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Nonlinear nonhomogeneous Dirichlet problems with singular and convection terms

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Abstract

We consider a nonlinear Dirichlet problem driven by a general nonhomogeneous differential operator and with a reaction exhibiting the combined effects of a parametric singular term plus a Carathéodory perturbation $f(z, x, y)$ which is only locally defined in $x \in \mathbb{R}$. Using the frozen variable method, we prove the existence of a positive smooth solution, when the parameter is small.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper we study the following nonhomogeneous parametric singular Dirichlet problem with gradient dependence (convection):

$$\begin{cases} -\operatorname{div} a(Du(z)) = \lambda u(z)^{-\eta} + f(z, u(z), Du(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \qquad u > 0, \qquad \lambda > 0, \qquad 0 < \eta < 1. \end{cases} \tag{P_λ}$$

In this problem, the map $a : \mathbb{R}^N \mapsto \mathbb{R}^N$ in the differential operator is continuous and strictly monotone (thus, maximal monotone too) and satisfies certain other growth and regularity conditions listed in hypotheses H_0 below (see Sect. 2). These conditions are general and provide a broad framework in which we can fit many differential operators of interest, such as the p-Laplacian and the (p, q)-Laplacian (that is, the sum of a p-Laplacian and of q-Laplacian). In the reaction (right-hand side) of (P_λ), we have the competing effects of a parametric singular term $u \mapsto \lambda u^{-\eta}$ with $\lambda > 0$ being the parameter and of a perturbation $u \mapsto f(z, u, Du)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $y \in \mathbb{R}^N$ the function $z \mapsto f(z, x, y)$ is measurable and for a.a. $z \in \Omega$, $(x, y) \mapsto f(z, x, y)$ is continuous). So, this perturbation is gradient dependent and on $f(z, \cdot, y)$ we do not impose any growth condition. Instead we assume that near zero the function $x \mapsto f(z, x, y)$ exhibits a kind of oscillatory behavior.

The gradient dependence of the reaction means that problem (P_λ) is not variational and so eventually our method of proof is going to be topological. We use the so-called

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"frozen variable method". According to this approach, in the perturbation $f(z, x, y)$, we fix ("freeze") the y-variable. This leads to a variational problem, which a priori can be solved using tools from the critical point theory. However, the presence of the singular term leads to an energy functional which is not C^1 and so we have difficulty in applying the minimax theorems of critical point theory. So, we need to find a way to bypass the singularity and deal with C^1-functionals. This is done by considering the purely singular problem (that is, $f \equiv 0$), which we show that for every $\lambda > 0$ it has a unique positive smooth solution which converges to zero in $C_0^1(\overline{\Omega})$ as $\lambda \to 0^+$. We use this solution and its properties and truncation techniques, to show that, for all small values of the parameter $\lambda > 0$, the "frozen problem" has at least one positive smooth solution. In order to use topological tools (fixed point theory), we need to find a canonical way to choose such a positive solution. This is done by showing that the frozen problem has a smallest positive solution (minimal positive solution). Then we show that the minimal solution map satisfies all the requirements of Leray–Schauder alternative principle (see Sect. 2) and so we can produce a positive solution for (P_λ) when $\lambda > 0$ is small.

Recently there have been published some existence results for nonlinear problem with convection. We mention the work of Bai [1], Bai–Gasinski–Papageorgiou [2], Candito–Gasinski–Papageorgiou [4], Faraci–Motreanu–Puglisi [5], Gasinski–Krech–Papageorgiou [6], Gasinski–Papageorgiou [10], Gasinski–Winkert [11], Liu–Papageorgiou [18], Papageorgiou–Rădulescu–Repovš [20], Papageorgiou–Vetro–Vetro [24], Tanaka [31]. However, none of the aforementioned works involves singular terms. The only work examining the combined effects of singular and convection terms, is the recent paper of Papageorgiou–Rădulescu–Repovš [22], which deals with a nonparametric Neumann problem driven by the p-Laplacian. Nonlinear singular Dirichlet problems were also investigated in the paper of Papageorgiou–Winkert [25] for different settings and conditions.

2 Mathematical background—hypotheses

The main spaces in the analysis of our problem (P_λ), are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \left. \frac{\partial u}{\partial n} \right|_{\partial\Omega} < 0 \right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Let X be a Banach space and $\xi : X \mapsto X$. We say that $\xi(\cdot)$ is "compact" if it is continuous and for every $B \subseteq X$ bounded, the set $\overline{\xi(B)} \subseteq X$ is compact. The "Leray–Schauder alternative principle" asserts the following.

Theorem 1 *If X is a Banach space, $\xi : X \mapsto X$ is compact and*

$$\mathcal{D}(\xi) = \big\{ u \in X : u = t\xi(u) \text{ for some } 0 < t < 1 \big\},$$

then the following alternative holds:

(a) $\mathcal{D}(\xi)$ *is unbounded; or*

(b) $\xi(\cdot)$ *has a fixed point.*

Let $\beta \in C^1(0,\infty)$ with $\beta(t) > 0$ for all $t > 0$ and assume that

$$0 < \hat{c} \le \frac{\beta'(t)t}{\beta(t)} \le c_0,$$

$$c_1 t^{p-1} \le \beta(t) \le c_2\bigl(t^{s-1} + t^{p-1}\bigr),$$

for all $t > 0$, some $1 \le s < p < N$, $0 < c_1 < c_2$.

Then our hypotheses on the map $a(\cdot)$ are the following.

H_0: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and

    (i) $a_0 \in C^1(0,\infty)$, $t \mapsto a_0(t)t$ is strictly increasing on $(0,\infty)$, $a_0(t)t \to 0^+$ as $t \to 0^+$ and $\lim_{t \to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1$;

    (ii) there exists $c_3 > 0$ such that

$$\bigl|\nabla a(y)\bigr| \le c_3 \frac{\beta(|y|)}{|y|} \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\};$$

    (iii) $\frac{\beta(|y|)}{|y|}|\xi|^2 \le (\nabla a(y)\xi, \xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$;

    (iv) if $G_0(t) = \int_0^t a_0(s)s\,ds$, $t \ge 0$, then there exist $q \in (1,p)$ and $c_* > 0$ such that

$$\limsup_{t \to 0^+} \frac{qG_0(t)}{t^q} \le c_*.$$

Remark 1 Such conditions on the differential operator were used in the context of singular or convection problems, also by Papageorgiou–Rădulescu–Repovš [23] and Candito–Gasinski–Papageorgiou [4]. Hypotheses H_0(i), (ii), (iii) are motivated by the nonlinear regularity theory of Lieberman [17] and the nonlinear maximum principle of Pucci–Serrin [28] (p. 111). Hypothesis H_0(iv) serves the needs of our problem and it is mild. As we will see in the examples listed below, it is satisfied in all cases of interest.

Clearly the above hypotheses imply that the primitive $G_0(\cdot)$ is strictly convex and strictly increasing. Let $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Evidently $G \in C^1(\mathbb{R}^N, \mathbb{R})$, it is convex and we have

$$\nabla G(y) = G_0'\bigl(|y|\bigr)\frac{y}{|y|} = a_0\bigl(|y|\bigr)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\},$$

$$\nabla G(0) = 0.$$

So, $G(\cdot)$ is the primitive of $a(\cdot)$ and on account of the convexity of $G(\cdot)$, we have

$$G(y) \le \bigl(a(y), y\bigr)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{1}$$

Hypotheses H_0 lead easily to the following properties of the map $a(\cdot)$.

