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Qualitative analysis of nonlinear coupled pantograph differential equations of fractional order with integral boundary conditions

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Abstract

In this research article, we develop a qualitative analysis to a class of nonlinear coupled system of fractional delay differential equations (FDDEs). Under the integral boundary conditions, existence and uniqueness for the solution of this system are carried out. With the help of Leray–Schauder and Banach fixed point theorem, we establish indispensable results. Also, some results affiliated to Ulam–Hyers (UH) stability for the system under investigation are presented. To validate the results, illustrative examples are given at the end of the manuscript.

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1 Introduction

Theoretical and applied aspects of fractional calculus can be found comprehensively in the literature. This area of mathematics has numerous applications of modeling various phenomena and processes in nonlinear oscillations of earthquakes, nanotechnology, engineering discipline, and other different scientific fields (see [1–5]). These fields played a vital role for the birth of some new subareas of research like fractional dynamics, fractional fuzzy calculus, etc. Different aspects related to qualitative and numerical analysis corresponding to boundary and initial value problems of fractional order differential equations (FODEs) have been widely explored. The literature related to this area contains significant contributions (we refer to [6–9]).

There is a rapidly growing body of literature on a delay type of differential equations including discrete, continuous, and proportional delay. The area of delay FODEs has many applications in mathematical modeling of real world problems and processes. Among the delay problems, there are those containing a proportional delay, are called pantograph differential equations (PDEs). The existing work about PDEs indicates that such problems have many real world applications. Initially, pantograph was introduced to govern

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the drawing and scaling aspects of a device. Later on this device was more refined and now scientists use it in electric trains [10, 11], material modeling [12], and in modeling lasers, especially quantum dot lasers [13]. Basically, PDEs form a subclass of differential equations called delay, which gives change in terms of dependent variable at previous time [14]. Not long ago, in this area, some beneficial work has been carried out (see [15–17]).

FODEs have been extensively studied with respect to different angles. Among these, stability analysis in the UH sense is an important aspect that gained proper attention from researchers [18, 19]. Based on the fundamental definition of UH stability of a system, the notion was later modified to more general types, and their results were successfully applied to various problems (we refer to [20, 21] and the references therein). The UH-type stability has been developed very well in the sense of differential equations in the last twenty years [22–24].

To the best of our knowledge, qualitative study to a coupled system of FDDEs under integral boundary conditions has not been investigated properly. Therefore, using results from fixed point theory, we study the qualitative aspects of a coupled system of FDDEs under integral conditions given as follows:

$$\begin{cases} D_{+0}^\beta r(t) = -f_1(t, r(\lambda t), s(t), D^p s(t)), \\ D_{+0}^\gamma s(t) = -f_2(t, r(t), s(\lambda t), D^q r(t)), \\ r(0) = 0, \quad r(1) = \int_0^1 \phi(t)r(t) dt, \\ s(0) = 0, \quad s(1) = \int_0^1 \phi(t)s(t) dt, \end{cases} \tag{1}$$

where $t \in I_1 = [0, 1]$, $\beta, \gamma \in (1, 2]$, and $\lambda, p, q \in (0, 1) = I_2$. Also the functions $f_1, f_2 : I_1 \times R^3 \rightarrow R$ are nonlinear continuous functions, $\phi : I_2 \rightarrow [0, \infty)$ is a bounded function, and D is a Riemann–Liouville derivative. Further, the Banach space is defined by $(W, \|\cdot\|)$ with the norm $\|w\| = \max_{t \in I_1} |w|$. Consequently, $U = W \times W$ is a Banach space with norms denoted by $\|(w_1, w_2)\| = \|w_1\| + \|w_2\|$ or $\|(w_1, w_2)\| = \max\{\|w_1\|, \|w_2\|\}$.

2 Preliminaries

Related results and definitions for the current work, which may be found in [25–28], are discussed here.

Definition 1 Integral with fractional order $\gamma \in R^+$ for a function $x \in L[0, 1]$ is defined as follows:

$$I_{+0}^\gamma x(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-z)^{\gamma-1} x(z) dz, \quad \gamma > 0,$$

provided that the integral exists on the right-hand side on $(0, \infty)$.

