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Low Mach number limit for the compressible Navier–Stokes equations with density-dependent viscosity and vorticity-slip boundary condition

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Abstract

In this paper, we consider the three-dimensional compressible Navier–Stokes equations with density-dependent viscosity and vorticity-slip boundary condition in a bounded smooth domain. The main idea is to derive the uniform estimates for both time and the Mach number. The difficulty is dealing with density-dependent viscosity terms carefully. With the uniform estimates, we can verify the low Mach limit of the global strong solutions of compressible Navier–Stokes equations and the global existence and uniqueness of the strong solution of incompressible Navier–Stokes equations around a steady state.

Keywords: Navier–Stokes equations; Density-dependent viscosity; Bounded domain; Low Mach number limit; Global existence

1 Introduction

In this paper, we study the low Mach number limit for the initial-boundary value problem of the following three-dimensional compressible Navier–Stokes equations in a bounded domain $\Omega \subset \mathcal{R}^3$ with a smooth boundary:

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1}$$

$$(\rho u)_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\epsilon^2} \nabla p(\rho) = \operatorname{div}\big(2\mu(\rho) \mathrm{D}(\mathbf{u})\big) + \nabla\big(\lambda(\rho) \operatorname{div} \mathbf{u}\big), \tag{2}$$

where ρ and $\mathbf{u} = (u_1, u_2, u_3)$ denote the density of the fluid and the velocity, respectively, with $\mathrm{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^t)/2$. The functions $\mu(\rho) = \rho^\alpha$ $(\alpha > 0)$ and $\lambda(\rho) = \rho^\beta$ $(\beta > 0)$ are the shear and bulk viscosity coefficients of the fluid, respectively, satisfying $\mu(\rho) > 0$ and $\mu(\rho) + \frac{2}{3}\lambda(\rho) > 0$. This condition makes sense in the case that ρ is far away from the vacuum, for instance, the shallow water waves. The constant $\epsilon \in (0, 1]$ is the Mach number. The pressure $p(\rho)$ satisfies the barotropic law, namely,

$$p(\rho) = a\rho^{\gamma}, \tag{3}$$

where $a > 0$ and $\gamma > 1$ are constants.

Formally, as the Mach number ϵ vanishes, the solution to (1)–(2) will converge to the one of the following incompressible Navier–Stokes equations:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla \pi = 0, \tag{4}$$

$$\operatorname{div} \mathbf{u} = 0. \tag{5}$$

It is known as the low Mach number limit. Since the large parameter ϵ^{-2} appears in (2), this limit process is singular. The fact that both the uniform estimates in Mach number and the convergence to the incompressible model are usually difficult to obtain creates a serious difficulty for the rigorous justification of this limit.

The low Mach number limit of local smooth solutions to the Navier–Stokes equations (or the Euler equations) in $\mathcal{R}^n$ or $\mathcal{T}^n$ with "well-prepared" initial data was proved by Klainerman and Majda in [17, 18]. They established the general framework for studying the low Mach number limit for local strong or smooth solutions. For bounded domain, Lions and Masmoudi [19] investigated the low Mach number limit for the weak solutions to the Navier–Stokes equations with the "vorticity-slip" boundary condition, that is, on the boundary $\partial\Omega \subset \mathcal{R}^n$,

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad \operatorname{curl} \mathbf{u} = 0 \quad \text{for } n = 2, \quad \text{or} \tag{6}$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad \mathbf{n} \times \operatorname{curl} \mathbf{u} = 0 \quad \text{for } n = 3, \tag{7}$$

where $\operatorname{curl} \mathbf{u} = (\partial_2 u_1, -\partial_1 u_2)^t$ for $n = 2$ and $\operatorname{curl} \mathbf{u} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^t$ for $n = 3$. There are abundant results about the low Mach number limit for local solutions to the isentropic Navier–Stokes equations, the reader may refer to [6–10, 23, 24] and the references therein, for instance.

The low Mach number limit for global solutions to the isentropic Navier–Stokes equations have been considered by many authors; see [3, 14, 21, 22]. Compared with the study of the low Mach number limit for local solutions, one must get the uniform estimates with respect to both the Mach number ϵ and $t \in [0, +\infty)$. Thus this is challenging. D. Hoff [14] verified the low Mach number limit for the global solutions in $\mathcal{R}^3 \times [0, +\infty)$ with general large initial data. For bounded domain, H. Bessaih [3] investigated the low Mach number limit of regular solutions to the compressible Navier–Stokes equations with no-slip boundary conditions and slightly compressible initial data. In [21], Ou obtained the low Mach number limit of regular solutions to the compressible Navier–Stokes equations (1)–(2) with slightly compressible initial data in a 2-D bounded domain with the "vorticity-slip" boundary condition (6). [22] investigated the low Mach number limit of strong solutions to 3-D Navier–Stokes equations with Navier's slip boundary condition for all time.

Concerning with the low mach number limit of the compressible non-isentropic Navier–Stokes equations, many results was presented in [2, 5, 12, 13, 15, 16, 20], and the references therein. After learning this progress on the low mach number limit carefully, we find the fact that most of it was concerned with the constant viscosity coefficients.

The purpose of this paper is to verify rigorously the corresponding low Mach number limit for all time of the 3-D isentropic Navier–Stokes equations with density-dependent viscosity and the "vorticity-slip" boundary condition. We establish the uniform estimates of strong solutions with respect to the Mach number and justify rigorously the low Mach number limit for all time when the non-constant viscosity coefficients are present, in contrast with [21]. Because the viscosity depends on the density, the uniform estimates of strong solutions are much more difficult to obtain.

To verify the low Mach number limit, we shall consider the density varies slightly around a constant state, namely,

$$\rho = 1 + \epsilon\sigma.$$

We reformulate the problem (1)–(2) as

$$\sigma_t + \operatorname{div}(\sigma \mathbf{u}) + \frac{1}{\epsilon} \operatorname{div} \mathbf{u} = 0, \tag{8}$$

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \frac{1}{\epsilon} p'(1 + \epsilon\sigma)\nabla\sigma = \operatorname{div}\big(2\mu(\rho)\mathrm{D}(\mathbf{u})\big) + \nabla\big(\lambda(\rho) \operatorname{div} \mathbf{u}\big). \tag{9}$$

The initial data for the system (1)–(2) are defined as

$$\rho(0,x) = \rho_0(x), \qquad \mathbf{u}(0,x) = \mathbf{u}_0(x), \quad x \in \Omega. \tag{10}$$

We impose the "vorticity-slip" boundary condition for the velocity, that is, on the boundary $\partial\Omega$ of $\Omega \subset \mathcal{R}^3$,

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad \mathbf{n} \times \operatorname{curl} \mathbf{u} = 0, \tag{11}$$

where $\operatorname{curl} \mathbf{u} = (\partial_2 \mathbf{u}_3 - \partial_3 \mathbf{u}_2, \partial_3 \mathbf{u}_1 - \partial_1 \mathbf{u}_3, \partial_1 \mathbf{u}_2 - \partial_2 \mathbf{u}_1)^t$ and $\mathbf{n}$ is the unit outer normal vector to the boundary.

We state the main results of this paper as follows.

Theorem 1.1 (Global-in-time existence) *Let $\epsilon \in (0,1]$ be a fixed constant and $\Omega \subset \mathcal{R}^3$ be a simply connected, bounded domain with smooth boundary $\partial\Omega$. Suppose that the initial datum (σ_0, u_0) satisfies the following conditions*:

$$\big\|(\sigma_0, \mathbf{u}_0)\big\|_{\mathbf{H}^2} + \big\|(\sigma_t, \mathbf{u}_t)(0)\big\|_{H^1} \le m, \tag{12}$$

with $\int_\Omega \sigma_0 \, dx = 0$ and $1 + \epsilon\sigma_0 \le m$ for some positive constant m. Assume the following compatibility conditions are satisfied:

$$\partial_t^i \mathbf{u}(0) \cdot \mathbf{n} = \mathbf{n} \times \partial_t^i \operatorname{curl} \mathbf{u}(0) = 0 \quad \text{on } \partial\Omega, i = 0, 1. \tag{13}$$

Then, for any $\epsilon \in (0, \epsilon_1]$ with $\epsilon_1 \in (0,1)$ being a constant, the initial-boundary value problem (8)–(11) admits a unique solution $(\sigma, \mathbf{u}, \mathbf{H})$ in $\Omega \times \mathcal{R}^+$, satisfying

$$\sigma \in C\big(\mathcal{R}^+, H^2\big), \qquad \mathbf{u} \in C\big(\mathcal{R}^+, H^2\big) \cap L^2\big(\mathcal{R}^+; H^3\big),$$

$$\sigma_t \in C(\mathcal{R}^+, H^1), \qquad \mathbf{u}_t \in C(\mathcal{R}^+, H^1) \cap L^2(R^+; H^2),$$

where $R^+ = [0, +\infty)$. *Moreover, the uniform estimates are satisfied*:

$$\sup_{0\le s\le t} \left(\|(\sigma, \mathbf{u})(s)\|_{H^2} + \|(\sigma_t, \mathbf{u}_t)(s)\|_{H^1} \right) \le Cm, \quad \forall t \in \mathcal{R}^+, \tag{14}$$

where C *is a positive constant independent of* $\epsilon \in (0, \epsilon_1]$ *and* $t \in [0, +\infty)$.

