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Ground state and nodal solutions for critical Kirchhoff–Schrödinger–Poisson systems with an asymptotically 3-linear growth nonlinearity

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Abstract

In this paper, we consider the existence of a least energy nodal solution and a ground state solution, energy doubling property and asymptotic behavior of solutions of the following critical problem:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u + V(x)u + \lambda\phi u = |u|^4u + kf(u), & x\in\mathbb{R}^3, \\ -\Delta\phi = u^2, & x\in\mathbb{R}^3. \end{cases}$$

By nodal Nehari manifold method, for each $b > 0$, we obtain a least energy nodal solution u_b and a ground-state solution v_b to this problem when $k \gg 1$, where the nonlinear function $f \in C(\mathbb{R},\mathbb{R})$. We also give an analysis on the behavior of u_b as the parameter $b \to 0$.

Keywords: Kirchhoff–Schrödinger–Poisson systems; Nodal solution; Ground state solution; Nehari manifold

1 Introduction and main results

Our goal of this paper is to consider the existence of nodal solution and ground state solution of the following Kirchhoff–Schrödinger–Poisson system:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u + V(x)u + \lambda\phi u = f(x,u), & x\in\mathbb{R}^3, \\ -\Delta\phi = u^2, & x\in\mathbb{R}^3, \end{cases}$$

where $V(x)$ is a smooth function and $b > 0, \lambda > 0$. When $a = 1$, $b = 0$, Kirchhoff–Schrödinger–Poisson equation reduces to the undermentioned Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u), & x\in\mathbb{R}^3, \\ -\Delta\phi = u^2, & x\in\mathbb{R}^3. \end{cases} \tag{1.1}$$

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System (1.1) is derived from the time-varying Schrödinger equation, which describes the interaction of quantum (non-relativistic) particles with the electromagnetic field generated by motion. On the other hand, recently a great attention has been given to the so-called Kirchhoff equations

$$-\left(a+b\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u=f(x,u), \tag{1.2}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain or $\Omega=\mathbb{R}^N$, $a>0$, $b>0$ and u satisfies some boundary conditions. Problem (1.2) is related to the stationary analogue of the Kirchhoff–Schrödinger type equation

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u=f(x,u), \tag{1.3}$$

which was introduced by Kirchhoff [6] as a generalization of the well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{p_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\,dx\right)=f(x,u), \tag{1.4}$$

for free vibration of elastic strings. The Kirchhoff's model takes into account the length variation of the string produced by the transverse vibration, so the nonlocal term appears. For more mathematical and physical background on Schrödinger–Poisson systems or Kirchhoff-type problems, we refer the readers to [1, 2, 13] and the references therein.

The appearance of nonlocal term not only makes it playing an important role in many physical applications, but also brings some difficulties and challenges in mathematical analysis. This fact makes the study of Kirchhoff–Schrödinger–Poisson system or similar problems particularly interesting. A lot of interesting results on the existence of nonlocal problems were obtained recently in, for example, [4, 5, 7–9, 11, 13–17, 21, 25, 27–29] and the cited references. We especially refer to the paper [10] for the existence of ground state positive solutions of Kirchhoff–Schrödinger-type equations with singular exponential nonlinearities in $\mathbb{R}^N$.

In the past few years, many researchers began to search for nodal solutions to Kirchhoff–Schrödinger-type equations or similar problems and got some interesting results. Zhong and Tang [28] considered the following subcritical Schrödinger–Poisson system:

$$\begin{cases}-\Delta u+V(x)u+k\phi u=|u|^2u+\lambda f(u), & x\in\mathbb{R}^3,\\ -\Delta\phi=u^2, & x\in\mathbb{R}^3,\end{cases} \tag{1.5}$$

where the nonlinearity $f(u)$ satisfies 3-linear growth condition at infinity and linear growth at zero. With the help of the nodal Nehari manifold, they studied the existence and asymptotic behavior of least energy nodal solution to system (1.5).

Wang [18] studied the existence of a least energy sign-changing solution for the following Kirchhoff-type equation:

$$\begin{cases}-(a+b\int_{\Omega}|\nabla u|^2\,dx)\Delta u=|u|^4u+\lambda f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $\lambda, a, b > 0$ are fixed parameters. $f(x,\cdot)$ is continuously differentiable for a.e. $x \in \Omega$. By using the constraint variational method and the degree theory, he got the existence of a least energy nodal solution to the Kirchhoff-type equation.

Wang, Zhang, and Guan [20] studied the following Schrödinger–Poisson system with critical growth:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = |u|^4 u + \mu f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\mu, \lambda > 0$, $f \in C^1(\mathbb{R},\mathbb{R})$. They got the existence and asymptotic behavior of a least energy sign-changing solution to the above system.

Motivated by the above references, in this paper, we study the existence of both ground state and least energy nodal solution for the following critical Kirchhoff–Schrödinger–Poisson system with asymptotically 3-linear growth nonlinearity:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx)\Delta u + V(x)u + \lambda\phi u = |u|^4 u + kf(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.7}$$

where a, b, k, λ are positive real numbers. Similar to [22], we suppose that $V \in C(\mathbb{R}^3,\mathbb{R}^+)$ and satisfies that $E \hookrightarrow\hookrightarrow L^p(\mathbb{R}^3)$ (compact embedding) for $2 < p < 6$, and $E \hookrightarrow L^6(\mathbb{R}^3)$ is continuous, where E is a Hilbert space defined by

$$E = \begin{cases} H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}, & \text{if } V(x) \text{ is a constant}, \\ \{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\,\mathrm{d}x < \infty\}, & \text{if } V(x) \text{ is not a constant} \end{cases}$$

with the inner product defined by

$$\langle u, v\rangle = \int_{\mathbb{R}^3} \bigl(a\nabla u \cdot \nabla v + V(x)uv\bigr)\,dx, \quad \forall u, v \in E$$

and the norm $\|\cdot\|$:

$$\|u\|^2 = \int_{\mathbb{R}^3} \bigl(a|\nabla u|^2 + V(x)u^2\bigr)\,dx.$$

As for the function f, we assume $f \in C(\mathbb{R},\mathbb{R})$ and satisfies the following hypotheses:

(f_1) $f(t)\cdot t > 0$ for $t \neq 0$;

(f_2) $\lim_{t\to\infty} \frac{f(t)}{t^3} = 1$ and $\frac{f(t)}{t^3} < 1$ for all $t \in \mathbb{R}\setminus\{0\}$;

(f_3) $\frac{f(t)}{|t|^3}$ is an increasing function in $(-\infty,0)$ and $(0,+\infty)$.

Remark 1.1 We note that under conditions (f_1)–(f_3), it is easy to see

$$\lim_{t\to 0}\frac{f(t)}{t} = 0. \tag{1.8}$$

The function $f(t) = \frac{t^5}{1+t^2}$ is an example satisfying all conditions (f_1)–(f_3).

It is well known that the equation $-\Delta\phi = u^2$ can be solved as

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}\,dy. \tag{1.9}$$

So system (1.7) is merely a single equation on u:

$$-\left(a + b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\Delta u + V(x)u + \lambda\phi_u u = |u|^4 u + kf(u), \quad x\in\mathbb{R}^3. \tag{1.10}$$

Based on the results above, the energy functional associated with system (1.7) and so with (1.10) is defined by

$$J_k^b(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 + V(x)u^2\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2\,dx - k\int_{\mathbb{R}^3}F(u)\,dx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6\,dx$$

for any $u\in E$. Moreover, under our conditions, $J_k^b(u)$ belongs to $C^1(E,\mathbb{R})$, and the Fréchet derivative of J_k^b is

$$\left\langle\left(J_k^b\right)'(u), v\right\rangle = \int_{\mathbb{R}^3}\left(a\nabla u\cdot\nabla v + V(x)uv\right)dx + b\left(\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\left(\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\,dx\right) + \lambda\int_{\mathbb{R}^3}\phi_u uv\,dx - k\int_{\mathbb{R}^3}f(u)v\,dx - \int_{\mathbb{R}^3}|u|^4 uv\,dx$$

for any $u, v\in E$.

As it is well known, if $u\in E$ is a solution of system (1.7) and $u^{\pm}\neq 0$, then u is a nodal solution of system (1.7), where

$$u^+ = \max\left\{u(x), 0\right\}, \qquad u^- = \min\left\{u(x), 0\right\}.$$

Note that, since system (1.7) involved pure critical nonlinearity $|u|^4 u$, it will prevent us from using the standard arguments as in [3, 12, 19, 22]. Hence, we need to show some techniques to overcome the lack of compactness in $E\hookrightarrow L^6(\mathbb{R}^3)$.

The main results can be stated as follows.

Theorem 1.1 *Suppose that* (f_1)–(f_3) *are satisfied. Then there exists* $k^\star > 0$ *such that, for all* $k\geq k^\star$, *system* (1.7) *has a least energy nodal solution* u_b, *which has precisely two nodal domains.*

Remark 1.2 The least energy nodal solution u_b is a solution of (1.7) satisfying

$$J_k^b(u_b) = \inf_{u\in\mathcal{M}_k^b} J_k^b(u),$$

where $\mathcal{M}_k^b$ is defined by (2.1) in the next section. We recall that the nodal of a continuous function $u:\mathbb{R}^3\to\mathbb{R}$ is the surface $u^{-1}(0)$. Every connected component of $\mathbb{R}^3\setminus u^{-1}(0)$ is called a nodal domain.

Theorem 1.2 *Suppose that* (f_1)–(f_3) *are satisfied. Then there exists* $k^{\star\star} > 0$ *such that, for all* $k \geq k^{\star\star}$, *the* $c^* > 0$ *is achieved and*

$$J_k^b(u_b) > 2c^*,$$

where $c^* = \inf_{u \in \mathcal{N}_k^b} J_k^b(u)$, $\mathcal{N}_k^b = \{u \in H \backslash \{0\} | \langle (J_k^b)'(u), u \rangle = 0\}$, *and* u_b *is the least energy nodal solution obtained in Theorem* 1.1. *In particular,* $c^* > 0$ *is achieved either by a positive or a negative function* v_b *which is a ground state solution of system* (1.7).

Theorem 1.3 *Suppose that* (f_1)–(f_3) *are satisfied. Then there exists* $k^{\star\star\star} > 0$ *such that, for all* $k \geq k^{\star\star\star}$, *for any least energy nodal solution sequence* $\{u_{b_n}\}$ *with* $b_n \to 0$ *as* $n \to \infty$, *there exists a subsequence, still denoted by* $\{u_{b_n}\}$, *such that* u_{b_n} *converges to* u_0 *weakly in* E *as* $n \to \infty$, *where* u_0 *is a least energy nodal solution of the following problem*:

$$\begin{cases} -a\Delta u + V(x)u + \lambda\phi u = |u|^4 u + kf(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.11}$$

Comparing with the literature works, the above three results can be regarded as a generalization of those in [12, 19, 20]. As for Kirchhoff–Schrödinger–Poisson equation, to the best of our knowledge, few results involved the existence and asymptotic behavior of ground state nodal solutions in case of critical growth. It is worth noting that the Brower degree method used in [20, 23] is strictly dependent on the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$, so we have to find new ways to solve our model where we only allow $f \in C(\mathbb{R}, \mathbb{R})$. On the other hand, in our modeling, both of the nonlocal terms $\int_{\mathbb{R}^3} |\nabla u|^2 \, dx$ and ϕu appear, we need to overcome the difficulties caused by the nonlocal terms under a uniform variational framework. It is also due to the lack of compactness embedded in full space that we cannot use the method in [18]. Thankfully, after appropriate modifications, the deformation lemma used in [12] can be applied to get the existence of a least energy nodal solution of the Kirchhoff–Schrödinger–Poisson system.

2 Some technical lemmas

To fix some notations, the letter C, C_i will be repeatedly used to denote various positive constants whose exact values are irrelevant. $|\cdot|_p$ denote the norm in $L^p(\mathbb{R}^3)$ for $p > 1$.

We first list some properties of ϕ_u for our use, one can find the details in [14, 26].

