mathfrak{J}_c^{\zeta, \nu^*} \phi(s)$ is reduced to the standard Riemann–Liouville integral $\mathcal{R} \mathfrak{J}_0^{\nu^*} \phi(s)$. Moreover, the Riemann–Liouville conformable derivative for ϕ of order ν^* along with $\zeta \in (0, 1]$ is formulated as

$$\begin{aligned} \mathcal{R}C \mathfrak{D}_c^{\zeta, \nu^*} \phi(s) &= \mathfrak{D}_c^{\zeta, k} (\mathcal{R}C \mathfrak{J}_c^{\zeta, k-\nu^*} \phi)(s) \\ &= \frac{\mathfrak{D}_c^{\zeta, k}}{\Gamma(k - \nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{k-\nu^*-1} \phi(q) \frac{dq}{(q-c)^{1-\zeta}}, \end{aligned}$$

where $k = [\text{Re}(\nu^*)] + 1$ and also $\mathfrak{D}_c^{\zeta, k} = \overbrace{\mathfrak{D}_c^\zeta \mathfrak{D}_c^\zeta \dots \mathfrak{D}_c^\zeta}^{k \text{ times}}$ so that \mathfrak{D}_c^ζ illustrates the left conformable derivative with $\zeta \in (0, 1]$ [44]. In similar way, one can simply see that, if $c = 0$ and $\zeta = 1$, then $\mathcal{R}C \mathfrak{D}_c^{\zeta, \nu^*} \phi(s)$ is reduced to the standard Riemann–Liouville derivative $\mathcal{R} \mathfrak{D}_0^{\nu^*} \phi(s)$. In the rest, we intend to recall the definition of a similar notion in the framework of the Caputo. To do this, build $\mathcal{L}_\zeta(c) := \{h_* : [c, d] \rightarrow \mathbb{R} : \mathfrak{J}_c^\zeta h_*(s) \text{ exists for each } s \in [c, d]\}$ for $\zeta \in (0, 1]$ and take

$$\mathbb{I}_\zeta([c, d]) := \{\phi : [c, d] \rightarrow \mathbb{R} : \phi(s) = \mathfrak{J}_c^\zeta h_*(s) + \phi(c), \text{ for some } h_* \in \mathcal{L}_\zeta(c)\},$$

where $\mathfrak{J}_c^\zeta h_*(s) = \int_c^s h_*(q) d\zeta(q, c) = \int_c^s h_*(q) \frac{dq}{(q-c)^{1-\zeta}}$ is the left conformable integral of h_* [42]. For $k \in \mathbb{N}$, define $\mathcal{C}_{c, \zeta}^k([c, d]) := \{\phi : [c, d] \rightarrow \mathbb{R} : \mathfrak{D}_c^{\zeta, k-1} \phi \in \mathbb{I}_\zeta([c, d])\}$. In this phase, the Caputo conformable derivative for $\phi \in \mathcal{C}_{c, \zeta}^k([c, d])$ of order ν^* along with $\zeta \in (0, 1]$ is illustrated as

$$\begin{aligned} \mathcal{C}C \mathfrak{D}_c^{\zeta, \nu^*} \phi(s) &= \mathcal{R}C \mathfrak{J}_c^{\zeta, k-\nu^*} (\mathfrak{D}_c^{\zeta, k} \phi)(s) \\ &= \frac{1}{\Gamma(k - \nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{k-\nu^*-1} \mathfrak{D}_c^{\zeta, k} \phi(q) \frac{dq}{(q-c)^{1-\zeta}} \end{aligned}$$

such that $k = [\text{Re}(v^*)] + 1$ [44]. It is clear that ${}^{CC}\mathcal{D}_c^{\zeta, v^*} \phi(s) = {}^C\mathcal{D}_0^{v^*} \phi(s)$ when we have $c = 0$ and $\zeta = 1$.

Lemma 1 ([44]) *Take $\text{Re}(v^*) > 0$, $\text{Re}(w^*) > 0$ and $\text{Re}(\sigma) > 0$. Then, for $\zeta \in (0, 1]$ and for each $s > c$, the following hold:*

- (i1) ${}^{\mathcal{RC}}\mathcal{J}_c^{\zeta, v^*} ({}^{\mathcal{RC}}\mathcal{J}_c^{\zeta, w^*} \phi)(s) = ({}^{\mathcal{RC}}\mathcal{J}_c^{\zeta, v^* + w^*} \phi)(s)$,
- (i2) ${}^{\mathcal{RC}}\mathcal{J}_c^{\zeta, v^*} (s - c)^{\zeta(\sigma - 1)}(z) = \frac{1}{\zeta^{v^*}} \frac{\Gamma(\sigma)}{\Gamma(\sigma + v^*)} (z - c)^{\zeta(\sigma + v^* - 1)}$,
- (i3) ${}^{\mathcal{RC}}\mathcal{D}_c^{\zeta, v^*} (s - c)^{\zeta(\sigma - 1)}(z) = \zeta^{v^*} \frac{\Gamma(\sigma)}{\Gamma(\sigma - v^*)} (z - c)^{\zeta(\sigma - v^* - 1)}$,
- (i4) ${}^{\mathcal{RC}}\mathcal{D}_c^{\zeta, v^*} ({}^{\mathcal{RC}}\mathcal{J}_c^{\zeta, w^*} \phi)(s) = ({}^{\mathcal{RC}}\mathcal{J}_c^{\zeta, w^* - v^*} \phi)(s) (\text{Re}(v^*) < \text{Re}(w^*))$.

Lemma 2 ([46]) *Take $k - 1 < \text{Re}(v^*) < k$ and $\phi \in C_{c, \zeta}^k([c, d])$. Then, for $\zeta \in (0, 1]$, the following identity is valid:*

$${}^{\mathcal{RC}}\mathcal{J}_c^{\zeta, v^*} ({}^{CC}\mathcal{D}_c^{\zeta, v^*} \phi)(s) = \phi(s) - \sum_{j=0}^{k-1} \frac{\mathcal{D}_c^{\zeta j} \phi(c)}{\zeta^j j!} (s - c)^{j\zeta}.$$

In the light of the above lemma, one can deduce that the general solution of the homogeneous equation $({}^{CC}\mathcal{D}_c^{\zeta, v^*} \phi)(s) = 0$ is obtained as follows:

$$\phi(s) = \sum_{j=0}^{k-1} \hat{r}_j^* (s - c)^{j\zeta} = \hat{r}_0^* + \hat{r}_1^* (s - c)^\zeta + \hat{r}_2^* (s - c)^{2\zeta} + \dots + \hat{r}_{k-1}^* (s - c)^{(k-1)\zeta},$$

such that $k - 1 < \text{Re}(v^*) < k$ and $\hat{r}_0^*, \hat{r}_1^*, \dots, \hat{r}_{k-1}^* \in \mathbb{R}$.

In the sequel, we intend to devote the rest of this section to reviewing some primary definitions and key properties on the set-valued maps. To achieve this goal, we regard the normed space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$. In addition to this, we introduce the notations $\mathcal{P}(\mathcal{Y})$, $\mathcal{P}_{cls}(\mathcal{Y})$, $\mathcal{P}_{bnd}(\mathcal{Y})$, $\mathcal{P}_{cmp}(\mathcal{Y})$ and $\mathcal{P}_{cvx}(\mathcal{Y})$ for the illustration of the collection of all nonempty subsets, all closed subsets, all bounded subsets, all compact subsets and all convex subsets of \mathcal{Y} , respectively. An element $\phi^* \in \mathcal{Y}$ is defined to be a fixed point for $\tilde{\mathcal{D}} : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{Y})$ when we have $\phi^* \in \tilde{\mathcal{D}}(\phi^*)$ [57]. In this case, we illustrate the set of all fixed points of $\tilde{\mathcal{D}}$ by symbol $FIX(\tilde{\mathcal{D}})$ [57]. In the subsequent text, the Pompeiu–Hausdorff metric $\mathbb{P}\mathbb{H}_{d_{\mathcal{Y}}} : \mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Y}) \rightarrow \mathbb{R} \cup \{\infty\}$ is given by

$$\mathbb{P}\mathbb{H}_{d_{\mathcal{Y}}}(\mathcal{E}_1, \mathcal{E}_2) = \max \left\{ \sup_{b_1 \in \mathcal{E}_1} d_{\mathcal{Y}}(b_1, \mathcal{E}_2), \sup_{b_2 \in \mathcal{E}_2} d_{\mathcal{Y}}(\mathcal{E}_1, b_2) \right\}$$