Remark 1.1 To simplify the statement, we use the notation "$\mathbf{u}_t(0)$" to signify the quantity $\mathbf{u}_t|_{t=0} := -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - p'(1+\epsilon\sigma_0)\nabla\sigma_0/\epsilon + \operatorname{div}(2\mu(1+\epsilon\sigma_0)\mathrm{D}(\mathbf{u}_0)) + \nabla(\lambda(1+\epsilon\sigma_0)\operatorname{div}\mathbf{u}_0)$ obtained from the equation (9). The notation "$\partial_t^i \mathbf{u}(0)$" is given by differentiating (9) $i-1$ times with respect to t and then letting $t = 0$. The same rule applies to the notations $\partial_t^i \sigma(0)$.

Assume that the assumptions in Theorem 1.1 are satisfied. Then one can get the local existence of the initial-boundary problem (8)–(11) by the method of characteristics, the Galerkin method and the Schauder fixed point theorem, that is, there exists a $T^\star > 0$, such that for $T \le T^\star$ the problem (8)–(11) admits a solution satisfying

$$\sigma \in C([0, T], H^2), \qquad (\mathbf{u}, \mathbf{H}) \in C([0, T], H^2) \cap L^2(0, T; H^3),$$
$$\sigma_t \in C([0, T], H^1), \qquad (\mathbf{u}_t, \mathbf{H}_t) \in C([0, T], H^1) \cap L^2(0, T; H^2).$$

The boundary conditions (11) are "complementing" boundary conditions in the sense of Agmon–Douglis–Nirenberg [1]. The local existence result can be proved by the frame in [22], so we omit the details of the proof here.

Theorem 1.2 (Incompressible limit) *Let the assumptions in Theorem* 1.1 *be satisfied, and* $\mathbf{u}$ *be the global strong solution established in Theorem* (1.1). *Suppose that the initial data* $\mathbf{u}_0 \to \mathbf{v}_0$ *as* $\epsilon \to 0$ *in* H^s *for any* $0 \le s < 2$. *Then we have* $\mathbf{u} \to \mathbf{v}$ *in* $C(\bar{\mathcal{R}}^+_{\mathrm{loc}}, H^s)$ *as* $\epsilon \to 0$, *for any* $0 \le s < 2$. *Moreover, there exists a function* $P(x,t)$, *such that* $(\mathbf{v}, P)$ *is the unique global strong solution to the following initial-boundary value problem of incompressible Navier–Stokes equations*:

$$\operatorname{div}\mathbf{v} = 0,$$
$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla P = \mu\Delta\mathbf{v},$$
$$\mathbf{v}\cdot\mathbf{n} = \mathbf{n}\times\operatorname{curl}\mathbf{v} = 0 \quad \text{on } \partial\Omega,$$
$$\mathbf{v}|_{t=0} = \mathbf{v}_0(x), \quad x \in \Omega.$$

Before ending this section, we introduce the notations throughout this paper. We use the constant C to denote various positive constants independent of ϵ and t, use the constant C_η to emphasize the dependence on η. Moreover, we denote by H^m and $\|\cdot\|_{H^m}$ the Sobolev space $H^m(\Omega) \equiv W^{m,2}(\Omega)$ and its norm, by L^p and $\|\cdot\|_{L^p}$ the Lebesgue space $L^p(\Omega)$ and its norm.

2 Preliminaries

In this paper, we will use the following lemmas frequently.

Lemma 2.1 (See [4]) *Let Ω be a bounded domain in $\mathcal{R}^N$ with smooth boundary $\partial\Omega$ and outward normal n. Then there exists a constant $C > 0$ independent of u, such that*

$$\|u\|_{H^s(\Omega)} \leq C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\operatorname{curl} u\|_{H^{s-1}(\Omega)} + \|u \cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \|u\|_{H^{s-1}(\Omega)}\big), \tag{15}$$

for any $u \in H^s(\Omega)^N$.

Lemma 2.2 (See [26]) *Let Ω be a bounded domain in $\mathcal{R}^N$ with smooth boundary ∂U and outward normal n. Then there exists a constant $C > 0$ independent of u, such that*

$$\|u\|_{H^s(\Omega)} \leq C\big(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\operatorname{curl} u\|_{H^{s-1}(\Omega)} + \|u \times n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \|u\|_{H^{s-1}(\Omega)}\big), \tag{16}$$

for all $u \in H^s(\Omega)^N$.

Lemma 2.3 (See [11]) *Let $\Omega \subset \mathcal{R}^3$ be a open bounded domain with C^2 boundary $\partial\Omega$. Moreover, we assume that Ω is simply connected and non-axisymmetric. Then, for any $u \in H^1(\Omega)$ satisfying $u \cdot n|_{\partial\Omega} = 0$, one has*

$$\|u\|_{H^1(\Omega)} \leq C\big(\|D(u)\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}\big) \tag{17}$$

and

$$\|\nabla u\|_{L^2(\Omega)} \leq C\big(\|\operatorname{div} u\|_{L^2(\Omega)} + \|\operatorname{curl} u\|_{L^2(\Omega)}\big), \tag{18}$$

where C is a constant independent of u.

Lemma 2.4 (See [4]) *Assume $f \in C([0,T]; W^{k,p}(\Omega, \mathcal{R}^N))$ with*

$$k > \frac{N}{p} + 1 \quad \text{and} \quad 1 \leq p \leq +\infty.$$

Then the problem

$$\frac{du}{dt}(x,t) = f\big(u(x,t),t\big), \qquad u(x,0) = x$$

has a solution $u \in C^1([0,T]; D^{k,p}(\Omega))$, where

$$D^{k,p}(\Omega) = \big\{\eta \in W^{k,p}(\Omega) \mid \eta \text{ is a bijective from } \overline{\Omega} \text{ onto } \overline{\Omega}, \eta^{-1} \in W^{k,p}(\Omega)\big\}.$$

Lemma 2.5 (See [4]) *Let $k \geq 2$ be an integer, and let $1 \leq p \leq q \leq +\infty$ be such that $p < +\infty$ and $k > \frac{N}{p} + 1$. Let $f \in W^{k,p}(\Omega)$, then the mapping $g \longmapsto g \circ f$ is continuous from $D^{k,p}(\Omega)$ into $W^{k,p}(\Omega)$.*

3 Energy estimates

In order to extend the local solution of the initial-boundary value problem (8)–(11) globally in time, we shall establish a differential inequality which provides us the uniform estimates of solutions for both time and the Mach number. Suppose that $(\sigma, \mathbf{u})$ is the local solution to the initial-boundary value problem (8)–(11) in $\Omega \times (0,T)$, for $0 < T < \infty$. Moreover, we assume that $1/c \le \rho = 1 + \epsilon\sigma \le c$ for some constant $c > 1$. Then the viscosity coefficients can be estimated as follows: $1/c^\alpha \le \mu(\rho) = \rho^\alpha \le c^\alpha$ and $1/c^\beta \le \lambda(\rho) = \rho^\beta \le c^\beta$.

3.1 L^2 estimate

Lemma 3.1 *For the solution to* (8)–(11), *we have*

$$\frac{d}{dt}\big\|\big(\sqrt{p'(\rho)}\sigma, \sqrt{\rho}\mathbf{u}\big)\big\|_{L^2}^2 + \gamma_1\|\mathbf{u}\|_{H^1}^2 \le \epsilon\|\sigma_t\|_{L^2}^2 + C\|\sigma\|_{H^1}^2\big(\|\mathbf{u}\|_{H^2}^2 + \|\sigma\|_{H^1}^2\big),$$

where γ_1 is a positive constant independent of ϵ.

Proof We integrate the product of (8) and $p'(\rho)\sigma$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\sqrt{p'(\rho)}\sigma\big\|_{L^2}^2 + \frac{1}{\epsilon}\int_\Omega p'(\rho)\operatorname{div}\mathbf{u}\sigma\,dx \\
&\quad = -\int_\Omega p'(\rho)\sigma\operatorname{div}(\sigma\mathbf{u})\,dx + \frac{1}{2}\int_\Omega p''(\rho)\epsilon\sigma_t\sigma^2\,dx \\
&\quad \le \epsilon\|\sigma_t\|_{L^2}^2 + \eta\|\mathbf{u}\|_{H^1}^2 + C_\eta\|\sigma\|_{H^1}^4.
\end{aligned}$$

Due to the boundary conditions (11) and Lemma 2.3, we have

$$\begin{aligned}
&-\int_\Omega\big(2\mu(\rho)\operatorname{div}\big(\mathrm{D}(\mathbf{u})\big) + \lambda(\rho)\nabla\operatorname{div}\mathbf{u}\big)\cdot\mathbf{u}\,dx \\
&\quad = -\int_\Omega\big(\big(2\mu(\rho)+\lambda(\rho)\big)\nabla\operatorname{div}\mathbf{u} - \mu(\rho)\operatorname{curl}\operatorname{curl}\mathbf{u}\big)\cdot\mathbf{u}\,dx \\
&\quad = \int_\Omega\big(\big(2\mu(\rho)+\lambda(\rho)\big)|\operatorname{div}\mathbf{u}|^2 + \mu(\rho)|\operatorname{curl}\mathbf{u}|^2\big)\,dx \\
&\qquad + \int_\Omega\big[\nabla\big(2\mu(\rho)+\lambda(\rho)\big)\operatorname{div}\mathbf{u} + \nabla\big(\mu(\rho)\big)\times\operatorname{curl}\mathbf{u}\big]\cdot\mathbf{u}\,dx \\
&\quad \ge \iota_0\|\mathbf{u}\|_{H^1}^2 + \int_\Omega\big[(\nabla\big(2\mu(\rho)+\lambda(\rho)\big)\operatorname{div}\mathbf{u} + \nabla\big(\mu(\rho)\big)\operatorname{curl}\mathbf{u}\big]\cdot\mathbf{u}\,dx.
\end{aligned}$$