Definition 2 Riemann–Liouville-type fractional order $\gamma \in R^+$ derivative for a function $x \in L[0, 1]$ is recalled as follows:

$$D_{+0}^\gamma x(t) = \frac{1}{\Gamma(m-\gamma)} \left(\frac{d}{dt}\right)^m \int_0^t (t-z)^{m-\gamma-1} x(z) dz,$$

where $m > \gamma$ and the integral on R^+ is pointwise defined.

Lemma 1 For the FODE,

$$D_{+0}^\gamma x(t) = 0, \quad \gamma \in (m - 1, m]$$

is expressed as

$$x(t) = \sum_{k=1}^m d_k t^{\gamma-k}, \quad k = 1, 2, \dots, m, \text{ where } d_k \in R,$$

where $m = [\gamma] + 1$.

Lemma 2 For the FODE, the following result holds:

$$I_{+0}^\gamma [D_{+0}^\gamma x(t)] = x(t) + \sum_{k=1}^m d_k t^{\gamma-k}, \quad k = 1, 2, 3, \dots, m,$$

with $d_k \in R$ and $m = [\gamma] + 1$.

Definition 3 ([29]) For the operators $S_1, S_2 \ni S_1, S_2 : U \rightarrow W$, such that

$$\begin{cases} r(t) - S_1(r, s)(t) = 0, \\ s(t) - S_2(r, s)(t) = 0 \end{cases} \tag{2}$$

is UH-type stable if $\ell_i > 0$ ($i = 1, 2, 3, 4$), $\Delta_i > 0$ ($i = 1, 2$) real numbers and for every $(\hat{r}, \hat{s}) \in U$, we have

$$\begin{cases} \|\hat{r} - S_1(\hat{r}, \hat{s})\| \leq \Delta_1, \\ \|\hat{s} - S_2(\hat{r}, \hat{s})\| \leq \Delta_2 \end{cases} \tag{3}$$

$\exists(\bar{r}, \bar{s}) \in U$ of (2), \ni

$$\begin{cases} \|\hat{r} - \bar{r}\| \leq \ell_1 \Delta_1 + \ell_2 \Delta_2, \\ \|\hat{s} - \bar{s}\| \leq \ell_3 \Delta_1 + \ell_4 \Delta_2. \end{cases} \tag{4}$$

Definition 4 If β_i ($i = 1, 2, 3, \dots, n$) are eigenvalues of \mathcal{M} of order $n \times n$, with spectral radius of $\Upsilon(\mathcal{M})$ defined by

$$\Upsilon(\mathcal{M}) = \max\{|\beta_i| \text{ for } i = 1, 2, \dots, n\}.$$

Moreover, if $\Upsilon(\mathcal{M}) < 1$, then \mathcal{M} converges to 0.

Theorem 1 ([29]) For the two operators $S_1, S_2 \ni S_1, S_2 : U \rightarrow W \ni$

$$\begin{cases} \|S_1(r, s) - S_1(\hat{r}, \hat{s})\| \leq \ell_1 \|r - \hat{r}\| + \ell_2 \|s - \hat{s}\|, \\ \|S_2(r, s) - S_2(\hat{r}, \hat{s})\| \leq \ell_3 \|r - \hat{r}\| + \ell_4 \|s - \hat{s}\| \\ \forall(r, s)(\hat{r}, \hat{s}) \in U \end{cases} \tag{5}$$

and if the following matrix

$$\mathcal{M} = \begin{bmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{bmatrix} \tag{6}$$

tends to zero, subsequently fixed points of (2) are UH-type stable.

Some needed assumptions for our work are given as follows:

$$(M_1) \quad \forall r, \bar{r}, s, \bar{s} \in C(I_1, R) \quad \forall t \in I_1 \quad \exists L_{f_1} > 0 \ni$$

$$|f_1(t, r(\lambda t), s(t), D^p s(t)) - f_1(t, \bar{r}(\lambda t), \bar{s}(t), D^p \bar{s}(t))| \leq L_{f_1} (|r - \bar{r}| + 2|s - \bar{s}|).$$

$$(M_2) \quad \forall r, \bar{r}, s, \bar{s} \in C(I_1, R) \quad \forall t \in I_1 \quad \exists L_{f_2} > 0 \ni$$

$$|f_2(t, r(t), s(\lambda t), D^q r(t)) - f_2(t, \bar{r}(t), \bar{s}(\lambda t), D^q \bar{r}(t))| \leq L_{f_2} (2|r - \bar{r}| + |s - \bar{s}|).$$