so that $d_{\mathcal{Y}}(\mathcal{E}_1, b_2) = \inf_{b_1 \in \mathcal{E}_1} d_{\mathcal{Y}}(b_1, b_2)$ and $d_{\mathcal{Y}}(b_1, \mathcal{E}_2) = \inf_{b_2 \in \mathcal{E}_2} d_{\mathcal{Y}}(b_1, b_2)$ [57]. We say that $\tilde{\mathcal{D}} : \mathcal{Y} \rightarrow \mathcal{P}_{cls}(\mathcal{Y})$ is Lipschitzian furnished with real constant $\hat{c} > 0$ whenever the inequality $\mathbb{P}\mathbb{H}_{d_{\mathcal{Y}}}(\tilde{\mathcal{D}}(\phi), \tilde{\mathcal{D}}(\phi')) \leq \hat{c} d_{\mathcal{Y}}(\phi, \phi')$ is valid for each $\phi, \phi' \in \mathcal{Y}$. Notice that a Lipschitz map $\tilde{\mathcal{D}}$ is defined to be a contraction if $\hat{c} \in (0, 1)$ [57]. The multifunction $\tilde{\mathcal{D}}$ is called completely continuous if $\tilde{\mathcal{D}}(K)$ is relatively compact for any $K \in \mathcal{P}_{bnd}(\mathcal{Y})$ and also $\tilde{\mathcal{D}} : [0, 1] \rightarrow \mathcal{P}_{cls}(\mathbb{R})$ is measurable if $z \mapsto d_{\mathcal{Y}}(a, \tilde{\mathcal{D}}(z))$ is measurable for each $a \in \mathbb{R}$ [57, 58]. In addition to the above notions, we say that $\tilde{\mathcal{D}}$ possesses an upper semi-continuity specification if for each $\phi^* \in \mathcal{Y}$, the set $\tilde{\mathcal{D}}(\phi^*)$ belongs to $\mathcal{P}_{cls}(\mathcal{Y})$ and, for every open set \mathbb{V} which contains $\tilde{\mathcal{D}}(\phi^*)$, there exists a neighborhood \mathcal{U}_0^* of ϕ^* so that $\tilde{\mathcal{D}}(\mathcal{U}_0^*) \subseteq \mathbb{V}$ [57].

The graph of $\tilde{\mathcal{D}} : \mathcal{Y} \rightarrow \mathcal{P}_{cls}(\mathcal{X})$ is regarded by $\mathbb{G}\mathbb{R}(\tilde{\mathcal{D}}) = \{(\phi, x) \in \mathcal{Y} \times \mathcal{X} : x \in \tilde{\mathcal{D}}(\phi)\}$. Also, $\mathbb{G}\mathbb{R}(\tilde{\mathcal{D}})$ is called closed if for both convergent sequences $\{\phi_n\}_{n \geq 1}$ in \mathcal{Y} and $\{x_n\}_{n \geq 1}$ in \mathcal{X}

along with $\phi_n \rightarrow \phi_0, x_n \rightarrow x_0$ and $x_n \in \tilde{\mathfrak{D}}(\phi_n)$, we have $x_0 \in \tilde{\mathfrak{D}}(\phi_0)$ [57, 58]. With due attention to [57], it is concluded that, if $\tilde{\mathfrak{D}} : \mathfrak{Y} \rightarrow \mathcal{P}_{cls}(\mathfrak{X})$ is a set-valued map having the upper semi-continuity property, then $\mathbb{G}\mathbb{R}(\tilde{\mathfrak{D}})$ is a closed subset of $\mathfrak{Y} \times \mathfrak{X}$. In the opposite direction, if $\tilde{\mathfrak{D}}$ possesses the complete continuity and closed graph specifications, in this case, $\tilde{\mathfrak{D}}$ has an upper semi-continuity property [57]. Moreover, it is clear that $\tilde{\mathfrak{D}}$ is convex-valued if $\tilde{\mathfrak{D}}(\phi) \in \mathcal{P}_{cvx}(\mathfrak{Y})$ for any $\phi \in \mathfrak{Y}$. We illustrate the family of all existing selections of $\tilde{\mathfrak{D}}$ at $\phi \in \mathcal{C}_{\mathbb{R}}([0, 1])$ as

$$(\text{SEL})_{\tilde{\mathfrak{D}}, \phi} := \{ \hat{k} \in \mathcal{L}_{\mathbb{R}}^1([0, 1]) : \hat{k}(s) \in \tilde{\mathfrak{D}}(s, \phi(s)) \}$$

for each $s \in [0, 1]$ (a.e.) [57, 58]. It is necessary to pay attention to the fact that by assuming $\tilde{\mathfrak{D}}$ to be an arbitrary multi-valued function, then, for any $\phi \in \mathcal{C}_{\mathfrak{Y}}([0, 1])$, we find that $(\text{SEL})_{\tilde{\mathfrak{D}}, \phi}$ is nonempty if $\dim(\mathfrak{Y})$ is finite [57]. The multi-valued map $\tilde{\mathfrak{D}} : [0, 1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is supposed to be Carathéodory whenever $s \mapsto \tilde{\mathfrak{D}}(s, \phi)$ is measurable for every $\phi \in \mathbb{R}$ and $\phi \mapsto \tilde{\mathfrak{D}}(s, \phi)$ is upper semi-continuous for all $\phi \in [0, 1]$ (a.e.) [57, 58]. In addition, a Carathéodory map $\tilde{\mathfrak{D}} : [0, 1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is defined to be \mathcal{L}^1 -Carathéodory if for any $\gamma > 0$, a function $\varphi_{\gamma} \in \mathcal{L}_{\mathbb{R}^+}^1([0, 1])$ exists provided that $\|\tilde{\mathfrak{D}}(s, \phi)\| = \sup_{p \in \tilde{\mathfrak{D}}(s, \phi)} |p| \leq \varphi_{\gamma}(s)$ for each $|\phi| \leq \gamma$ and for almost any $s \in [0, 1]$ [57, 58].

A set \mathcal{A} is defined to be $(\mathfrak{L} \otimes \mathfrak{B})$ -measurable whenever \mathcal{A} is contained in the σ -algebra generated by all sets $\mathcal{M} \times \mathcal{Q}$ in which \mathcal{M} denotes Lebesgue measurable subset in $[0, M]$ and \mathcal{Q} stands for the Borel measurable subset of \mathbb{R} [58]. A subset \mathcal{A} of $\mathcal{L}_{\mathbb{R}}^1([0, 1])$ is supposed to be decomposable whenever for each $\phi_1, \phi_2 \in \mathcal{A}$ and $\mathcal{M} \subset [0, M]$, an inclusion $\phi_2 \chi_{\mathcal{M}} + \phi_1 \chi_{([0, M] - \mathcal{M})} \in \mathcal{A}$ holds so that χ indicates the characteristic function [58]. Now, the multifunction $\tilde{\mathfrak{D}} : \mathfrak{Y} \rightarrow \mathcal{P}_{cls}(\mathfrak{Y})$ is supposed to be lower semi-continuous (l.s.c.) if $\{ \phi \in \mathfrak{Y} : \tilde{\mathfrak{D}}(\phi) \cap B \neq \emptyset \}$ is an open set for every open subset B of \mathfrak{Y} [58]. Now, we regard \mathfrak{Y} as a separable Banach space and $\tilde{\mathfrak{D}} : \mathfrak{Y} \rightarrow \mathcal{P}(\mathcal{L}_{\mathbb{R}}^1([0, M]))$ as an arbitrary set-valued map. Then $\tilde{\mathfrak{D}}$ is an operator having (BC)-property if $\tilde{\mathfrak{D}}$ is lower semi-continuous and it possesses nonempty closed and decomposable values [58]. For $\tilde{\mathfrak{D}} : [0, M] \times \mathbb{R} \rightarrow \mathcal{P}_{cmp}(\mathbb{R})$, we assign a set-valued operator $\mathfrak{N} : \mathfrak{Y} \rightarrow \mathcal{P}(\mathcal{L}_{\mathbb{R}}^1([0, M]))$ by taking

$$\mathfrak{N}(x) = \{ \phi \in \mathcal{L}_{\mathbb{R}}^1([0, M]) : \phi(s) \in \tilde{\mathfrak{D}}(s, \phi(s)) \text{ for a.e. } s \in [0, M] \}.$$

Then \mathfrak{N} is said to be a Niemytzki operator associated with $\tilde{\mathfrak{D}}$ [58]. Moreover, $\tilde{\mathfrak{D}} : [0, M] \times \mathbb{R} \rightarrow \mathcal{P}_{cmp}(\mathbb{R})$ is said to be of lower semi-continuous type (l.s.c. type) whenever its relevant Niemytzki operator \mathfrak{N} is lower semi-continuous and involves nonempty closed decomposable values [58]. The next theorems are regarded as our required tools for verifying desired results in the current research.