Integrating the product of (9) and u, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \frac{1}{\epsilon}\int_\Omega p'(\rho)\mathbf{u}\cdot\nabla\sigma\,dx + \Big\|\sqrt{\big(2\mu(\rho)+\lambda(\rho)\big)}\operatorname{div}\mathbf{u}\Big\|_{L^2}^2 + \Big\|\sqrt{\mu(\rho)}\operatorname{curl}\mathbf{u}\Big\|_{L^2}^2 \\
&\quad = -\int_\Omega\big[\nabla\big(2\mu(\rho)+\lambda(\rho)\big)\operatorname{div}\mathbf{u} + \nabla\big(\mu(\rho)\big)\times\operatorname{curl}\mathbf{u}\big]\cdot\mathbf{u}\,dx \\
&\qquad + \int_\Omega\big[\nabla\big(2\mu(\rho)\big)\cdot\mathrm{D}(\mathbf{u}) + \nabla\big(\lambda(\rho)\big)\operatorname{div}\mathbf{u}\big]\cdot\mathbf{u}\,dx \\
&\quad \le \eta\|\mathbf{u}\|_{H^1}^2 + C_\eta\|\mathbf{u}\|_{H^2}^2\|\sigma\|_{H^1}^2.
\end{aligned}$$

Using (11) again and integration by parts, we have

$$\begin{aligned}&\frac{1}{\epsilon}\int_\Omega p'(\rho)\operatorname{div}\mathbf{u}\sigma\,dx+\frac{1}{\epsilon}\int_\Omega p'(\rho)\mathbf{u}\cdot\nabla\sigma\,dx\\&\quad=-\int_\Omega p''(\rho)\nabla\sigma\cdot\mathbf{u}\sigma\,dx\le\eta\|\mathbf{u}\|_{H^1}^2+C_\eta\|\sigma\|_{H^1}^4.\end{aligned}\tag{19}$$

Summing up the above equalities and choosing η small enough, we get the lemma. □

3.2 Estimates of first order derivatives

Lemma 3.2 *For the solution to (8)–(11), we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\big\|\sqrt{2\mu(\rho)+\lambda(\rho)}\operatorname{div}\mathbf{u}\big\|_{L^2}^2+\big\|\sqrt{\mu(\rho)}\operatorname{curl}\mathbf{u}\big\|_{L^2}^2\Big]\\&\qquad+\frac{d}{dt}\int_\Omega\rho\mathbf{u}_t\cdot\mathbf{u}\,dx+\frac{1}{2}\big\|\sqrt{p'(\rho)}\sigma_t\big\|_{L^2}^2\\&\quad\le C\|\mathbf{u}_t\|_{H^1}^2+\eta\|\mathbf{u}\|_{H^2}^2\\&\qquad+C_\eta\big[\|\mathbf{u}\|_{H^2}^2\big(\|\mathbf{u}\|_{H^2}^2+\|\sigma_t\|_{H^1}^2+\|\sigma\|_{H^2}^2\big)+\|\sigma\|_{H^2}^2\big(\|\mathbf{u}_t\|_{H^1}^2+\|\sigma_t\|_{H^1}^2\big)\big],\end{aligned}$$

where η is to be determined later.

Proof First, by differentiating (9) with respect to t, we have

$$\begin{aligned}&(\rho\mathbf{u}_t)_t+\epsilon\sigma_t\mathbf{u}\cdot\nabla\mathbf{u}+\rho(\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}_t)+p''(\rho)\sigma_t\nabla\sigma+\frac{1}{\epsilon}p'(\rho)\nabla\sigma_t\\&\quad=\operatorname{div}\big(2\mu(\rho)D(\mathbf{u}_t)\big)+\nabla\big(\lambda(\rho)\operatorname{div}\mathbf{u}_t\big)\\&\qquad+\operatorname{div}\big(2\mu'(\rho)\epsilon\sigma_tD(\mathbf{u})\big)+\nabla\big(\lambda'(\rho)\epsilon\sigma_t\operatorname{div}\mathbf{u}\big).\end{aligned}\tag{20}$$

Multiplying (20) by $\mathbf{u}$ in L^2, integrating by parts and using the boundary conditions (11), we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\big\|\sqrt{2\mu(\rho)+\lambda(\rho)}\operatorname{div}\mathbf{u}\big\|_{L^2}^2+\big\|\sqrt{\mu(\rho)}\operatorname{curl}\mathbf{u}\big\|_{L^2}^2\Big]\\&\qquad+\frac{d}{dt}\int_\Omega\rho\mathbf{u}_t\mathbf{u}\,dx+\frac{1}{\epsilon}\int_\Omega p'(\rho)\nabla\sigma_t\cdot\mathbf{u}\,dx\\&\quad=\int_\Omega\rho\mathbf{u}_t^2\,dx-\int_\Omega\rho_t(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\,dx-\int_\Omega\rho(\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}_t)\cdot\mathbf{u}\,dx\\&\qquad-\int_\Omega p''(\rho)\sigma_t\nabla\sigma\cdot\mathbf{u}\,dx+\int_\Omega\operatorname{div}\big(2\mu'(\rho)\epsilon\sigma_tD(\mathbf{u})\big)+\nabla\big(\lambda'(\rho)\epsilon\sigma_t\operatorname{div}\mathbf{u}\big)\cdot\mathbf{u}\,dx\\&\qquad-\int_\Omega\nabla\big(2\mu(\rho)+\lambda(\rho)\big)\cdot\mathbf{u}\operatorname{div}\mathbf{u}_t\,dx-\int_\Omega\nabla\big(\mu(\rho)\big)\times\operatorname{curl}\mathbf{u}_t\cdot\mathbf{u}\,dx\\&\qquad+\frac{1}{2}\int_\Omega\partial_t\big(2\mu(\rho)+\lambda(\rho)\big)|\operatorname{div}\mathbf{u}|^2\,dx+\frac{1}{2}\int_\Omega\partial_t\big(\mu(\rho)\big)|\operatorname{curl}\mathbf{u}|^2\,dx\\&\quad\le C\|\mathbf{u}_t\|_{L^2}^2+\eta\|\mathbf{u}\|_{H^2}^2\\&\qquad+C_\eta\big[\|\mathbf{u}\|_{H^2}^2\big(\|\mathbf{u}\|_{H^2}^2+\|\sigma_t\|_{H^1}^2+\|\sigma\|_{H^2}^2\big)+\|\sigma\|_{H^2}^2\big(\|\mathbf{u}_t\|_{H^1}^2+\|\sigma_t\|_{H^1}^2\big)\big].\end{aligned}$$

We multiply (8) by $p'(\rho)\sigma_t$, integrate by parts and use the boundary conditions (11) again to infer that

$$\begin{aligned}
&\big\|\sqrt{p'(\rho)\sigma_t}\big\|_{L^2}^2 - \frac{1}{\epsilon}\int_\Omega p'(\rho)\nabla\sigma_t\cdot\mathbf{u}\,dx \\
&\quad= -\int_\Omega p'(\rho)\sigma_t \operatorname{div}(\sigma\mathbf{u})\,dx + \int_\Omega \nabla\big(p'(\rho)\big)\cdot\mathbf{u}\sigma_t\,dx \\
&\quad\le \eta\|\sigma_t\|_{L^2}^2 + C_\eta\|\sigma\|_{H^1}^2\|\mathbf{u}\|_{H^1}^2.
\end{aligned}$$

Summing up the above estimates, we obtain the above lemma. □

Lemma 3.3 *For the solution to* (8)–(11), *we have*

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\sigma\|_{L^2}^2 + \big\|\sqrt{p'(1)^{-1}}\sqrt{2\mu(\rho)+\lambda(\rho)}\nabla\operatorname{div}\mathbf{u}\big\|_{L^2}^2 \\
&\quad\le C\|\mathbf{u}_t\|_{L^2}^2 + \eta\|\mathbf{u}\|_{H^1}^2 + C_\eta\|\sigma\|_{H^2}^4 + C\|\mathbf{u}\|_{H^2}^2\|\sigma\|_{H^2}^2, \quad 0<\eta<1,
\end{aligned}$$

where η is to be determined later.

