Ahmed et al. *Boundary Value Problems* (2020) 2020:120
https://doi.org/10.1186/s13661-020-01418-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Approximate controllability of noninstantaneous impulsive Hilfer fractional integrodifferential equations with fractional Brownian motion

Hamdy M. Ahmed[1*], Mahmoud M. El-Borai[2], A.S. Okb El Bab[3] and M. Elsaid Ramadan[3,4]

*Correspondence:
hamdy_17eg@yahoo.com
[1]Higher Institute of Engineering, El-Shorouk Academy, El-Shorouk City, Cairo, Egypt
Full list of author information is available at the end of the article

Abstract
We introduce the investigation of approximate controllability for a new class of nonlocal and noninstantaneous impulsive Hilfer fractional neutral stochastic integrodifferential equations with fractional Brownian motion. An appropriate set of sufficient conditions is derived for the considered system to be approximately controllable. For the main results, we use fractional calculus, stochastic analysis, fractional power of operators and Sadovskii's fixed point theorem. At the end, an example is also given to show the applicability of our obtained theory.

MSC: 93B05; 26A33; 93C10; 60H10; 34K40; 60G22

Keywords: Hilfer fractional neutral stochastic differential equations; Nonlocal conditions; Approximate controllability; Fractional Brownian motion; Noninstantaneous impulsive; Sadovskii fixed point theorem

1 Introduction

Fractional differential equations have received great attention due to their applications in many important applied fields such as population dynamics, heat conduction in materials with memory, seepage flow in porous media, autonomous mobile robots, fluid dynamics, traffic models, electro magnetic, aeronautics, economics, and diffusion theory; see for instance [1–3]. Moreover, stochastic perturbation is unavoidable in nature and hence it is important and necessary to consider stochastic effect into the investigation of fractional differential equations. Recently, stochastic fractional differential equations driven by fractional Brownian motion have been considered greatly by research community in various aspects due to its salient features for real world problems (see [4–10]). The theory of impulsive differential equations and impulsive differential inclusions has wide applications in control, electrical engineering, mechanics and biology [11]. In general, the classical instantaneous impulses cannot describe certain dynamics of evolution processes. For example, when we consider the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes. In fact, the above situation can be characterized by a new case of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval called noninstantaneous impulsive differential equations. Hernández and O'Regan [12] and Pierri et al. [13] introduced some initial value problems for a new class of non-instantaneous impulsive differential equations to describe some certain dynamic change of evolution processes in the pharmacotherapy (as therapy using pharmaceutical drugs). Very recently, Pierri et al. [14] studied the existence of global solutions for a class of impulsive abstract differential equations with non-instantaneous impulses. On the other hand, controllability results for linear and nonlinear integer order differential systems was studied by several authors. The concept of controllability is an important part of mathematical control theory. Generally speaking, controllability means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability problems for different kinds of dynamical systems have been studied by several authors (see [15–18]) and the references therein. Thus, the dynamical systems must be treated by the weaker concept of controllability, namely approximate controllability. Many authors studied the approximate controllability, for example, Sakthivel et al. studied the approximate controllability of nonlinear fractional dynamical systems (see [19]). Sakthivel et al. obtained sufficient conditions for the approximate controllability of fractional nonlinear differential inclusions (see [20]). Sakthivel et al. obtained sufficient conditions for the approximate controllability of fractional stochastic differential inclusions with nonlocal conditions (see [21]). Debbouche and Torres established sufficient conditions for the approximate controllability of fractional delay dynamic inclusions with nonlocal control conditions (see [22]). Ahmed studied the approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space (see [23]). Muthukumar and Rajivganthi obtained sufficient conditions for the approximate controllability of second-order neutral stochastic differential equations with infinite delay and Poisson jumps (see [24]). Yan and Jia established sufficient conditions for the approximate controllability of partial fractional neutral stochastic functional integrodifferential inclusions with state-dependent delay (see [25]). Yana and Lu studied the approximate controllability of a multi-valued fractional impulsive stochastic partial integrodifferential equation with infinite delay (see [26]). Very recently, a new set of sufficient conditions are established in [27] for the approximate controllability of a class of semilinear Hilfer fractional differential control inclusions in Banach spaces by using the fractional calculus, fixed point technique, semigroup theory and multi-valued analysis. Moreover, up to now no work has been reported yet regarding the approximate controllability results for noninstantaneous impulsive Hilfer fractional stochastic integrodifferential equations with fractional Brownian motion, which motivates the present study. The purpose of this paper is to study the approximate controllability of noninstantaneous impulsive semilinear Hilfer fractional stochastic integrodifferential equations with fractional Brownian motion and nonlocal conditions in a Hilbert space of the form