Theorem 3 (Bohnenblust–Karlin theorem, [59]) *Regard \mathfrak{Y} as a Banach space and $\mathbb{E} \neq \emptyset$ as a subset contained in \mathfrak{Y} which is convex, bounded and closed. Assume that $\tilde{\mathfrak{D}} : \mathbb{E} \rightarrow \mathcal{P}_{cls,cmp}(\mathfrak{Y})$ is upper semi-continuous provided that $\tilde{\mathfrak{D}}(\mathbb{E}) \subset \mathbb{E}$ and $\tilde{\mathfrak{D}}(\mathbb{E})$ is compact. Then $\tilde{\mathfrak{D}}$ possesses a fixed point.*

Theorem 4 (Closed graph theorem, [60]) *Regard \mathfrak{Y} as a separable Banach space, $\tilde{\mathfrak{D}} : [0, 1] \times \mathfrak{Y} \rightarrow \mathcal{P}_{cmp,cvx}(\mathfrak{Y})$ as an \mathcal{L}^1 -Carathéodory multifunction and $\Upsilon^* : \mathcal{L}_{\mathfrak{Y}}^1([0, 1]) \rightarrow \mathcal{C}_{\mathfrak{Y}}([0, 1])$ as a linear continuous map. In this phase, $\Upsilon^* \circ (\text{SEL})_{\tilde{\mathfrak{D}}} : \mathcal{C}_{\mathfrak{Y}}([0, 1]) \rightarrow$*

$\mathcal{P}_{cmp,cvx}(\mathcal{C}_{\mathfrak{N}}([0, 1]))$ is another operator in $\mathcal{C}_{\mathfrak{N}}([0, 1]) \times \mathcal{C}_{\mathfrak{N}}([0, 1])$ by $\phi \mapsto (\Upsilon^* \circ (\text{SEL})_{\tilde{\mathfrak{D}}})(\phi) = \Upsilon^*((\text{SEL})_{\tilde{\mathfrak{D}},\phi})$ involving the closed graph specification.

Theorem 5 (Martelli theorem, [61]) *The space \mathfrak{N} is supposed to be Banach space and the set-valued map $\tilde{\mathfrak{D}} : \mathbb{E} \rightarrow \mathcal{P}_{bnd,cls,cvx}(\mathfrak{N})$ is assumed to be completely continuous. If the set $\Delta = \{\phi \in \mathfrak{N} : \eta\phi \in \tilde{\mathfrak{D}}(\phi), \eta > 1\}$ is bounded, then a fixed point exists for $\tilde{\mathfrak{D}}$.*

Theorem 6 (Nonlinear alternative theorem for Kakutani mappings, [62]) *Regard \mathfrak{N} as a Banach space and $\mathbb{E} \neq \emptyset$ as a subset contained in \mathfrak{N} which is convex and closed. Also, let \mathfrak{U} be an open subset contained in \mathbb{E} and $0 \in \mathfrak{U}$. By assuming $\tilde{\mathfrak{D}} : \bar{\mathfrak{U}} \rightarrow \mathcal{P}_{cmp,cvx}(\mathbb{E})$ as a compact and upper semi-continuous mapping, we have*

- (i) *a fixed point exists for $\tilde{\mathfrak{D}}$ in $\bar{\mathfrak{U}}$, or*
- (ii) *$v^* \in \text{partial}\mathfrak{U}$ and $\eta \in (0, 1)$ exist for which $v^* \in \eta\tilde{\mathfrak{D}}(v^*)$.*

Theorem 7 (Bressan and Colombo theorem, [63]) *The Banach space \mathfrak{N} is supposed to be separable and $\tilde{\mathfrak{D}} : \mathfrak{N} \rightarrow \mathcal{P}(\mathcal{L}^1_{\mathbb{R}}([0, M]))$ is a set-valued map having (BC)-property. Then $\tilde{\mathfrak{D}}$ possesses a continuous selection, i.e. a continuous map $\check{g} : \mathfrak{N} \rightarrow \mathcal{L}^1_{\mathbb{R}}([0, M])$ exists provided that $\check{g}(y) \in \tilde{\mathfrak{D}}(y)$ for any $y \in \mathfrak{N}$.*

Theorem 8 (Covitz and Nadler theorem, [64]) *Regard \mathfrak{N} as a Banach space. If $\tilde{\mathfrak{D}} : \mathfrak{N} \rightarrow \mathcal{P}_{cls}(\mathbb{E})$ is a contraction, then $\text{FIX}(\tilde{\mathfrak{D}})$ is a nonempty set.*

3 Main results

After reviewing and introducing some auxiliary concepts in previous sections, we proceed to deduce desired existence theorems. To arrive at this goal, we regard the norm $\|\phi\|_{\mathfrak{N}} = \sup_{s \in [c, M]} |\phi(s)|$ on the space $\mathfrak{N} = \{\phi(s) : \phi(s) \in \mathcal{C}_{\mathbb{R}}([c, M])\}$. Then $(\mathfrak{N}, \|\cdot\|_{\mathfrak{N}})$ is a Banach space. Besides, keep in mind the following for convenience in the computations:

$$\tilde{\Omega} = \mu_1^*(M - c)^{\zeta} + \mu_2^* \frac{(\sigma - c)^{\zeta(1+\theta^*)}}{\zeta^{\theta^*} \Gamma(2 + \theta^*)} \neq 0, \tag{2}$$

$$\tilde{X} = \frac{(M - c)^{\zeta v^*}}{\zeta^{v^*} \Gamma(1 + v^*)} + \frac{(M - c)^{\zeta}}{|\tilde{\Omega}|} \left[|\mu_1^*| \frac{(M - c)^{\zeta v^*}}{\zeta^{v^*} \Gamma(1 + v^*)} + |\mu_2^*| \frac{(\sigma - c)^{\zeta(v^*+\theta^*)}}{\zeta^{(v^*+\theta^*)} \Gamma(1 + v^* + \theta^*)} \right]. \tag{3}$$

In the next result, we derive an integral construction for the solution of the proposed three-point Caputo conformable pantograph BVP (1).

Lemma 9 *Regard $\check{h} \in \mathfrak{N}$. In this phase, ϕ_0 is regarded as a solution for the fractional linear differential equation in the Caputo conformable settings*

$${}^{CC}\mathcal{D}_c^{\zeta, v^*} \phi(s) = \check{h}(s), \quad (s \in [c, M], c \geq 0) \tag{4}$$

subject to three-point Riemann–Liouville conformable integral boundary conditions

$$\phi(c) = 0, \quad \mu_1^* \phi(M) + \mu_2^* {}^{RC}\mathcal{I}_c^{\zeta, \theta^*} \phi(\sigma) = \xi^*, \tag{5}$$

iff ϕ_0 satisfies integral equation

$$\phi(s) = \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s - c)^{\zeta} - (q - c)^{\zeta}}{\zeta} \right)^{v^*-1} \check{h}(q) \frac{dq}{(q - c)^{1-\zeta}}$$

$$\begin{aligned}
 & + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{h}(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\
 & \left. - \frac{\mu_2^*}{\Gamma(\nu^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} \check{h}(q) \frac{dq}{(q-c)^{1-\zeta}} \right], \tag{6}
 \end{aligned}$$

where a nonzero constant $\tilde{\Omega}$ is defined by (2).

Proof First, we regard ϕ_0 as a function which satisfies the Caputo conformable equation (4). Then we see that ${}^{CC}\mathcal{D}_c^{\zeta, \nu^*} \phi_0(s) = \check{h}(s)$. Now, we integrate both sides of the latter equation in the ν^* -th order Riemann–Liouville conformable settings. We have

$$\phi_0(s) = \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{h}(q) \frac{dq}{(q-c)^{1-\zeta}} + \hat{r}_0^* + \hat{r}_1^* (s-c)^\zeta \tag{7}$$

so that we wish to find constant coefficients $\hat{r}_0^*, \hat{r}_1^* \in \mathbb{R}$. Prior to seeking these constants, by taking the integral of the Riemann–Liouville conformable type with respect to s on both sides of (7), we obtain

$$\begin{aligned}
 {}^{\mathcal{R}C}\mathcal{J}_c^{\zeta, \theta^*} \phi_0(s) & = \frac{1}{\Gamma(\nu^* + \theta^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} \check{h}(q) \frac{dq}{(q-c)^{1-\zeta}} \\
 & + \hat{r}_0^* \frac{(s-c)^{\zeta \theta^*}}{\zeta^{\theta^*} \Gamma(1 + \theta^*)} + \hat{r}_1^* \frac{(s-c)^{\zeta(1 + \theta^*)}}{\zeta^{\theta^*} \Gamma(2 + \theta^*)}.
 \end{aligned}$$

The first boundary condition causes \hat{r}_0^* to be zero. Now, according to the second integral boundary condition, we get

$$\begin{aligned}
 \hat{r}_1^* & = \frac{1}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{h}(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\
 & \left. - \frac{\mu_2^*}{\Gamma(\nu^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} \check{h}(q) \frac{dq}{(q-c)^{1-\zeta}} \right].
 \end{aligned}$$

By inserting the obtained values \hat{r}_0^* and \hat{r}_1^* into (7), we obtain

$$\begin{aligned}
 \phi_0(s) & = \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{h}(q) \frac{dq}{(q-c)^{1-\zeta}} \\
 & + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{h}(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\
 & \left. - \frac{\mu_2^*}{\Gamma(\nu^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} \check{h}(q) \frac{dq}{(q-c)^{1-\zeta}} \right],
 \end{aligned}$$

indicating that ϕ_0 satisfies (6). In the reverse direction, we can simply verify that ϕ_0 satisfies the given three-point Caputo conformable problem (4)–(5) whenever ϕ_0 satisfies the integral equation (6). \square