$$(M_3) \quad \text{For positive real number } \mathfrak{M} \text{ and } \forall t \in I_1 \ni,$$

$$|\phi(t)| \leq \mathfrak{M}.$$

$$(M_4) \quad \text{For positive real number } \mathfrak{A} \text{ and } \forall t \in I_1 \ni,$$

$$\max \left\{ \int_0^1 \phi(t)r(t) dt, \int_0^1 \phi(t)s(t) dt \right\} \leq \mathfrak{A}.$$

$$(M_5) \quad \text{For some positive real numbers } C_{f_1}, D_{f_1} \text{ and } M_{f_1}, M_{f_2} \ni$$

$$|f_1(t, r(\lambda t), s(t), D^p s(t))| \leq C_{f_1}|r| + 2D_{f_1}|s| + M_{f_1},$$

$$|f_2(t, r(t), s(\lambda t), D^q r(t))| \leq 2C_{f_2}|r| + D_{f_2}|s| + M_{f_2}.$$

Theorem 2 ([30]) *Let E be a closed convex subset of a Banach space U . Let A be an open subset of E with $0 \in A$, and let $\mathbb{F} : \bar{A} \rightarrow E$ be a continuous and compact mapping. Then either the mapping \mathbb{F} has a fixed point in \bar{A} , or $\exists x \in \partial A$ and $\kappa \in (0, 1)$ with $x = \kappa \mathbb{F}x$.*

3 Results about the existence of solutions

Theorem 3 *Let $r(t) \in C(I_1, R)$ and $u(t) \in L(I_1, R)$, the solution for the problem*

$$\begin{aligned} D_{+0}^\beta r(t) + u(t) &= 0, \quad t \in I_1, \beta \in (1, 2], \\ r(0) &= 0, \quad r(1) = \int_0^1 \phi(t)r(t) dt \end{aligned} \tag{7}$$

is as follows:

$$r(t) = \int_0^1 G_1(t, z)u(z) dz + t^{\beta-1} \int_0^1 \phi(t)r(t) dt, \tag{8}$$

where $G_1(t, z)$ is given below:

$$\begin{cases} \frac{t^{\beta-1}(1-z)^{\beta-1}-(t-z)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq z \leq t \leq 1, \\ \frac{t^{\beta-1}(1-z)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq z \leq t \leq 1. \end{cases} \tag{9}$$

Proof Taking integral to both sides of (7) and using Lemma (2) implies

$$\begin{aligned} r(t) &= -I^\beta u(t) + d_1 t^{\beta-1} + d_2 t^{\beta-2} \\ &= -\int_0^t \frac{(t-z)^{\beta-1} u(z) dz}{\Gamma(\beta)} + d_1 t^{\beta-1} + d_2 t^{\beta-2}. \end{aligned} \tag{10}$$

With the help of boundary conditions in (7), we obtain

$$\begin{aligned} d_2 &= 0, \\ d_1 &= \int_0^1 \phi(t)r(t) dt + \frac{1}{\Gamma(\beta)} \int_0^1 (1-z)^{\beta-1} u(z) dz. \end{aligned}$$

Hence, (10) implies

$$\begin{aligned} r(t) &= -\frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} u(z) dz + t^{\beta-1} \int_0^1 \phi(t)r(t) dt \\ &\quad + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-z)^{\beta-1} u(z) dz \\ &= \int_0^1 G_1(t, z)u(z) dz + t^{\beta-1} \int_0^1 \phi(t)r(t) dt. \end{aligned} \tag{11}$$

In the same way, for $s(t) \in C(I_1, R)$ and $u(t) \in L(I_1, R)$, the solution for the problem

$$\begin{aligned} D_{+0}^\gamma s(t) + u(t) &= 0, \quad t \in I_1, \gamma \in (1, 2], \\ s(0) &= 0, \quad s(1) = \int_0^1 \phi(t)s(t) dt \end{aligned} \tag{12}$$

can be obtained in the form

$$\begin{aligned} s(t) &= -\frac{1}{\Gamma(\gamma)} \int_0^t (t-z)^{\gamma-1} u(z) dz + t^{\gamma-1} \int_0^1 \phi(t)s(t) dt \\ &\quad + \frac{t^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 (1-z)^{\gamma-1} u(z) dz \\ &= \int_0^1 G_2(t, z)u(z) dz + t^{\gamma-1} \int_0^1 \phi(t)s(t) dt, \end{aligned} \tag{13}$$

where $G_2(t, z)$ is given below:

$$\begin{cases} \frac{t^{\gamma-1}(1-z)^{\gamma-1}-(t-z)^{\gamma-1}}{\Gamma(\gamma)}, & 0 \leq z \leq t \leq 1, \\ \frac{t^{\gamma-1}(1-z)^{\gamma-1}}{\Gamma(\gamma)}, & 0 \leq z \leq t \leq 1. \end{cases} \tag{14}$$