$$\begin{cases} D_{0+}^{\nu,\mu}[x(t)+F(t,x(t),x(b_1(t)),\dots,x(b_m(t)))]+Ax(t) \\ \quad = Bu(t)+\int_0^t G(s,x(s),x(a_1(s)),\dots,x(a_k(s)))\,d\omega(s) \\ \qquad + \sigma(t,x(t),x(c_1(t)),\dots,x(c_p(t)))\frac{dB^H}{dt}, \quad t\in(s_i,t_{i+1}], i\in[0,m], \\ x(t)=g_i(t,x(t)), \quad t\in(t_i,s_i], i\in[1,m], \\ I_{0+}^{(1-\nu)(1-\mu)}x(0)+\xi(x)=x_0, \end{cases} \tag{1.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative with $0 \leq \nu \leq 1$, $0 < \mu < 1$, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t)$, $t \geq 0$, on a separable Hilbert space X with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and the control function $u(\cdot)$ is given in $L_2(J, U)$, the Hilbert space of admissible control functions with U a Hilbert space, $J = (0.T]$. The symbol B stands for a bounded linear from U into X, t_i, s_i are fixed number satisfying $0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < s_{m-1} < t_m \leq s_m \leq t_{m+1} = T$ and g_i is noninstantaneous impulsive function for all $i = 1, 2, \ldots, m$. Let K be another separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_K$ and norm $\| \cdot \|_K$. Suppose $\{\omega(t)\}_{t\geq 0}$ is Q-Wiener process defined on $(\Omega, \Upsilon, \{\Upsilon_t\}_{t\geq 0}, P)$ with values in Hilbert space K and $\{B^H(t)\}_{t\geq 0}$ is Q-fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$ defined on $(\Omega, \Upsilon, \{\Upsilon_t\}_{t\geq 0}, P)$ with values in Hilbert space Y. We are also employing the same notation $\| \cdot \|$ for the norm in X, K, Y, $L(K, X)$ and $L(Y, X)$ where $L(K, X)$ and $L(Y, X)$ denote, respectively, the space of all bounded linear operators from K into X and Y into X. The functions F, G, σ, g_i and ξ are given functions to be defined later.

The main contributions of this paper are summarized as follows:

- The study of approximate controllability of noninstantaneous impulsive semilinear Hilfer fractional stochastic integrodifferential equations with fractional Brownian motion and nonlocal conditions described in the form (1.1) is an untreated topic in the literature and this is an additional motivation for writing this paper.
- Using methods of functional analysis, a set of sufficient conditions are proposed for approximate controllability.
- The results are established with the use of semigroup theory, fractional calculus and stochastic analysis.
- The application is demonstrated through an example of stochastic control Hilfer fractional partial differential equation with fractional Brownian motion.

2 Preliminaries

In this section, some definitions and results are given which will be used throughout this paper.

Definition 2.1 ([28]) The left-sided Riemann–Liouville fractional integral of order $\mu > 0$ with the lower limit a for a function $f : [a, \infty) \to R$ is defined as

$$I_{a+}^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_a^t \frac{f(s)}{(t-s)^{1-\mu}} \, ds, \quad t > a, \mu > 0,$$

provided that the right side is pointwise defined on $[a, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (Hilfer fractional derivative [29]) The left-sided Hilfer fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ of function $f(t)$ is defined as

$$D_{a+}^{\nu,\mu} f(t) = I_{a+}^{\nu(1-\mu)} \frac{d}{dt} I_{a+}^{(1-\nu)(1-\mu)} f(t),$$

where $D := \frac{d}{dt}$.

Let (Ω, Υ, P) be a complete probability space equipped with a normal filtration $\Upsilon_t, t \in [0,T]$ where Υ_t is the σ-algebra generated by random variables $\{\omega(s), B^H(s), s \in [0,T]\}$ and all P-null sets.

Suppose that $\{\beta^H(t), t \in [0,T]\}$ is the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. That is, β^H is a centered Gaussian process with covariance function $R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ (see [30]).

Consider the Wiener process $\omega = \omega(t)$, $t \in [0,T]$ defined by $\omega(t) = \beta^H((K_H^*)^{-1} 1_{[0,T]})$ then ω is a Wiener process. Moreover, β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s)\, d\omega(s)$$

where $K_H(t,s)$ is the kernel given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (\nu - s)^{H-\frac{3}{2}} \nu^{H-\frac{1}{2}}\, du$$

for $s < t$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}, \quad p > 0, q > 0.$$

We put $K_H(t,s) = 0$ if $t \leq s$.

We will denote by ζ the reproducing kernel Hilbert space of the fBm. In fact ζ is the closure of set of indicator functions $\{1_{[0,T]}, t \in [0,T]\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]} \rangle_\zeta = R_H(t,s)$.

The mapping $1_{[0,T]} \to \beta^H(t)$ can be extended to an isometry from ζ onto the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ under this isometry.