In this position, we deal with several existence criteria for the proposed pantograph fractional BVP (1) in the Caputo conformable settings. With due attention to Lemma 9, a function $\phi \in \mathcal{AC}_{\mathbb{R}}^2([c, M])$ is regarded as a solution for the pantograph inclusion problem

(1) in the Caputo conformable frame if ϕ satisfies the given boundary conditions (1) and also a function $\check{g} \in \mathcal{L}^1_{\mathbb{R}}([c, M])$ exists with $\check{g} \in \tilde{\mathcal{D}}(s, \phi(s), \phi(\lambda^*s))$ for any (a.e.) $s \in [c, M]$ and

$$\begin{aligned} \phi(s) = & \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \\ & + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\ & \left. - \frac{\mu_2^*}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*+\theta^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right]. \end{aligned} \tag{8}$$

3.1 The upper semi-continuity case

Here, we assume that values of the set-valued map $\tilde{\mathcal{D}}$ belong to $\mathcal{P}_{cvx}(\mathfrak{Y})$. The first existence criterion is derived due to both Bohnenblust–Karlin’s theorem, Theorem 3, and the closed graph theorem, Theorem 4.

Theorem 10 *Let the following be valid:*

- (HP₁) $\tilde{\mathcal{D}} : [c, M] \times \mathfrak{Y} \times \mathfrak{Y} \rightarrow \mathcal{P}_{cmp, cvx}(\mathfrak{Y})$ is Carathéodory.
- (HP₂) For each $\mu > 0$, a function $\varphi_\mu \in \mathcal{L}^1_{\mathbb{R}^+}([c, M])$ exists provided that

$$\| \tilde{\mathcal{D}}(s, \phi, \bar{\phi}) \| = \sup \{ |\check{g}| : \check{g} \in \tilde{\mathcal{D}}(s, \phi, \bar{\phi}) \} \leq \varphi_\mu(s)$$

for any $\phi, \bar{\phi} \in \mathfrak{Y}$ with $\|\phi\|_{\mathfrak{Y}}, \|\bar{\phi}\|_{\mathfrak{Y}} \leq \mu$ and for a.e. $s \in [c, M]$.

Then at least one solution exists on $[c, M]$ for three-point Caputo conformable pantograph BVP (1) if

$$\begin{aligned} \Theta \left\{ \frac{(M-c)^{\zeta(v^*-1)}}{\zeta^{(v^*-1)}\Gamma(v^*)} + \frac{(M-c)^\zeta}{|\tilde{\Omega}|} \left[|\mu_1^*| \frac{(M-c)^{\zeta(v^*-1)}}{\zeta^{(v^*-1)}\Gamma(v^*)} \right. \right. \\ \left. \left. + |\mu_2^*| \frac{(\sigma-c)^{\zeta(v^*+\theta^*-1)}}{\zeta^{(v^*+\theta^*-1)}\Gamma(v^*+\theta^*)} \right] \right\} < 1, \end{aligned} \tag{9}$$

where $\liminf_{\mu \rightarrow \infty} \int_c^M \frac{\varphi_\mu(q)}{\mu} dq = \Theta < \infty$.

Proof To transform the given Caputo conformable pantograph BVP (1) into a well-known fixed point problem, we regard a multifunction $\Psi : \mathfrak{Y} \rightarrow \mathcal{P}(\mathfrak{Y})$ formulated by

$$\Psi(\phi) = \begin{cases} \psi \in \mathfrak{Y} : \\ \psi(s) = \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \\ \quad + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \right. \\ \quad \times \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} - \frac{\mu_2^*}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*+\theta^*-1} \\ \quad \times \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \left. \right] \quad (\check{g} \in \mathbb{S}EL_{\tilde{\mathcal{D}}, \phi}). \end{cases} \tag{10}$$

We claim that Ψ satisfies all existing hypotheses of Theorem 3 and so Ψ possesses a fixed point which is regarded as a solution function for the proposed Caputo conformable pantograph BVP (1). In the first stage, we are going to check the convexity of $\Psi(\phi)$ for each

$\phi \in \mathfrak{N}$. For this purpose, let $\psi_1, \psi_2 \in \Psi(\phi)$. Then there are two functions $\check{g}_1, \check{g}_2 \in \text{SEL}_{\check{\mathfrak{D}}, \phi}$ so that, for any $s \in [c, M]$, we get

$$\begin{aligned} \psi_i(s) &= \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}_i(q) \frac{dq}{(q-c)^{1-\zeta}} \\ &\quad + \frac{(s-c)^\zeta}{\check{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}_i(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\ &\quad \left. - \frac{\mu_2^*}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*+\theta^*-1} \check{g}_i(q) \frac{dq}{(q-c)^{1-\zeta}} \right] \quad (i = 1, 2). \end{aligned}$$

Take $0 \leq \kappa \leq 1$. In this phase, for any $s \in [c, M]$, one may write

$$\begin{aligned} &[\kappa\psi_1 + (1-\kappa)\psi_2](s) \\ &= \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} [\kappa\check{g}_1(q) + (1-\kappa)\check{g}_2(q)] \frac{dq}{(q-c)^{1-\zeta}} \\ &\quad + \frac{(s-c)^\zeta}{\check{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \right. \\ &\quad \times [\kappa\check{g}_1(q) + (1-\kappa)\check{g}_2(q)] \frac{dq}{(q-c)^{1-\zeta}} - \frac{\mu_2^*}{\Gamma(v^* + \theta^*)} \\ &\quad \left. \times \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*+\theta^*-1} [\kappa\check{g}_1(q) + (1-\kappa)\check{g}_2(q)] \frac{dq}{(q-c)^{1-\zeta}} \right]. \end{aligned}$$

As $\text{SEL}_{\check{\mathfrak{D}}, \phi}$ is convex ($\check{\mathfrak{D}}$ is convex-valued), so it is deduced that $[\kappa\psi_1 + (1-\kappa)\psi_2] \in \Psi(\phi)$. Next, we verify that Ψ is a bounded operator on \mathcal{B}_μ , where $\mathcal{B}_\mu = \{\phi \in \mathfrak{N} : \|\phi\|_{\mathfrak{N}} \leq \mu\}$ for every constant $\mu > 0$. Obviously, \mathcal{B}_μ is a convex bounded and closed set belonging to \mathfrak{N} . We claim that $\mu \in \mathbb{R}^+$ exists so that $\Psi(\mathcal{B}_\mu) \subseteq \mathcal{B}_\mu$. To confirm this claim, we assume that, for any $\mu \in \mathbb{R}^+$, there is a function $\phi_\mu \in \mathcal{B}_\mu$ and $\psi_\mu \in \Psi(\phi_\mu)$ with $\|\Psi(\phi_\mu)\|_{\mathfrak{N}} > \mu$ and

$$\begin{aligned} \psi_\mu(s) &= \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}_\mu(q) \frac{dq}{(q-c)^{1-\zeta}} \\ &\quad + \frac{(s-c)^\zeta}{\check{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}_\mu(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\ &\quad \left. - \frac{\mu_2^*}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*+\theta^*-1} \check{g}_\mu(q) \frac{dq}{(q-c)^{1-\zeta}} \right] \end{aligned}$$

for $\check{g}_\mu \in \text{SEL}_{\check{\mathfrak{D}}, \phi}$. Then, for any $s \in [c, M]$, we get

$$\begin{aligned} |\Psi(\phi_\mu)(s)| &\leq \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} |\check{g}_\mu(q)| \frac{dq}{(q-c)^{1-\zeta}} \\ &\quad + \frac{(s-c)^\zeta}{|\check{\Omega}|} \left[|\xi^*| + \frac{|\mu_1^*|}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} |\check{g}_\mu(q)| \frac{dq}{(q-c)^{1-\zeta}} \right. \\ &\quad \left. + \frac{|\mu_2^*|}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*+\theta^*-1} |\check{g}_\mu(q)| \frac{dq}{(q-c)^{1-\zeta}} \right]. \end{aligned}$$

In view of hypothesis (HP_2) and taking the supremum, we obtain

$$\begin{aligned} \mu < \|\Omega(\phi_\mu)\|_{\mathfrak{B}} &\leq \frac{(M-c)^\zeta}{|\tilde{\Omega}|} |\xi^*| + \left\{ \frac{(M-c)^{\zeta(v^*-1)}}{\zeta^{(v^*-1)}\Gamma(v^*)} \right. \\ &\quad \left. + \frac{(M-c)^\zeta}{|\tilde{\Omega}|} \left[|\mu_1^*| \frac{(M-c)^{\zeta(v^*-1)}}{\zeta^{(v^*-1)}\Gamma(v^*)} + |\mu_2^*| \frac{(\sigma-c)^{\zeta(v^*+\theta^*-1)}}{\zeta^{(v^*+\theta^*-1)}\Gamma(v^*+\theta^*)} \right] \right\} \\ &\quad \times \int_c^M \varphi_\mu(rq) \, dq. \end{aligned} \tag{11}$$

In the following, we multiply both sides of (11) by $1/\mu$ and take the lower limit of it when μ goes to infinity. Then we find that

$$1 \leq \Theta \left\{ \frac{(M-c)^{\zeta(v^*-1)}}{\zeta^{(v^*-1)}\Gamma(v^*)} + \frac{(M-c)^\zeta}{|\tilde{\Omega}|} \left[|\mu_1^*| \frac{(M-c)^{\zeta(v^*-1)}}{\zeta^{(v^*-1)}\Gamma(v^*)} + |\mu_2^*| \frac{(\sigma-c)^{\zeta(v^*+\theta^*-1)}}{\zeta^{(v^*+\theta^*-1)}\Gamma(v^*+\theta^*)} \right] \right\},$$

and this is a contradiction by considering the condition (9). Therefore there is $\mu \in \mathbb{R}^+$ provided that $\Psi(\mathcal{B}_\mu) \subseteq \mathcal{B}_\mu$. This means that Ψ is a set-valued map from \mathcal{B}_μ to \mathcal{B}_μ .