Proof Applying ∇ to (8), multiplying the resulting equation by $\nabla\sigma$, integrating in L^2, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\sigma\|_{L^2}^2 + \frac{1}{\epsilon}\int_\Omega \nabla\operatorname{div}\mathbf{u}\cdot\nabla\sigma\,dx \\
&\quad= -\int_\Omega \big((\mathbf{u}\cdot\nabla)\nabla\sigma + \nabla\mathbf{u}\nabla\sigma + \nabla\sigma\operatorname{div}\mathbf{u} + \sigma\nabla\operatorname{div}\mathbf{u}\big)\nabla\sigma\,dx \\
&\quad\le \eta\big(\|\mathbf{u}\|_{H^1}^2 + \|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2\big) + C_\eta\|\sigma\|_{H^2}^4.
\end{aligned}$$

Now, we apply $\langle(9), p'(\rho)^{-1}\nabla\operatorname{div}\mathbf{u}\rangle$ to derive that

$$\begin{aligned}
&\big\|\sqrt{p'(\rho)^{-1}}\sqrt{2\mu(\rho)+\lambda(\rho)}\nabla\operatorname{div}\mathbf{u}\big\|_{L^2}^2 - \frac{1}{\epsilon}\int_\Omega \nabla\operatorname{div}\mathbf{u}\cdot\nabla\sigma\,dx \\
&\quad= \int_\Omega \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\cdot p'(\rho)^{-1}\nabla\operatorname{div}\mathbf{u}\,dx \\
&\qquad+ \int_\Omega p'(\rho)^{-1}\mu(\rho)\operatorname{curl}\operatorname{curl}\mathbf{u}\cdot\nabla\operatorname{div}\mathbf{u}\,dx \\
&\qquad- \int_\Omega \big[2\nabla\mu(\rho)\cdot D(\mathbf{u}) + \nabla\lambda(\rho)\cdot\operatorname{div}\mathbf{u}\big]\cdot p'(\rho)^{-1}\nabla\operatorname{div}\mathbf{u}\,dx \\
&\quad\le \eta\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2 + C_\eta\big(\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^2}^2(\|\mathbf{u}\|_{H^2}^2 + \|\sigma\|_{H^2}^2)\big),
\end{aligned}$$

where with the aid of $\operatorname{curl}\nabla = 0$ and $\operatorname{curl}\mathbf{u}\times\mathbf{n}|_{\partial\Omega} = 0$,

$$\begin{aligned}
&\int_\Omega p'(\rho)^{-1}\mu(\rho)\operatorname{curl}\operatorname{curl}\mathbf{u}\cdot\nabla\operatorname{div}\mathbf{u}\,dx \\
&\quad= -\int_\Omega \nabla\big[p'(\rho)^{-1}\mu(\rho)\big]\times\operatorname{curl}\mathbf{u}\cdot\nabla\operatorname{div}\mathbf{u}\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\partial\Omega} p'(\rho)^{-1}\mu(\rho)(\mathbf{n}\times \operatorname{curl}\mathbf{u})\cdot\nabla \operatorname{div}\mathbf{u}\,dS \\
&\le \eta\|\nabla \operatorname{div}\mathbf{u}\|_{L^2}^2 + C_\eta\|\mathbf{u}\|_{H^2}^2\|\sigma\|_{H^2}^2.
\end{aligned}
$$

Putting the above estimates together, we get this lemma. □

Lemma 3.4 *For the solution to* (8)–(11), *we have*

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\sqrt{p'(1)}\sigma_t\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\big) + \gamma_2\|\mathbf{u}_t\|_{H^1}^2 \\
&\quad\le \eta\|\mathbf{u}\|_{H^1}^2 + C_\eta\|\sigma_t\|_{H^1}^4 \\
&\qquad + C\big[\|\sigma_t\|_{H^1}^2\big(\|\mathbf{u}\|_{H^2}^2 + \|\sigma\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^2}^2\big) + \|\mathbf{u}_t\|_{H^1}^2\big\|(\mathbf{u},\sigma)\big\|_{H^2}^2 + \|\sigma\|_{H^2}^4\big],
\end{aligned} \tag{21}
$$

where $0<\eta<1$ *is to be determined later, and* γ_2 *is a positive constant independent of* ϵ.

Proof Applying ∂_t to (8), multiplying the resulting equation by $p'(1)\sigma_t$, integrating in L^2, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{p'(1)}\sigma_t\|_{L^2}^2 + \frac{p'(1)}{\epsilon}\int_\Omega \sigma_t \operatorname{div}\mathbf{u}_t\,dx \\
&\quad = -p'(1)\int_\Omega (\mathbf{u}\cdot\nabla\sigma_t + \mathbf{u}_t\cdot\nabla\sigma + \sigma_t\operatorname{div}\mathbf{u} + \sigma\operatorname{div}\mathbf{u}_t)\sigma_t\,dx \\
&\quad \le \eta\big(\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^1}^2\big) + C_\eta\big(\|\sigma\|_{H^1}^4 + \|\sigma_t\|_{H^1}^4\big).
\end{aligned}
$$

Applying ∂_t to (9), we have

$$
\begin{aligned}
&\rho(\partial_{tt}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_t) + \frac{1}{\epsilon}p'(1)\nabla\sigma_t \\
&\quad = \operatorname{div}\big(2\mu(\rho)\mathrm{D}(\mathbf{u}_t)\big) + \nabla\big(\lambda(\rho)\operatorname{div}\mathbf{u}_t\big) \\
&\qquad - \rho_t(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \rho\mathbf{u}_t\cdot\nabla\mathbf{u} + \partial_t\left[\frac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\right] \\
&\qquad + \operatorname{div}\big(2\partial_t\mu(\rho)\mathrm{D}(\mathbf{u})\big) + \nabla\big(\partial_t\lambda(\rho)\operatorname{div}\mathbf{u}\big).
\end{aligned} \tag{22}
$$

Taking $\langle(22),\mathbf{u}_t\rangle$ and using the boundary conditions (11), we find that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \big\|\sqrt{2\mu(\rho)+\lambda(\rho)}\operatorname{div}(\mathbf{u}_t)\big\|_{L^2}^2 \\
&\qquad + \big\|\sqrt{\mu(\rho)}\operatorname{curl}\mathbf{u}_t\big\|_{L^2}^2 + \frac{p'(1)}{\epsilon}\int_\Omega \nabla\sigma_t\cdot\mathbf{u}_t\,dx \\
&\quad = \int_\Omega\left[\frac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\right]_t\cdot\mathbf{u}_t\,dx - \int_\Omega\big[\epsilon\sigma_t(\mathbf{u}\cdot\nabla)\mathbf{u} + \rho(\mathbf{u}_t\cdot\nabla)\mathbf{u}\big]\cdot\mathbf{u}_t\,dx \\
&\qquad - \int_\Omega\big[\nabla\big(2\mu(\rho)+\lambda(\rho)\big)\operatorname{div}\mathbf{u}_t + \nabla\big(\mu(\rho)\big)\times\operatorname{curl}\mathbf{u}_t\big]\cdot\mathbf{u}_t\,dx \\
&\qquad - \int_\Omega \partial_t\big(2\mu(\rho)+\lambda(\rho)\big)|\operatorname{div}\mathbf{u}_t|^2 + \partial_t\big(\mu(\rho)\big)|\operatorname{curl}\mathbf{u}_t|^2\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_\Omega \big[\operatorname{div}\big(2\partial_t\mu(\rho)\mathrm{D}(\mathbf{u})\big)+\nabla\big(\partial_t\lambda(\rho)\operatorname{div}\mathbf{u}\big)\big]\cdot\mathbf{u}_t\,dx\\
&\le \eta\|\mathbf{u}_t\|_{H^1}^2+C_\eta\big(\|\sigma_t\|_{H^1}^2\big(\big\|(\mathbf{u},\sigma)\big\|_{H^2}^2+\|\mathbf{u}_t\|_{H^2}^2\big)+\|\mathbf{u}_t\|_{H^1}^2\big\|(\mathbf{u},\sigma)\big\|_{H^2}^2\big).
\end{aligned}
$$

Hence, by choosing η appropriately small and using Korn's inequality, we obtain the estimate (21). □

Next, we estimate the vorticity of the velocity, which is denoted by $\omega = \operatorname{curl}\mathbf{u}$. By virtue of (8) and (9), it is easy to see that ω satisfies the following systems:

$$
\begin{aligned}
&\rho\omega_t+\rho\mathbf{u}\cdot\nabla\omega-\mu(\rho)\Delta\omega=g,\\
&\omega\times\mathbf{n}=0\quad\text{on }\partial\Omega,
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
g={}&-\rho\omega\operatorname{div}\mathbf{u}-\frac{\epsilon}{\rho}\nabla\sigma\times\big(\operatorname{div}\big(\mu(\rho)\nabla\mathbf{u}\big)+\nabla\big[\lambda(\rho)\operatorname{div}\mathbf{u}\big]\big)\\
&+\operatorname{curl}\big(\mu(\rho)\big)\Delta\mathbf{u}+\big(\nabla\mu(\rho)\big)\cdot\nabla\operatorname{curl}\mathbf{u}.
\end{aligned}
$$

Then we have the following.