Proposition 2.1 *For any* $u \in E$, *we have*

(i) *there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}^3} \phi_u u^2 \, dx \leq C\|u\|^4 \quad \forall u \in E;$$

(ii) $\phi_u \geq 0$, $\forall u \in E$;

(iii) $\phi_{tu} = t^2\phi_u$, $\forall t > 0$ *and* $u \in E$;

(iv) *if* $u_n \rightharpoonup u$ *in* E, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$ *and*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx \to \int_{\mathbb{R}^3} \phi_u u^2 \, dx.$$

For fixed $u \in E$ with $u^{\pm} \neq 0$, the function $\psi_u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and the mapping $W_u : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ are well defined by

$$\psi_u(s,t) = J_k^b\big(su^+ + tu^-\big),$$
$$W_u(s,t) = \big(\big\langle \big(J_k^b\big)'\big(su^+ + tu^-\big), su^+\big\rangle, \big\langle \big(J_k^b\big)'\big(su^+ + tu^-\big), tu^-\big\rangle\big),$$

and

$$\mathcal{M}_k^b = \big\{u \in E, u^{\pm} \neq 0 \text{ and } \big\langle \big(J_k^b\big)'(u), u^+\big\rangle = \big\langle \big(J_k^b\big)'(u), u^-\big\rangle = 0\big\}. \tag{2.1}$$

Lemma 2.1 *Assume that* (f_1)–(f_3) *are satisfied, if* $u \in E$ *with* $u^{\pm} \neq 0$, *then* ψ_u *has the following properties*:

(i) *The pair* (s,t) *is a critical point of* ψ_u *with* $s,t > 0 \Leftrightarrow su^+ + tu^- \in \mathcal{M}_k^b$;

(ii) *The function* ψ_u *has a unique critical point* (s_u, t_u) *on* $(0,\infty) \times (0,\infty)$, *which is also the unique maximum point of* ψ_u *on* $[0,\infty) \times [0,\infty)$; *Furthermore, if* $\langle (J_k^b)'(u), u^{\pm}\rangle \leq 0$, *then* $0 < s_u, t_u \leq 1$.

Proof (i) By the definition of ψ_u, we have that

$$\begin{aligned}\nabla\psi_u(s,t) &= \left(\frac{\partial\psi_u}{\partial s}, \frac{\partial\psi_u}{\partial t}\right)\\ &= \left(\frac{1}{s}\big\langle \big(J_k^b\big)'\big(su^+ + tu^-\big), su^+\big\rangle, \frac{1}{t}\big\langle \big(J_k^b\big)'\big(su^+ + tu^-\big), tu^-\big\rangle\right).\end{aligned}$$

From the definition, item (i) is obvious.

(ii) It is easy to see

$$\begin{aligned}&\big\langle \big(J_k^b\big)'\big(su^+ + tu^-\big), su^+\big\rangle\\ &\quad= s^2\big\|u^+\big\|^2 + bs^4\left(\int_{\mathbb{R}^3}\big|\nabla u^+\big|^2\,dx\right)^2 + bs^2t^2\left(\int_{\mathbb{R}^3}\big|\nabla u^+\big|^2\,dx\right)\left(\int_{\mathbb{R}^3}\big|\nabla u^-\big|^2\,dx\right)\\ &\qquad+ s^4\lambda\int_{\mathbb{R}^3}\phi_{u^+}\big|u^+\big|^2\,dx + s^2t^2\lambda\int_{\mathbb{R}^3}\phi_{u^-}\big|u^+\big|^2\,dx - s^6\int_{\mathbb{R}^3}\big|u^+\big|^6\,dx\\ &\qquad- k\int_{\mathbb{R}^3}f\big(su^+\big)su^+\,dx\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\big\langle \big(J_k^b\big)'\big(su^+ + tu^-\big), tu^-\big\rangle\\ &\quad= t^2\big\|u^-\big\|^2 + bt^4\left(\int_{\mathbb{R}^3}\big|\nabla u^-\big|^2\,dx\right)^2\\ &\qquad+ bs^2t^2\int_{\mathbb{R}^3}\big|\nabla u^+\big|^2\,dx\int_{\mathbb{R}^3}\big|\nabla u^-\big|^2\,dx + t^4\lambda\int_{\mathbb{R}^3}\phi_{u^-}\big|u^-\big|^2\,dx\\ &\qquad+ s^2t^2\lambda\int_{\mathbb{R}^3}\phi_{u^+}\big|u^-\big|^2\,dx - t^6\int_{\mathbb{R}^3}\big|u^-\big|^6\,dx - k\int_{\mathbb{R}^3}f\big(tu^-\big)tu^-\,dx.\end{aligned} \tag{2.3}$$

From (f_1) and (f_2), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ satisfying

$$\left|f(t)\right| \leq \varepsilon |t| + C_\varepsilon |t|^4 \tag{2.4}$$

for all $t \in \mathbb{R}$. From the Sobolev embedding theorem it follows that

$$\begin{aligned}
&\left\langle \left(J_k^b\right)'\left(su^+ + tu^-\right), su^+\right\rangle \\
&\quad \geq s^2\left\|u^+\right\|^2 - s^6 \int_{\mathbb{R}^3} \left|u^+\right|^6 dx - k\varepsilon s^2 \int_{\mathbb{R}^3} \left|u^+\right|^2 dx - kC_\varepsilon s^q \int_{\mathbb{R}^3} \left|u^+\right|^5 dx \\
&\quad \geq s^2\left\|u^+\right\|^2 - C_1 s^6 \left\|u^+\right\|^6 - k\varepsilon C_2 s^2 \left\|u^+\right\|^2 - kC_\varepsilon C_3 s^q \left\|u^+\right\|^5 \\
&\quad \geq (1 - k\varepsilon C_4) s^2 \left\|u^+\right\|^2 - C_4 s^6 \left\|u^+\right\|^6 - kC_4 s^5 \left\|u^+\right\|^5.
\end{aligned}$$

By choosing $\varepsilon > 0$ such that $(1 - k\varepsilon C_4) > 0$, we can infer that

$$\left\langle \left(J_k^b\right)'\left(su^+ + tu^-\right), su^+\right\rangle > 0$$

for $0 < s \ll 1$ and all $t \geq 0$. Similarly, there holds

$$\left\langle \left(J_k^b\right)'\left(su^+ + tu^-\right), tu^-\right\rangle > 0$$

for $0 < t \ll 1$ and all $s \geq 0$. Hence, there exists $\delta_1 > 0$ such that

$$\left\langle \left(J_k^b\right)'\left(\delta_1 u^+ + tu^-\right), \delta_1 u^+\right\rangle > 0, \qquad \left\langle \left(J_k^b\right)'\left(su^+ + \delta_1 u^-\right), \delta_1 u^-\right\rangle > 0 \tag{2.5}$$

for all $s \geq 0$, $t \geq 0$. It is worth noting that assumption (f_1) implies

$$F(t) \geq 0, \quad t \in \mathbb{R}. \tag{2.6}$$

Thus, choosing $s = \delta_2' > \delta_1$, it follows that, for $t \in [\delta_1, \delta_2']$ and $\delta_2' \gg 1$,

$$\begin{aligned}
&\left\langle \left(J_k^b\right)'\left(\delta_2' u^+ + tu^-\right), \delta_2' u^+\right\rangle \\
&\quad \leq \left(\delta_2'\right)^2 \left\|u^+\right\|^2 + b\left(\delta_2'\right)^4 \left\|u^+\right\|^4 + b\left(\delta_2'\right)^4 \left\|u^+\right\|^2 \left\|u^-\right\|^2 \\
&\qquad + \left(\delta_2'\right)^4 \lambda \int_{\mathbb{R}^3} \phi_{u^+} \left|u^+\right|^2 dx + \left(\delta_2'\right)^4 \lambda \int_{\mathbb{R}^3} \phi_{u^-} \left|u^+\right|^2 dx - \left(\delta_2'\right)^6 \int_{\mathbb{R}^3} \left|u^+\right|^6 dx \\
&\quad \leq 0.
\end{aligned}$$

Analogously, one can show that

$$\begin{aligned}
&\left\langle \left(J_k^b\right)'\left(su^+ + tu^-\right), tu^-\right\rangle \\
&\quad \leq t^2 \left\|u^-\right\|^2 + bt^4 \left\|u^-\right\|^4 + bs^2 t^2 \left\|u^+\right\|^2 \left\|u^-\right\|^2 \\
&\qquad + t^4 \lambda \int_{\mathbb{R}^3} \phi_{u^-} \left|u^-\right|^2 dx + s^2 t^2 \lambda \int_{\mathbb{R}^3} \phi_{u^+} \left|u^-\right|^2 dx - t^6 \int_{\mathbb{R}^3} \left|u^-\right|^6 dx.
\end{aligned}$$

Choosing $\delta_2 > \delta_2' \gg 1$, we deduce

$$\langle (J_k^b)'(\delta_2 u^+ + tu^-), \delta_2 u^+\rangle < 0, \qquad \langle (J_k^b)'(su^+ + \delta_2 u^-), \delta_2 u^-\rangle < 0 \tag{2.7}$$

for all $s, t \in [\delta_1, \delta_2]$.

From (2.5) and (2.7), the assumptions of Miranda's theorem (see Lemma 2.4 in [7]) are satisfied. Thus there is $(s_u, t_u) \in (0,\infty) \times (0,\infty)$ satisfying $W_u(s_u, t_u) = (0,0)$. So $s_u u^+ + t_u u^- \in \mathcal{M}_k^b$.

Now we turn to proving that the pair (s_u, t_u) is unique. We first suppose that $u \in \mathcal{M}_k^b$, thus

$$\begin{aligned}
&\|u^+\|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx\right)^2 + b\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2\,dx + \lambda\int_{\mathbb{R}^3} \phi_{u^+}|u^+|^2\,dx \\
&\quad + \lambda\int_{\mathbb{R}^3} \phi_{u^-}|u^+|^2\,dx = \int_{\mathbb{R}^3} |u^+|^6\,dx + k\int_{\mathbb{R}^3} f(u^+)u^+\,dx
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\|u^-\|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u^-|^2\,dx\right)^2 + b\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2\,dx + \lambda\int_{\mathbb{R}^3} \phi_{u^-}|u^-|^2\,dx \\
&\quad + \lambda\int_{\mathbb{R}^3} \phi_{u^+}|u^-|^2\,dx = \int_{\mathbb{R}^3} |u^-|^6\,dx + k\int_{\mathbb{R}^3} f(u^-)u^-\,dx.
\end{aligned} \tag{2.9}$$

We will show that the pair $(s_u, t_u) = (1,1)$ is the unique one such that $s_u u^+ + t_u u^- \in \mathcal{M}_k^b$. Let (s_0, t_0) be a pair of numbers such that $s_0 u^+ + t_0 u^- \in \mathcal{M}_k^b$ with $0 < s_0 \le t_0$. We have

$$\begin{aligned}
&s_0^2\|u^+\|^2 + bs_0^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx\right)^2 + bs_0^2t_0^2\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2\,dx \\
&\quad + s_0^4\lambda\int_{\mathbb{R}^3} \phi_{u^+}|u^+|^2\,dx + s_0^2t_0^2\lambda\int_{\mathbb{R}^3} \phi_{u^-}|u^+|^2\,dx \\
&\quad = s_0^6\int_{\mathbb{R}^3} |u^+|^6\,dx + k\int_{\mathbb{R}^3} f(s_0u^+)s_0u^+\,dx
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&t_0^2\|u^-\|^2 + bt_0^4\left(\int_{\mathbb{R}^3} |\nabla u^-|^2\,dx\right)^2 + bs_0^2t_0^2\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2\,dx \\
&\quad + t_0^4\lambda\int_{\mathbb{R}^3} \phi_{u^-}|u^-|^2\,dx + s_0^2t_0^2\lambda\int_{\mathbb{R}^3} \phi_{u^+}|u^-|^2\,dx \\
&\quad = t_0^6\int_{\mathbb{R}^3} |u^-|^6\,dx + k\int_{\mathbb{R}^3} f(t_0u^-)t_0u^-\,dx.
\end{aligned} \tag{2.11}$$

By comparing (2.9) and (2.11), we deduce

$$\begin{aligned}
&\frac{\|u^-\|^2}{t_0^2} + b\left(\int_{\mathbb{R}^3} |\nabla u^-|^2\,dx\right)^2 + b\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2\,dx \\
&\quad + \lambda\int_{\mathbb{R}^3} \phi_{u^-}|u^-|^2\,dx + \lambda\int_{\mathbb{R}^3} \phi_{u^+}|u^-|^2\,dx
\end{aligned}$$

$$\geq t_0^2 \int_{\mathbb{R}^3} |u^-|^6 \, dx + k \int_{\mathbb{R}^3} \left[\frac{f(t_0 u^-)}{(t_0 u^-)^3} \right] (u^-)^4 \, dx. \tag{2.12}$$

Combining (2.9) with (2.12), one has that

$$\left(\frac{1}{t_0^2} - 1 \right) \|u^-\|^2 \geq (t_0^2 - 1) \int_{\mathbb{R}^3} |u^-|^6 \, dx + k \int_{\mathbb{R}^3} \left[\frac{f(t_0 u^-)}{(t_0 u^-)^3} - \frac{f(u^-)}{(u^-)^3} \right] (u^-)^4 \, dx.$$

By using assumption (f_3), we get $t_0 \leq 1$. Analogously, from (2.8), (2.10), and $0 < s_0 \leq t_0$,

$$\left(\frac{1}{s_0^2} - 1 \right) \|u^+\|^2 \leq (s_0^2 - 1) \int_{\mathbb{R}^3} |u^+|^6 \, dx + k \int_{\mathbb{R}^3} \left[\frac{f(s_0 u^+)}{(s_0 u^+)^3} - \frac{f(u^+)}{(u^+)^3} \right] (u^+)^4 \, dx.$$

By using assumption (f_3), we get $s_0 \geq 1$. Consequently, $s_0 = t_0 = 1$.