In the sequel, we check that $\Psi(\phi)$ is equi-continuous. Let ϕ be arbitrary member belonging to \mathcal{B}_μ and $\psi \in \Psi(\phi)$. In this case, there exists $\check{g} \in \mathbb{S}EL_{\tilde{\Omega},\phi}$ so that, for each $s \in [c, M]$, we have

$$\begin{aligned} \psi(s) &= \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \\ &\quad + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\ &\quad \left. - \frac{\mu_2^*}{\Gamma(v^*+\theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*+\theta^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right]. \end{aligned}$$

Therefore for any $s', s'' \in [c, M]$ with $s' < s''$, we get

$$\begin{aligned} &|\psi(s'') - \psi(s')| \\ &\leq \left| \frac{1}{\Gamma(v^*)} \int_c^{s''} \left(\frac{(s''-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\ &\quad \left. - \frac{1}{\Gamma(v^*)} \int_c^{s'} \left(\frac{(s'-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right| \\ &\quad + \frac{(s''-c)^\zeta - (s'-c)^\zeta}{|\tilde{\Omega}|} \left[|\xi^*| + \frac{|\mu_1^*|}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \right. \\ &\quad \left. \times |\check{g}(q)| \frac{dq}{(q-c)^{1-\zeta}} \right] \\ &\leq \left| \frac{1}{\Gamma(v^*)} \int_c^{s'} \left\{ \left(\frac{(s''-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} - \left(\frac{(s'-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \right\} \right. \\ &\quad \left. \times \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right| + \left| \frac{1}{\Gamma(v^*)} \int_{s'}^{s''} \left(\frac{(s''-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right| \\ &\quad + \frac{(s''-c)^\zeta - (s'-c)^\zeta}{|\tilde{\Omega}|} \left[|\xi^*| + \frac{|\mu_1^*|}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \right. \end{aligned}$$

$$\times \left[\check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right].$$

As $s' \rightarrow s''$, we realize that the RHS of the latter inequality approaches 0 without any dependence to $\phi \in \mathcal{B}_\mu$. This points to the fact that Ψ is equi-continuous. By virtue of the well-known Ascoli–Arzelá theorem, we deduce that the set-valued map Ψ possesses a complete continuity specification.

Eventually, we verify that Ψ possesses a closed graph. To reach this goal, let $\phi_n \rightarrow \phi_*$, $\psi_n \in \Psi(\phi_n)$ and $\psi_n \rightarrow \psi_*$. Also, choose $\check{g}_n \in \mathbb{S}\mathbb{E}\mathbb{L}_{\tilde{\Omega}, \phi_n}$. Our aim is to prove $\psi_* \in \Psi(\phi_*)$. Hence, for each $s \in [c, M]$, we have

$$\begin{aligned} \psi_n(s) &= \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{g}_n(q) \frac{dq}{(q-c)^{1-\zeta}} \\ &\quad + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{g}_n(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\ &\quad \left. - \frac{\mu_2^*}{\Gamma(\nu^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} \check{g}_n(q) \frac{dq}{(q-c)^{1-\zeta}} \right]. \end{aligned}$$

In this case, we want to prove that a function $\check{g}_* \in \mathbb{S}\mathbb{E}\mathbb{L}_{\tilde{\Omega}, \phi_*}$ exists so that, for each $s \in [c, M]$,

$$\begin{aligned} \psi_*(s) &= \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{g}_*(q) \frac{dq}{(q-c)^{1-\zeta}} \\ &\quad + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{g}_*(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\ &\quad \left. - \frac{\mu_2^*}{\Gamma(\nu^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} \check{g}_*(q) \frac{dq}{(q-c)^{1-\zeta}} \right]. \end{aligned}$$

To achieve this purpose, we define a new continuous linear operator $\Upsilon^* : \mathcal{L}_{\mathbb{R}^+}^1([c, M]) \rightarrow \mathfrak{Y}$ illustrated by

$$\begin{aligned} \check{g} \mapsto \Upsilon^*(\check{g})(s) &= \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \\ &\quad + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\ &\quad \left. - \frac{\mu_2^*}{\Gamma(\nu^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right]. \end{aligned}$$

It is evident that $\|\psi_n - \psi_*\|_{\mathfrak{Y}} \rightarrow 0$ as $n \rightarrow \infty$. So in the light of Theorem 4, we realize that $\Upsilon^* \circ \mathbb{S}\mathbb{E}\mathbb{L}_{\tilde{\Omega}, \phi}$ is a closed graph operator. Furthermore, $\psi_n(s) \in \Upsilon^*(\mathbb{S}\mathbb{E}\mathbb{L}_{\tilde{\Omega}, \phi_n})$. As $\phi_n \rightarrow \phi_*$, Theorem 4 yields

$$\begin{aligned} \psi_*(s) &= \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{g}_*(q) \frac{dq}{(q-c)^{1-\zeta}} \\ &\quad + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} \check{g}_*(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \end{aligned}$$

$$- \frac{\mu_2^*}{\Gamma(\nu^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma - c)^\zeta - (q - c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} \check{g}_*(q) \frac{dq}{(q - c)^{1-\zeta}} \Big]$$

for some $\check{g}_* \in \text{SEL}_{\tilde{\mathcal{D}}, \phi_*}$. Consequently, we realize that Ψ is a compact and upper semi-continuous multifunction furnished with closed and convex values. Hence, by considering Theorem 3, we realize that Ψ possesses a fixed point, which is the same solution as for the proposed three-point Caputo conformable pantograph inclusion problem (1). This completes the proof. \square

Our second criterion is derived with the help of Martelli’s fixed point result given by Theorem 5.

Theorem 11 *Let the following be valid:*

(HP₃) $\tilde{\mathcal{D}} : [c, M] \times \mathfrak{Y} \times \mathfrak{Y} \rightarrow \mathcal{P}_{\text{bnd,cls,conv}}(\mathfrak{Y})$ is Carathéodory;

(HP₄) a function $\chi \in \mathfrak{Y}$ exists so that $\|\tilde{\mathcal{D}}(s, \phi, \bar{\phi})\|_{\mathfrak{Y}} \leq \chi(s)$ for all a.e. $s \in [c, M]$ and each $\phi, \bar{\phi} \in \mathfrak{Y}$.

Then the three-point Caputo conformable pantograph inclusion problem (1) possesses at least one solution on $[c, M]$.

Proof Let us regard Ψ as given in Theorem 10. Then, in a similar manner, we can simply confirm the convexity and the complete continuity of the operator Ψ . Thus, it just remains to check the boundedness of the set $\Delta = \{\eta\phi \in \Psi(\phi), \eta > 1\}$. To investigate this, let $\phi \in \Delta$. Hence $\eta\phi \in \Psi(\phi)$ for some $\eta > 1$ and a function $\check{g} \in \text{SEL}_{\tilde{\mathcal{D}}, \phi}$ exists provided that

$$\begin{aligned} \eta\phi(s) = & \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s - c)^\zeta - (q - c)^\zeta}{\zeta} \right)^{\nu^* - 1} \check{g}(q) \frac{dq}{(q - c)^{1-\zeta}} \\ & + \frac{(s - c)^\zeta}{|\tilde{\Omega}|} \left[\xi^* - \frac{\mu_1^*}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M - c)^\zeta - (q - c)^\zeta}{\zeta} \right)^{\nu^* - 1} \check{g}(q) \frac{dq}{(q - c)^{1-\zeta}} \right. \\ & \left. - \frac{\mu_2^*}{\Gamma(\nu^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma - c)^\zeta - (q - c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} \check{g}(q) \frac{dq}{(q - c)^{1-\zeta}} \right]. \end{aligned}$$