□

Corollary 1 *In light of Theorem 3, the solution of system (1) is given as follows:*

$$\left\{ \begin{aligned} r(t) &= t^{\beta-1} \int_0^1 \phi(t)r(t) dt \\ &\quad - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} f_1(z, r(\lambda z), s(z), D^\beta s(z)) dz \\ &\quad + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-z)^{\beta-1} f_1(z, r(\lambda z), s(z), D^\beta s(z)) dz \\ &= \int_0^1 G_1(t, z) f_1(z, r(\lambda z), s(z), D^\beta s(z)) dz \\ &\quad + t^{\beta-1} \int_0^1 \phi(t)r(t) dt, \\ s(t) &= t^{\gamma-1} \int_0^1 \phi(t)s(t) dt \\ &\quad - \frac{1}{\Gamma(\gamma)} \int_0^t (t-z)^{\gamma-1} f_2(z, r(z), s(\lambda z), D^\gamma r(z)) dz \\ &\quad + \frac{t^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 (1-z)^{\gamma-1} f_2(z, r(z), s(\lambda z), D^\gamma r(z)) dz \\ &= \int_0^1 G_2(t, z) f_2(z, r(z), s(\lambda z), D^\gamma r(z)) dz \\ &\quad + t^{\gamma-1} \int_0^1 \phi(t)s(t) dt, \end{aligned} \right. \tag{15}$$

where $G(t, z) = (G_1(t, z), G_2(t, z))$ is called Green's function of (1).

Theorem 4 *Let f_1, f_2, f_3 be continuous functions, subsequently $(r, s) \in U$ is a solution of (1) if and only if (r, s) is the solution of the integral equations given in (15).*

Proof Assuming that (r, s) is the solution of (15), then differentiating both sides of (15), we get (1). Anyway, if (r, s) is a solution of (1), then (r, s) is a solution of (15). □

Let $\mathbb{F}_1, \mathbb{F}_2 : U \rightarrow U$ such that

$$\begin{aligned} \mathbb{F}_1(r, s) &= t^{\beta-1} \int_0^1 \phi(t)r(t) dt \\ &\quad - \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} f_1(z, r(\lambda z), s(z), D^\beta s(z)) dz \\ &\quad + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-z)^{\beta-1} f_1(z, r(\lambda z), s(z), D^\beta s(z)) dz \\ &= \int_0^1 G_1(t, z) f_1(z, r(\lambda z), s(z), D^\beta s(z)) dz \\ &\quad + t^{\beta-1} \int_0^1 \phi(t)r(t) dt, \\ \mathbb{F}_2(r, s) &= t^{\gamma-1} \int_0^1 \phi(t)s(t) dt \\ &\quad - \frac{1}{\Gamma(\gamma)} \int_0^t (t-z)^{\gamma-1} f_2(z, r(z), s(\lambda z), D^\gamma r(z)) dz \\ &\quad + \frac{t^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 (1-z)^{\gamma-1} f_2(y, r(z), s(\lambda z), D^\gamma r(z)) dz \\ &= \int_0^1 G_2(t, z) f_2(z, r(z), s(\lambda z), D^\gamma r(z)) dz + t^{\gamma-1} \int_0^1 \phi(t)s(t) dt \end{aligned}$$

and $\mathbb{F}(r, s) = \begin{pmatrix} \mathbb{F}_1(r, s) \\ \mathbb{F}_2(r, s) \end{pmatrix}$. Thus, solutions of (15) are fixed points of \mathbb{F} .