Lemma 3.5

$$
\frac{d}{dt}\|\sqrt{\rho}\omega\|_{L^2}^2+\big\|\sqrt{\mu(\rho)}\operatorname{curl}\omega\big\|_{L^2}^2\le\eta\|\omega\|_{L^2}^2+C_\eta\|\mathbf{u}\|_{H^2}^2\big(\|\mathbf{u}\|_{H^2}^2+\|\sigma\|_{H^2}^2\big),
\tag{24}
$$

where $0<\eta<1$ *is a positive constant which is to be determined.*

Proof Multiplying $(23)_1$ by ω, with the aid of the boundary condition $(23)_2$ we infer that

$$
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\omega\|_{L^2}^2+\big\|\sqrt{\mu(\rho)}\operatorname{curl}\omega\big\|_{L^2}^2=\int_\Omega g\cdot\omega\,dx-\int_\Omega\nabla\mu(\rho)\times\operatorname{curl}\omega\cdot\omega\,dx,
\tag{25}
$$

where

$$
\begin{aligned}
-\int_\Omega\mu(\rho)\Delta\omega\cdot\omega\,dx&=\int_\Omega\mu(\rho)\operatorname{curl}\operatorname{curl}\omega\cdot\omega\,dx\\
&=\int_\Omega\mu(\rho)|\operatorname{curl}\omega|^2\,dx+\int_\Omega\nabla\big(\mu(\rho)\big)\times\operatorname{curl}\omega\cdot\omega\,dx\\
&\quad+\int_{\partial\Omega}\mu(\rho)(\mathbf{n}\times\operatorname{curl}\mathbf{u})\cdot\omega\,dS.
\end{aligned}
$$

With the aid of Lemma 2.2, it is easy to verify that

$$
\begin{aligned}
\int_\Omega g\omega\,dx&\le\eta\|\omega\|_{H^1}^2+C_\eta\|\mathbf{u}\|_{H^2}^2\big(\|\mathbf{u}\|_{H^2}^2+\|\sigma\|_{H^2}^2\big)\\
&\le\eta\big(\|\omega\|_{L^2}^2+\|\operatorname{curl}\omega\|_{L^2}^2\big)+C_\eta\|\mathbf{u}\|_{H^2}^2\big(\|\mathbf{u}\|_{H^2}^2+\|\sigma\|_{H^2}^2\big).
\end{aligned}
$$

Using Holder's inequality and Young's inequality, we have

$$\int_\Omega \nabla\mu(\rho)\times \operatorname{curl}\omega\cdot\omega\,dx \le \eta\|\operatorname{curl}\omega\|_{L^2}^2 + C_\eta\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^2}^2.$$

Inserting the above two inequalities into (25) and choosing η appropriately small, we get the above lemma. □

Definition 3.1 Now, defining two functions:

$$\begin{aligned} &\Psi_1(t) := \int_\Omega \rho\mathbf{u}\cdot\mathbf{u}_t\,dx + \|(\sigma_t,\mathbf{u}_t)\|_{L^2}^2 + \|(\sigma,\mathbf{u})\|_{H^1}^2, \\ &\Phi_1(t) := \|\sigma_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^2 + \|(\operatorname{curl}\operatorname{curl}\mathbf{u},\nabla\operatorname{div}\mathbf{u})\|_{L^2}^2 + \|\mathbf{u}_t\|_{H^1}^2. \end{aligned} \tag{26}$$

we conclude from Lemmas 3.1–3.5 that, for small ϵ, there is a positive constant C_1, such that

$$\begin{aligned} \frac{d}{dt}\Psi_1(t) + \Phi_1(t) \le{}& C_1\big(\|\sigma_t\|_{H^1}^2\|(\mathbf{u}_t,\sigma_t)\|_{H^1}^2 + \|\sigma\|_{H^2}^2\big(\|\sigma\|_{H^2}^2 + \|\sigma_t\|_{H^1}^2\big) \\ &+ \|\mathbf{u}\|_{H^2}^2\big(\|(\mathbf{u}_t,\sigma_t)\|_{H^1}^2 + \|(\sigma,\mathbf{u})\|_{H^2}^2\big)\big). \end{aligned} \tag{27}$$

3.3 Boundedness of second order derivatives

First, we show the following lemma.

Lemma 3.6 *For the solution to* (8)–(11), *we have*

$$\begin{aligned} &\frac{d}{dt}\big\|\sqrt{2\mu(\rho)+\lambda(\rho)}\nabla\operatorname{div}\mathbf{u}\big\|_{L^2}^2 - 2\frac{d}{dt}\int_\Omega \rho\mathbf{u}_t\cdot\nabla\operatorname{div}\mathbf{u}\,dx + \big\|\sqrt{p'(1)}\nabla\sigma_t\big\|_{L^2}^2 \\ &\quad\le \eta\|\nabla\operatorname{div}\mathbf{u}_t\|_{L^2}^2 + C_\eta\|\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2 \\ &\qquad + C\big[\|\sigma\|_{H^2}^2\|(\sigma_t,\mathbf{u}_t)\|_{H^1}^2 + \|\mathbf{u}\|_{H^3}^2\big(\|(\mathbf{u},\sigma)\|_{H^2}^2 + \|(\sigma_t,\mathbf{u}_t)\|_{H^1}^2\big)\big], \end{aligned} \tag{28}$$

where $0<\eta<1$ *is a small positive constant which is to be determined.*

Proof Differentiating (9) with respect to t, we have

$$\begin{aligned} &\rho\big(\mathbf{u}_{tt} + (\mathbf{u}\cdot\nabla)\mathbf{u}_t\big) + \frac{1}{\epsilon}p'(1)\nabla\sigma_t \\ &\quad= \big(2\mu(\rho)+\lambda(\rho)\big)\nabla\operatorname{div}\mathbf{u}_t - \mu(\rho)\operatorname{curl}\operatorname{curl}\mathbf{u}_t \\ &\qquad - \rho_t\big(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) - \rho\mathbf{u}_t\cdot\nabla\mathbf{u} + \left[\frac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\right]_t \\ &\qquad + \big(2\mu(\rho)+\lambda(\rho)\big)_t\nabla\operatorname{div}\mathbf{u} - \mu(\rho)_t\operatorname{curl}\operatorname{curl}\mathbf{u} \\ &\qquad + 2\nabla\mu(\rho)\cdot D(u_t) + \operatorname{div}\mathbf{u}_t\nabla\lambda(\rho) + 2\nabla\big(\mu(\rho)\big)_t\cdot D(u) + \operatorname{div}\mathbf{u}\nabla\big(\lambda(\rho)\big)_t. \end{aligned} \tag{29}$$

Multiplying (29) by $\nabla \operatorname{div} \mathbf{u}$ and integrating in L^2, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigl\|\sqrt{2\mu(\rho)+\lambda(\rho)}\nabla \operatorname{div}\mathbf{u}\bigr\|_{L^2}^2 - \frac{d}{dt}\int_\Omega \rho\mathbf{u}_t\cdot\nabla\operatorname{div}\mathbf{u}\,dx - \frac{p'(1)}{\epsilon}\int_\Omega \nabla\sigma_t\cdot\nabla\operatorname{div}\mathbf{u}\,dx \\
&= \int_\Omega\biggl[\biggl(\frac{p'(1+\epsilon\sigma)-p'(1)}{\epsilon}\nabla\sigma\biggr)_t + \epsilon\sigma_t\mathbf{u}\cdot\nabla\mathbf{u} + \rho\bigl((\mathbf{u}_t\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}_t\bigr)\biggr]\cdot\nabla\operatorname{div}\mathbf{u}\,dx \\
&\quad - \int_\Omega \rho\mathbf{u}_t\cdot\nabla\operatorname{div}\mathbf{u}_t\,dx \\
&\quad + \int_\Omega\biggl[-\frac{1}{2}\bigl(2\mu(\rho)+\lambda(\rho)\bigr)_t\nabla\operatorname{div}\mathbf{u} + \mu(\rho)_t\operatorname{curl}\operatorname{curl}\mathbf{u}\biggr]\cdot\nabla\operatorname{div}\mathbf{u}\,dx \\
&\quad + \int_\Omega \mu(\rho)\operatorname{curl}\operatorname{curl}\mathbf{u}_t\cdot\nabla\operatorname{div}\mathbf{u}\,dx \\
&\quad - \int_\Omega\bigl[2\nabla\mu(\rho)\cdot D(u_t) + \nabla\lambda(\rho)\cdot\nabla\mathbf{u}_t + 2\nabla\bigl(\mu(\rho)\bigr)_t\cdot D(u) + \nabla\bigl(\lambda(\rho)\bigr)_t\cdot\nabla\mathbf{u}\bigr] \\
&\quad \cdot\nabla\operatorname{div}\mathbf{u}\,dx \\
&\le \eta\bigl(\bigl\|(\nabla\operatorname{div}\mathbf{u}_t, \nabla\operatorname{div}\mathbf{u}, \nabla\sigma_t)\bigr\|_{L^2}^2\bigr) + C_\eta\bigl(\|\mathbf{u}_t\|_{L^2}^2 + \|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2\bigr) \\
&\quad + C_\eta\bigl[\|\sigma\|_{H^2}^2\bigl\|(\mathbf{u}_t,\sigma_t)\bigr\|_{H^1}^2 + \|\mathbf{u}\|_{H^3}^2\bigl(\bigl\|(\mathbf{u}_t,\sigma_t)\bigr\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^2\bigr)\bigr],
\end{aligned}
$$

where we have used the following estimate:

$$
\begin{aligned}
&\int_\Omega \mu(\rho)\operatorname{curl}\operatorname{curl}\mathbf{u}_t\cdot\nabla\operatorname{div}\mathbf{u}\,dx \\
&\le \biggl|\int_{\partial\Omega}\mu(\rho)(\mathbf{n}\times\operatorname{curl}\mathbf{u}_t)\cdot\nabla\operatorname{div}\mathbf{u}\,dS\biggr| + \biggl|\int_\Omega \nabla\mu(\rho)\times\operatorname{curl}\mathbf{u}_t\cdot\nabla\operatorname{div}\mathbf{u}\,dx\biggr| \\
&\le \eta\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2 + C_\eta\|\mathbf{u}_t\|_{H^1}^2\|\sigma\|_{H^2}^2.
\end{aligned}
$$