Hence, by considering (HP₄) and for any $s \in [c, M]$ and some $\eta > 1$, we have

$$\begin{aligned} |\phi(s)| \leq & \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s - c)^\zeta - (q - c)^\zeta}{\zeta} \right)^{\nu^* - 1} |\check{g}(q)| \frac{dq}{(q - c)^{1-\zeta}} \\ & + \frac{(s - c)^\zeta}{|\tilde{\Omega}|} \left[|\xi^*| + \frac{|\mu_1^*|}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M - c)^\zeta - (q - c)^\zeta}{\zeta} \right)^{\nu^* - 1} |\check{g}(q)| \frac{dq}{(q - c)^{1-\zeta}} \right. \\ & \left. + \frac{|\mu_2^*|}{\Gamma(\nu^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma - c)^\zeta - (q - c)^\zeta}{\zeta} \right)^{\nu^* + \theta^* - 1} |\check{g}(q)| \frac{dq}{(q - c)^{1-\zeta}} \right] \end{aligned}$$

and so we obtain

$$\begin{aligned} \|\phi\|_{\mathfrak{Y}} \leq & \left\{ \frac{(M - c)^{\zeta\nu^*}}{\zeta^{\nu^*} \Gamma(1 + \nu^*)} + \frac{(M - c)^\zeta}{|\tilde{\Omega}|} \left[|\mu_1^*| \frac{(M - c)^{\zeta\nu^*}}{\zeta^{\nu^*} \Gamma(1 + \nu^*)} \right. \right. \\ & \left. \left. + |\mu_2^*| \frac{(\sigma - c)^{\zeta(\nu^* + \theta^*)}}{\zeta^{(\nu^* + \theta^*)} \Gamma(1 + \nu^* + \theta^*)} \right] \right\} \|\chi\|_{\mathfrak{Y}} + \frac{(M - c)^\zeta}{|\tilde{\Omega}|} |\xi^*| < \infty. \end{aligned}$$

Thus we find the set Δ is bounded. Finally, with due attention to Theorem 5, we conclude that Ψ possesses at least one fixed point which is regarded as a solution for the proposed three-point Caputo conformable pantograph inclusion problem (1) on $[c, M]$. \square

The next criterion in this regard is obtained by the nonlinear alternative theorem about Kakutani mappings (Theorem 6).

Theorem 12 *Suppose that the hypothesis (HP₁) is valid. Further, assume that both following hypotheses are valid too:*

(HP₅) *there are a nondecreasing continuous function $\Phi_1 : [0, \infty) \rightarrow (0, \infty)$ and a continuous function $\Phi_2 \in C_{\mathbb{R}^+}([c, M])$ provided that*

$$\|\tilde{\mathcal{D}}(s, \phi, \bar{\phi})\| = \sup\{|\check{g}| : \check{g} \in \tilde{\mathcal{D}}(s, \phi, \bar{\phi})\} \leq \Phi_2(s)\Phi_1(|\phi|)$$

for each $(s, \phi, \bar{\phi}) \in [c, M] \times \mathfrak{Y} \times \mathfrak{Y}$;

(HP₆) *a number $M \in \mathbb{R}^+$ exists provided that*

$$\frac{M}{\|\Phi_2\|_{\mathfrak{Y}} \Phi_1(M)\tilde{X} + \frac{(M-c)^\zeta}{|\tilde{\Omega}|} |\xi^*|} > 1,$$

where \tilde{X} is given in (3) and $\|\Phi_2\|_{\mathfrak{Y}} = \sup_{s \in [c, M]} |\Phi_2(s)|$.

Then a solution exists on the interval $[c, M]$ for the proposed three-point Caputo conformable pantograph inclusion problem (1).

Proof Let $\phi \in \eta\Psi(\phi)$ for some $\eta \in (0, 1)$, where Ψ is the same operator considered in the proof of Theorem 10. Our intention is to show that an open set $\mathfrak{U} \in C_{\mathbb{R}}([c, M])$ exists with $\phi \notin \eta\Psi(\phi)$ for each $\eta \in (0, 1)$ and all $\phi \in \partial\mathfrak{U}$. To check this, we assume that $\eta \in (0, 1)$ and $\phi \in \eta\Psi(\phi)$. Then there is a function $\check{g} \in \mathcal{L}_{\mathbb{R}}^1([c, M])$ with $\check{g} \in \text{SEL}_{\tilde{\mathcal{D}}, \phi}$ so that

$$\begin{aligned} \phi(s) = & \frac{\eta}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \\ & + \frac{\eta(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\ & \left. - \frac{\eta\mu_2^*}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^* + \theta^* - 1} \check{g}(q) \frac{dq}{(q-c)^{1-\zeta}} \right]. \end{aligned}$$

According to hypothesis (HP₅), for every $s \in [c, M]$ and some $\eta \in (0, 1)$, we may write

$$\begin{aligned} \|\phi\|_{\mathfrak{Y}} \leq & \frac{(M-c)^\zeta}{|\tilde{\Omega}|} |\xi^*| + \|\Phi_2\|_{\mathfrak{Y}} \Phi_1(\|\phi\|_{\mathfrak{Y}}) \times \left\{ \frac{(M-c)^\zeta v^*}{\zeta^{v^*} \Gamma(1+v^*)} \right. \\ & \left. + \frac{(M-c)^\zeta}{|\tilde{\Omega}|} \left[|\mu_1^*| \frac{(M-c)^\zeta v^*}{\zeta^{v^*} \Gamma(1+v^*)} + |\mu_2^*| \frac{(\sigma-c)^\zeta (v^* + \theta^*)}{\zeta^{(v^* + \theta^*)} \Gamma(1+v^* + \theta^*)} \right] \right\}. \end{aligned}$$

This yields

$$\frac{\|\phi\|_{\mathfrak{Y}}}{\|\Phi_2\|_{\mathfrak{Y}} \Phi_1(\|\phi\|_{\mathfrak{Y}})\tilde{X} + \frac{(M-c)^\zeta}{|\tilde{\Omega}|} |\xi^*|} \leq 1.$$

With due attention to condition (HP_6) , there is a number \mathbb{M} so that $\mathbb{M} \neq \|\phi\|_{\mathfrak{Y}}$. Let us assume

$$\mathfrak{U} = \{ \phi \in C_{\mathbb{R}}([c, M]) : \|\phi\|_{\mathfrak{Y}} < \mathbb{M} \}.$$

By proceeding similar to the proof of Theorem 10, it is easily verified that $\Psi : \bar{\mathfrak{U}} \rightarrow \mathcal{P}(\mathfrak{Y})$ is a compact and upper semi-continuous multifunction having closed and convex values. So we observe that there exists no $\phi \in \partial\mathfrak{U}$ so that $\phi \in \eta\Psi(\phi)$ for some $\eta \in (0, 1)$ in view of the choice of \mathfrak{U} . Hence, by Theorem 6 one concludes that Ψ is a multifunction including a fixed point $\phi \in \mathfrak{U}$ and eventually we find that the proposed three-point Caputo conformable pantograph inclusion BVP (1) involves a solution on $[c, M]$. \square

3.2 The lower semi-continuity case

In the current position, we derive other existence criterion in the lower semi-continuous phase. Here, the set-valued map $\tilde{\mathfrak{D}}$ has not necessarily convex values. We discuss the next result by applying nonlinear alternative of Leray–Schauder along with the selection result due to Colombo and Bressan (Theorem 7) for all lower semi-continuous mappings having decomposable values.

Theorem 13 *Let the hypotheses (HP_5) and (HP_6) along with the following condition be valid:*

(HP_7) the nonempty set-valued map $\tilde{\mathfrak{D}} : [c, M] \times \mathfrak{Y} \times \mathfrak{Y} \rightarrow \mathcal{P}_{comp}(\mathfrak{Y})$ is supposed to be compact-valued such that $(s, \phi, \bar{\phi}) \mapsto \tilde{\mathfrak{D}}(s, \phi, \bar{\phi})$ is $(\mathfrak{L} \otimes \mathfrak{B} \otimes \mathfrak{B})$ -measurable and $\phi \mapsto \tilde{\mathfrak{D}}(s, \phi, \bar{\phi})$ is lower semi-continuous for any $s \in [c, M]$.

In this case, at least one solution exists on $[c, M]$ for the proposed three-point Caputo conformable pantograph inclusion problem (1).