Theorem 5 *Under assumptions $(M_4), (M_5)$ and if the functions $f_1, f_2 : I_1 \times R^3 \rightarrow R$ are continuous, then $\mathbb{F}(E) \subset E$ and $\mathbb{F} : E \rightarrow E$ is completely continuous.*

Proof Let

$$\widehat{\Lambda}_1 = \frac{2(C_{f_1} + 2D_{f_1})}{\Gamma(\beta + 1)} + \frac{2(2C_{f_2} + D_{f_2})}{\Gamma(\gamma + 1)}$$

and

$$\widehat{\Lambda}_2 = \frac{2M_{f_1}}{\Gamma(\beta + 1)} + \frac{2M_{f_2}}{\Gamma(\gamma + 1)} + 2\mathfrak{A}.$$

We define \mathfrak{B} a closed subset of $E \ni$:

$$\mathfrak{B} = \{(r, s) \in E : \|(r, s)\| \leq \rho\}, \quad \text{where } \rho \geq \max \left\{ \frac{\widehat{\Lambda}_2}{1 - \widehat{\Lambda}_1} \right\}.$$

For arbitrary $(r, s) \in \mathfrak{B}$, we have

$$\begin{aligned} |\mathbb{F}_1(r, s)| &\leq t^{\beta-1} \left| \int_0^1 \phi(t)r(t) dt \right| \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} |f_1(z, r(\lambda z), s(z), D^p s(z))| dz \\ &\quad + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-z)^{\beta-1} |f_1(z, r(\lambda z), s(z), D^p s(z))| dz \\ &\leq (C_{f_1}|r| + 2D_{f_1}|s| + M_{f_1}) \left(\frac{t^\beta}{\Gamma(\beta + 1)} + \frac{t^{\beta-1}}{\Gamma(\beta + 1)} \right) \\ &\quad + \mathfrak{A}t^{\beta-1} \\ &\leq \frac{2(C_{f_1}\rho + 2D_{f_1}\rho + M_{f_1})}{\Gamma(\beta + 1)} + \mathfrak{A} \\ &\leq \frac{\rho}{2}. \end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned} |\mathbb{F}_2(r, s)| &\leq \frac{2(2C_{f_2}\rho + D_{f_2}\rho + M_{f_2})}{\Gamma(\gamma + 1)} + \mathfrak{A} \\ &\leq \frac{\rho}{2}. \end{aligned} \tag{17}$$

Therefore, from (16) and (17), one has

$$|\mathbb{F}(r, s)| \leq \rho. \tag{18}$$

Thus (18) leads to the uniform boundedness of \mathbb{F} on B .

To show that $\mathbb{F} : E \rightarrow E$ is completely continuous, take $t_1, t_2 \in I_1 \ni 0 \leq t_1 \leq t_2 \leq 1$, and using $(M_4), (M_5)$, we have $|\mathbb{F}_1(r(t_1), s(t_1)) - \mathbb{F}_1(r(t_2), s(t_2))|$

$$\leq \frac{1}{\Gamma(\beta)} \int_0^{t_1} ((t_1 - z)^{\beta-1} - (t_2 - z)^{\beta-1}) |f_1(z, r(\lambda z), s(z), D^p s(z))| dz$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - z)^{\beta-1} |f_1(z, r(\lambda z), s(z), D^\rho s(z))| dz \\
 & + \frac{(t_1^{\beta-1} - t_2^{\beta-1})}{\Gamma(\beta)} \int_0^1 (1 - z)^{\beta-1} |f_1(z, r(\lambda z), s(z), D^\rho s(z))| dz \\
 & + (t_1^{\beta-1} - t_2^{\beta-1}) \mathfrak{A} \\
 & \leq \frac{(C_{f_1} \rho + 2D_{f_1} \rho + M_{f_1})}{\Gamma(\beta + 1)} ((t_1^\beta - t_2^\beta) + (t_2 - t_1)^\beta + (t_1^{\beta-1} - t_2^{\beta-1})) \\
 & + \mathfrak{A}(t_1^{\beta-1} - t_2^{\beta-1}). \tag{19}
 \end{aligned}$$

Obviously, the right-hand side in inequality (19) tends to zero on $t_1 \rightarrow t_2$. Also \mathbb{F}_1 is bounded and continuous. Thus it is uniformly bounded. Hence

$$\|\mathbb{F}_1(r(t_1), s(t_1)) - \mathbb{F}_1(r(t_2), s(t_2))\| \rightarrow 0 \text{ as } t_1 \text{ tends to } t_2.$$

In a similar way, one can also show that

$$\|\mathbb{F}_2(r(t_1), s(t_1)) - \mathbb{F}_2(r(t_2), s(t_2))\| \rightarrow 0 \text{ as } t_1 \text{ tends to } t_2.$$