In the case $u \notin \mathcal{M}_k^b$, we suppose that there are (s_1, t_1), (s_2, t_2) such that

$$u_1 = s_1 u^+ + t_1 u^- \in \mathcal{M}_k^b, \qquad u_2 = s_2 u^+ + t_2 u^- \in \mathcal{M}_k^b.$$

Thus,

$$u_2 = \left(\frac{s_2}{s_1} \right) s_1 u^+ + \left(\frac{t_2}{t_1} \right) t_1 u^- = \left(\frac{s_2}{s_1} \right) u_1^+ + \left(\frac{t_2}{t_1} \right) u_1^- \in \mathcal{M}_k^b.$$

According to $u_1 \in \mathcal{M}_k^b$ and the fact of the previous case, one has that

$$\frac{s_2}{s_1} = \frac{t_2}{t_1} = 1.$$

Thus $s_1 = s_2$, $t_1 = t_2$. Therefore (s_u, t_u) is the unique critical point of ψ_u in $(0, \infty) \times (0, \infty)$.

In the following, we show that the critical point (s_u, t_u) of ψ_u is its unique maximum point on $[0, +\infty) \times [0, +\infty)$. By definition

$$\begin{aligned} \psi_u(s,t) = {} & \frac{s^2}{2} \|u^+\|^2 + \frac{bs^4}{4} \left(\int_{\mathbb{R}^3} |\nabla u^+|^2 \, dx \right)^2 + \frac{s^4}{4} \lambda \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^2 \, dx - \frac{s^6}{6} \int_{\mathbb{R}^3} |u^+|^6 \, dx \\ & - \int_{\mathbb{R}^3} F(su^+) \, dx + \frac{t^2}{2} \|u^-\| + \frac{bt^4}{4} \left(\int_{\mathbb{R}^3} |\nabla u^-|^2 \, dx \right)^2 + \frac{t^4}{4} \lambda \int_{\mathbb{R}^3} \phi_{u^-} |u^-|^2 \, dx \\ & - \frac{t^6}{6} \int_{\mathbb{R}^3} |u^+|^6 \, dx - \int_{\mathbb{R}^3} F(tu^-) \, dx + \frac{s^2 t^2}{4} \lambda \int_{\mathbb{R}^3} \phi_{u^-} |u^+|^2 \, dx \\ & + \frac{s^2 t^2}{4} \lambda \int_{\mathbb{R}^3} \phi_{u^+} |u^-|^2 \, dx + \frac{bs^2 t^2}{2} \int_{\mathbb{R}^3} |\nabla u^+|^2 \, dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2 \, dx. \end{aligned}$$

Now (2.6) implies that

$$\lim_{|(s,t)| \to \infty} \psi_u(s,t) = -\infty.$$

By contradiction, we suppose that the boundary point $(0, t_0)$ is a maximum point of ψ_u with $t_0 \geq 0$. By direct computation, it follows that

$$(\psi_u)_s'(s, t_0) = s \|u^+\|^2 + bs^3 \left(\int_{\mathbb{R}^3} |\nabla u^+|^2 \, dx \right)^2 + ast_0^2 \left(\int_{\mathbb{R}^3} |\nabla u^+|^2 \, dx \right) \left(\int_{\mathbb{R}^3} |\nabla u^-|^2 \, dx \right)$$

$$
\begin{aligned}
&+ s^3\lambda \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^2\,dx + \frac{st_0^2}{2}\lambda \int_{\mathbb{R}^3} \phi_{u^+} |u^-|^2\,dx \\
&+ \frac{st_0^2}{2}\lambda \int_{\mathbb{R}^3} \phi_{u^-} |u^+|^2\,dx - s^5 \int_{\mathbb{R}^3} |u^+|^6\,dx - \int_{\mathbb{R}^3} f(su^+)u^+\,dx \\
> 0
\end{aligned}
$$

when $s \ll 1$. It follows that ψ_u is an increasing function with respect to s when $s \ll 1$, which is a contradiction. Analogously, ψ_u cannot achieve its global maximum on the boundary point $(s,0)$ with $s \geq 0$.

In the remainder of our proof, we will prove that $0 < s_u, t_u \leq 1$ when $\langle (J_k^b)'(u), u^{\pm} \rangle \leq 0$. Suppose $s_u \geq t_u > 0$. One has

$$
\begin{aligned}
& s_u^2 \|u^+\|^2 + bs_u^4 \left(\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \right)^2 + bs_u^4 \int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2\,dx \\
&\quad + s_u^4 \lambda \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^2\,dx + s_u^4 \lambda \int_{\mathbb{R}^3} \phi_{u^-} |u^+|^2\,dx \\
&\geq s_u^2 \|u^+\|^2 + bs_u^4 \left(\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \right)^2 + bs_u^2 t_u^2 \int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2\,dx \\
&\quad + s_u^4 \lambda \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^2\,dx + s_u^2 t_u^2 \lambda \int_{\mathbb{R}^3} \phi_{u^-} |u^+|^2\,dx \\
&= s_u^6 \int_{\mathbb{R}^3} |u^+|^6\,dx + k \int_{\mathbb{R}^3} f(s_u u^+) s_u u^+\,dx.
\end{aligned}
\tag{2.13}
$$

In view of $\langle (J_k^b)'(u), u^+ \rangle \leq 0$, one has that

$$
\begin{aligned}
& \|u^+\|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \right)^2 + b \int_{\mathbb{R}^3} |\nabla u^+|^2\,dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2\,dx + \lambda \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^2\,dx \\
&\quad + \lambda \int_{\mathbb{R}^3} \phi_{u^-} |u^+|^2\,dx \leq \int_{\mathbb{R}^3} |u^+|^6\,dx + k \int_{\mathbb{R}^3} f(u^+) u^+\,dx.
\end{aligned}
\tag{2.14}
$$

By comparing (2.13) and (2.14), it follows that

$$
\left(\frac{1}{s_u^2} - 1 \right) \|u^+\|^2 \geq (s_u^2 - 1) \int_{\mathbb{R}^3} |u^+|^6\,dx + k \int_{\mathbb{R}^3} \left[\frac{f(s_u u^+)}{(s_u u^+)^3} - \frac{f(u^+)}{(u^+)^3} \right] (u^+)^4\,dx.
$$

It implies $s_u \leq 1$. Therefore $0 < s_u, t_u \leq 1$. □

Lemma 2.2 *If $u \in \mathcal{M}_k^b$, then $tu \notin \mathcal{M}_k^b$ for every $t > 0$, $t \neq 1$. More precisely,*

$$
\begin{aligned}
&\left\langle \left(J_k^b\right)'(tu), tu^{\pm} \right\rangle > 0 \quad \text{for } t \in (0,1), \\
&\left\langle \left(J_k^b\right)'(tu), tu^{\pm} \right\rangle < 0 \quad \text{for } t > 1.
\end{aligned}
$$

Proof From (2.2) and $u \in \mathcal{M}_k^b$, we have that

$$
\begin{aligned}
\left\langle \left(J_k^b\right)'(tu), tu^+ \right\rangle &= t^2\left(1 - t^2\right) \|u^+\|^2 + t^4\left(1 - t^2\right) \int_{\mathbb{R}^3} |u^+|^6\,dx \\
&\quad + kt^4 \int_{\mathbb{R}^3} \left(f\left(u^+\right) - \frac{f(tu^+)}{t^3} \right) u^+\,dx.
\end{aligned}
$$

According to (f_3), when $0 < t < 1$,

$$\left\langle \left(J_k^b\right)'(tu), tu^+ \right\rangle > 0,$$

while in the case $t > 1$,

$$\left\langle \left(J_k^b\right)'(tu), tu^+ \right\rangle < 0.$$

Similarly, it is easy to get

$$\left\langle \left(J_k^b\right)'(tu), tu^- \right\rangle > 0 \quad \text{for } t \in (0,1), \qquad \left\langle \left(J_k^b\right)'(tu), tu^- \right\rangle < 0 \quad \text{for } t > 1.$$

The proof is complete. □

Lemma 2.3 *Let* $c_b^k = \inf_{u \in \mathcal{M}_k^b} J_k^b(u)$, *then we have that*

$$\lim_{k \to \infty} c_b^k = 0.$$

Proof For any $u \in \mathcal{M}_k^b$, we can deduce

$$\begin{aligned} &\|u^\pm\|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u^\pm|^2 \, dx\right)^2 + b\int_{\mathbb{R}^3} |\nabla u^+|^2 \, dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2 \, dx + \lambda \int_{\mathbb{R}^3} \phi_{u^\pm} |u^\pm|^2 \, dx \\ &\quad + \lambda \int_{\mathbb{R}^3} \phi_{u^\mp} |u^\pm|^2 \, dx = k \int_{\mathbb{R}^3} f(u^\pm) u^\pm \, dx + \int_{\mathbb{R}^3} |u^\pm|^6 \, dx. \end{aligned}$$

Hence, in view of (2.4), it follows that

$$\begin{aligned} \|u^\pm\|^2 &\le k \int_{\mathbb{R}^3} f(u^\pm) u^\pm \, dx + \int_{\mathbb{R}^3} |u^\pm|^6 \, dx \\ &\le k\varepsilon C_1 \|u^\pm\|^2 + kC_2 \|u^\pm\|^5 + C_3 \|u^\pm\|^6. \end{aligned}$$

Therefore, we have that

$$(1 - k\varepsilon C_1)\|u^\pm\|^2 \le kC_2 \|u^\pm\|^5 + C_3 \|u^\pm\|^6.$$

We now choose ε small enough such that $(1 - k\varepsilon C_1) > 0$, so there is $\rho > 0$ such that

$$\|u^\pm\| \ge \rho \tag{2.15}$$

for all $u \in \mathcal{M}_k^b$. For any $u \in \mathcal{M}_k^b$, in view of the definition of $\mathcal{M}_k^b$, $\langle (J_k^b)'(u), u\rangle = 0$. From assumption (f_3), we have

$$f(t)t - 4F(t) \ge 0, \tag{2.16}$$

and $f(t)t - 4F(t)$ is increasing in $(0, +\infty)$ and decreasing in $(-\infty, 0)$. Hence, one gets

$$J_k^b(u) = J_k^b(u) - \frac{1}{4}\left\langle \left(J_k^b\right)'(u), u \right\rangle$$

$$= \frac{1}{4}\|u\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} |u|^6\,dx + \frac{k}{4}\int_{\mathbb{R}^3}\big[f(u)u - 4F(u)\big]\,dx$$
$$\geq \frac{1}{4}\|u\|^2$$

for any $u \in \mathcal{M}_k^b$.

From the above discussion, we can see that $c_b^k = \inf_{u\in\mathcal{M}_k^b} J_k^b(u)$ is well defined.