Lemma 2 *The mapping $a(\cdot)$ is continuous, strictly monotone (hence maximal monotone too) and*

(a) $|a(y)| \leq c_4(|y|^{s-1} + |y|^{p-1})$ *for some $c_4 > 0$, all $y \in \mathbb{R}^N$;*

(b) $\frac{c_1}{p-1}|y|^p \leq (a(y), y)_{\mathbb{R}^N}$ *for all $y \in \mathbb{R}^N$.*

Using this lemma and (1) we are led to the following growth restrictions on $G(\cdot)$.

Corollary 3 *We have $\frac{c_1}{p-1}|y|^p \leq G(y) \leq c_5(1 + |y|^p)$ for some $c_5 > 0$, all $y \in \mathbb{R}^N$.*

Examples The following maps $a(\cdot)$ satisfy hypotheses H_0 (see Papageorgiou–Rădulescu [19]):

(i) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$.

This map corresponds to the p-Laplace differential operator defined by

$$\Delta_p u = \mathrm{div}\left(|Du|^{p-2}Du\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

(ii) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p$.

This map corresponds to the (p, q)-Laplace differential operator defined by

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Such operators arise often in the mathematical models of physical processes and recently there have been published several works dealing with equations driven by such operators. We mention the works of Bobkov–Tanaka [3], Papageorgiou–Zhang [26, 27], Rădulescu [29], Ragusa–Tachikawa [30].

(iii) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}} y$ with $1 < p < \infty$.

This map corresponds to the extended capillary differential operator defined by

$$\mathrm{div}\left((1 + |Du|^2)^{\frac{p-2}{2}} Du\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Let $A : W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \ (\frac{1}{p} + \frac{1}{p'} = 1)$ be the nonlinear operator defined by

$$\langle A(u), h \rangle = \int_\Omega \left(a(Du), Dh\right)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

From Gasinski–Papageorgiou [9] (Problem 2.192, p. 279), we have the following properties for this operator.

Proposition 4 *The operator $A(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$$\text{``} u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0$$

$$\Rightarrow \quad u_n \to u \quad \text{in } W_0^{1,p}(\Omega).\text{''}$$

In the sequel by $\hat{\lambda}_1(r)$ we denote the first (principal) eigenvalue of $(-\Delta_r, W_0^{1,r}(\Omega))$ $(1 < r < \infty)$. We know that

- $\hat{\lambda}_1(r) > 0,$ it is simple and isolated.

- $\hat{\lambda}_1(r) = \inf\left\{ \dfrac{\|Du\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0 \right\}.$ (2)

In (2) the infimum is realized on the corresponding one-dimensional eigenspace the elements of which do not change sign. By $\hat{u}_1(r)$ we denote the L^r-normalized (that is, $\|\hat{u}_1(r)\|_r = 1$), positive eigenfunction corresponding to $\hat{\lambda}_1(r)$. The nonlinear regularity theory and the nonlinear maximum principle (see Gasinski–Papageorgiou [8], pp. 738–739), imply $\hat{u}_1(r) \in \text{int } C_+$. So, we have $\|D\hat{u}_1(r)\|_r^r = \hat{\lambda}_1(r)$. We mention that every eigenvalue $\hat{\lambda}$ of $(-\Delta_r, W_0^{1,r}(\Omega))$ distinct from $\hat{\lambda}_1(r)$ has eigenfunctions which are nodal (sign-changing).

Given $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \qquad u = u^+ - u^-, \qquad |u| = u^+ + u^-.$$

If $u, v : \Omega \mapsto \mathbb{R}$ are measurable functions and $u \leq v$, then we define

$$[u, v] = \left\{ h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega \right\}.$$

A set $S \subseteq W_0^{1,p}(\Omega)$ is said to be the downward directed, if for every $u, v \in S$, we can find $y \in S$ such that $y \leq u$ and $y \leq v$.

Now we introduce the hypotheses on perturbation $f(z, x, y)$.

H_1: $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a Carathéodory function, $f(z, 0, y) = 0$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ and

    (i) there exist $0 < \delta_0 < \theta$ and $c_6 > 0$ such that

$$\left| f(z, x, y) \right| \leq c_6\left(1 + |y|^{p-1}\right) \quad \text{for a.a. } z \in \Omega, \text{all } 0 \leq x \leq \theta, \text{all } y \in \mathbb{R}^N,$$

$$\theta^{-\eta} + f(z, \theta, y) \leq 0, \quad \text{for a.a. } z \in \Omega, \text{all } y \in \mathbb{R}^N,$$

$$f(z, x, y) \geq 0 \quad \text{for a.a. } z \in \Omega, \text{all } 0 \leq x \leq \delta_0, \text{all } y \in \mathbb{R}^N;$$

    (ii) for every $M > 0$, there exists $\eta_M \in L^\infty(\Omega)$ such that

$$\eta_M(z) \geq \frac{c_*}{q}\hat{\lambda}_1(q) \quad \text{for a.a. } z \in \Omega, \qquad \eta_M \not\equiv \frac{c_*}{q}\hat{\lambda}_1(q),$$

$$\liminf_{x \to 0^+} \frac{f(z, x, y)}{x^{q-1}} \geq \eta_M(z) \quad \text{uniformly for a.a. } z \in \Omega, \text{all } |y| \leq M;$$

    (iii) for a.a. $z \in \Omega$, all $0 \leq x \leq \theta$, all $y \in \mathbb{R}^N$ and all $0 < t < 1$, we have

$$f\left(z, \frac{1}{t}x, y\right) \leq \frac{1}{t^{p-1}}f(z, x, y).$$

Remark 2 The above hypotheses concern only the behavior of $f(z, \cdot, y)$ near zero. No global growth condition is imposed on $f(z, \cdot, y)$. Hypothesis H_1(i) dictates an oscillatory behavior for $f(z, \cdot, y)$. It starts positive and by the time we have reached $x = \theta$, the perturbation

$f(z, \cdot, y)$ has become negative. In the case of the equation driven by the q-Laplacian, hypothesis H_1(ii) is a nonuniform nonresonance condition at zero. Hypothesis H_1(iii) is satisfied if for a.a. $z \in \Omega$ and all $y \in \mathbb{R}^N$ the quotient function $x \mapsto \frac{f(z,x,y)}{x^{p-1}}$ is nonincreasing on $(0, \theta]$.

Example The following function satisfies hypotheses H_1 above. For the sake of simplicity, we drop the z-dependence. We have

$$f(x, y) = \eta\big(x^{p-1} - 2x^{r-1}\big)\big(1 + |y|^{p-1}\big) \quad \text{for all } x \geq 0, \text{all } y \in \mathbb{R}^N,$$

with $\eta > \max\{\frac{c_*}{q}\hat{\lambda}_1(q), 1\}$, $1 < q < r$. For $\theta = 1$ hypothesis H_1(i) is satisfied.

As we already mentioned in the Introduction, due to the presence of the singular term, we have an energy functional which is not C^1 and this prevents us from using the tools of critical point theory. So, we need to find a way to bypass the singularity and deal with C^1-functionals. For this reason in the next section we deal with the purely singular problem ($f \equiv 0$). A solution of this problem will help us isolate the singularity.