Consequently, $\mathbb{F} : E \rightarrow E$ is equicontinuous and so by Arzelá–Ascoli theorem, \mathbb{F} is completely continuous. □

Theorem 6 *Under assumptions (M_1) – (M_3) and if $d < 1$, then system (1) has a unique solution, where*

$$d = \max \left\{ \frac{2L_{f_1}}{\Gamma(\beta + 1)} + \mathfrak{M}, \frac{2L_{f_2}}{\Gamma(\gamma + 1)} + \mathfrak{M} \right\}.$$

Proof Taking $r, \bar{r}, s, \bar{s} \in \mathbb{W}$, and for every $t \in I_1$, let

$$\begin{aligned}
 \|\mathbb{F}_1(r, s) - \mathbb{F}_1(\bar{r}, \bar{s})\| & \leq \max_{t \in I_1} \frac{1}{\Gamma(\beta)} \int_0^t (t - z)^{\beta-1} |f_1(z, r(\lambda z), s(z), D^\rho s(z)) \\
 & \quad - f_1(z, \bar{r}(\lambda z), \bar{s}(z), D^\rho \bar{s}(z))| dz + \max_{t \in I_1} \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1 - z)^{\beta-1} \\
 & \quad \times |f_1(z, r(\lambda z), s(z), D^\rho s(z)) - f_1(z, \bar{r}(\lambda z), \bar{s}(z), D^\rho \bar{s}(z))| dz \\
 & \quad + \max_{t \in I_1} t^{\beta-1} \left| \int_0^1 \phi(t)r(t) dt - \int_0^1 \phi(t)\bar{r}(t) dt \right| \\
 & \leq \frac{2L_{f_1}}{\Gamma(\beta + 1)} (\|r - \bar{r}\| + 2\|s - \bar{s}\|) + \mathfrak{M}\|r - \bar{r}\| \\
 & \leq d_1 (\|r - \bar{r}\| + \|s - \bar{s}\|), \tag{20}
 \end{aligned}$$

where

$$d_1 = \frac{2L_{f_1}}{\Gamma(\beta + 1)} + \mathfrak{M}.$$

Similarly,

$$\|\mathbb{F}_2(r, s) - \mathbb{F}_2(\bar{r}, \bar{s})\| \leq d_2(\|r - \bar{r}\| + \|s - \bar{s}\|), \tag{21}$$

where

$$d_2 = \frac{2L_{f_2}}{\Gamma(\gamma + 1)} + \mathfrak{M}.$$

Hence, from (20) and (21), one has

$$\begin{aligned} \|\mathbb{F}(r, s) - \mathbb{F}(\bar{r}, \bar{s})\| &\leq \max(d_1, d_2)(\|r - \bar{r}\| + \|s - \bar{s}\|) \\ &= d(\|r - \bar{r}\| + \|s - \bar{s}\|), \end{aligned} \tag{22}$$

where $d = \max_{t \in I_1} \{d_1, d_2\}$. Hence, \mathbb{F} is a contraction, thus \mathbb{F} has a unique fixed point, which implies that the concerned system (1) has a unique solution. \square

Theorem 7 *Under assumptions $(M_4), (M_5)$ along with continuous functions $f_1, f_2 : I_1 \times \mathbb{R}^3 \rightarrow \mathbb{R}$, system (1) possesses at least one solution in*

$$\{(r, s) \in E : \|(r, s)\| \leq \rho\}, \quad \text{where } \rho > \max\left(\frac{\widehat{\Lambda}_2}{1 - \widehat{\Lambda}_1}\right). \tag{23}$$

Proof Let $A = \{(r, s) \in E : \|(r, s)\| < \rho\}$, $\rho > \max(\frac{\widehat{\Lambda}_2}{1 - \widehat{\Lambda}_1})$. In light of Theorem (5), the mapping $\mathbb{F} : E \rightarrow E$ is completely continuous. Consider $(r, s) \in A \ni \|(r, s)\| < \rho$. Then in view of Theorem (5), $\mathbb{F}(r, s) \leq \rho$ implies $\mathbb{F}(r, s) \in \bar{A}$. Hence $\mathbb{F} : \bar{A} \rightarrow \bar{A}$.