Proof From both conditions (HP_5) and (HP_7) , we immediately deduce that $\tilde{\mathfrak{D}}$ is of lower semi-continuous type. In this case, the selection result attributed to Colombo and Bressan (Theorem 7) implies that a continuous function $y : \mathcal{AC}_{\mathbb{R}}^1([c, M]) \rightarrow \mathcal{L}_{\mathbb{R}}^1([c, M])$ exists so that $y(\phi) \in \tilde{\mathfrak{N}}(\phi)$ for each element $y \in C_{\mathbb{R}}([c, M])$, where $\tilde{\mathfrak{N}}(\phi) : C_{\mathbb{R}}([c, M]) \rightarrow \mathcal{L}_{\mathbb{R}}^1([c, M])$ stands for the Nemytskii operator associated with $\tilde{\mathfrak{D}}$ given by

$$\tilde{\mathfrak{N}}(\phi) = \{ \check{g} \in \mathcal{L}_{\mathbb{R}}^1([c, M]) : \check{g} \in \tilde{\mathfrak{D}}(s, \phi(s), \bar{\phi}(\lambda^*s)) \text{ for a.e. } s \in [c, M] \}.$$

In this moment, we regard the following reformulated BVP:

$$\begin{cases} {}^{CC}\mathfrak{D}_c^{\zeta, \nu^*} \phi(s) = y(\phi(s)) & (s \in [c, M], c \geq 0), \\ \phi(c) = 0, \quad \mu_1^* \phi(M) + \mu_2^* {}^{\mathcal{RC}}\mathfrak{I}_c^{\zeta, \theta^*} \phi(\sigma) = \xi^*, & \sigma \in (c, M). \end{cases} \tag{12}$$

Notice that, if $\phi \in \mathcal{AC}_{\mathbb{R}}^2([c, M])$ is regarded as a solution of problem (12), then ϕ will be as a solution of main inclusion problem (1). Define an operator $\bar{\Psi}$ as follows:

$$\begin{aligned} \bar{\Psi}(\phi) = & \frac{1}{\Gamma(\nu^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} y(\phi(q)) \frac{dq}{(q-c)^{1-\zeta}} \\ & + \frac{(s-c)^\zeta}{\bar{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(\nu^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{\nu^*-1} y(\phi(q)) \frac{dq}{(q-c)^{1-\zeta}} \right] \end{aligned}$$

$$-\frac{\mu_2^*}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma - c)^\zeta - (q - c)^\zeta}{\zeta} \right)^{v^* + \theta^* - 1} y(\phi(q)) \frac{dq}{(q - c)^{1-\zeta}}.$$

In this way, the aforementioned Caputo conformable problem (12) is reduced to a standard fixed point problem. Finally, one can simply prove that the newly-defined operator $\bar{\Psi}$ is completely continuous and continuous. The remaining proof is implemented as one in Theorem 12 and thus we omit it again. This finishes the proof process and yields the required existence result. \square

3.3 The Lipschitzian case

Here, we discuss the existence criterion when $\tilde{\mathfrak{D}}$ has non-convex values. To reach the desired purpose, we utilize a fixed point result attributed to Covitz and Nadler (Theorem 8) on set-valued maps.

Theorem 14 *Let the following be valid:*

- (HP₈) *the set-valued map $\tilde{\mathfrak{D}} : [c, M] \times \mathfrak{Y} \times \mathfrak{Y} \rightarrow \mathcal{P}_{cmp}(\mathfrak{Y})$ is such that, for each $\phi, \bar{\phi} \in \mathfrak{Y}$, $\tilde{\mathfrak{D}}(\cdot, \phi, \bar{\phi}) : [c, M] \rightarrow \mathcal{P}_{cmp}(\mathfrak{Y})$ is measurable;*
- (HP₉) *there is a function $y \in C_{\mathbb{R}^+}([c, M])$ with $d_{\mathfrak{Y}}(0, \tilde{\mathfrak{D}}(s, 0, 0)) \leq y(s)$ for almost all $s \in [c, M]$ such that*

$$\mathbb{P}\mathbb{H}_{d_{\mathfrak{Y}}}(\tilde{\mathfrak{D}}(s, \phi_1, \bar{\phi}_1), \tilde{\mathfrak{D}}(s, \phi_2, \bar{\phi}_2)) \leq y(s)(|\phi_1 - \phi_2| + |\bar{\phi}_1 - \bar{\phi}_2|)$$

for almost all $s \in [c, M]$ and $\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2 \in \mathfrak{Y}$.

Then the three-point Caputo conformable pantograph inclusion problem (1) possesses at least one solution on interval $[c, M]$ so that

$$2\tilde{X}\|y\|_{\mathfrak{Y}} < 1, \tag{13}$$

where \tilde{X} is illustrated by (3) and $\|y\|_{\mathfrak{Y}} = \sup_{s \in [c, M]} |y(s)|$.

Proof We again regard $\Psi : \mathfrak{Y} \rightarrow \mathcal{P}(\mathfrak{Y})$ similar to the one defined in the proof of Theorem 10. In this case, the three-point Caputo conformable pantograph inclusion problem (1) is transformed into a standard fixed point problem. At first, we verify that $\Psi(\phi) \neq \emptyset$ for any $\phi \in \mathfrak{Y}$ and also is closed set for every $\check{g} \in \text{SEL}_{\tilde{\mathfrak{D}}, \phi}$. To see this, it is clear that $\tilde{\mathfrak{D}}(\cdot, \phi(\cdot), \bar{\phi}(\cdot))$ is measurable in view of the measurable selection theorem ([65], Theorem III.6) and so a measurable selection $\check{g} \in \mathcal{L}_{\mathbb{R}}^1([c, M])$ exists and thus $\tilde{\mathfrak{D}}$ is integrable bounded. This means that $\text{SEL}_{\tilde{\mathfrak{D}}, \phi} \neq \emptyset$. In addition, $\Psi(\phi) \in \mathcal{P}_{cls}(\mathfrak{Y})$ for each $\phi \in \mathfrak{Y}$ as is verified in Theorem 10. Thus $\Psi(\phi)$ is a closed set for each $\phi \in \mathfrak{Y}$. In the sequel, we show that there is a constant $\hat{c} < 1$ so that

$$\mathbb{P}\mathbb{H}_{d_{\mathfrak{Y}}}(\Psi(\phi_1), \Psi(\phi_2)) \leq \hat{c}(\|\phi_1 - \phi_2\|_{\mathfrak{Y}} + \|\bar{\phi}_1 - \bar{\phi}_2\|_{\mathfrak{Y}})$$

for any $\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2 \in \mathfrak{Y}$. To confirm this, let $\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2 \in \mathfrak{Y}$ and $\psi_1(s) \in \Psi(\phi)$. Hence, for each $s \in [c, M]$, there exists $\check{g}_1(s) \in \tilde{\mathfrak{D}}(s, \phi_1(s), \bar{\phi}_1(s))$ so that

$$\psi_1(s) = \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s - c)^\zeta - (q - c)^\zeta}{\zeta} \right)^{v^* - 1} \check{g}_1(q) \frac{dq}{(q - c)^{1-\zeta}}$$

$$\begin{aligned}
 & + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}_1(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\
 & \left. - \frac{\mu_2^*}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^* + \theta^* - 1} \check{g}_1(q) \frac{dq}{(q-c)^{1-\zeta}} \right].
 \end{aligned}$$

In view of the assumption (HP₉), we get

$$\mathbb{P}\mathbb{H}_{d,\mathfrak{Y}}(\tilde{\mathfrak{D}}(s, \phi_1, \bar{\phi}_1), \tilde{\mathfrak{D}}(s, \phi_2, \bar{\phi}_2)) \leq \gamma(s)(|\phi_1(s) - \phi_2(s)| + |\bar{\phi}_1(s) - \bar{\phi}_2(s)|).$$

Thus, there is a function $h_* \in \tilde{\mathfrak{D}}(s, \phi_2(s), \bar{\phi}_2(s))$ provided that

$$|\check{g}_1(s) - h_*(s)| \leq \gamma(s)(|\phi_1(s) - \phi_2(s)| + |\bar{\phi}_1(s) - \bar{\phi}_2(s)|).$$

Define a new multifunction $\mathcal{A}^* : [c, M] \rightarrow \mathcal{P}(\mathfrak{Y})$ formulated by

$$\mathcal{A}^*(s) = \{h_* \in \mathbb{R} : |\check{g}_1(s) - h_*(s)| \leq \gamma(s)(|\phi_1(s) - \phi_2(s)| + |\bar{\phi}_1(s) - \bar{\phi}_2(s)|)\}.$$

We know that the set-valued map $\mathcal{A}^*(s) \cap \tilde{\mathfrak{D}}(s, \phi_2(s), \bar{\phi}_2(s))$ is measurable (Proposition III.4 [65]). Hence, there is \check{g}_2 which is regarded as a measurable selection for \mathcal{A}^* . In consequence, $\check{g}_2(s) \in \tilde{\mathfrak{D}}(s, \phi_2(s), \bar{\phi}_2(s))$ and for each $s \in [c, M]$, we have

$$|\check{g}_1(s) - \check{g}_2(s)| \leq \gamma(s)(|\phi_1(s) - \phi_2(s)| + |\bar{\phi}_1(s) - \bar{\phi}_2(s)|).$$