Let $u \in E$ with $u^{\pm} \neq 0$ be fixed. According to Lemma 2.1, for each $k > 0$, there exist $s_k, t_k > 0$ such that $s_k u^+ + t_k u^- \in \mathcal{M}_k^b$. Hence, by (2.6), the Sobolev embedding theorem and Proposition 2.1, we have

$$\begin{aligned}
0 \leq c^k &= \inf_{u\in\mathcal{M}_k^b} J_k^b(u) \leq J_k^b\big(s_k u^+ + t_k u^-\big) \\
&\leq \frac{1}{2}\big\|s_k u^+ + t_k u^-\big\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}\big|\nabla\big(s_k u^+ + t_k u^-\big)\big|^2\,dx\right)^2 \\
&\quad + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{s_k u^+ + t_k u^-}\big|s_k u^+ + t_k u^-\big|^2\,dx \\
&\leq \frac{s_k^2}{2}\big\|u^+\big\|^2 + \frac{t_k^2}{2}\big\|u^-\big\|^2 + Cs_k^4\big\|u^+\big\|^4 + Ct_k^4\big\|u^-\big\|^4
\end{aligned}$$

for some constants $C > 0$. We now define

$$\Phi_u = \big\{(s_k, t_k) \in [0,\infty) \times [0,\infty) : W_u(s_k, t_k) = (0,0), k > 0\big\}.$$

Hence we have that

$$\begin{aligned}
&s_k^6\int_{\mathbb{R}^3}\big|u^+\big|^6\,dx + t_k^6\int_{\mathbb{R}^3}\big|u^-\big|^6\,dx \\
&\quad\leq s_k^6\int_{\mathbb{R}^3}\big|u^+\big|^6\,dx + t_k^6\int_{\mathbb{R}^3}\big|u^-\big|^6\,dx + k\int_{\mathbb{R}^3} f\big(s_k u^+\big)s_k u^+\,dx + k\int_{\mathbb{R}^3} f\big(t_k u^-\big)t_k u^-\,dx \\
&\quad= \big\|s_k u^+ + t_k u^-\big\|^2 + b\left(\int_{\mathbb{R}^3}\big|\nabla\big(s_k u^+ + t_k u^-\big)\big|^2\,dx\right)^2 + \lambda\int_{\mathbb{R}^3}\phi_{s_k u^+ + t_k u^-}\big|s_k u^+ + t_k u^-\big|^2\,dx \\
&\quad\leq s_k^2\big\|u^+\big\|^2 + t_k^2\big\|u^-\big\|^2 + Cs_k^4\big\|u^+\big\|^4 + Ct_k^4\big\|u^-\big\|^4.
\end{aligned}$$

It follows that Φ_u is a bounded set. We suppose that $k_n \to \infty$ as $n \to \infty$. For $(s_{k_n}, t_{k_n}) \in \Phi_u$, there exist s_0 and t_0 such that

$$(s_{k_n}, t_{k_n}) \to (s_0, t_0)$$

as $n \to \infty$ (in the subsequence sense). We suppose that $s_0 > 0$ or $t_0 > 0$. Thanks to $s_{k_n}u^+ + t_{k_n}u^- \in \mathcal{M}_b^{k_n}$, we get

$$\begin{aligned}
&\big\|s_{k_n}u^+ + t_{k_n}u^-\big\|^2 + b\left(\int_{\mathbb{R}^3}\big|\nabla\big(s_{k_n}u^+ + t_{k_n}u^-\big)\big|^2\,dx\right)^2 \\
&\qquad + \lambda\int_{\mathbb{R}^3}\phi_{s_{k_n}u^+ + t_{k_n}u^-}\big|s_{k_n}u^+ + t_{k_n}u^-\big|^2\,dx \\
&\quad= \int_{\mathbb{R}^3}\big|s_{k_n}u^+ + t_{k_n}u^-\big|^6\,dx + k_n\int_{\mathbb{R}^3} f\big(s_{k_n}u^+ + t_{k_n}u^-\big)\big(s_{k_n}u^+ + t_{k_n}u^-\big)\,dx. \qquad (2.17)
\end{aligned}$$

According to $s_{k_n}u^+ \to s_0u^+$ and $t_{k_n}u^- \to t_0u^-$ in E, $\int_{\mathbb{R}^3}|\nabla(s_{k_n}u^+ + t_{k_n}u^-)|^2\,dx \le \|s_{k_n}u^+ + t_{k_n}u^-\|^2$, (2.4) and (2.6), so as $n \to \infty$, there holds

$$\int_{\mathbb{R}^3} f\left(s_{k_n}u^+ + t_{k_n}u^-\right)\left(s_{k_n}u^+ + t_{k_n}u^-\right)dx \to \int_{\mathbb{R}^3} f\left(s_0u^+ + t_0u^-\right)\left(s_0u^+ + t_0u^-\right)dx > 0.$$

Because $k_n \to \infty$ as $n \to \infty$ and $\{s_{k_n}u^+ + t_{k_n}u^-\}$ is bounded in E, following the Sobolev embedding theorem, we have a contradiction with equality (2.17). Thus, $s_0 = t_0 = 0$, and so $\lim_{k\to\infty} c_b^k = 0$. □

Lemma 2.4 *There exists $k^\star > 0$ such that, for all $k \ge k^\star$, the infimum c_b^k is achieved.*

Proof In view of the definition of c_b^k, we deduce that there exists a sequence $\{u_n\} \subset \mathcal{M}_b^k$ satisfying

$$\lim_{n\to\infty} J_k^b(u_n) = c_b^k.$$

Following from (2.8) and (2.9), $\{u_n\}$ is bounded in E. So in the subsequence sense, there exists $u_b = u_b^+ + u_b^- \in E$ such that $u_n \rightharpoonup u_b$. Since the embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is compact for $p \in (2,6)$, we deduce

$$\begin{aligned} &u_n \to u_b \quad \text{in } L^p\left(\mathbb{R}^3\right), \forall p \in (2,6), \\ &u_n(x) \to u_b(x) \quad \text{a.e. } x \in \mathbb{R}^3. \end{aligned}$$

Then we have

$$\begin{aligned} &u_n^\pm \rightharpoonup u_b^\pm \quad \text{in } E, \\ &u_n^\pm \to u_b^\pm \quad \text{in } L^p\left(\mathbb{R}^3\right), \\ &u_n^\pm(x) \to u_b^\pm(x) \quad \text{a.e. } x \in \mathbb{R}^3. \end{aligned}$$

Denote $\beta := \frac{(S)^{\frac{3}{2}}}{3}$, where

$$S := \inf_{u\in E\setminus\{0\}} \frac{\|u\|^2}{(\int_{\mathbb{R}^3}|u|^6\,dx)^{\frac{1}{3}}}.$$

The Sobolev embedding theorem insures that $\beta > 0$. Lemma 2.3 implies that there exists $k^\star > 0$ such that $c_b^k < \beta$ for all $k \ge k^\star$. Fix $k \ge k^\star$, in view of Lemma 2.1, we have

$$J_k^b\left(su_n^+ + tu_n^-\right) \le J_k^b(u_n)$$

for all $s, t \in [0, +\infty)$. Because $u_n^\pm \rightharpoonup u_b^\pm$ in E, E is a Hilbert space, we can deduce

$$\left\|u_n^\pm\right\|^2 - \left\|u_n^\pm - u_b^\pm\right\|^2 = 2\left(u_n^\pm, u_b^\pm\right) - \left\|u_b^\pm\right\|^2,$$

where we can assume that the sequence $\{\|u_n^\pm\|\}$ is convergent, so we have

$$\lim_{n\to\infty}\left\|u_n^\pm\right\|^2 = \lim_{n\to\infty}\left\|u_n^\pm - u_b^\pm\right\|^2 + \left\|u_b^\pm\right\|^2.$$

Obviously, we can let $n \to \infty$ in both sides of the above equation. On the other hand, by (2.4) we have

$$\int_{\mathbb{R}^3} F(su_n^{\pm})\,dx \to \int_{\mathbb{R}^3} F(su_b^{\pm})\,dx.$$

Thus, we get

$$\begin{aligned}
&\liminf_{n\to\infty} J_k^b(su_n^+ + tu_n^-)\\
&\quad\ge \frac{s^2}{2}\lim_{n\to\infty}\left(\|u_n^+ - u_b^+\|^2 + \|u_b^+\|^2\right) + \frac{t^2}{2}\lim_{n\to\infty}\left(\|u_n^- - u_b^-\|^2 + \|u_b^-\|^2\right)\\
&\qquad + \frac{bs^2}{4}\left[\liminf_{n\to\infty}\left(\int_{\mathbb{R}^3}|\nabla u_n^+|^2\,dx\right)\right]^2 + \frac{bt^2}{4}\left[\liminf_{n\to\infty}\left(\int_{\mathbb{R}^3}|\nabla u_n^-|^2\,dx\right)\right]^2\\
&\qquad + \frac{bs^2t^2}{4}\liminf_{n\to\infty}\left(\int_{\mathbb{R}^3}|\nabla u_n^+|^2\,dx\right)\cdot\liminf_{n\to\infty}\left(\int_{\mathbb{R}^3}|\nabla u_n^-|^2\,dx\right)\\
&\qquad + \frac{\lambda s^4}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n^+}|u_n^+|^2\,dx + \frac{\lambda t^4}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n^-}|u_n^-|^2\,dx\\
&\qquad - \frac{s^6}{6}\lim_{n\to\infty}\left(|u_n^+ - u_b^+|_6^6 + |u_b^+|_6^6\right) - \frac{t^6}{6}\lim_{n\to\infty}\left(|u_n^- - u_b^-|_6^6 + |u_b^-|_6^6\right)\\
&\qquad - k\int_{\mathbb{R}^3}F(su_b^+)\,dx - k\int_{\mathbb{R}^3}F(tu_b^-)\,dx\\
&\qquad + \frac{\lambda s^2t^2}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n^+}|u_n^-|^2\,dx + \frac{\lambda s^2t^2}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n^-}|u_n^+|^2\,dx.
\end{aligned}$$

By using Fatou's lemma, there holds

$$\begin{aligned}
\liminf_{n\to\infty} J_k^b(su_n^+ + tu_n^-) &\ge J_k^b(su_b^+ + tu_b^-) + \frac{s^2}{2}\lim_{n\to\infty}\|u_n^+ - u_b^+\|^2 + \frac{t^2}{2}\lim_{n\to\infty}\|u_n^- - u_b^-\|^2\\
&\quad - \frac{s^6}{6}\lim_{n\to\infty}|u_n^+ - u_b^+|_6^6 - \frac{t^6}{6}\lim_{n\to\infty}|u_n^- - u_b^-|_6^6\\
&= J_k^b(su_b^+ + tu_b^-) + \frac{s^2}{2}A_1 - \frac{s^6}{6}B_1 + \frac{t^2}{2}A_2 - \frac{t^6}{6}B_2,
\end{aligned}$$

where

$$\begin{aligned}
&A_1 = \lim_{n\to\infty}\|u_n^+ - u_b^+\|^2, \qquad A_2 = \lim_{n\to\infty}\|u_n^- - u_b^-\|^2,\\
&B_1 = \lim_{n\to\infty}|u_n^+ - u_b^+|_6^6, \qquad B_2 = \lim_{n\to\infty}|u_n^- - u_b^-|_6^6.
\end{aligned}$$

From the above fact, one has that

$$J_k^b(su_b^+ + tu_b^-) + \frac{s^2}{2}A_1 - \frac{s^6}{6}B_1 + \frac{t^2}{2}A_2 - \frac{t^6}{6}B_2 \le c_b^k \tag{2.18}$$

for all $s \ge 0$, $t \ge 0$.

Claim 1. $u_b^{\pm} \neq 0$. In fact, by contradiction, if $u_b^+ = 0$, we divide it into two cases.

Case 1: $B_1 = 0$. In this case, if $A_1 = 0$, in view of the fact (2.15), we obtain $\|u_b^+\| > 0$, which is absurd. If $A_1 > 0$, we let $t = 0$ in (2.18) that $\frac{s^2}{2}A_1 \le c_b^k$ for all $s \ge 0$, which is false.

Case 2: $B_1 > 0$. In this case, by the definition of S, we deduce

$$\beta = \frac{(S)^{\frac{3}{2}}}{3} \le \frac{1}{3}\left(\frac{A_1}{(B_1)^{\frac{1}{3}}}\right)^{\frac{3}{2}}.$$

On the other hand,

$$\frac{1}{3}\left(\frac{A_1}{(B_1)^{\frac{1}{3}}}\right)^{\frac{3}{2}} = \max_{s\ge 0}\left\{\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1\right\}.$$

Thanks to $c_b^k < \beta$, by substituting $t = 0$ into (2.18), we have that

$$\beta \le \max_{s\ge 0}\left\{\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1\right\} \le c_b^k < \beta,$$

which is a contradiction. Thus $u_b^+ \neq 0$. Similarly, we also get $u_b^- \neq 0$. Therefore $u_b^\pm \neq 0$ as claimed.