3 Purely singular problem

In this section we examine the following purely singular problem:

$$-\operatorname{div} a\big(Du(z)\big) = \lambda u(z)^{-\eta} \quad \text{in } \Omega, u|_{\partial\Omega} = 0, u > 0. \tag{Q_λ}$$

Proposition 5 *If hypotheses* H_0 *hold, then for every* $\lambda > 0$ *problem* (Q_λ) *admits a unique solution* $\overline{u} \in \operatorname{int} C_+$ *and* $\overline{u}_\lambda \to 0$ *in* $C_0^1(\overline{\Omega})$ *as* $\lambda \to 0^+$.

Proof Let $g \in L^p(\Omega)$ and $\varepsilon > 0$. We consider the following auxiliary Dirichlet problem:

$$-\operatorname{div} a\big(Du(z)\big) = \frac{\lambda}{(|g(z)| + \varepsilon)^\eta} \quad \text{in } \Omega, u|_{\partial\Omega} = 0, u \geq 0. \tag{Au_λ}$$

Recall that $A : W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ is the operator defined by

$$\big\langle A(u), h \big\rangle = \int_\Omega \big(a(Du)\big), Dh\big)_{\mathbb{R}^N} \, dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

On account of Proposition 4, the operator $A(\cdot)$ is continuous and maximal monotone. In addition, from Lemma 2(b) we see that $A(\cdot)$ is coercive. Therefore $A(\cdot)$ is surjective (see Papageorgiou–Rădulescu–Repovš [21], Corollary 2.8.7, p. 135). So, we can find $\hat{u}_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$A(\hat{u}_\varepsilon) = \frac{\lambda}{(|g(z)| + \varepsilon)^\eta}.$$

Moreover, the strict monotonicity of $A(\cdot)$ implies that this solution is unique. Note that $\frac{\lambda}{(|g(\cdot)| + \varepsilon)^\eta} \in L^\infty(\Omega)_+$. So, the nonlinear regularity theory of Lieberman [17] and the nonlinear maximum principle of Pucci–Serrin [28] (p. 111, 120) imply that $\hat{u}_\varepsilon \in \operatorname{int} C_+$.

We consider the solution map $\gamma_\varepsilon : L^p(\Omega) \mapsto L^p(\Omega)$ defined by

$$\gamma_\varepsilon(g) = \hat{u}_\varepsilon.$$

Consider the following perturbation of problem (Q_λ):

$$-\operatorname{div} a\big(Du(z)\big) = \frac{\lambda}{(u(z) + \varepsilon)^\eta} \quad \text{in } \Omega, u|_{\partial\Omega} = 0, u \geq 0. \tag{Q_λ^ε}$$

Evidently a fixed point of $\gamma_\varepsilon(\cdot)$ is a solution of (Q_λ^ε). Clearly $\gamma_\varepsilon(\cdot)$ is continuous. Moreover, using Lemma 2(b) we have

$$\frac{c_1}{p-1}\|D\hat{u}_\varepsilon\|_p^p \leq \int_\Omega \frac{\lambda}{(|g(z)| + \varepsilon)^\eta}\hat{u}_\varepsilon \, dz \leq \frac{\lambda}{\varepsilon^\eta}c_7\|\hat{u}_\varepsilon\| \quad \text{for some } c_7 > 0$$

$$\Rightarrow \quad \gamma_\varepsilon\big(L^p(\Omega)\big) \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

The fact that $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ compactly implies that

$$\gamma_\varepsilon\big(L^p(\Omega)\big) \subseteq L^p(\Omega) \text{ is relatively compact.}$$

Then, by the Schauder–Tychonov fixed point theorem (see Papageorgiou–Rădulescu–Repovš [21], Theorem 4.3.21, p. 298), we can find $\widetilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\gamma_\varepsilon(\widetilde{u}_\varepsilon) = \widetilde{u}_\varepsilon.$$

This is the unique positive solution of (Q_λ^ε) and since $\gamma_\varepsilon(\cdot)$ has values in $\operatorname{int} C_+$, we infer that $\widetilde{u}_\varepsilon \in \operatorname{int} C_+$.

Claim: $\{\widetilde{u}_\varepsilon\}_{\varepsilon \in (0,1]} \subseteq \operatorname{int} C_+$ is nonincreasing.

Let $0 < \varepsilon' < \varepsilon$. We have

$$-\operatorname{div} a(D\widetilde{u}_{\varepsilon'}) = \frac{\lambda}{(\widetilde{u}_{\varepsilon'} + \varepsilon')^\eta} \geq \frac{\lambda}{(\widetilde{u}_{\varepsilon'} + \varepsilon)^\eta} \quad \text{in } \Omega. \tag{3}$$

We introduce the Carathéodory function $k_\varepsilon : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$k_\varepsilon(z, x) = \begin{cases} \frac{\lambda}{(x^+ + \varepsilon)^\eta} & \text{if } x \leq \widetilde{u}_{\varepsilon'}(z), \\ \frac{\lambda}{(\widetilde{u}_{\varepsilon'}(z) + \varepsilon)^\eta} & \text{if } \widetilde{u}_{\varepsilon'}(z) < x. \end{cases} \tag{4}$$

We set $K_\varepsilon(z, x) = \int_0^x k_\varepsilon(z, s) \, ds$ and consider the C^1-functional $\psi_\varepsilon : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ defined by

$$\psi_\varepsilon(u) = \int_\Omega G(Du) \, dz - \int_\Omega K_\varepsilon(z, u) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From Corollary 3 and (4), we see that the functional $\psi_\varepsilon(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we see that $\psi_\varepsilon(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $\widetilde{u}_\varepsilon^* \in W_0^{1,p}(\Omega)$ such that

$$\psi_\varepsilon(\widetilde{u}_\varepsilon^*) = \min\{\psi_\varepsilon(u) : u \in W_0^{1,p}(\Omega)\}. \tag{5}$$

From hypothesis H_0(iv) and Corollary 3, we have

$$G(y) \le c_8\left(|y|^q + |y|^p\right) \quad \text{for some } c_8 > 0, \text{all } y \in \mathbb{R}^N. \tag{6}$$

For $t \in (0,1)$ we have

$$\psi_\varepsilon(t\widetilde{u}_{\varepsilon'}) \le \frac{t^p}{p}\|D\widetilde{u}_{\varepsilon'}\|_p^p + \frac{t^q}{q}\|D\widetilde{u}_{\varepsilon'}\|_q^q - \frac{t^{1-\eta}}{1-\eta}\int_\Omega \widetilde{u}_{\varepsilon'}^{1-\eta}\,dz \quad \text{(see (6))}.$$

Since $0 < \eta < 1 < q < p$, for $t \in (0,1)$ small, we see that

$$\psi_\varepsilon(t\widetilde{u}_{\varepsilon'}) < 0$$
$$\Rightarrow \quad \psi_\varepsilon\left(\widetilde{u}_\varepsilon^*\right) < 0 = \psi_\varepsilon(0) \quad \text{(see (5))}$$
$$\Rightarrow \quad \widetilde{u}_\varepsilon^* \ne 0.$$

From (5) we have

$$\psi_\varepsilon\left(\widetilde{u}_\varepsilon^*\right) = 0$$
$$\Rightarrow \quad \left\langle A\left(\widetilde{u}_\varepsilon^*\right), h\right\rangle = \int_\Omega k_\varepsilon\left(z, \widetilde{u}_\varepsilon^*\right)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{7}$$