Hence

$$\begin{aligned}
 \psi_2(s) & = \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}_2(q) \frac{dq}{(q-c)^{1-\zeta}} \\
 & + \frac{(s-c)^\zeta}{\tilde{\Omega}} \left[\xi^* - \frac{\mu_1^*}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} \check{g}_2(q) \frac{dq}{(q-c)^{1-\zeta}} \right. \\
 & \left. - \frac{\mu_2^*}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^* + \theta^* - 1} \check{g}_2(q) \frac{dq}{(q-c)^{1-\zeta}} \right]
 \end{aligned}$$

and so

$$\begin{aligned}
 & |\psi_1(s) - \psi_2(s)| \\
 & \leq \frac{1}{\Gamma(v^*)} \int_c^s \left(\frac{(s-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} |\check{g}_1(q) - \check{g}_2(q)| \frac{dq}{(q-c)^{1-\zeta}} \\
 & + \frac{(s-c)^\zeta}{|\tilde{\Omega}|} \left[\frac{|\mu_1^*|}{\Gamma(v^*)} \int_c^M \left(\frac{(M-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^*-1} |\check{g}_1(q) - \check{g}_2(q)| \frac{dq}{(q-c)^{1-\zeta}} \right. \\
 & \left. + \frac{|\mu_2^*|}{\Gamma(v^* + \theta^*)} \int_c^\sigma \left(\frac{(\sigma-c)^\zeta - (q-c)^\zeta}{\zeta} \right)^{v^* + \theta^* - 1} |\check{g}_1(q) - \check{g}_2(q)| \frac{dq}{(q-c)^{1-\zeta}} \right].
 \end{aligned}$$

This implies that

$$\|\psi_1 - \psi_2\|_{\mathfrak{Y}} \leq 2 \left\{ \frac{(M-c)^\zeta v^*}{\zeta v^* \Gamma(1 + v^*)} + \frac{(M-c)^\zeta}{|\tilde{\Omega}|} \left[|\mu_1^*| \frac{(M-c)^\zeta v^*}{\zeta v^* \Gamma(1 + v^*)} \right. \right.$$

$$+ \left| \mu_2^* \frac{(\sigma - c)^{\zeta(v^* + \theta^*)}}{\zeta(v^* + \theta^*) \Gamma(1 + v^* + \theta^*)} \right\} \|y\|_{\mathfrak{Y}} \|\phi_1 - \phi_2\|_{\mathfrak{Y}}.$$

In a similar way, by interchanging the roles of ϕ_1 and ϕ_2 , the following holds:

$$\mathbb{P}\mathbb{H}_{d_{\mathfrak{Y}}}(\Psi(\phi_1), \Psi(\phi_2)) \leq 2\tilde{X} \|y\|_{\mathfrak{Y}} \|\phi_1 - \phi_2\|_{\mathfrak{Y}}.$$

Then, in the light of the condition (13), we find that Ψ is a contraction. In conclusion, with the help of a fixed point result attributed to Nadler and Covitz (Theorem 8), we deduce that Ψ involves a fixed point which is regarded as a solution for the proposed three-point Caputo conformable pantograph inclusion problem (1). This completes the proof. \square

4 Examples

The last part of the present research is devoted to proposing two numerical simulative examples to demonstrate the consistency of our findings. For this purpose, we formulate a general three-point Caputo conformable pantograph inclusion problem as follows:

$$\begin{cases} {}^{CC}\mathfrak{D}_c^{1/2, 3/2} \phi(s) \in \tilde{\mathfrak{D}}(s, \phi(s), \phi(s/4)), & (s \in [0, 1]) \\ \phi(0) = 0, & 0.7\phi(1) + 1.3{}^{\mathcal{RC}}\mathfrak{I}_c^{1/2, 1/2} \phi(1/2) = 2, \end{cases} \tag{14}$$

so that $\zeta = 1/2, v^* = 3/2, \theta^* = 1/2, \sigma = 1/2, \mu_1^* = 0.7, \mu_2^* = 1.3, \xi^* = 2, \lambda^* = 1/4, c = 0$ and $M = 1$. By some computations, we get $\tilde{\Omega} \simeq 1.52234$ and $\tilde{X} \simeq 3.95999$. With due attention to the above data, we design two examples in the following frameworks.

Example 1 In view of the above Caputo conformable pantograph inclusion problem (14), assume that $\tilde{\mathfrak{D}}(s, \phi(s), \phi(s/4))$ is a set-valued map formulated by

$$\tilde{\mathfrak{D}}(s, \phi, \bar{\phi}) = \left[\frac{2 \sin |\phi| + s|\bar{\phi}| + 1/3}{5(16 + \phi^2)}, \frac{e^{-\phi^2} |\bar{\phi}|}{7(16 + \phi^2)} \right]. \tag{15}$$

For each $\mu > 0$, we have $\|\tilde{\mathfrak{D}}(s, \phi, \bar{\phi})\| \leq \frac{\mu s}{5} + 7/15 = \varphi_{\mu}(s)$ with $\|\phi\|_{\mathfrak{Y}}, \|\bar{\phi}\|_{\mathfrak{Y}} \leq \mu$ for a.e. $s \in [0, 1]$ and also $\liminf_{\mu \rightarrow \infty} \int_0^1 \frac{\varphi_{\mu}(q)}{\mu} dq = \Theta = 1/10$. Furthermore, it is simple to investigate that $\tilde{\mathfrak{D}}(s, \phi, \bar{\phi})$ is Carathéodory. On the other hand, since

$$\Theta \left\{ \frac{(M - c)^{\zeta(v^* - 1)}}{\zeta(v^* - 1) \Gamma(v^*)} + \frac{(M - c)^{\zeta}}{|\tilde{\Omega}|} \left[\left| \mu_1^* \frac{(M - c)^{\zeta(v^* - 1)}}{\zeta(v^* - 1) \Gamma(v^*)} + \left| \mu_2^* \frac{(\sigma - c)^{\zeta(v^* + \theta^* - 1)}}{\zeta(v^* + \theta^* - 1) \Gamma(v^* + \theta^*)} \right| \right] \right\} \simeq 0.293337 < 1,$$

therefore we see that all hypotheses of Theorem 10 are valid about this problem. This implies that three-point Caputo conformable pantograph inclusion problem (14) along with the set-valued map $\tilde{\mathfrak{D}}(s, \phi, \bar{\phi})$ defined in (15) has at least one solution on $s \in [0, 1]$.

Example 2 In view of the above Caputo conformable pantograph inclusion problem (14), assume that $\tilde{\mathfrak{D}}(s, \phi(s), \phi(s/4))$ is a set-valued map formulated by

$$\tilde{\mathfrak{D}}(s, \phi, \bar{\phi}) = \left[\frac{1}{3\sqrt{81 + s^2}} \left(\frac{\sin |\phi| + \tan^{-1} |\bar{\phi}| + 1}{1 + |\phi| + |\bar{\phi}|} \right), \frac{s}{11e^{2s}} \left(\frac{|\phi|}{|\phi| + 1} + \frac{|\bar{\phi}|}{|\bar{\phi}| + 1} \right) \right]. \tag{16}$$

It is evident that $\tilde{\mathfrak{D}}$ given in (16) is measurable for all $\phi, \bar{\phi} \in \mathfrak{N}$. Now, we get

$$\mathbb{P}H_{d_{\mathfrak{N}}}(\tilde{\mathfrak{D}}(s, \phi_1, \bar{\phi}_1), \tilde{\mathfrak{D}}(s, \phi_2, \bar{\phi}_2)) \leq \frac{s}{11e^{2s}}(|\phi_1 - \phi_2| + |\bar{\phi}_1 - \bar{\phi}_2|),$$

for a.e. all $s \in [0, 1]$ and $\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2 \in \mathfrak{N}$. Here, we set $\gamma(s) = \frac{s}{11e^{2s}}$. In this case, $\|\gamma\|_{\mathfrak{N}} = 1/11$ and $d_{\mathfrak{N}}(0, \tilde{\mathfrak{D}}(s, 0, 0)) \leq \gamma(s)$ for almost all $s \in [0, 1]$. In addition, we have

$$2\tilde{X}\|\gamma\|_{\mathfrak{N}} \simeq 0.719998 < 1.$$

As one can see, all hypotheses of Theorem 14 are valid. Then at least one solution exists for the proposed three-point Caputo conformable pantograph inclusion problem (14) along with $\tilde{\mathfrak{D}}(s, \phi, \bar{\phi})$ defined in (16) on the interval $s \in [0, 1]$.

5 Conclusion

Over the years, the human beings have needed to be acquainted with various natural phenomena more and more. One possible way to achieve this aim is to utilize the techniques and tools available in mathematics and particularly the mathematical operators in modeling of different processes. In the current manuscript, we reformulate and investigate the well-known pantograph differential equation by applying newly-defined conformable operators in both Caputo and Riemann–Liouville settings simultaneously for the first time. In fact, we derive required existence criteria of solutions corresponded to inclusion version of three-point Caputo conformable pantograph BVP subject to Riemann–Liouville conformable integral conditions. To achieve this aim, we establish our main results in some cases including the lower semi-continuous, the upper semi-continuous and the Lipschitz set-valued maps. Eventually, the last part of the present research is devoted to proposing two numerical simulative examples to demonstrate the consistency of our findings.

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Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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