Claim 2. $B_1 = B_2 = 0$. We only prove $B_1 = 0$. By contradiction, we suppose that $B_1 > 0$. We have two cases.

Case 1: $B_2 > 0$. Let s_a and t_b be the numbers such that

$$\frac{s_a^2}{2}A_1 - \frac{s_a^6}{6}B_1 = \max_{s\ge 0}\left\{\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1\right\},$$
$$\frac{t_b^2}{2}A_2 - \frac{t_b^6}{6}B_2 = \max_{t\ge 0}\left\{\frac{t^2}{2}A_2 - \frac{t^6}{6}B_2\right\}.$$

Since ψ_u is continuous, we have $(s_u, t_u) \in [0, s_a] \times [0, t_b]$ satisfying

$$\psi_u(s_u, t_u) = \max_{(s,t)\in[0,s_a]\times[0,t_b]} \psi_u(s,t).$$

Note that if $0 < t \ll 1$, we deduce

$$\psi_u(s,0) = J_k^b(su_b^+) < J_k^b(su_b^+) + J_k^b(tu_b^-) \le J_k^b(su_b^+ + tu_b^-) = \psi_u(s,t)$$

for all $s \in [0, s_a]$. Thus there is $t_0 \in [0, t_b]$ such that

$$\psi_u(s,0) \le \psi_u(s,t_0)$$

for all $s \in [0, s_a]$. It follows that any point of the form $(s, 0)$ with $0 \le s \le s_a$ is not the maximizer of ψ_u. Thus, $(s_u, t_u) \notin [0, s_a] \times \{0\}$. Similarly, it shows that $(s_u, t_u) \notin \{0\} \times [0, t_b]$. By direct computation, we get

$$\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1 > 0, \tag{2.19}$$
$$\frac{t^2}{2}A_2 - \frac{t^6}{6}B_2 > 0 \tag{2.20}$$

for all $s \in (0, s_a]$, $t \in (0, t_b]$. Hence there hold

$$\beta \le \frac{s_a^2}{2}A_1 - \frac{s_a^6}{6}B_1 + \frac{t^2}{2}A_2 - \frac{t^6}{6}B_2,$$
$$\beta \le \frac{t_b^2}{2}A_2 - \frac{t_b^6}{6}B_2 + \frac{s^2}{2}A_1 - \frac{s^6}{6}B_1$$

for all $s \in [0, s_a]$, $t \in [0, t_b]$. In view of (2.18), it follows that

$$\psi_u(s, t_b) \le 0, \qquad \psi_u(s_a, t) \le 0$$

for all $s \in [0, s_a]$, $t \in [0, t_b]$. That is, $(s_u, t_u) \notin \{s_a\} \times [0, t_b]$ and $(s_u, t_u) \notin \times [0, s_a] \times \{t_b\}$. Hence, we can deduce that $(s_u, t_u) \in (0, s_a) \times (0, t_b)$. By Lemma 2.1, it follows that (s_u, t_u) is a critical point of ψ_u. Thus, $s_u u^+ + t_u u^- \in \mathcal{M}_k^b$. By (2.18), (2.19), and (2.20), we deduce

$$\begin{aligned} c_b^k &\ge J_k^b\bigl(s_u u_b^+ + t_u u_b^-\bigr) + \frac{s_u^2}{2}A_1 - \frac{s_u^6}{6}B_1 + \frac{t_u^2}{2}A_2 - \frac{t_u^6}{6}B_2 \\ &> J_k^b\bigl(s_u u_b^+ + t_u u_b^-\bigr) \\ &\ge c_b^k. \end{aligned}$$

It is impossible. The proof of Case 1 is completed.

Case 2: $B_2 = 0$. From the definition of J_k^b, it is easy to show that there exists $t_0 \in [0, \infty)$ such that $J_k^b(su_b^+ + tu_b^-) \le 0$ for all $(s, t) \in [0, s_a] \times [t_0, \infty)$. Thus, there is $(s_u, t_u) \in [0, s_a] \times [0, \infty)$ satisfying

$$\psi_u(s_u, t_u) = \max_{(s,t)\in[0,s_a]\times[0,\infty)} \psi_u(s, t).$$

We need to prove that $(s_u, t_u) \in (0, s_a) \times (0, \infty)$. Similarly, it is noticed that $\psi_u(s, 0) < \psi_u(s, t)$ for $s \in [0, s_a]$ and $0 < t \ll 1$, that is, $(s_u, t_u) \notin [0, s_a] \times \{0\}$. Also, for s small enough, we get $\psi_u(0, t) < \psi_u(s, t)$ for $t \in [0, \infty)$, that is, $(s_u, t_u) \notin \{0\} \times [0, \infty)$. We note that

$$\beta \le \frac{s_a^2}{2}A_1 - \frac{s_a^6}{6}B_1 + \frac{t^2}{2}A_2$$

for all $t \in [0, \infty)$. Thus also from (2.20) and $B_2 = 0$, we have $\psi_u(s_a, t) \le 0$ for all $t \in [0, \infty)$. Hence, $(s_u, t_u) \notin \{s_a\} \times [0, \infty)$. That is, (s_u, t_u) is an inner maximizer of ψ_u in $[0, s_a) \times [0, \infty)$. So $s_u u^+ + t_u u^- \in \mathcal{M}_k^b$. Hence, by using (2.19), we obtain

$$\begin{aligned} c_b^k &\ge J_k^b\bigl(s_u u_b^+ + t_u u_b^-\bigr) + \frac{s_u^2}{2}A_1 - \frac{s_u^6}{6}B_1 + \frac{t_u^2}{2}A_2 - \frac{t_u^6}{6}B_2 \\ &> J_k^b\bigl(s_u u_b^+ + t_u u_b^-\bigr) \\ &\ge c_b^k, \end{aligned}$$

which is a contradiction. It is similar for $B_2 = 0$. From the above discussion, we know that Claim 2 is true.

Claim 3. c_b^k is achieved. Since $u_b^{\pm} \neq 0$, by Lemma 2.1, there are $s_u, t_u > 0$ such that $\widetilde{u} := s_u u_b^+ + t_u u_b^- \in \mathcal{M}_b^k$. On the other hand, $u_n \rightharpoonup u_b$ in E, then $\int_{\mathbb{R}^3}(V(x)u_n^2)\,dx \to \int_{\mathbb{R}^3}(V(x)u_b^2)\,dx$ and $\liminf_{n\to\infty}\|u_n\| \geq \|u_b\|$, so we get

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx \geq \int_{\mathbb{R}^3}|\nabla u_b|^2\,dx.$$

On the other hand, by (2.4), we deduce

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} f\bigl(u_n^{\pm}\bigr)u_n^{\pm}\,dx = \int_{\mathbb{R}^3} f(u_b)^{\pm}u_b^{\pm}\,dx.$$

Thanks to Proposition 2.1, we get

$$\begin{aligned}
\bigl\langle \bigl(J_k^b\bigr)'(u_b), u_b^{\pm}\bigr\rangle &\leq \liminf_{n\to\infty}\bigl\|u_n^{\pm}\bigr\|^2 + b\biggl[\liminf_{n\to\infty}\biggl(\int_{\mathbb{R}^3}\bigl|\nabla u_n^{\pm}\bigr|^2\,dx\biggr)\biggr]^2 \\
&\quad + b\liminf_{n\to\infty}\biggl(\int_{\mathbb{R}^3}\bigl|\nabla u_n^{+}\bigr|^2\,dx\biggr)\cdot\liminf_{n\to\infty}\biggl(\int_{\mathbb{R}^3}\bigl|\nabla u_n^{-}\bigr|^2\,dx\biggr) \\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}\bigl|u_n^{\pm}\bigr|^2\,dx - \lim_{n\to\infty}\int_{\mathbb{R}^3} f\bigl(u_n^{\pm}\bigr)u_n^{\pm}\,dx - \lim_{n\to\infty}\int_{\mathbb{R}^3}\bigl|u_n^{\pm}\bigr|^6 \\
&\leq \lim_{n\to\infty}\bigl\langle \bigl(J_k^b\bigr)'(u_n), u_n^{\pm}\bigr\rangle = 0.
\end{aligned}$$

Therefore from Lemma 2.2 we have that $0 < s_u, t_u \leq 1$. Since $u_n \in \mathcal{M}_b^k$, $B_1 = B_2 = 0$ and $\|u\|$ is lower semicontinuous, it follows that

$$\begin{aligned}
c_b^k &\leq J_k^b(\widetilde{u})) - \frac{1}{4}\bigl\langle\bigl(J_k^b\bigr)'(\widetilde{u}), \widetilde{u}\bigr\rangle \\
&= \frac{1}{4}\|\widetilde{u}\|^2 + \frac{1}{12}|\widetilde{u}|_6^6 + \frac{k}{4}\int_{\mathbb{R}^3}\bigl[f(\widetilde{u})\widetilde{u} - 4F(\widetilde{u})\bigr]\,dx \\
&= \frac{1}{4}\bigl(\bigl\|s_{u_b}u_b^+\bigr\|^2 + \bigl\|t_{u_b}u_b^-\bigr\|^2\bigr) + \frac{1}{12}\bigl(\bigl|s_{u_b}u_b^+\bigr|_6^6 + \bigl|t_{u_b}u_b^-\bigr|_6^6\bigr) \\
&\quad + \frac{k}{4}\int_{\mathbb{R}^3}\bigl[f\bigl(s_{u_b}u_b^+\bigr)\bigl(s_{u_b}u_b^+\bigr) - 4F\bigl(s_{u_b}u_b^+\bigr)\bigr]\,dx \\
&\quad + \frac{k}{4}\int_{\mathbb{R}^3}\bigl[f\bigl(x, t_{u_b}u_b^-\bigr)\bigl(t_{u_b}u_b^-\bigr) - 4F\bigl(x, t_{u_b}u_b^-\bigr)\bigr]\,dx.
\end{aligned}$$

By using $0 < s_{u_b}, t_{u_b} \leq 1$, $f(t)t - 4F(t)$ is increasing in $(0, +\infty)$ and decreasing in $(-\infty, 0)$, we have

$$\begin{aligned}
c_b^k &\leq \frac{1}{4}\|u_b\|^2 + \frac{1}{12}|u_b|_6^6 + \frac{k}{4}\int_{\mathbb{R}^3}\bigl[f(u_b)u_b - 4F(u_b)\bigr]\,dx \\
&\leq \liminf_{n\to\infty}\biggl[J_k^b(u_n) - \frac{1}{4}\bigl\langle\bigl(J_k^b\bigr)'(u_n), u_n\bigr\rangle\biggr] \\
&= \liminf_{n\to\infty} J_k^b(u_n) \\
&= c_b^k.
\end{aligned}$$

Therefore the infimum c_b^k is achieved by $u_b = u_b^+ + u_b^- \in \mathcal{M}_b^k$. □

3 The proof of the main results

In this section, we prove the main results. Firstly, we prove Theorem 1.1. In fact, thanks to Lemma 2.4, we should prove that the minimizer u_b for c_b^k is indeed a nodal solution of system (1.7), but $\mathcal{M}_k^b$ is not a smooth manifold, we will apply a new method to complete our certification.

3.1 The proof of Theorem 1.1

Proof Since $u_b \in \mathcal{M}_b^k$ and $J_k^b(u_b^+ + u_b^-) = c_b^k$, we have $\langle (J_k^b)'(u_b), u_b^+\rangle = \langle (J_k^b)'(u_b), u_b^-\rangle = 0$. By Lemma 2.1, for $(s,t) \in (\mathbb{R}_+ \times \mathbb{R}_+)\backslash(1,1)$, we have

$$J_k^b\big(su_b^+ + tu_b^-\big) < J_k^b\big(u_b^+ + u_b^-\big) = c_b^k. \tag{3.1}$$

If $(J_k^b)'(u_b) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that

$$\big\|\big(J_k^b\big)'(v)\big\| \geq \theta \quad \text{for all } \|v - u_b\| \leq 3\delta.$$

We know by result (2.15), if $u \in \mathcal{M}_k^b$, there exists $L > 0$ such that $\|u_b^{\pm}\| > L$, and we can assume $6\delta < L$. Let $Q := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g(s,t) = su_b^+ + tu_b^-$, $(s,t) \in Q$. In view of (3.1), it is easy to see that

$$\overline{c}_b^k := \max_{\partial Q} I \circ g < c_b^k. \tag{3.2}$$

Let $\varepsilon := \min\{(c_b^k - \overline{c}_b^k)/4, \theta\delta/8\}$ and $S_\delta := B(u_b, \delta)$, according to Lemma 2.3 of [24], there exists a deformation $\eta \in C([0,1] \times E, E)$ satisfying

(a) $\eta(t,v) = v$ if $t = 0$, or $v \notin (J_k^b)^{-1}([c_b^k - 2\varepsilon, c_b^k + 2\varepsilon]) \cap S_{2\delta}$;
(b) $\eta(1, (J_k^b)^{c_b^k+\varepsilon} \cap S_\delta) \subset (J_k^b)^{c_b^k-\varepsilon}$;
(c) $J_k^b(\eta(1,v)) \leq J_k^b(v)$ for all $v \in E$;
(d) $J_k^b(\eta(\cdot,v))$ is nonincreasing for every $v \in E$.