In (7) first we choose $h = -(\widetilde{u}_\varepsilon^*)^- \in W_0^{1,p}(\Omega)$. We have

$$\frac{c_1}{p-1}\left\|D\left(\widetilde{u}_\varepsilon^*\right)^-\right\|_p^p \le 0 \quad \left(\text{see Lemma 2(b) and (4)}\right)$$
$$\Rightarrow \quad \widetilde{u}_\varepsilon^* \ge 0, \qquad \widetilde{u}_\varepsilon^* \ne 0.$$

Next we test (7) with $h = (\widetilde{u}_\varepsilon^* - \widetilde{u}_{\varepsilon'})^+ \in W_0^{1,p}(\Omega)$. We have

$$\left\langle A\left(\widetilde{u}_\varepsilon^*\right), \left(\widetilde{u}_\varepsilon^* - \widetilde{u}_{\varepsilon'}\right)^+\right\rangle = \int_\Omega \frac{\lambda}{(\widetilde{u}_{\varepsilon'} + \varepsilon)^\eta}\left(\widetilde{u}_\varepsilon^* - \widetilde{u}_{\varepsilon'}\right)^+ dz \quad \text{(see (4))}$$
$$\le \left\langle A(\widetilde{u}_{\varepsilon'}), \left(\widetilde{u}_\varepsilon^* - \widetilde{u}_{\varepsilon'}\right)^+\right\rangle \quad \text{(see (3))}$$
$$\Rightarrow \quad \widetilde{u}_\varepsilon^* \le \widetilde{u}_{\varepsilon'}.$$

So, we have proved that

$$\widetilde{u}_\varepsilon^* \in [0, \widetilde{u}_{\varepsilon'}], \qquad \widetilde{u}_\varepsilon^* \ne 0. \tag{8}$$

From (8), (4) and (7) it follows that

$$\widetilde{u}_\varepsilon^* = \widetilde{u}_\varepsilon$$
$$\Rightarrow \quad \widetilde{u}_\varepsilon \le \widetilde{u}_{\varepsilon'}.$$

This proves the claim.

Next, let $\varepsilon_n = \frac{1}{n}$ and $\widetilde{u}_n = \widetilde{u}_{\varepsilon_n} \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$. We have

$$\langle A(\widetilde{u}_n), h \rangle = \int_\Omega \frac{\lambda}{(\widetilde{u}_n + \frac{1}{n})^\eta} h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{all } n \in \mathbb{N}, \tag{9}$$

$$\widetilde{u}_1 \le \widetilde{u}_n \quad \text{for all } n \in \mathbb{N} \text{ (by the claim).} \tag{10}$$

Consider the Banach space $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ (with the supremum norm). This is an ordered Banach space with positive cone

$$K_+ = \left\{ u \in C_0(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega} \right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} K_+ = \{u \in C_+ : c_u \hat{d} \le u \text{ for some } c_u > 0\},$$

with $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. Since $\widetilde{u}_1 \in \operatorname{int} C_+$, we can find $0 < c_9 < c_{10}$ such that

$$c_9 \hat{d} \le \widetilde{u}_1 \le c_{10} \hat{d} \quad \text{(see Guo–Webb [13], Lemma 2.3).} \tag{11}$$

Let $s > N$ and note that $\hat{u}_1(p)^{1/s} \in K_+$. Using Proposition 4.1.22, p. 274, of Papageorgiou–Rădulescu–Repovš [21], we can find $c_{11} > 0$ such that

$$\begin{aligned} &\hat{u}_1(p)^{1/s} \le c_{11} \hat{d} \\ &\Rightarrow \quad \hat{d}^{-\eta} \le c_{12} \hat{u}_1(p)^{-\eta/s} \quad \text{for some } c_{12} > 0. \end{aligned} \tag{12}$$

Since $0 < \eta < 1$, the Lemma in Lazer–McKenna [16] implies that $\hat{u}_1(p)^{-\eta/s} \in L^s(\Omega)$. Therefore

$$\hat{d}^{-\eta} \in L^s(\Omega) \quad \text{(see (12)).} \tag{13}$$

From (11) we have

$$\begin{aligned} &\widetilde{u}_1^{-\eta} \le c_{13} \hat{d}^{-\eta} \quad \text{for some } c_{13} > 0 \\ &\Rightarrow \quad \widetilde{u}_1^{-\eta} \in L^s(\Omega) \quad \text{(see (13)).} \end{aligned} \tag{14}$$

We return to (9) and use $h = \widetilde{u}_n \in W_0^{1,p}(\Omega)$. Then from Lemma 2(b) we have

$$\begin{aligned} &\frac{c_1}{p-1} \|D\widetilde{u}_n\|_p^p \le \int_\Omega \frac{\lambda}{(\widetilde{u}_n + \frac{1}{n})^\eta} \widetilde{u}_n \, dz \le \lambda c_{14} \|\widetilde{u}_n\|^{1-\eta} \quad \text{for some } c_{14} > 0, \text{all } n \in \mathbb{N} \\ &\Rightarrow \quad \{\widetilde{u}_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \end{aligned}$$

So, we may assume that

$$\widetilde{u}_n \xrightarrow{w} \overline{u}_\lambda \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad \widetilde{u}_n \to \overline{u}_\lambda \quad \text{in } L^p(\Omega). \tag{15}$$

Note that

$$0 \leq \frac{\lambda}{(\widetilde{u}_n + \frac{1}{n})^\eta} \leq \frac{\lambda}{\widetilde{u}_1^\eta} \in L^s(\Omega) \quad \text{(see (14))}. \tag{16}$$

On account of (15), we can also say (at least for a subsequence) that

$$\frac{\lambda}{(\widetilde{u}_n(z) + \frac{1}{n})^\eta} \to \frac{\lambda}{\overline{u}_\lambda(z)^\eta} \quad \text{for a.a. } z \in \Omega. \tag{17}$$

From (16) and (17) it follows that

$$\frac{\lambda}{(\widetilde{u}_n + \frac{1}{n})^\eta} \xrightarrow{w} \frac{\lambda}{\overline{u}_\lambda^\eta} \quad \text{in } L^s(\Omega) \tag{18}$$

(see Gasinski–Papageorgiou [9], Problem 1.19, p. 38).

Again we return to (9) and choose $h = \widetilde{u}_n - \overline{u}_\lambda \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (15) and (18). We obtain

$$\lim_{n\to\infty} \langle A(\widetilde{u}_n), \widetilde{u}_n - \overline{u}_\lambda \rangle = 0$$

$$\Rightarrow \quad \widetilde{u}_n \to \overline{u}_\lambda \text{ in } W_0^{1,p}(\Omega) \quad \text{(see Proposition 4)}. \tag{19}$$

Therefore if in (9) we pass to the limit as $n \to \infty$ and use (19), then

$$\left\{ \begin{array}{ll} \langle A(\overline{u}_\lambda), h \rangle = \int_\Omega \frac{\lambda}{\overline{u}_\lambda^\eta} h \, dz & \text{for all } h \in W_0^{1,p}(\Omega), \\ \widetilde{u}_1 \leq \overline{u}_\lambda & \text{(see (10))}. \end{array} \right\} \tag{20}$$

From (20) we infer that

$\overline{u}_\lambda$ is a positive solution of (Q_λ).