Take $(r, s) \in \partial A$ and $\kappa \in (0, 1) \ni (r, s) = \kappa \mathbb{F}(r, s)$. Subsequently, in light of assumptions $(M_4), (M_5)$ and for $t \in I_1$, we get

$$\begin{aligned} |r(t)| &\leq \kappa \left(t^{\beta-1} \left| \int_0^1 \phi(t)r(t) dt \right| \right. \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} |f_1(z, r(\lambda z), s(z), D^\rho s(z))| dz \\ &\quad + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-z)^{\beta-1} |f_1(z, r(\lambda z), s(z), D^\rho s(z))| dz \\ &\leq (C_{f_1}|r| + 2D_{f_1}|s| + M_{f_1}) \left(\frac{t^\beta}{\Gamma(\beta + 1)} + \frac{t^{\beta-1}}{\Gamma(\beta + 1)} \right) + \mathfrak{M}t^{\beta-1} \\ &\leq \kappa \left(\frac{2(C_{f_1}\rho + 2D_{f_1}\rho + M_{f_1})}{\Gamma(\beta + 1)} + \mathfrak{M} \right) \\ &< \frac{\kappa\rho}{2}, \end{aligned}$$

which implies that $\|r\| < \frac{\kappa\rho}{2}$. In a similar way, one can also show that $\|s\| < \frac{\kappa\rho}{2}$. Therefore, $\|(r, s)\| < \kappa\rho$ but $\|(r, s)\| = \rho$, which implies $(r, s) \notin \partial A$. Hence, by Lemma 2, \mathbb{F} has at least one fixed point $(r, s) \in \bar{A}$. \square

4 Results regarding stability

Theorem 8 *If the matrix \mathcal{M} converges to 0 and assumptions (M_1) – (M_3) , $d < 1$ hold, then the results of (1) are UH-type stable.*

Proof Taking $(r, s), (\bar{r}, \bar{s}) \in U$ arbitrary solutions and for every $t \in I_1$, we have

$$\begin{aligned} \|\mathbb{F}_1(r, s) - \mathbb{F}_1(\bar{r}, \bar{s})\| &\leq \max_{t \in I_1} \frac{1}{\Gamma(\beta)} \int_0^t (t-z)^{\beta-1} |f_1(z, r(\lambda z), s(z), D^\beta s(z)) \\ &\quad - f_1(z, \bar{r}(\lambda z), \bar{s}(z), D^\beta \bar{s}(z))| dz + \max_{t \in I_1} \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-z)^{\beta-1} \\ &\quad \times |f_1(z, r(\lambda z), s(z), D^\beta s(z)) - f_1(z, \bar{r}(\lambda z), \bar{s}(z), D^\beta \bar{s}(z))| dz \\ &\quad + \max_{t \in I_1} t^{\beta-1} \left| \int_0^1 \phi(t)r(t) dt - \int_0^1 \phi(t)\bar{r}(t) dt \right| \\ &\leq \frac{2L_{f_1}}{\Gamma(\beta+1)} (\|r - \bar{r}\| + 2\|s - \bar{s}\|) + \mathfrak{M}\|r - \bar{r}\| \\ &\leq \ell_1 \|r - \bar{r}\| + \ell_2 \|s - \bar{s}\|, \end{aligned} \tag{24}$$

where

$$\ell_1 = \frac{2L_{f_1}}{\Gamma(\beta+1)} + \mathfrak{M}, \quad \ell_2 = \frac{4L_{f_1}}{\Gamma(\beta+1)}.$$

In a similar way, we may have

$$\|\mathbb{F}_2(r, s) - \mathbb{F}_2(\bar{r}, \bar{s})\| \leq \ell_3 \|r - \bar{r}\| + \ell_4 \|s - \bar{s}\|, \tag{25}$$

where

$$\ell_3 = \frac{2L_{f_2}}{\Gamma(\gamma+1)} + \mathfrak{M} \quad \text{and} \quad \ell_4 = \frac{4L_{f_2}}{\Gamma(\gamma+1)}.$$

So, from (24) and (25), we get

$$\begin{aligned} \|\mathbb{F}_1(r, s) - \mathbb{F}_1(\bar{r}, \bar{s})\| &\leq \ell_1 \|r - \bar{r}\| + \ell_2 \|s - \bar{s}\|, \\ \|\mathbb{F}_2(r, s) - \mathbb{F}_2(\bar{r}, \bar{s})\| &\leq \ell_3 \|r - \bar{r}\| + \ell_4 \|s - \bar{s}\|. \end{aligned} \tag{26}$$