Similarly, we take $\langle\nabla(8), p'(1)\nabla\sigma_t\rangle$ to infer that

$$
\begin{aligned}
&\bigl\|\sqrt{p'(1)}\nabla\sigma_t\bigr\|_{L^2}^2 + \frac{p'(1)}{\epsilon}\int_\Omega \nabla\sigma_t\cdot\nabla\operatorname{div}\mathbf{u}\,dx \\
&= -p'(1)\int_\Omega\bigl((\mathbf{u}\cdot\nabla)\nabla\sigma + \nabla\mathbf{u}\nabla\sigma + \nabla\sigma\operatorname{div}\mathbf{u} + \sigma\nabla\operatorname{div}\mathbf{u}\bigr)\cdot\nabla\sigma_t\,dx \\
&\le \eta\|\nabla\sigma_t\|_{L^2}^2 + C_\eta\|\mathbf{u}\|_{H^2}^2\|\sigma\|_{H^2}^2.
\end{aligned}
$$

Summing up the above inequalities together and choosing η small, we get the above estimate. □

Lemma 3.7 *We have*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2\sigma\|_{L^2}^2 + \bigl\|\sqrt{p'(\rho)^{-1}}\sqrt{2\mu(\rho)+\lambda(\rho)}\nabla^2\operatorname{div}\mathbf{u}\bigr\|_{L^2}^2 \\
&\le \eta\|\mathbf{u}\|_{H^3}^2 + C_\eta\|\sigma\|_{H^2}^4 + C\bigl(\bigl\|\nabla^2\operatorname{curl}\mathbf{u}\bigr\|_{L^2}^2 + \|\mathbf{u}_t\|_{H^1}^2\bigr) \\
&\quad + C\bigl(\|\mathbf{u}\|_{H^2}^4 + \|\sigma\|_{H^2}^2(\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^3}^2)\bigr),
\end{aligned}
\tag{30}
$$

where $0 < \eta < 1$ is a small positive constant which is to be determined.

Proof We differentiate (8) twice with respect to x to have

$$\nabla^2\sigma_t + \mathbf{u}\cdot\nabla\big(\nabla^2\sigma\big) + 2\nabla\mathbf{u}\cdot\nabla(\nabla\sigma) + \nabla^2\mathbf{u}\cdot\nabla\sigma + \nabla^2(\sigma\operatorname{div}\mathbf{u}) + \frac{1}{\epsilon}\nabla^2\operatorname{div}\mathbf{u} = 0. \tag{31}$$

Taking $\langle(31), p'(1)\nabla^2\sigma\rangle$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\sqrt{p'(1)}\nabla^2\sigma\big\|_{L^2}^2 + \frac{p'(1)}{\epsilon}\int_\Omega \nabla^2\operatorname{div}\mathbf{u}\nabla^2\sigma\,dx \\
&\quad = -p'(1)\int_\Omega\big[\big(\mathbf{u}\cdot\nabla\big(\nabla^2\sigma\big) + 2\nabla\mathbf{u}\cdot\nabla(\nabla\sigma) + \nabla^2\mathbf{u}\cdot\nabla\sigma + \nabla^2(\sigma\operatorname{div}\mathbf{u})\big)\big]\nabla^2\sigma\,dx \\
&\quad \le \eta\|\mathbf{u}\|_{H^3}^2 + C_\eta\|\sigma\|_{H^2}^4.
\end{aligned}$$

Then we apply ∇ to (9) to get

$$\begin{aligned}
&\big(2\mu(\rho) + \lambda(\rho)\big)\nabla^2\operatorname{div}\mathbf{u} - \mu(\rho)\nabla\operatorname{curl}\operatorname{curl}\mathbf{u} - \frac{1}{\epsilon}p'(1)\nabla^2\sigma \\
&\quad = \nabla\left[\frac{p'(1+\epsilon\sigma) - p'(1)}{\epsilon}\nabla\sigma\right] + \rho\big(\nabla\mathbf{u}_t + \nabla\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla^2\mathbf{u}\big) + \epsilon\nabla\sigma(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) \\
&\qquad - \nabla\big[2\nabla\mu(\rho)\cdot D(u) + \nabla\big(\lambda(\rho)\big)\operatorname{div}\mathbf{u}\big] \\
&\qquad - \nabla\big(2\mu(\rho) + \lambda(\rho)\big)\nabla\operatorname{div}\mathbf{u} + \nabla\mu(\rho)\operatorname{curl}\operatorname{curl}\mathbf{u},
\end{aligned}$$

which, by multiplying $\nabla^2\operatorname{div}\mathbf{u}$ in L^2, gives

$$\begin{aligned}
&\big\|\sqrt{2\mu(\rho) + \lambda(\rho)}\nabla^2\operatorname{div}\mathbf{u}\big\|_{L^2}^2 - \frac{p'(1)}{\epsilon}\int_\Omega\nabla^2\operatorname{div}\mathbf{u}\nabla^2\sigma\,dx \\
&\quad \le \eta\big\|\nabla^2\operatorname{div}\mathbf{u}\big\|_{L^2}^2 + C_\eta\big(\big\|\nabla^2\operatorname{curl}\mathbf{u}\big\|_{L^2}^2 + \|\mathbf{u}_t\|_{H^1}^2\big) \\
&\qquad + C_\eta\big[\|\sigma\|_{H^2}^2\big(\|\sigma\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2 + \|\mathbf{u}_t\|_{H^1}^2\big) + \|\mathbf{u}\|_{H^2}^4\big],
\end{aligned}$$

where we have used the fact that $\|\nabla\operatorname{curl}\operatorname{curl}\mathbf{u}\|_{L^2} \le \|\nabla^2\operatorname{curl}\mathbf{u}\|_{L^2}$.

Summing up the above two inequalities together and choosing η suitably small, we get the estimate (30). □

Lemma 3.8 *For the solution to* (8)–(11), *we have*

$$\begin{aligned}
&\frac{d}{dt}\big(\|\operatorname{div}\mathbf{u}_t\|_{L^2}^2 + \big\|\sqrt{\rho^{-1}p'(1)}\nabla\sigma_t\big\|_{L^2}^2\big) + \big\|\sqrt{\rho^{-1}}\sqrt{2\mu(\rho) + \lambda(\rho)}\nabla\operatorname{div}\mathbf{u}_t\big\|_{L^2}^2 \\
&\quad \le \eta\big(\|\mathbf{u}_t\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2\big) + C\|\nabla\operatorname{curl}\mathbf{u}_t\|_{L^2}^2 + C_\eta\big(\|\sigma\|_{H^2}^2\|\mathbf{u}_t\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2 \\
&\qquad + \|\mathbf{u}\|_{H^2}^4 + \|\sigma_t\|_{H^1}^2\big(\|\mathbf{u}_t\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2 + \|\sigma\|_{H^2}^2 + \|\sigma_t\|_{H^1}^2\big)\big), \quad 0 < \eta < 1,
\end{aligned}$$

where η *is a small positive constant which is to be determined.*

Proof It is obvious that $\mathbf{u}_{tt}\cdot \mathrm{n}|_{\partial\Omega} = 0$, thus

$$\int_\Omega \mathbf{u}_{tt}\nabla\operatorname{div}\mathbf{u}_t\,dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega|\operatorname{div}\mathbf{u}_t|^2\,dx.$$

We take $\langle(29), \rho^{-1}\nabla \operatorname{div} \mathbf{u}_t\rangle$ and $\langle\partial_t\nabla(8), \rho^{-1}p'(1)\nabla\sigma_t\rangle$, summing up the resulting equations to obtain the above lemma. Here we use the following estimate:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \partial_t\big(\rho^{-1}p'(1)\big)|\nabla\sigma_t|^2\,dx - \int_\Omega \rho^{-1}p'(1)\nabla\sigma_t\cdot\mathbf{u}\cdot\nabla^2\sigma_t\,dx \\
&\quad= \frac{1}{2}\int_\Omega \partial_t\big(\rho^{-1}p'(1)\big)|\nabla\sigma_t|^2\,dx + \frac{1}{2}\int_\Omega \operatorname{div}\big(\rho^{-1}p'(1)\mathbf{u}\big)|\nabla\sigma_t|^2\,dx \\
&\quad= -\frac{1}{2}\int_\Omega \rho_t\rho^{-2}p'(1)|\nabla\sigma_t|^2\,dx + \frac{1}{2}\int_\Omega \rho^{-2}p'(1)\operatorname{div}(\rho\mathbf{u})|\nabla\sigma_t|^2\,dx \\
&\quad= -\frac{1}{2}\int_\Omega \big[\rho_t + \operatorname{div}(\rho\mathbf{u})\big]\rho^{-2}p'(1)|\nabla\sigma_t|^2\,dx \\
&\quad= 0.
\end{aligned}$$

□

Next, we estimate the derivatives of curl $\mathbf{u}$.

Lemma 3.9

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}\omega_t\|_{L^2}^2 + \gamma_3\|\operatorname{curl}\omega_t\|_{L^2}^2 \\
&\quad\le \eta\|\omega_t\|_{L^2}^2 + C_\eta\Big(\big(\|\sigma_t\|_{H^1}^2 + \|\sigma\|_{H^2}^2\big)\big(\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^3}^2\big) + \|\mathbf{u}_t\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2\Big) \\
&\qquad + C\|\sigma\|_{H^2}^4\|\mathbf{u}\|_{H^2}^2.
\end{aligned}$$

where $\gamma_3 > 0$ is a positive constant and $0 < \eta < 1$ is a small positive constant which is to be determined.