Let $\theta = \frac{\lambda}{\overline{u}_\lambda^\eta} \in L^s(\Omega)$ and consider the linear Dirichlet problem

$$-\Delta y(z) = \theta(z) \quad \text{in } \Omega, y|_{\partial\Omega} = 0.$$

From Theorem 9.15, p. 241, of Gilbarg–Trudinger [12], we know that this problem has a unique solution $y \in W^{2,s}(\Omega)$, $y \geq 0$, $y \neq 0$. Since $s > N$, by the Sobolev embedding theorem, we have $W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$ continuously with $\alpha = 1 - \frac{N}{s} \in (0,1)$. Let $w = Dy \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$. We have

$$-\operatorname{div}\big(a\big(D\overline{u}_\lambda(z)\big) - w(z)\big) = 0, \quad \text{in } \Omega, \overline{u}_\lambda|_{\partial\Omega} = 0.$$

Then the nonlinear regularity theory of Lieberman [17] implies that $\overline{u}_\lambda \in C_+ \setminus \{0\}$. Using the nonlinear maximum principle of Pucci–Serrin [28] (pp. 111, 120), we infer that $\overline{u}_\lambda \in \operatorname{int} C_+$.

From (20) and (14) and Theorem 7.1, p. 286, of Ladyzhenskaya–Uraltseva [15] (see also Gasinski–Papageorgiou [8], p. 737), we know that there exists $c_{15} > 0$ such that

$$\|\overline{u}_\lambda\|_\infty \leq c_{15} \quad \text{for all } 0 < \lambda \leq 1.$$

Then the nonlinear regularity theory of Lieberman [17], says that we can find $\alpha \in (0,1)$ and $c_{16} > 0$ such that

$$\overline{u}_\lambda \in C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}), \qquad \|\overline{u}_\lambda\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_{16} \quad \text{for all } \lambda \in (0,1]. \tag{21}$$

Since $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ compactly, from (14), (20) and (21), we conclude that

$$\overline{u}_\lambda \to 0 \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } \lambda \to 0^+.$$

This proof is now complete.　　　　　　　　　　　　　　　　　　　　　　　□

4 The frozen variable method

In this section we develop the method described in the Introduction (the frozen variable method).

So, fix $v \in C_0^1(\overline{\Omega})$ and consider the following Dirichlet problem (the "frozen problem"):

$$\begin{cases} -\operatorname{div} a(Du(z)) = \lambda u(z)^{-\eta} + f(z, u(z), Dv(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \qquad u \geq 0, \qquad \lambda > 0, \qquad 0 < \eta < 1. \end{cases} \tag{P_λ^v}$$

Since we have fixed the gradient variable in f, the resulting problem (P_λ^v) has a variational structure. However, as we already mentioned in the Introduction, the presence of the singular term $u \mapsto \lambda u^{-\eta}$ leads to an energy functional which is not C^1 and so we cannot use the minimax theorems of the critical point theory. To remedy this situation, we use the solution $\overline{u}_\lambda$ of (Q_λ) to bypass the singularity and deal with a C^1-functional.

Let $M \geq \|Dv\|_\infty$. On account of hypotheses H_1(ii), given $\varepsilon > 0$ small, we can find $\delta \in (0, \delta_0]$ such that

$$f(z, x, y) \geq \big(\eta_M(z) - \varepsilon\big)x^{q-1} \quad \text{for a.a } z \in \Omega, \text{all } 0 \leq x \leq \delta, \text{all } |y| \leq M. \tag{22}$$

Proposition 6 *If hypotheses H_0, H_1 hold, then we can find $\hat{\lambda}_* > 0$ such that for all $\lambda \in (0, \hat{\lambda}_*]$ problem (P_λ^v) admits a positive solution $\hat{u}_v \in \operatorname{int} C_+$.*

Proof Proposition 5 implies that we can find $\hat{\lambda}_* \in (0,1]$ such that

$$0 \leq \overline{u}_\lambda(z) \leq \delta \quad \text{for all } z \in \overline{\Omega}, \text{all } 0 < \lambda \leq \hat{\lambda}_*. \tag{23}$$

We introduce the Carathéodory function $\tau_v^\lambda : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$\tau_v^\lambda(z, x) = \begin{cases} \lambda \overline{u}_\lambda(z)^{-\eta} + f(z, \overline{u}_\lambda(z), Dv(z)) & \text{if } x < \overline{u}_\lambda(z), \\ \lambda x^{-\eta} + f(z, x, Dv(z)) & \text{if } \overline{u}_\lambda(z) \leq x \leq \theta, \\ \lambda \theta^{-\eta} + f(z, \theta, Dv(z)) & \text{if } \theta \leq x; \end{cases} \tag{24}$$

recall that $\delta_0 < \theta$, see hypothesis H_1(i). We set $T_v^\lambda(z, x) = \int_0^x \tau_v^\lambda(z, s)\, ds$ and consider the C^1-functional $\varphi_v^\lambda : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$ defined by

$$\varphi_v^\lambda(u) = \int_\Omega G(Du)\, dz - \int_\Omega T_v^\lambda(z, u)\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Corollary 3 and (24) imply that $\varphi_\nu^\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_\nu \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\nu^\lambda(\hat{u}_\nu) = \min\{\varphi_\nu^\lambda(u) : u \in W_0^{1,p}(\Omega)\}. \tag{25}$$

Hypothesis $H_0(iv)$ implies that, given $\varepsilon' \in (0, \varepsilon]$, we can find $\delta' \in (0, \delta]$ such that

$$G(y) \leq \frac{c_* + \varepsilon'}{q}|y|^q \quad \text{for all } |y| \leq \delta'. \tag{26}$$

Let $t \in (0, 1)$ be small such that

$$0 \leq t\hat{u}_1(q)(z) \leq \min\{\overline{u}_\lambda(z), \delta_0\} \quad \text{for all } z \in \overline{\Omega}$$

(recall that $\overline{u}_\lambda \in \operatorname{int} C_+$ and use Proposition 4.1.22, p. 274, of Papageorgiou–Rădulescu–Repovš [21]). We have

$$\varphi_\nu^\lambda\big(t\hat{u}_1(q)\big) \leq \frac{t^p}{p}\big\|D\hat{u}_1(q)\big\|_p^p + \frac{t^q}{q}\big(c_* + \varepsilon'\big)\hat{\lambda}_1(q) - \frac{t^q}{q}\int_\Omega \big(\eta_M(z) - \varepsilon\big)\hat{u}_1(q)^q\, dz$$

$$\text{(see (26), (22), (23), (24))}$$

$$\leq \frac{t^p}{p}\big\|D\hat{u}_1(q)\big\|_p^p + \frac{t^q}{q}\left(\int_\Omega \big(c_*\hat{\lambda}_1(q) - \eta_M(z)\big)\hat{u}_1(q)^q\, dz + 2\varepsilon\right)$$

$$\big(\text{recall } \big\|\hat{u}_1(q)\big\|_q = 1\big).$$

Note that

$$\mu = \int_\Omega \big(\eta_M(z) - c_*\hat{\lambda}_1(q)\big)\hat{u}_1^q\, dz > 0$$

(see hypothesis $H_1(ii)$ and recall $\hat{u}_1(q) \in \operatorname{int} C_+$).