We remind that, for a functional $\Phi : E \to \mathbb{R}$, the level set Φ^μ is defined by $\Phi^\mu = \{u \in E : \Phi(u) \leq \mu\}$. Firstly, we need to prove that

$$\max_{(s,t)\in\bar{Q}} J_k^b\big(\eta\big(1, g(s,t)\big)\big) < c_b^k. \tag{3.3}$$

In fact, it follows from Lemma 2.1 that $J_k^b(g(s,t)) \leq c_b^k < c_b^k + \varepsilon$. That is,

$$g(s,t) \in \big(J_k^b\big)^{c_b^k+\varepsilon}.$$

On the other hand, from (a) and (d), we get

$$J_k^b\big(\eta(1,v)\big) \leq J_k^b\big(\eta(0,v)\big) = J_k^b(v), \quad \forall v \in E. \tag{3.4}$$

For $(s,t) \in Q$, when $s \neq 1$ or $t \neq 1$, according to (3.1) and (3.4),

$$J_k^b\big(\eta\big(1, g(s,t)\big)\big) \leq J_k^b\big(g(s,t)\big) < c_b^k.$$

If $s = 1$ and $t = 1$, that is, $g(1,1) = u_b$, so that it holds $g(1,1) \in (J_k^b)^{c_b^k+\varepsilon} \cap S_\delta$, then by (b)

$$J_k^b\big(\eta\big(1, g(1,1)\big)\big) \le c_b^k - \varepsilon < c_b^k.$$

Thus (3.3) holds. In the following, we prove that $\eta(1, g(Q)) \cap \mathcal{M}_b^k \neq \emptyset$, which contradicts the definition of c_b^k. Let $\varphi(s,t) := \eta(1, g(s,t))$ and

$$\Psi(s,t) := \left(\frac{1}{s}\big\langle (J_k^b)'\big(\varphi(s,t)\big), \big(\varphi(s,t)\big)^+\big\rangle, \frac{1}{t}\big\langle (J_k^b)'\big(\varphi(s,t)\big), \big(\varphi(s,t)\big)^-\big\rangle\right).$$

The claim holds if there exists $(s_0,t_0) \in Q$ such that $\Psi(s_0,t_0) = (0,0)$. Since

$$\begin{aligned}\big\|g(s,t) - u_b\big\|^2 &= \big\|(s-1)u_b^+ + (t-1)u_b^-\big\|^2 \\ &\ge |s-1|^2\big\|u_b^+\big\|^2 \\ &> |s-1|^2(6\delta)^2,\end{aligned}$$

and $|s-1|^2(6\delta)^2 > 4\delta^2 \Leftrightarrow s < 2/3$ or $s > 4/3$, using the item (a) above and the range of s, for $s = \frac{1}{2}$ and for every $t \in [\frac{1}{2}, \frac{3}{2}]$, we have $g(\frac{1}{2}, t) \notin S_{2\delta}$. So from (a) we have $\varphi(\frac{1}{2}, t) = g(\frac{1}{2}, t)$. Thus

$$\Psi\left(\frac{1}{2}, t\right) = \left(2\left\langle (J_k^b)'\left(\frac{1}{2}u_b^+ + tu_b^-\right), \frac{1}{2}u_b^+\right\rangle, \frac{1}{t}\left\langle (J_k^b)'\left(\frac{1}{2}u_b^+ + tu_b^-\right), tu^-\right\rangle\right).$$

By Lemma 2.2, we know that

$$\begin{aligned}&\left\langle (J_k^b)'\left(\frac{1}{2}u_b^+ + tu_b^-\right), \frac{1}{2}u_b^+\right\rangle \\ &\quad= \left\langle (J_k^b)'\left(\frac{1}{2}u_b^+\right), \frac{1}{2}u_b^+\right\rangle + \frac{t^2 b}{4}\int_{\mathbb{R}^3}\big|\nabla u_b^-\big|^2\,dx \cdot \int_{\mathbb{R}^3}\big|\nabla u_b^+\big|^2\,dx + \frac{t^2 b}{4}\int_{\mathbb{R}^3}\phi_{u_b^-}\big|u_b^+\big|^2\,dx \\ &\quad\ge \left\langle (J_k^b)'\left(\frac{1}{2}u_b\right), \frac{1}{2}u_b^+\right\rangle > 0,\end{aligned}$$

from which we obtain

$$\left\langle (J_k^b)'\left(\frac{1}{2}u_b^+ + tu_b^-\right), \frac{1}{2}u_b^+\right\rangle > 0 \quad \text{for every } t \in \left[\frac{1}{2}, \frac{3}{2}\right]. \tag{3.5}$$

Similarly, for $s = \frac{3}{2}$ and for every $t \in [\frac{1}{2}, \frac{3}{2}]$, we have $\varphi(\frac{3}{2}, t) = g(\frac{3}{2}, t)$, so that

$$\begin{aligned}&\left\langle (J_k^b)'\left(\frac{3}{2}u_b^+ + tu_b^-\right), \frac{3}{2}u_b^+\right\rangle \\ &\quad= \left\langle (J_k^b)'\left(\frac{3}{2}u_b^+\right), \frac{3}{2}u_b^+\right\rangle + \frac{9t^2}{4}b\int_{\mathbb{R}^3}\big|\nabla u_b^-\big|^2\,dx \cdot \int_{\mathbb{R}^3}\big|\nabla u_b^+\big|^2\,dx + \frac{9t^2}{4}b\int_{\mathbb{R}^3}\phi_{u_b^-}\big|u_b^+\big|^2\,dx \\ &\quad\le \left\langle (J_k^b)'\left(\frac{3}{2}u_b\right), \frac{3}{2}u_b^+\right\rangle < 0,\end{aligned}$$

so that

$$\left\langle (J_k^b)'\left(\frac{3}{2}u_b^+ + tu_b^-\right), \frac{3}{2}u_b^+\right\rangle < 0 \quad \text{for every } t \in \left[\frac{1}{2}, \frac{3}{2}\right]. \tag{3.6}$$

Similarly, we have

$$\left\langle (J_k^b)'\left(su_b^+ + \frac{1}{2}u_b^-\right), \frac{1}{2}u_b^-\right\rangle > 0 \quad \text{for every } s \in \left[\frac{1}{2}, \frac{3}{2}\right], \tag{3.7}$$

$$\left\langle (J_k^b)'\left(su_b^+ + \frac{3}{2}u_b^-\right), \frac{3}{2}u_b^-\right\rangle < 0 \quad \text{for every } s \in \left[\frac{1}{2}, \frac{3}{2}\right]. \tag{3.8}$$

Since Ψ is continuous on Q, according to (3.5)–(3.7), by Miranda's theorem (Lemma 2.4 [7]), we have $\Psi(s_0, t_0) = 0$ for some $(s_0, t_0) \in Q$, so $\eta(1, g(s_0, t_0)) = \varphi(s_0, t_0) \in \mathcal{M}_b^k$. By (3.3), we have a contradiction. From the above discussion, we conclude that u_b is a nodal solution for system (1.7).

Finally, we prove that u_b has exactly two nodal domains. To this end, we first write u_b as

$$u_b = u_1 + u_2 + u_3$$

with $u_1 \geq 0$, $u_2 \leq 0$. Set $\Omega_i = \{x \in \mathbb{R}^3 : u_i(x) \neq 0\}$. We further assume $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, 3$. Since u_b is a nodal solution, we suppose the nodal domains $\Omega_1 \neq \emptyset$, $\Omega_2 \neq \emptyset$. By contradiction, we suppose u_b possesses more than two nodal domains, then we have $u_3 \neq 0$ and so $\Omega_3 \neq \emptyset$. Setting $v := u_1 + u_2$, we easily see that $v^\pm \neq 0$. So, there exists a positive pair (s_v, t_v) such that

$$s_v u_1 + t_v u_2 \in \mathcal{M}_b^k.$$

Thus,

$$J_k^b(s_v u_1 + t_v u_2) \geq c_b^k.$$

Moreover, using the fact that $\langle (J_k^b)'(u_b), u_i\rangle = 0$, from the definition, we get $\langle (J_k^b)'(v), v^\pm\rangle \leq 0$. So, thanks to Lemma 2.1, we have that

$$(s_v, t_v) \in (0, 1] \times (0, 1].$$

By direct calculation,

$$\begin{aligned}
0 &= \left\langle (J_k^b)'(u_b), u_3\right\rangle \\
&= \|u_3\|^2 + b\int_{\mathbb{R}^3} |\nabla u_1|^2\,dx \cdot \int_{\mathbb{R}^N} |\nabla u_3|^2\,dx \\
&\quad + b\int_{\mathbb{R}^3} |\nabla u_2|^2\,dx \cdot \int_{\mathbb{R}^N} |\nabla u_3|^2\,dx + b\left(\int_{\mathbb{R}^N} |\nabla u_3|^2\,dx\right)^2 \\
&\quad + \lambda\int_{\mathbb{R}^3} \phi_{u_1}|u_3|^2\,dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_{u_2}|u_3|^2\,dx + \lambda\int_{\mathbb{R}^3} \phi_{u_3}|u_3|^2\,dx \\
&\quad - \lambda\int_{\mathbb{R}^3} |u_3|^6\,dx - \frac{k}{4}\int_{\mathbb{R}^3} f(u_3)u_3\,dx \\
&= \left\langle (J_k^b)'(u_3), u_3\right\rangle + b\int_{\mathbb{R}^3} \left(|\nabla u_1|^2 + |\nabla u_2|^2\right)dx \cdot \int_{\mathbb{R}^N} |\nabla u_3|^2\,dx \\
&\quad + \lambda\int_{\mathbb{R}^3} \phi_{u_1}|u_3|^2\,dx + \lambda\int_{\mathbb{R}^3} \phi_{u_2}|u_3|^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&< 4J_k^b(u_3) + b\int_{\mathbb{R}^3}|\nabla u_1|^2\,dx\cdot\int_{\mathbb{R}^N}|\nabla u_3|^2\,dx + b\int_{\mathbb{R}^3}|\nabla u_2|^2\,dx\cdot\int_{\mathbb{R}^N}|\nabla u_3|^2\,dx \\
&\quad + \lambda\int_{\mathbb{R}^3}\phi_{u_1}|u_3|^2\,dx + \lambda\int_{\mathbb{R}^3}\phi_{u_2}|u_3|^2\,dx, \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
-\big\langle (J_k^b)'(u_3), u_3\big\rangle &= b\int_{\mathbb{R}^3}\big(|\nabla u_1|^2 + |\nabla u_2|^2\big)\,dx\cdot\int_{\mathbb{R}^N}|\nabla u_3|^2\,dx \\
&\quad + \lambda\int_{\mathbb{R}^3}\phi_{u_1}|u_3|^2\,dx + \lambda\int_{\mathbb{R}^3}\phi_{u_2}|u_3|^2\,dx. \qquad (3.10)
\end{aligned}
$$