Proof Firstly, we apply ∂_t to $(23)_1$ to see that

$$\rho(\omega_{tt} + \mathbf{u}\cdot\nabla\omega_t) - \mu(\rho)\triangle\omega_t = h, \tag{32}$$

where

$$h := -\epsilon\sigma_t(\omega_t + \mathbf{u}\cdot\nabla\omega) - \rho\mathbf{u}_t\nabla\omega + \partial_t\big(\mu(\rho)\big)\triangle\omega + g_t,$$

with

$$\begin{aligned}
|g_t| \le{}& C\Big(\epsilon|\sigma_t||\nabla u|^2 + |\nabla u_t||\nabla u| + \epsilon^2|\sigma_t||\nabla\sigma|\big(|\nabla\sigma||\nabla\mathbf{u}| + \big|\nabla^2 u\big|\big) \\
&+ |\nabla\sigma_t|\big(|\nabla\sigma||\nabla\mathbf{u}| + \big|\nabla^2 u\big|\big) + \epsilon|\nabla\sigma|\big(|\nabla\sigma_t||\nabla\mathbf{u}| + \epsilon|\nabla\sigma||\nabla u_t| + \big|\nabla^2 u_t\big|\big) \\
&+ \epsilon\big(|\nabla\sigma_t|\big|\nabla^2 u\big| + |\nabla\sigma|\big|\nabla^2 u_t\big|\big)\Big).
\end{aligned}$$

Obviously, the boundary condition for (32) reads

$$\omega_t\times\mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{33}$$

Therefore, by virtue of (33) and integration by parts,

$$\begin{aligned}
&-\int_\Omega \mu(\rho)\Delta\omega_t\cdot\omega_t\,dx\\
&\quad=\int_\Omega \mu(\rho)\operatorname{curl}\operatorname{curl}\omega_t\cdot\omega_t\,dx\\
&\quad=\int_\Omega \mu(\rho)|\operatorname{curl}\omega_t|^2\,dx+\int_\Omega \nabla\mu(\rho)\times\operatorname{curl}\omega_t\cdot\omega_t\,dx+\int_{\partial\Omega}\mu(\rho)\operatorname{curl}\omega_t(\omega_t\times\mathbf{n})\,dS.
\end{aligned}$$

Multiplying (32) by ω_t in L^2, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\omega_t\|_{L^2}^2+\big\|\sqrt{\mu(\rho)}\operatorname{curl}\omega_t\big\|_{L^2}^2\\
&\quad=\int_\Omega h\cdot\omega_t\,dx-\int_\Omega\nabla\mu(\rho)\times\nabla\omega_t\cdot\omega_t\,dx\\
&\quad\le\|\omega_t\|_{H^1}\big[\|g_t\|_{L^2}+\|\sigma_t\|_{H^1}\big(\|\mathbf{u}_t\|_{H^2}+\|\mathbf{u}\|_{H^2}\|\mathbf{u}\|_{H^3}\big)\\
&\qquad+\|\mathbf{u}_t\|_{H^1}\|\mathbf{u}\|_{H^3}+\|\sigma_t\|_{H^1}\|\mathbf{u}\|_{H^3}+\|\sigma\|_{H^2}\|\mathbf{u}_t\|_{H^2}\big].
\end{aligned}\tag{34}$$

□

With the aid of Lemma 2.2 and (33), we have

$$\|\omega_t\|_{H^1}^2\le C\big(\|\operatorname{curl}\omega_t\|_{L^2}^2+\|\omega_t\|_{L^2}^2\big).$$

Using Young's inequality and the above equality, we obtain the following lemma.

Lemma 3.10 *For the solution to* (8)–(11), *we have*

$$\begin{aligned}
&\frac{d}{dt}\big\|\sqrt{\mu(\rho)}\nabla\omega\big\|_{L^2}^2+\|\sqrt{\rho}\omega_t\|_{L^2}^2+\frac{1}{20}\big\|\sqrt{\mu(\rho)}\Delta\omega\big\|_{L^2}^2\\
&\quad\le\eta\big(\|\omega_t\|_{L^2}^2+\|\operatorname{curl}\omega\|_{L^2}^2\big)+C_\eta\big(\|\mathbf{u}\|_{H^3}^2\big(\|\mathbf{u}\|_{H^2}^2+\|\sigma\|_{H^2}^2+\|\sigma_t\|_{H^1}^2\big)+\|\mathbf{u}_t\|_{H^2}^2\|\sigma\|_{H^2}^2\big),
\end{aligned}$$

where $0<\eta<1$ *is a small positive constant which is to be determined.*

Proof We take $\langle(23)_1,\omega_t-\delta\triangle\omega\rangle$ (in which δ is a positive constant to be determined later) to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\sqrt{\delta\rho+\mu(\rho)}\operatorname{curl}\omega\big\|_{L^2}^2+\|\sqrt{\rho}\omega_t\|_{L^2}^2+\delta\big\|\sqrt{\mu(\rho)}\triangle\omega\big\|_{L^2}^2\\
&\quad=\int_\Omega g\cdot(\omega_t-\delta\triangle\omega)\,dx+\frac{1}{2}\int_\Omega\partial_t\big(\delta\rho+\mu(\rho)\big)|\operatorname{curl}\omega|^2\,dx\\
&\qquad+\int_\Omega\nabla\big(\delta\rho+\mu(\rho)\big)\times\operatorname{curl}\omega\cdot\omega_t\,dx+\int_\Omega\rho\mathbf{u}\cdot\nabla\omega\cdot(\omega_t-\delta\triangle\omega)\,dx\\
&\quad\le\eta\big(\|\omega_t\|_{L^2}^2+\|\triangle\omega\|_{L^2}^2\big)+C_\eta\big(\|\mathbf{u}\|_{H^2}^4+\|\sigma\|_{H^2}^2\big(\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}_t\|_{H^1}^2\big)\big)\\
&\qquad+\eta\|\operatorname{curl}\omega\|_{L^2}^2+C_\eta\big(\|\sigma_t\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}_t\|_{H^2}^2\big),
\end{aligned}$$

where we use the following estimate:

$$
\begin{aligned}
&-\int_\Omega \big(\delta\rho + \mu(\rho)\big)\Delta\omega \cdot \omega_t\,dx \\
&\quad= \int_\Omega \big(\delta\rho + \mu(\rho)\big)\operatorname{curl}\operatorname{curl}\omega \cdot \omega_t\,dx \\
&\quad= \int_\Omega \big(\delta\rho + \mu(\rho)\big)\operatorname{curl}\omega \cdot \operatorname{curl}\omega_t\,dx + \int_\Omega \nabla\big(\delta\rho + \mu(\rho)\big) \times \operatorname{curl}\omega \cdot \omega_t\,dx \\
&\qquad+ \int_{\partial\Omega} \big(\delta\rho + \mu(\rho)\big)\operatorname{curl}\omega \cdot (\omega_t \times \mathbf{n})\,dS \\
&\quad= \frac{1}{2}\frac{d}{dt}\big\|\sqrt{\delta\rho + \mu(\rho)}\operatorname{curl}\omega\big\|_{L^2}^2 - \frac{1}{2}\int_\Omega \partial_t\big(\delta\rho + \mu(\rho)\big)|\operatorname{curl}\omega|^2\,dx \\
&\qquad+ \int_\Omega \nabla\big(\delta\rho + \mu(\rho)\big) \times \operatorname{curl}\omega \cdot \omega_t\,dx.
\end{aligned}
$$

Thus, we choose δ and η suitably small to conclude the lemma.

In order to close the estimates, we have to estimate $\|\sigma\|_{H^2}$. To this end, we obtain from the continuity equation (8) and the boundary condition $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$

$$
\frac{d}{dt}\int_\Omega \sigma\,dx = -\int_{\partial\Omega}\left(\sigma + \frac{1}{\epsilon}\right)\mathbf{u}\cdot\mathbf{n}\,dS = 0,
$$

thus

$$
\int_\Omega \sigma\,dx = \int_\Omega \sigma_0\,dx = 0.
$$

From Eqs. (9) and Poincare's inequality, we have

$$
\begin{aligned}
\|\sigma\|_{H^2}^2 &\le C\|\nabla\sigma\|_{H^1}^2 \\
&\le C\epsilon^2\big(\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^3}^2\big) \\
&\quad+ C\epsilon\big[\|\sigma\|_{H^2}^2\big(\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^3}^2 + \|\mathbf{u}\|_{H^2}^4\big) + \|\mathbf{u}\|_{H^2}^4\big].
\end{aligned}
\tag{35}
$$

In addition, in order to control the terms $\|\mathbf{u}_t\|_{H^2}$ and $\|\mathbf{u}\|_{H^3}$, we use the following fact which is obtained from Lemmas 2.1–2.2 and the boundary condition (11):

$$
\begin{aligned}
&\|\mathbf{u}\|_{H^3} \le C\big(\|\operatorname{div} u\|_{H^2} + \|\operatorname{curl}\mathbf{u}\|_{H^2} + \|\mathbf{u}\|_{H^2}\big), \\
&\|\operatorname{curl}\mathbf{u}\|_{H^2} \le C\big(\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|_{H^1} + \|\operatorname{curl}\mathbf{u}\|_{H^1}\big), \\
&\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|_{H^1} \le C\big(\|\triangle\operatorname{curl} u\|_{L^2} + \|\operatorname{curl}\operatorname{curl}\mathbf{u}\times\mathbf{n}\|_{H^{1/2}(\partial\Omega)} + \|\operatorname{curl}\mathbf{u}\|_{L^2}\big), \\
&\|\mathbf{u}_t\|_{H^2} \le C\big(\|\operatorname{div} u_t\|_{H^1} + \|\operatorname{curl}\mathbf{u}_t\|_{H^1} + \|\mathbf{u}_t\|_{H^1}\big), \\
&\|\operatorname{curl}\mathbf{u}_t\|_{H^1} \le C\big(\|\operatorname{curl}\operatorname{curl}\mathbf{u}_t\|_{L^2} + \|\operatorname{curl}\mathbf{u}_t\|_{L^2}\big),
\end{aligned}
\tag{36}
$$

where the estimate of the term $\|\operatorname{curl}\operatorname{curl}\mathbf{u}\times\mathbf{n}\|_{H^{1/2}(\partial\Omega)}$ is crucial for the proof. We can estimate it by the strategy in [22] as follows: In order to derive an estimate near the boundary, we firstly construct the local coordinates by the isothermal coordinates $\lambda(\psi,\varphi)$, where $\lambda(\psi,\varphi)$ satisfies

$$\lambda_\psi \cdot \lambda_\psi > 0, \qquad \lambda_\varphi \cdot \lambda_\varphi > 0 \quad \text{and} \quad \lambda_\psi \cdot \lambda_\varphi = 0.$$