We have

$$\varphi_\nu^\lambda\big(t\hat{u}_1(q)\big) \leq c_{17}t^p - c_{18}t^q \quad \text{for some } c_{17}, c_{18} > 0 \text{ (with } \varepsilon > 0 \text{ small)}.$$

Since $q < p$, choosing $t \in (0, 1)$ even smaller, we have

$$\varphi_\nu^\lambda\big(t\hat{u}_1(q)\big) < 0$$

$$\Rightarrow \quad \varphi_\nu^\lambda(\hat{u}_\nu) < 0 = \varphi_\nu^\lambda(0) \quad \text{(see (25))}$$

$$\Rightarrow \quad \hat{u}_\nu \neq 0.$$

From (25) we have

$$\big(\varphi_\nu^\lambda\big)'(\hat{u}_\nu) = 0$$

$$\Rightarrow \quad \big\langle A(\hat{u}_\nu), h\big\rangle = \int_\Omega \tau_\nu^\lambda(z, \hat{u}_\nu)h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{27}$$

In (27) first we choose $h = (\overline{u}_\lambda - \hat{u}_v)^+ \in W_0^{1,p}(\Omega)$. Then

$$\langle A(\hat{u}_v), (\overline{u}_\lambda - \hat{u}_v)^+\rangle$$

$$= \int_\Omega \left(\lambda \overline{u}_\lambda^{-\eta} + f(z, \overline{u}_\lambda, Dv)\right)(\overline{u}_\lambda - \hat{u}_v)^+ \, dz \quad \text{(see (24))}$$

$$\geq \int_\Omega \lambda \overline{u}_\lambda^{-\eta}(\overline{u}_\lambda - \hat{u}_v)^+ \, dz$$

$$\left(\text{since } f(z, \overline{u}_\lambda, Dv) \geq 0 \quad \text{(see (23)) and hypothesis } H_1(ii)\right)$$

$$= \langle A(\overline{u}_\lambda), (\overline{u}_\lambda - \hat{u}_v)^+\rangle \quad \text{(see Proposition 5)}$$

$$\Rightarrow \quad \overline{u}_\lambda \leq \hat{u}_v.$$

Next, in (27) we choose $h = (\hat{u}_v - \theta)^+ \in W_0^{1,p}(\Omega)$. Then

$$\langle A(\hat{u}_v), (\hat{u}_v - \theta)^+\rangle$$

$$= \int_\Omega \left(\lambda \theta^{-\eta} + f(z, \theta, Dv)\right)(\hat{u}_v - \theta)^+ \, dz \quad \text{(see (24))}$$

$$\leq \int_\Omega \left(\theta^{-\eta} + f(z, \theta, Dv)\right)(\hat{u}_v - \theta)^+ \, dz \quad \text{(recall } 0 < \lambda \leq \hat{\lambda}_* \leq 1)$$

$$\leq 0 \quad \left(\text{see hypothesis } H_1(i)\right)$$

$$\Rightarrow \quad \hat{u}_v \leq \theta$$

(by the weak comparison principle; see Pucci–Serrin [28], Theorem 3.4.1, p. 61).

So, we have proved that

$$\hat{u}_v \in [\overline{u}_\lambda, \theta]. \tag{28}$$

From (28), (24) and (27), we infer that $\hat{u}_v$ is a positive solution of (P_λ^v). Since $\overline{u}_\lambda^{-\eta} \in L^s(\Omega)$ ($s > N$) (see (20) and (14)), reasoning as in the proof of Proposition 5 (see the part of the proof after (20)), we infer that $\hat{u}_v \in \text{int}\, C_+$.

This proof is now complete. $\square$

Let S_v^λ be the set of positive solutions of the "frozen problem" (P_λ^v). We have just proved that for $v \in C_0^1(\overline{\Omega})$ and $0 < \lambda \leq \hat{\lambda}_*$ we have

$$\emptyset \neq S_v^\lambda \subseteq \text{int}\, C_+.$$

We will show that each of these solution sets has a smallest element. In this way we have a canonical procedure to choose an element from S_v^λ as $v \in C_0^1(\overline{\Omega})$ varies. So, we define the minimal solution map on which we will use Theorem 1 (the Leray–Schauder alternative principle).

To produce a minimal element for the solution set S_v^λ, we need the following result providing a lower bound for the set S_v^λ.

Proposition 7 *If hypotheses* H_0, H_1 *hold and* $0 < \lambda \leq \hat{\lambda}_*$, *then* $\overline{u}_\lambda \leq u$ *for all* $u \in S_v^\lambda$.

Proof Let $u \in S_\nu^\lambda$ and consider the Carathéodory function $\hat{\mu}_\lambda(z, x)$ defined on $\Omega \times \mathring{\mathbb{R}}_+ = \Omega \times (0, \infty)$ by

$$\hat{\mu}_\lambda(z, x) = \begin{cases} \lambda x^{-\eta} & \text{if } 0 < x \le u(z), \\ \lambda u(z)^{-\eta} & \text{if } u(z) < x. \end{cases} \tag{29}$$

We consider the following singular Dirichlet problem

$$-\operatorname{div} a(Du) = \hat{\mu}_\lambda(z, u) \quad \text{in } \Omega, u|_{\partial\Omega} = 0, u \ge 0. \tag{30}$$

Reasoning as in the proof of Proposition 5, we show that problem (30) has a solution $\tilde{u}_\lambda \in \operatorname{int} C_+$ and using (29), we show that

$$\tilde{u}_\lambda \in [0, u].$$

But then (29) and Proposition 5 imply that

$$\tilde{u}_\lambda = \overline{u}_\lambda$$

$$\Rightarrow \quad \overline{u}_\lambda \le u \quad \text{for all } u \in S_\nu^\lambda.$$

This proof is now complete. $\qquad\square$

Using this lower bound, we show the existence of a minimal element for the set S_ν^λ.

Proposition 8 *If hypotheses* H_0, H_1 *hold and* $0 < \lambda \le \hat{\lambda}_*$, *then problem* (P_λ^ν) *has a smallest positive solution* $u_\nu^* \in \operatorname{int} C_+$ *such that*

$$0 \le u_\nu^*(z) \le \theta \quad \text{for all } z \in \overline{\Omega}.$$

Proof From Proposition 18 of Papageorgiou–Rădulescu–Repovš [23], we know that S_ν^λ is downward directed. Then using Lemma 3.10, p. 178, of Hu–Papageorgiou [14], we can find $\{u_n\}_{n \in \mathbb{N}} \subseteq S_\nu^\lambda$ decreasing such that

$$\inf_{n \in \mathbb{N}} u_n = \inf S_\nu^\lambda.$$

From the proof of Proposition 6 we know that $S_\nu^\lambda \cap [0, \theta] \ne \emptyset$. So, without any loss of generality, we may assume that $\{u_n\}_{n \in \mathbb{N}} \subseteq [0, \theta]$. We have

$$\langle A(u_n), h \rangle = \int_\Omega \left(\lambda u_n^{-\eta} + f(z, u_n, Dv) \right) h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{all } n \in \mathbb{N}, \tag{31}$$

$$\overline{u}_\lambda \le u_n \le u_1 \quad \text{for all } n \in \mathbb{N} \text{ (see Proposition 7).} \tag{32}$$

In (31) we choose $h = u_n \in W_0^{1,p}(\Omega)$. Using Lemma 2(b), (32) and hypothesis H_1(i), we infer that

$$\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_\nu^* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_\nu^* \quad \text{in } L^p(\Omega). \tag{33}$$

In (31) we choose $h = u_n - u_\nu^* \in W_0^{1,p}(\Omega)$, pass to limit as $n \to \infty$ and use (33). We obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_\nu^* \rangle = 0 \quad \text{(see the proof Proposition 5)}$$

$$\Rightarrow \quad u_n \to u_\nu^* \quad \text{in } W_0^{1,p}(\Omega) \text{ (see Proposition 4).} \tag{34}$$