Then, by using (2.16), we get

$$
\begin{aligned}
c_b^k &\le J_k^b(s_v u_1 + t_v u_2) = J_k^b(s_v u_1 + t_v u_2) - \frac{1}{4}\big\langle (J_k^b)'(s_v u_1 + t_v u_2), s_v u_1 + t_v u_2\big\rangle \\
&= \frac{1}{4}\big(\|s_v u_1\|^2 + \|t_v u_2\|^2\big) + \frac{k}{4}\int_{\mathbb{R}^3}\big[f(s_v u_1)(s_v u_1) - 4F(s_v u_1)\big]\,dx \\
&\quad + \frac{k}{4}\int_{\mathbb{R}^3}\big[f(t_v u_2)(t_v u_2) - 4F(t_v u_2)\big]\,dx + \frac{s_v^6}{12}\int_{\mathbb{R}^3}|u_1|^6\,dx + \frac{t_v^6}{12}\int_{\mathbb{R}^3}|u_2|^6\,dx \\
&\le \frac{1}{4}\big(\|u_1\|^2 + \|u_2\|^2\big) + \frac{k}{4}\int_{\mathbb{R}^3}\big[f(u_1)u_1 - 4F(u_1)\big]\,dx \\
&\quad + \frac{k}{4}\int_{\mathbb{R}^3}\big[f(u_2)u_2 - 4F(u_2)\big]\,dx + \frac{1}{12}\int_{\mathbb{R}^3}|u_1|^6\,dx + \frac{1}{12}\int_{\mathbb{R}^3}|u_2|^6\,dx \\
&= J_k^b(u_1 + u_2) - \frac{1}{4}\big\langle (J_k^b)'(u_1 + u_2), (u_1 + u_2)\big\rangle.
\end{aligned}
$$

Similar to the computation of (3.10), from $\langle (J_k^b)'(u_b), u_b\rangle = 0$, there holds

$$
\begin{aligned}
&-\big\langle (J_k^b)'(u_1 + u_2), u_1 + u_2\big\rangle \\
&\quad = \big\langle (J_k^b)'(u_3), u_3\big\rangle + 2b\int_{\mathbb{R}^3}\big(|\nabla u_1|^2 + |\nabla u_2|^2\big)\,dx\cdot\int_{\mathbb{R}^3}|\nabla u_3|^2\,dx \\
&\quad\quad + \lambda\int_{\mathbb{R}^3}(\phi_{u_1} + \phi_{u_2})|u_3|^2\,dx + \lambda\int_{\mathbb{R}^3}\phi_{u_3}\big(|u_1|^2 + |u_2|^2\big)\,dx. \qquad (3.11)
\end{aligned}
$$

By using (3.9), (3.10), and (3.11), we get

$$
\begin{aligned}
c_b^k &\le J_k^b(u_1 + u_2) - \frac{1}{4}\big\langle (J_k^b)'(u_1 + u_2), (u_1 + u_2)\big\rangle \\
&= J_k^b(u_1 + u_2) + \frac{1}{4}\big\langle (J_k^b)'(u_b), u_3\big\rangle + \frac{b}{4}\int_{\mathbb{R}^3}|\nabla u_1|^2\,dx\cdot\int_{\mathbb{R}^N}|\nabla u_3|^2\,dx \\
&\quad + \frac{b}{4}\int_{\mathbb{R}^3}|\nabla u_2|^2\,dx\cdot\int_{\mathbb{R}^N}|\nabla u_3|^2\,dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_3}|u_1|^2\,dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_3}|u_2|^2\,dx \\
&< J_k^b(u_1) + J_k^b(u_2) + J_k^b(u_3) + \frac{b}{4}\int_{\mathbb{R}^N}\big(|\nabla u_2|^2 + |\nabla u_3|^2\big)\,dx\cdot\int_{\mathbb{R}^N}|\nabla u_1|^2\,dx \\
&\quad + \frac{b}{4}\int_{\mathbb{R}^N}\big(|\nabla u_1|^2 + |\nabla u_3|^2\big)\,dx\cdot\int_{\mathbb{R}^N}|\nabla u_2|^2\,dx \\
&\quad + \frac{b}{4}\int_{\mathbb{R}^N}\big(|\nabla u_1|^2 + |\nabla u_2|^2\big)\,dx\cdot\int_{\mathbb{R}^N}|\nabla u_3|^2\,dx
\end{aligned}
$$

$$+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_3}\left(|u_1|^2+|u_2|^2\right)dx+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_1+u_2}|u_3|^2\,dx$$
$$=J_k^b(u_b)=c_b^k.$$

So we get $u_3 = 0$ and u_b has exactly two nodal domains. □

3.2 The proof of Theorem 1.2

To prove Theorem 1.2, we should first prove that there exists a ground state solution of (1.7) for k large enough, and then to prove that the energy of sign-changing solution u_b is strictly larger than twice of that of the ground state solution.

Proof Similar to the proof of Lemma 2.4, we claim that there exists $k_1^\star > 0$ such that, for all $k \geq k_1^\star$, and $\forall b > 0$, there exists $v_b \in \mathcal{N}_b^k$ such that $J_k^b(v_b) = c^* > 0$. We give a brief proof of this claim.

We first list some results for the Nehari manifold $\mathcal{N}_b^k$. One can prove them by following the ideas as those in Lemma 2.4.

(i) If $v \in \mathcal{N}_b^k$, then $J_k^b(tv) \leq J_k^b(v)$ for all $t \geq 0$;
(ii) There exists $\rho > 0$ such that $\|v\| \geq \rho$ for all $v \in \mathcal{N}_b^k$;
(iii) There exists $M > 0$ such that $\|v\| \leq M$ for all $v \in \mathcal{N}_b^k$.

According to the definition of c^*, there is a sequence $\{v_n\} \subset \mathcal{N}_b^k$ such that $\lim_{n\to\infty} J_k^b(v_n) = c^*$. By property (iii), $\{v_n\}$ is bounded in E. In the subsequence sense, there exists $v_b \in E$ such that $v_n \rightharpoonup v_b$.

Denote $\beta := \frac{(S)^{\frac{3}{2}}}{3}$, where $S := \inf_{u\in E\setminus\{0\}} \frac{\|u\|^2}{(\int_{\mathbb{R}^3}|u|^6\,dx)^{\frac{1}{3}}}$. Similar to the proof of Lemma 2.3, there is $k^\star > 0$ such that $c^* < \beta$ for all $k \geq k^\star$. Therefore, $\liminf_{n\to\infty} J_k^b(tv_n) \geq J_k^b(tv_b) + \frac{t^2}{2}A - \frac{t^6}{6}B$, where $A = \lim_{n\to\infty}\|v_n - v_b\|^2$, $B = \lim_{n\to\infty}|v_n - v_b|_6^6$. From the above fact and property (i), we have

$$J_k^b(tv_b)+\frac{t^2}{2}A-\frac{t^6}{6}B\leq c^* \tag{3.12}$$

for all $t \geq 0$.

Firstly, we prove that $v_b \neq 0$. By contradiction, we suppose $v_b = 0$.

Case 1: $B = 0$. If $A = 0$, that is, $v_n \to v_b$ in E, then $v_b \in \mathcal{N}_b^k$, and so we have $\|v_b\| > \rho$ by property (ii), which contradicts our supposition. If $A > 0$, $\frac{t^2}{2}A \leq c^*$ for all $t \geq 0$, which is a contradiction.

Case 2: $B > 0$. According to the definition of S, we have that $\beta = \frac{(S)^{\frac{3}{2}}}{3} \leq \frac{1}{3}(\frac{A}{(B)^{\frac{1}{3}}})^{\frac{3}{2}}$. It is easy to see that

$$\frac{1}{3}\left(\frac{A}{(B)^{\frac{1}{3}}}\right)^{\frac{3}{2}}=\frac{\tilde{t}^2}{2}A-\frac{\tilde{t}^6}{6}B:=\max_{t\geq 0}\left\{\frac{t^2}{2}A-\frac{t^6}{6}B\right\},$$

so we have that

$$\beta\leq\max_{t\geq 0}\left\{\frac{t^2}{2}A-\frac{t^6}{6}B\right\}\leq c_b^k<\beta,$$

which is a contradiction.

Secondly, we claim that $B = 0$. By contradiction, we suppose that $B > 0$. Firstly, we can maximize $\psi_{v_b}(t) = J_k^b(tv_b)$ in $[0,\infty)$. Indeed, there exists $t_0 \in [0,\infty)$ such that $J_k^b(tv_b) \le 0$ for all $t \in [t_0,\infty)$. Let t_v be an inner maximizer of ψ_v in $[0,\infty)$. $J_k^b(\tilde{t}v_b) + \beta \le J_k^b(\tilde{t}v_b) + \frac{\tilde{t}^2}{2}A - \frac{\tilde{t}^6}{6}B \le c^* < \beta$ implies that $J_k^b(\tilde{t}v_b) < 0$. So $t_v \le \tilde{t}$ and $\frac{t_v^2}{2}A - \frac{t_v^6}{6}B > 0$. Thus from $t_v v_b \in \mathcal{N}_b^k$ we get a contradiction by

$$c^* \le J_k^b(t_v v_b) < J_k^b(t_v v_b) + \frac{t_v^2}{2}A - \frac{t_v^6}{6}B \le c^*.$$

Lastly, we prove that c^* is achieved by v_b. From the above arguments, we have $v_b \neq 0$ and $\widetilde{v} := t_v v_b \in \mathcal{N}_b^k$. Furthermore, because $v_n \rightharpoonup v_b$ in E and $v_n \in \mathcal{N}_b^k$, we have that $\langle (J_k^b)'(v_b), v_b\rangle \le 0$. Similar to Lemma 2.1, we have $0 < t_v \le 1$. Also as in the proof of Lemma 2.4, we have

$$\begin{aligned} c^* &\le J_k^b(\tilde{v}) - \frac{1}{4}\big\langle (J_k^b)'(\tilde{v}), \tilde{v}\big\rangle \\ &= \frac{1}{4}\|t_v v_b\|^2 + \frac{1}{12}|t_v v_b|_6^6 + \frac{k}{4}\int_{\mathbb{R}^3}\big[f(t_v v_b)t_v v_b - 4F(t_v v_b)\big]\,dx \\ &\le \frac{1}{4}\|v_b\|^2 + \frac{1}{12}|v_b|_6^6 + \frac{k}{4}\int_{\mathbb{R}^3}\big[f(v_b)v_b - 4F(v_b)\big]\,dx, \end{aligned}$$
$$\liminf_{n\to\infty}\left[J_k^b(v_n) - \frac{1}{4}\big\langle (J_k^b)'(v_n), v_n\big\rangle\right] = c^*.$$

Therefore, $t_v = 1$, and c^* is achieved by $v_b \in \mathcal{N}_b^k$.

By standard arguments, the critical points of the functional J_k^b on $\mathcal{N}_b^k$ are critical points of J_k^b in E, and we obtain $(J_k^b)'(v_b) = 0$, so v_b is a positive or negative solution. That is, v_b is a ground state solution of system (1.7). For all $k \ge k^\star$, and $\forall b > 0$, problem (1.7) has a least energy nodal solution u_b. Let

$$k^{\star\star} = \max\{k^\star, k_1^\star\}.$$

Suppose that $u_b = u^+ + u^-$. As in the proof of Lemma 2.1, there exist $s_{u^+}, t_{u^-} \in (0,1)$ such that

$$s_{u^+}u^+ \in \mathcal{N}_b^k, \quad t_{u^-}u^- \in \mathcal{N}_b^k.$$

Hence, by Lemma 2.1, we deduce

$$2c^* \le J_k^b\big(s_{u^+}u^+\big) + J_k^b\big(t_{u^-}u^-\big) \le J_k^b\big(s_{u^+}u^+ + t_{u^-}u^-\big) < J_k^b\big(u^+ + u^-\big) = c_b^k. \qquad \square$$

3.3 Proof of Theorem 1.3

At the end of the section, we give an analysis for the behavior of u_b as $b \to 0$. We regard $b > 0$ as a parameter in equation (1.7).

Proof For any $b > 0$, let $u_b \in E$ be the least energy nodal solution of system (1.7) obtained in Theorem 1.1. We will complete our proof with the following three assertions. We recall that u_{b_n} is a least energy nodal solution of system (1.7) with $b = b_n \to 0$ as $n \to \infty$.

Claim (a). As n is large enough, $\{u_{b_n}\}$ is bounded in E.