We cover the boundary $\partial\Omega$ by a finite number of bounded open sets $W^k \subset \mathcal{R}^3$, $k = 1,2,\dots,L$, such that, for any $x \in W^k \cap \Omega$,

$$x = \lambda^k(\psi,\varphi) + rn\big(\lambda^k(\psi,\varphi)\big) = \Lambda^k(\psi,\varphi,r),$$

where $\lambda^k(\psi,\varphi)$ is the isothermal coordinate and n is the unit outer normal to $\partial\Omega$. For simplicity, we will omit the superscript k in each W^k. Then we construct the orthonormal system corresponding to the local coordinates by

$$e_1 = \frac{\lambda_\psi}{|\lambda_\psi|}, \qquad e_2 = \frac{\lambda_\varphi}{|\lambda_\varphi|}, \qquad e_3 = n(\lambda) = e_1 \times e_2.$$

By direct calculations, we can use the fact that $J \in C^2$ and

$$\begin{aligned} J &= \det \operatorname{Jac} \Lambda = (\Lambda_\psi \times \Lambda_\varphi) \cdot e_3 \\ &= |\lambda_\psi||\lambda_\varphi| + r\big(|\lambda_\psi| n_\varphi \cdot e_2 + |\lambda_\varphi| n_\psi \cdot e_1\big) + r^2\big[(n_\psi \cdot e_1)(n_\varphi \cdot e_2) - (n_\psi \cdot e_2)(n_\varphi \cdot e_1)\big] > 0, \end{aligned}$$

for sufficiently small $r > 0$. Furthermore, we can derive some other relations:

$$\begin{aligned} &\operatorname{Jac}\big(\Lambda^{-1}\big) = (\operatorname{Jac}\Lambda)^{-1}, \\ &\big[\nabla\big(\Lambda^{-1}\big)^1\big] \circ \Lambda = \frac{1}{J}(\Lambda_\psi \times e_3), \\ &\big[\nabla\big(\Lambda^{-1}\big)^2\big] \circ \Lambda = \frac{1}{J}(e_3 \times \Lambda_\varphi), \\ &\big[\nabla\big(\Lambda^{-1}\big)^3\big] \circ \Lambda = \frac{1}{J}(\Lambda_\varphi \times \Lambda_\psi), \end{aligned}$$

where the notation '$\circ$' is the composite of operators. Set $y := (y_1,y_2,y_3) := (\psi,\varphi,r)$, $a_{ij} = ((\operatorname{Jac}\Lambda)^{-1})_{ij}$. Then $n = (a_{31},a_{32},a_{33})$, the tangential directions $\tau_i = (a_{i1},a_{i2},a_{i3})$ $(i = 1,2)$, and

$$a_{ij}a_{3j} = 0, \quad \text{for } i = 1,2.$$

We denote by D_i the partial derivative with respect to y_i in local coordinates. To be precise, D_3 is the normal derivative and D_i for $i = 1,2$ are the tangential derivatives in the original coordinates. Moreover, we have

$$\partial_{x_j} = a_{kj}D_k.$$

Next, we denote the vorticity near the boundary as $\widetilde{w} := (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)^t := w(t, \Lambda(y))$. So we get

$$\begin{aligned}
\operatorname{curl} w \cdot n &= (a_{k2}D_k\widetilde{w}_3 - a_{k3}D_k\widetilde{w}_2, a_{k3}D_k\widetilde{w}_1 - a_{k1}D_k\widetilde{w}_3, a_{k1}D_k\widetilde{w}_2 - a_{k2}D_k\widetilde{w}_1) \\
&\quad \cdot (a_{31}, a_{32}, a_{33}) \\
&= \big[(a_{32}a_{13} - a_{33}a_{12})D_1 + (a_{32}a_{23} - a_{33}a_{22})D_2\big]\widetilde{w}_1 \\
&\quad + \big[(a_{33}a_{11} - a_{31}a_{13})D_1 + (a_{33}a_{21} - a_{31}a_{23})D_2\big]\widetilde{w}_2 \\
&\quad + \big[(a_{31}a_{12} - a_{32}a_{11})D_1 + (a_{31}a_{22} - a_{32}a_{21})D_2\big]\widetilde{w}_3 \\
&= \sum_{i=1}^{2}(n \times \tau_i) \cdot D_i\widetilde{w} \\
&= \sum_{i=1}^{2}\big(D_i\big((n \times \tau_i) \cdot \widetilde{w}\big) - D_i(n \times \tau_i) \cdot \widetilde{w}\big) \\
&= \sum_{i=1}^{2}\big(D_i\big((n \times \widetilde{w}) \cdot \tau_i\big) - D_i(n \times \tau_i) \cdot \widetilde{w}\big).
\end{aligned}$$

Thus, with the boundary condition (11) we get the estimate

$$\begin{aligned}
\|\operatorname{curl} w \cdot n\|_{H^{\frac{1}{2}}(\partial\Omega)} &\le C\sum_{i=1}^{3}\|w_j\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\le C\|u\|_{H^2}.
\end{aligned} \tag{37}$$

□

Definition 3.2 We define

$$\Psi = \sum_{i=1}^{2}\Psi_i(t), \qquad \Phi = \sum_{i=1}^{2}\Phi_i(t), \tag{38}$$

where $\Psi_1(t)$ and $\Phi_1(t)$ are defined by (26), and

$$\begin{aligned}
\Psi_2(t) &= \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 - 2\int_\Omega \rho\mathbf{u}_t \cdot \nabla \operatorname{div} \mathbf{u}\,dx + \big\|\nabla^2\sigma\big\|_{L^2}^2 + \|\operatorname{div} \mathbf{u}_t\|_{L^2}^2 \\
&\quad + \|\nabla\sigma_t\|_{L^2}^2 + \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\operatorname{curl} \mathbf{u}_t\|_{L^2}^2, \\
\Phi_2(t) &= \big(\big\|\nabla^2 \operatorname{div} \mathbf{u}\big\|_{L^2}^2 + \|\nabla \operatorname{div} \mathbf{u}_t\|_{L^2}^2\big) + \|\nabla\sigma_t\|_{L^2}^2 + \|\nabla \operatorname{curl} \mathbf{u}_t\|_{L^2}^2 \\
&\quad + \big\|\nabla^2 \operatorname{curl} \mathbf{u}\big\|_{L^2}^2 + \|\sigma\|_{H^2}^2.
\end{aligned}$$

Combining Lemmas 3.1–3.10 with the estimates (35)–(37) and choosing a suitable constant C, and small enough constants ϵ and η, we finally conclude that

$$\frac{d}{dt}\Psi(t) + \Phi(t) \le c_0\Phi(t)\big(\Psi(t) + \Psi^2(t)\big), \tag{39}$$

where $c_0 \ge 1$ is a constant independent of ϵ.

Now, employing (39), and following the analysis in [25], we obtain the following uniform estimate.

Lemma 3.11 (Uniform estimate) *Let $\Omega \subset \mathcal{R}^3$ be a simply connected, bounded domain with smooth boundary $\partial\Omega$. Let (u,σ) be a solution to (8)–(11) in $\Omega \times (0,T)$ with $c^{-1} \le 1+\epsilon\sigma \le c$ for some $c > 1$, $\forall (x,t) \in \Omega \times (0,T)$, $\epsilon \in (0,\epsilon_1]$. Suppose that*

$$\Psi(0) \le \beta/(2c_0), \quad \beta \in \left(0, \frac{1}{2}\right].$$

Then we have

$$\Psi(t) \le \beta/(2c_0), \quad t \in [0,T].$$

4 Proof of Theorems 1.1 and 1.2

Now, recalling the definition (38) of $(\Psi(t), \Phi(t))$, we can use the uniform a priori estimate established in Lemma 3.11 to continue the local solution $(\sigma, \mathbf{u})$ globally in time by applying the standard extension techniques (see, for example, [27]), and obtain therefore a global solution. Furthermore, we can employ the uniform estimate given in Lemma 3.11 and the Arzelà–Ascoli theorem to easily show the strong convergence of $(\sigma, \mathbf{u})$ to the solution of the corresponding incompressible Navier–Stokes equation as $\epsilon \to 0$. This completes the proof of Theorems 1.1 and 1.2.

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Authors' contributions
All authors carried out the theoretical studies, participated in the design of the study, and drafted the manuscript. All authors read and approved the final manuscript.

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