Then, if in (31) we pass to the limit as $n \to \infty$ and use (34), we obtain

$$\langle A(u_\nu^*), h \rangle = \int_\Omega \left(\lambda (u_\nu^*)^{-\eta} + f(z, u_\nu^*, D\nu) \right) h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ and } \overline{u}_\lambda \leq u_\nu^*.$$

It follows that

$$u_\nu^* \in S_\nu^\lambda, \qquad u_\nu^* = \inf S_\nu^\lambda, \qquad 0 \leq u_\nu^*(z) \leq \theta \quad \text{for all } z \in \overline{\Omega}.$$

This proof is now complete. $\qquad\qquad\square$

Using Proposition 8, we define the minimal solution map

$$\xi_\lambda : C_0^1(\overline{\Omega}) \mapsto C_0^1(\overline{\Omega}), \quad \lambda \in (0, \hat{\lambda}_*],$$

by setting

$$\xi_\lambda(\nu) = u_\nu^* \quad \text{for all } \nu \in C_0^1(\overline{\Omega}).$$

Evidently a fixed point of $\xi_\lambda(\cdot)$ is a solution of problem (P_λ). To produce a fixed point of (P_λ), we will use Theorem 1 (the Leray–Schauder alternative principle). This is done in the next section.

5 Positive solution
To apply Theorem 1, we need to know that $\xi_\lambda(\cdot)$ is compact. The next proposition will be helpful in this respect.

Proposition 9 *If hypotheses* H_0, H_1 *hold,* $\nu_n \to \nu$ *in* $C_0^1(\overline{\Omega})$, $u \in S_\nu^\lambda \cap [0, \theta]$ *with* $\lambda \in (0, \hat{\lambda}_*]$, *then we can find* $u_n \in S_{\nu_n}^\lambda$, $n \in \mathbb{N}$ *such that*

$$u_n \to u \quad \text{in } C_0^1(\overline{\Omega}).$$

Proof Choosing $M \geq \sup_{n\in\mathbb{N}} \|D\nu_n\|_\infty$, we see that all the previous results are valid. We consider the following Dirichlet problem:

$$-\operatorname{div} a(Dy(z)) = \tau_{\nu_n}^\lambda(z, u(z)) \quad \text{in } \Omega, y|_{\partial\Omega} = 0 \text{ (see (24)).} \tag{35}$$

This problem has a unique solution $y_n \in W_0^{1,p}(\Omega)$. We know that $\overline{u}_\lambda^{-\eta} \in L^s(\Omega)$, $s > N$ (see (20) and (14)). Therefore $\{\tau_{v_n}^\lambda(\cdot, u(\cdot))\}_{n \in \mathbb{N}} \subseteq L^s(\Omega)$ is bounded (see (24)). Consider the linear Dirichlet problem

$$-\Delta e(z) = \tau_{v_n}^\lambda(z, u(z)) \quad \text{in } \Omega, e|_{\partial\Omega} = 0.$$

For each $n \in \mathbb{N}$, this problem has a unique solution $e_n \in W^{2,s}(\Omega)$ (see Gilbarg–Trudinger [12], Theorem 9.15, p. 241) and in addition we have

$$\|e_n\|_{W^{2,s}(\Omega)} \le c_{19} \quad \text{for some } c_{19} > 0, \text{all } n \in \mathbb{N} \tag{36}$$

(see Gilbarg–Trudinger [12], Lemma 9.17, p. 242). From the Sobolev embedding theorem we know that, for $\alpha = 1 - \frac{N}{s} \in (0,1)$, we have

$$W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega}) \text{ compactly.}$$

Hence from (36) it follows that

$$\{\hat{\gamma}_n = De_n\}_{n \in \mathbb{N}} \subseteq C^{0,\alpha}(\overline{\Omega}) \text{ is bounded.}$$

We rewrite (35) as

$$-\operatorname{div}\left(a(Dy) - \hat{\gamma}_n\right) = 0 \quad \text{on } \partial\Omega, y|_{\partial\Omega} = 0.$$

Then the nonlinear regularity theory of Lieberman [17] implies that we can find $\mu \in (0,1)$ and $c_{20} > 0$ such that

$$y_n \in C_0^{1,\mu}(\overline{\Omega}) \quad \text{and} \quad \|y_n\|_{C_0^{1,\mu}(\overline{\Omega})} \le c_{20} \quad \text{for all } n \in \mathbb{N}. \tag{37}$$

From (37) and the compact embedding of $C_0^{1,\mu}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$, it follows that

$$\{y_n\}_{n \in \mathbb{N}} \subseteq C_0^1(\overline{\Omega}) \text{ is relatively compact.}$$

Note that $\tau_{v_n}^\lambda(\cdot, u(\cdot)) \mapsto \tau_v^\lambda(\cdot, u(\cdot))$ in $L^s(\Omega)$. So, for the whole sequence, we have

$$y_n \to u \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty.$$

We set $y_n = y_n^0$ and consider the following Dirichlet problem:

$$-\operatorname{div} a(Dy(z)) = \tau_{v_n}^\lambda(z, y_n^0(z)) \quad \text{in } \Omega, y|_{\partial\Omega} = 0.$$

As above this problem has a unique solution $y_n^1 \in C_0^1(\overline{\Omega})$ and we have

$$y_n^1 \to u \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty.$$

We continue this way and generate a sequence $\{y_n^k\}_{k,n\in\mathbb{N}} \subseteq C_0^1(\overline{\Omega})$ such that

$$-\operatorname{div} a\big(Dy_n^k(z)\big) = \tau_{v_n}^\lambda\big(z, y_n^{k-1}(z)\big) \quad \text{in } \Omega, y_n^k|_{\partial\Omega} = 0, \tag{38}$$

$$y_n^k \to u \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty \text{ for all } k \in \mathbb{N}. \tag{39}$$

We will show that for every $n \in \mathbb{N}$

$$\{y_n^k\}_{k\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{40}$$

Arguing by contradiction, suppose that (40) is not true. By passing to a subsequence if necessary, we may assume that

$$\left\| y_n^k \right\| \to \infty \quad \text{as } k \to \infty. \tag{41}$$

We set $w_k = \frac{y_n^k}{\|y_n^k\|}$, $k \in \mathbb{N}$. Then $\|w_k\| = 1$ for all $k \in \mathbb{N}$ and so we may assume that

$$w_k \xrightarrow{w} w \quad \text{in } W_0^{1,p}(\Omega) \text{ and } w_k \to w \text{ in } L^p(\Omega) \text{ as } k \to \infty. \tag{42}$$

From (38) we have

$$\int_\Omega \left(\frac{a(Dy_n^k)}{\|y_n^k\|^{p-1}}, Dw_k \right)_{\mathbb{R}^N} dz = \int_\Omega \frac{\tau_{v_n}^\lambda(z, y_n^{k-1})}{\|y_n^k\|^{p-1}} w_k\, dz$$

$$\Rightarrow \quad \frac{c_1}{p-1}\|Dw_k\|_p^p \le \int_\Omega \frac{\tau_{v_n}^\lambda(z, y_n^{k-1})}{\|y_n^k\|^{p-1}} w_k\, dz \quad \big(\text{see Lemma 2(b)}\big). \tag{43}$$

On account of (24), (41) and (42), we have

$$\int_\Omega \frac{\tau_{v_n}^\lambda(z, y_n^{k-1})}{\|y_n^k\|^{p-1}} w_k\, dz \to 0 \quad \text{as } k \to \infty$$

$$\Rightarrow \quad w_k \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } k \to \infty \quad (\text{see } (43)),$$

which contradicts the fact that $\|w_k\| = 1$ for all $k \in \mathbb{N}$.