Choose a test function $\phi \in C_c^\infty(\mathbb{R}^3)$ with $\phi^\pm \neq 0$. From (2.7), for any $b \in [0,1]$, there exists a pair of positive numbers (k_1, k_2) such that

$$\left\langle \left(J_k^b\right)'\left(k_1\phi^+ + k_2\phi^-\right), k_1\phi^+\right\rangle < 0,$$

and

$$\left\langle \left(J_k^b\right)'\left(k_1\phi^+ + k_2\phi^-\right), k_2\phi^-\right\rangle < 0.$$

Thus, according to Lemma 2.1(ii), for any $b \in [0,1]$, there is a unique pair $s_\phi(b), t_\phi(b) \in (0,1] \times (0,1]$ such that

$$\overline{\phi} := s_\phi(b)k_1\phi^+ + t_\phi(b)k_2\phi^- \in \mathcal{M}_b^k. \tag{3.13}$$

Hence, for any $b \in [0,1]$, by using (2.4), we get

$$\begin{aligned}
J_k^b(u_b) &\leq J_k^b(\overline{\phi}) = J_k^b(\overline{\phi}) - \frac{1}{4}\left\langle \left(J_k^b\right)'(\overline{\phi}), \overline{\phi}\right\rangle \\
&= \frac{1}{4}\|\overline{\phi}\|^2 + \frac{k}{4}\int_{\mathbb{R}^3}\left[f(\overline{\phi})\overline{\phi} - 4F(\overline{\phi})\right]dx + \frac{1}{12}\int_{\mathbb{R}^3}|\overline{\phi}|^6\,dx \\
&\leq \frac{1}{4}\|\overline{\phi}\|^2 + \frac{k}{4}\int_{\mathbb{R}^3}\left(C_1\overline{\phi}^2 + C_2\overline{\phi}^q\right)dx + \frac{1}{12}\int_{\mathbb{R}^3}|\overline{\phi}|^6\,dx \\
&\leq \frac{1}{4}\left(k_1^2\left\|\phi^+\right\|^2 + k_2^2\left\|\phi^-\right\|^2\right) + \frac{k}{4}\int_{\mathbb{R}^3}\left(C_1{k_1}^2\left|\phi^+\right|^2 + C_1{k_2}^2\left|\phi^-\right|^2\right)dx \\
&\quad + \frac{k}{4}\int_{\mathbb{R}^3}\left(C_2{k_1}^5\left|\phi^+\right|^5 + C_2{k_2}^5\left|\phi^-\right|^5\right)dx + \frac{{k_1}^6}{12}\int_{\mathbb{R}^3}\left|\phi^+\right|^6 dx + \frac{{k_2}^6}{12}\int_{\mathbb{R}^3}\left|\phi^-\right|^6 dx \\
&:= C^*,
\end{aligned}$$

where $C^* > 0$ is a constant independent of b. So, as n is large enough, it follows that

$$C^* + 1 \geq J_{b_n}^k(u_{b_n}) = J_{b_n}^k(u_{b_n}) - \frac{1}{4}\left\langle \left(J_{b_n}^k\right)'(u_{b_n}), u_{b_n}\right\rangle \geq \frac{1}{4}\|u_{b_n}\|^2.$$

Therefore, we can deduce Claim (a) from the above inequality.

Claim (b). System (1.11) possesses a nodal solution u_0.

Since $\{u_{b_n}\}$ is bounded in E, in the subsequence sense, there exists $u_0 \in E$ such that

$$\begin{aligned}
&u_{b_n} \rightharpoonup u_0 \quad \text{in } E, \\
&u_{b_n} \to u_0 \quad \text{in } L^p\left(\mathbb{R}^3\right) \quad \text{for } p \in (2,6), \\
&u_{b_n} \to u_0 \quad \text{a.e. in } \mathbb{R}^3.
\end{aligned} \tag{3.14}$$

Thanks to $\{u_{b_n}\}$ being a least energy nodal solution of system (1.7) with $b = b_n$, we have that

$$\int_{\mathbb{R}^3} \big(a\nabla u_{b_n} \cdot \nabla v + V(x)u_{b_n}v\big)\,dx + b_n\left(\int_{\mathbb{R}^3} |\nabla u_{b_n}|^2\,dx\right)\left(\int_{\mathbb{R}^3} \nabla u_{b_n} \cdot \nabla v\,dx\right)$$
$$+ \lambda \int_{\mathbb{R}^3} \phi_{u_{b_n}} u_{b_n} v\,dx - k\int_{\mathbb{R}^3} f(u_{b_n})v\,dx - \int_{\mathbb{R}^3} |u_{b_n}|^4 u_{b_n} v\,dx = 0 \tag{3.15}$$

for any $v \in C_c^\infty(\mathbb{R}^3)$. Combining (3.14), (3.15) with Claim (a), we have that

$$\int_{\mathbb{R}^3} \big(a\nabla u_0 \cdot \nabla v + V(x)u_0 v\big)\,dx + \lambda \int_{\mathbb{R}^3} \phi_{u_0} u_0 v\,dx$$
$$- k\int_{\mathbb{R}^3} f(u_0)v\,dx - \int_{\mathbb{R}^3} |u_0|^4 u_0 v\,dx = 0$$

for any $v \in C_c^\infty(\mathbb{R}^3)$. It implies that u_0 is a weak solution of the Kirchhoff equation (1.11). We next deduce that $u_0^\pm \neq 0$. Since $u_{b_n} \in \mathcal{M}_{b_n}^k$, we have

$$\|u_{b_n}^\pm\|^2 + b_n\left(\int_{\mathbb{R}^3} |\nabla u_{b_n}^\pm|^2\,dx\right)^2 + b_n \int_{\mathbb{R}^3} \nabla u_{b_n}^\pm|^2\,dx \cdot \int_{\mathbb{R}^3} |\nabla u_{b_n}^\mp|^2\,dx)$$
$$+ \lambda \int_{\mathbb{R}^3} \phi_{u_{b_n}^\pm} |u_{b_n}^\pm|^2\,dx + \lambda \int_{\mathbb{R}^3} \phi_{u_{b_n}^\mp} |u_{b_n}^\pm|^2\,dx$$
$$= \int_{\mathbb{R}^3} |u_{b_n}^\pm|^6\,dx + k\int_{\mathbb{R}^3} f\big(u_{b_n}^\pm\big)u_{b_n}^\pm\,dx.$$

Hence, by using Claim (a) and the continuous embedding $E \hookrightarrow L^6(\mathbb{R}^3)$, we have u_{b_n} is bounded in $L^6(\mathbb{R}^3)$, thus there exists $k_2^\star > 0$ such that, for all $k \geq k_2^\star$, we have that

$$\rho \leq \|u_{b_n}^\pm\|^2 \leq \int_{\mathbb{R}^3} |u_{b_n}^\pm|^6\,dx + k\int_{\mathbb{R}^3} f\big(u_{b_n}^\pm\big)u_{b_n}^\pm\,dx \leq 2k\int_{\mathbb{R}^3} f\big(u_{b_n}^\pm\big)u_{b_n}^\pm\,dx.$$

By using (2.4), we have that

$$0 < \int_{\mathbb{R}^3} f\big(u_0^\pm\big)u_0^\pm\,dx.$$

Since u_0 is a solution of system (1.11), we have that

$$\|u_0^\pm\|^2 \geq k\int_{\mathbb{R}^3} f\big(x, u_0^\pm\big)u_0^\pm\,dx + \int_{\mathbb{R}^3} |u_0^\pm|^6\,dx \geq k\int_{\mathbb{R}^3} f\big(u_0^\pm\big)u_0^\pm\,dx > 0.$$

It implies $u_0^\pm \neq 0$.

Claim (c). Problem (1.11) possesses a least energy nodal solution v_0.

Similar to the proof of Theorem 1.1, there is $k_3^\star > 0$ such that, for all $k \geq k_3^\star$, problem (1.11) possesses a least energy nodal solution v_0, where $J_k^0(v_0) = c_k^0$ and $(J_k^0)'(v_0) = 0$. Let

$$k^{\star\star\star} = \max\big\{k^\star, k_2^\star, k_3^\star\big\}.$$

According to Lemma 2.1, there exists a positive pair $(s_{b_n}, t_{b_n}) \in (0,\infty) \times (0,\infty)$ such that $s_{b_n} v_0^+ + t_{b_n} v_0^- \in \mathcal{M}_{b_n}^k$. That is,

$$\begin{aligned} &s_{b_n}^2 \left\| v_0^+ \right\|^2 + \lambda s_{b_n}^4 \int_{\mathbb{R}^3} \phi_{v_0^+} \left| v_0^+ \right|^2 dx + \lambda s_{b_n}^2 t_{b_n}^2 \int_{\mathbb{R}^3} \phi_{v_0^-} \left| v_0^+ \right|^2 dx + b_n s_{b_n}^4 \left(\int_{\mathbb{R}^3} \left| \nabla v_0^+ \right|^2 dx \right)^2 \\ &\quad + b_n s_{b_n}^2 t_{b_n}^2 \int_{\mathbb{R}^3} \left| \nabla v_0^+ \right|^2 dx \cdot \int_{\mathbb{R}^3} \left| \nabla v_0^- \right|^2 dx \\ &\quad = s_{b_n}^6 \int_{\mathbb{R}^3} \left| v_0^+ \right|^6 dx + k \int_{\mathbb{R}^3} f\left(s_{b_n} v_0^+ \right) s_{b_n} v_0^+ \, dx \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &t_{b_n}^2 \left\| v_0^- \right\|^2 + \lambda t_{b_n}^4 \int_{\mathbb{R}^3} \phi_{v_0^-} \left| v_0^- \right|^2 dx + \lambda s_{b_n}^2 t_{b_n}^2 \int_{\mathbb{R}^3} \phi_{v_0^+} \left| v_0^- \right|^2 dx + b_n t_{b_n}^4 \left(\int_{\mathbb{R}^3} \left| \nabla v_0^- \right|^2 dx \right)^2 \\ &\quad + b_n s_{b_n}^2 t_{b_n}^2 \int_{\mathbb{R}^3} \left| \nabla v_0^+ \right|^2 dx \cdot \int_{\mathbb{R}^3} \left| \nabla v_0^- \right|^2 dx \\ &\quad = t_{b_n}^6 \int_{\mathbb{R}^3} \left| v_0^- \right|^6 dx + k \int_{\mathbb{R}^3} f\left(t_{b_n} v_0^- \right) t_{b_n} v_0^- \, dx. \end{aligned} \tag{3.17}$$

By recalling Claim (a), up to a subsequence, we can deduce $s_{b_n} \to s_0$ and $t_{b_n} \to t_0$, then it follows from (3.16) and (3.17) that

$$\begin{aligned} &s_0^2 \| v_0^+ \|^2 + \lambda s_0^4 \int_{\mathbb{R}^3} \phi_{v_0^+} | v_0^+ |^2 dx + \lambda s_0^2 t_0^2 \int_{\mathbb{R}^3} \phi_{v_0^-} | v_0^+ |^2 \\ &\quad = s_0^6 \int_{\mathbb{R}^3} | v_0^+ |^6 dx + k \int_{\mathbb{R}^3} f(s_0 v_0^+) s_0 v_0^+ dx \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} &t_0^2 \| v_0^- \|^2 + \lambda t_0^4 \int_{\mathbb{R}^3} \phi_{v_0^-} | v_0^- |^2 + \lambda s_0^2 t_0^2 \int_{\mathbb{R}^3} \phi_{v_0^+} | v_0^- |^2 \\ &\quad = t_0^6 \int_{\mathbb{R}^3} | v_0^- |^6 dx + k \int_{\mathbb{R}^3} f(t_0 v_0^-) t_0 v_0^- dx. \end{aligned} \tag{3.19}$$

Thanks to v_0 being a weak solution of problem (1.11), we get

$$\begin{aligned} &\left\| v_0^+ \right\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{v_0^+} | v_0^+ |^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{v_0^-} | v_0^+ |^2 dx \\ &\quad = \int_{\mathbb{R}^3} \left| v_0^+ \right|^6 dx + k \int_{\mathbb{R}^3} f\left(v_0^+ \right) v_0^+ \, dx \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\left\| v_0^- \right\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{v_0^-} | v_0^- |^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{v_0^+} | v_0^- |^2 dx \\ &\quad = \int_{\mathbb{R}^3} \left| v_0^- \right|^6 dx + k \int_{\mathbb{R}^3} f\left(v_0^- \right) v_0^- \, dx. \end{aligned} \tag{3.21}$$

By comparing formulas (3.18)–(3.21), it is obvious that $(s_0, t_0) = (1, 1)$. Similar to the proof of Lemma 2.1, we have

$$J_k^0(v_0) \le J_k^0(u_0) = \lim_{n\to\infty} J_k^{b_n}(u_{b_n}) \le \lim_{n\to\infty} J_k^{b_n}\left(s_{b_n}v_0^+ + t_{b_n}v_0^-\right) = J_k^0\left(v_0^+ + v_0^-\right) = J_k^0(v_0).$$

The above inequality implies that u_0 is a least energy nodal solution of problem (1.11). So far, we have proved Theorem 1.3. □

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