From (26), we can get the matrix \mathcal{M} as follows:

$$\mathcal{M} = \begin{bmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{bmatrix}.$$

Because it is given that the matrix tends to zero, the solution of (1) is UH-type stable. \square

5 Examples

Example 1 Consider the following system of FDDEs:

$$\begin{cases} D_{+0}^{1.3}r(t) = \frac{\cos t}{t^3+4} + \frac{r(0.5t)+s(t)}{e^{2t}(t^2+90)} + \frac{\sqrt{t}D^{0.5}s(t)}{89}, & t \in I_1, \\ D_{+0}^{1.4}s(t) = \frac{s(0.5t)+t^3}{(t+30)^2} + \frac{r(t)+5}{78(e^t+\sqrt{t})} + \frac{t \cos(D^{0.5}r(t))}{55}, & t \in I_1, \\ r(0) = 0, & r(1) = \int_0^1 \left(\frac{t}{20}\right)r(t), \\ s(0) = 0, & s(1) = \int_0^1 \left(\frac{t}{20}\right)s(t). \end{cases} \tag{27}$$

From the above equation, $\beta = 1.3, \gamma = 1.4, \lambda = 0.5, \mathfrak{M} = 0.05$. After the calculation, we have $L_{f_1} = 0.0112, L_{f_2} = 0.0182$,

$$d_1 = 0.0692, \quad d_2 = 0.0793.$$

As $\max\{d_1, d_2\} = 0.0793 < 1$. So, system (27) has an un-repeated solution under Theorem 6. Moreover, taking the values of ℓ_i ($i = 1, 2, 3, 4$), we have

$$\mathcal{M} = \begin{bmatrix} 0.0692 & 0.0384 \\ 0.0793 & 0.0586 \end{bmatrix}. \tag{28}$$

On calculation, the eigenvalues are $\beta_1 = 0.1193, \beta_2 = 0.0085$. Therefore $\Upsilon(\mathcal{M}) = 0.1193 < 1$. Thus the given system under delay term with respect to the given fractional order is HU stable by using Theorem 8.

Example 2 Consider the following system of FDDEs:

$$\begin{cases} D_{+0}^{1.3}r(t) = (3t - 5)^2 e^{-3t} + \frac{r(0.6t)}{(t^2+10)^4} + \frac{\cos|s(t)|-t^{\frac{1}{3}} \sin(D^{\frac{1}{3}}s(t))}{e^{6-t}}, & t \in I_1 \\ D_{+0}^{1.4}s(t) = \cos(t+3) + \frac{\sin|s(0.6t)|}{(e^t+18)^2} + \frac{r(t)+t^{\frac{1}{3}}(D^{\frac{1}{3}}r(t))}{(28-t)^3}, & t \in I_1 \\ r(0) = 0, & r(1) = \int_0^1 \left(\frac{t^2}{25}\right)r(t), \\ s(0) = 0, & s(1) = \int_0^1 \left(\frac{t^2}{25}\right)s(t). \end{cases} \tag{29}$$

On calculation, taking $\beta = 1.3, \gamma = 1.4, \lambda = 0.6, \mathfrak{M} = 0.04$. After calculation, we have $L_{f_1} = 0.0067, L_{f_2} = 0.0030$,

$$d_1 = 0.0515, \quad d_2 = 0.0448.$$

Now $\max\{d_1, d_2\} = 0.0515 < 1$. So (29) has a unique solution under Theorem 6. Moreover, after calculating the values of ℓ_i ($i = 1, 2, 3, 4$), we get the following matrix:

$$\mathcal{M} = \begin{bmatrix} 0.0515 & 0.0230 \\ 0.0448 & 0.0096 \end{bmatrix}. \tag{30}$$

The eigenvalues are $\beta_1 = 0.0689, \beta_2 = -0.0078$ of the above matrix. Clearly, $\Upsilon(\mathcal{M}) = 0.0689 < 1$. Hence the system under consideration is HU-type stable via Theorem 8.

6 Conclusion

We have performed a qualitative analysis of the solutions of (1). The problem has been studied with integral boundary conditions having proportional delay called pantograph. Through fixed point theory, the problem in hand has been investigated. Some results about the stability of UH have been discussed via nonlinear analysis. Then, some relevant examples were presented, the results were also discussed. The considered problem is random and includes many applied problems of fluid mechanics and dynamics as special cases [31].

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