Therefore (40) is true.

Then on account of (40) and the nonlinear regularity theory (see [17] and [21]), we infer that $\{y_n^k\}_{k\in\mathbb{N}} \subseteq C_0^1(\overline{\Omega})$ is relatively compact. Hence we may assume that

$$y_n^k \to u_n \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } k \to \infty. \tag{44}$$

From (38) and hypothesis H_1(i), we have

$$-\operatorname{div} a(Du_n) = \tau_{v_n}^\lambda(z, u_n) \quad \text{in } \Omega, u_n|_{\partial\Omega} = 0, \tag{45}$$

$$\overline{u}_\lambda \le u_n \le \theta \quad \text{for all } n \in \mathbb{N} \text{ (see (24))} \tag{46}$$

$$\Rightarrow \quad u_n \in S_{v_n}^\lambda \quad \text{for all } n \in \mathbb{N}.$$

From (45), (46) and the nonlinear regularity theory of Lieberman [17], it follows that $\{u_n\}_{n \in \mathbb{N}} \subseteq C_0^1(\overline{\Omega})$ is relatively compact. Then (39), (44) and the double limit lemma (see Gasinski–Papageorgiou [7], Problem 1.175, p. 61) imply that

$$u_n \to u \quad \text{in } C_0^1(\overline{\Omega}) \text{ with } u_n \in S_{v_n}^\lambda \text{ for all } n \in \mathbb{N}.$$

This proof is now complete.　　　　　　　　　　　　　　　　　　　　　　　　　　　　□

Using this proposition, we can establish the compactness of the minimal solution map $\xi_\lambda(\cdot)$.

Proposition 10 *If hypotheses* H_0, H_1 *hold and* $\lambda \in (0, \hat{\lambda}_*]$*, then the minimal solution map* $\xi_\lambda : C_0^1(\overline{\Omega}) \mapsto C_0^1(\overline{\Omega})$ *is compact.*

Proof Let $B \subseteq C_0^1(\overline{\Omega})$ be bounded. Let $v \in B$ and $u = \xi_\lambda(v)$. We have

$$-\operatorname{div} a(Du) = \lambda u^{-\eta} + f(z, u, Dv) \quad \text{in } \Omega, u|_{\partial\Omega} = 0, \overline{u}_\lambda \leq u \leq \theta$$

$$\Rightarrow \quad \frac{c_1}{p-1} \|Du\|_p^p \leq c_{21} \quad \text{for some } c_{21} > 0$$

$$\Rightarrow \quad \xi_\lambda(B) \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

From this as before using the nonlinear regularity theory of Lieberman [17], we infer that

$$\xi_\lambda(B) \subseteq C_0^1(\overline{\Omega}) \text{ is relatively compact.} \tag{47}$$

Next, we show that $\xi_\lambda(\cdot)$ is continuous. So, let $v_n \to v$ in $C_0^1(\overline{\Omega})$ and let $u_n = \xi_\lambda(v_n)$, $n \in \mathbb{N}$ and $u = \xi_\lambda(v)$. From (47) we see that $\{u_n\}_{n \in \mathbb{N}} \subseteq C_0^1(\overline{\Omega})$ is relatively compact. Hence we may assume that

$$u_n \to \widetilde{u} \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{48}$$

Evidently $\widetilde{u} \in S_v^\lambda$ and so we have

$$u \leq \widetilde{u}. \tag{49}$$

On the other hand, from Proposition 9, we know that we can find $\widetilde{u}_n \in S_{v_n}^\lambda$, $n \in \mathbb{N}$ such that

$$\widetilde{u}_n \to u \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{50}$$

We have $u_n \leq \widetilde{u}_n$ for all $n \in \mathbb{N}$ and so from (48) and (50), we obtain

$$\widetilde{u} \leq u$$

$$\Rightarrow \quad u = \widetilde{u} \quad (\text{see (49)}).$$

So, for the whole sequence we have

$$u_n \to u \quad \text{in } C_0^1(\overline{\Omega})$$

$$\Rightarrow \quad \xi_\lambda(\cdot) \text{ is continuous}$$

$$\Rightarrow \quad \xi_\lambda(\cdot) \text{ is compact.}$$

This proof is now complete. □

Let $\mathcal{D}_\lambda = \{u \in C_0^1(\overline{\Omega}) : u = t\xi_\lambda(u), 0 < t < 1\}$.

Proposition 11 *If hypotheses* H_0, H_1 *hold and* $\lambda \in (0, \hat{\lambda}_*]$, *then* $\mathcal{D}_\lambda \subseteq C_0^1(\overline{\Omega})$ *is bounded.*

Proof Let $u \in \mathcal{D}_\lambda$. We have

$$\frac{1}{t}u = \xi_\lambda(u) \quad \text{with } 0 < t < 1.$$

It follows that

$$\left\langle A\left(\frac{1}{t}u\right), h \right\rangle = \int_\Omega \left(\lambda \left(\frac{1}{t}u\right)^{-\eta} + f\left(z, \frac{1}{t}u, Du\right) \right) h \, dz \tag{51}$$

for all $h \in W_0^{1,p}(\Omega)$.

From Proposition 7, we know that

$$\overline{u}_\lambda \leq \frac{1}{t}u$$

$$\Rightarrow \quad t\overline{u}_\lambda \leq u \leq \theta \quad \left(\text{since } t \in (0, 1)\right). \tag{52}$$

In (51) we choose $h = u \in W_0^{1,p}(\Omega)$. Using Lemma 2(b), (52) (recall that $\overline{u}_\lambda^{-\eta} \in L^s(\Omega)$) and hypothesis H_1(i), we obtain

$$\frac{c_1}{p-1}\|Du\|_p^p \leq c_{22}\left(1 + \int_\Omega |f(z, u, Du)| \, dz\right) \quad \text{for some } c_{22} > 0$$

$$\leq c_{23}\left(1 + \|Du\|^{p-1}\right) \quad \text{for some } c_{23} > 0$$

$$\Rightarrow \quad \mathcal{D}_\lambda \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

Then as before via the nonlinear regularity theory of Lieberman [17], we see that $\mathcal{D}_\lambda \subseteq C_0^1(\overline{\Omega})$ is bounded (in fact relatively compact).

This proof is now complete. □

So we can use Theorem 1 on the map $\xi_\lambda(\cdot)$ and produce a fixed point which is a positive solution of problem (P_λ), for $\lambda \in (0, \hat{\lambda}_*]$. Concluding we can state the following existence theorem for problem (P_λ).

Theorem 12 *If hypotheses* H_0, H_1 *hold, then there exists* $\hat{\lambda}_* > 0$ *such that for all* $\lambda \in (0, \hat{\lambda}_*]$ *problem* (P_λ) *admits a positive solution*

$$\hat{u}_\lambda \in [\overline{u}_\lambda, \theta] \cap \operatorname{int} C_+.$$

Availability of data and materials
This paper focuses on theoretical analysis, not involving experiments and data.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

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