We recall that for $\psi, \varphi \in \zeta$ their scalar product in ζ is given by

$$\langle \psi, \varphi \rangle_\zeta = H(2H-1) \int_0^T \int_0^T \psi(s)\varphi(t) |t-s|^{2H-2}\, ds\, dt.$$

Let us consider the operator K^* from ζ to $L_2([0,T])$ defined by

$$\left(K_H^* \varphi\right)(s) = \int_s^T \varphi(t) \frac{\partial K_H}{\partial t}(t,s)\, dt.$$

Let $Q \in L(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace

$$\operatorname{Tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty, \quad \lambda_n \geq 0 \ (n = 1, 2, \ldots),$$

are non-negative real numbers and $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis in Y.

We define the infinite-dimensional fBm on Y with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where $\beta_n^H(t)$ are standard fBms mutually independent on (Ω, Υ, P). In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $L_2^0 := L_2^0(Y, X)$ of all Q-Hilbert Schmidt operators $\psi : Y \to X$. We recall that $\psi \in L(Y, X)$ is called a Q-Hilbert–Schmidt operator, if

$$\|\psi\|_{L_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty$$

and that the space L_2^0 equipped with the inner product $\langle \varphi, \psi \rangle_{L_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Let $\phi(s); s \in [0, T]$ be a function with values in $L_2^0(Y, X)$, the Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s)\, dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda}_n \phi(s) e_n \, d\beta_n^H = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda}_n K^*(\phi e_n)(s)\, d\beta_n(s), \tag{2.1}$$

where β_n is the standard Brownian motion.

Lemma 2.1 (see [4]) *If $\psi : [0, T] \to L_2^0(Y, X)$ satisfies $\int_0^T \|\psi(s)\|_{L_2^0}^2 < \infty$ then the above sum in (2.1) is well defined as X-valued random variable and we have*

$$E\left\| \int_0^t \psi(s)\, dB^H(s) \right\|^2 \le 2Ht^{2H-1} \int_0^t \left\| \psi(s) \right\|_{L_2^0}^2 ds.$$

We suppose that $0 \in \rho(A)$, the resolvent set of A, and $\|S(t)\| \le M$ for some constant $M \ge 1$ and every $t \ge 0$. We define the fractional power $A^{-\gamma}$ by

$$A^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} S(t)\, dt, \quad \gamma > 0.$$

For $\gamma \in (0, 1]$, A^γ is a closed linear operator on its domain $D(A^\gamma)$. Furthermore, the subspace $D(A^\gamma)$ is dense in X. We will introduce the following basic properties of A^γ.

Theorem 2.1 (see [31])

(1) *Let $0 < \gamma \le 1$, then $X_\gamma := D(A^\gamma)$ is a Banach space with the norm $\|x\|_\gamma = \|A^\gamma x\|$, $x \in X_\gamma$.*
(2) *If $0 < \beta < \gamma \le 1$, then $D(A^\gamma) \hookrightarrow D(A^\beta)$ and the embedding is compact whenever the resolvent operator of A is compact.*
(3) *For every $0 < \gamma \le 1$, there exists a positive constant C_γ such that*

$$\left\| A^\gamma S(t) \right\| \le \frac{C_\gamma}{t^\gamma}, \quad 0 < t \le T.$$

The collection of all strongly-measurable, square-integrable, X-valued random variables, denoted by $L_2(\Omega, X)$, is a Banach space equipped with norm

$$\left\| x(\cdot) \right\|_{L_2(\Omega,X)} = \left(E\left\| x(\cdot, \omega) \right\|^2 \right)^{\frac{1}{2}},$$

where the expectation, E is defined by $E(x) = \int_\Omega x(\omega)\, dP$.

Let $C(J, L_2(\Omega, X))$ be the Banach space of all continuous maps from J into $L_2(\Omega, X)$ satisfying the condition $\sup_{t\in J} E\|x(t)\|^2 < \infty$.

Define $\bar{C} = \{x : t^{(1-\nu)(1-\mu)}x(t) \in C(J, L_2(\Omega, X))\}$, with norm $\|\cdot\|_{\bar{C}}$ defined by

$$\|\cdot\|_{\bar{C}} = \Bigl(\sup_{t\in J} E\bigl|t^{(1-\nu)(1-\mu)}x(t)\bigr|^2\Bigr)^{\frac{1}{2}}.$$

Obviously, $\bar{C}$ is a Banach space.

We impose the following conditions on data of the problem:

$(H1)$ $F : J \times X^{m+1} \to X$ is a continuous function, and there exist a constant $\beta \in (0, 1)$ and $M_1, M_2 > 0$ such that the function $(-A)^\beta F$ satisfies the Lipschitz condition:

$$E\bigl\|A^\beta F(s_1, x_0, x_1, \ldots, x_m) - A^\beta F(s_2, y_0, y_1, \ldots, y_m)\bigr\|^2 \le M_1\Bigl(|s_1 - s_2| + \max_{i=0,1,\ldots,m} E\|x_i - y_i\|^2\Bigr),$$

for $0 \le s_1, s_2 \le b$, $x_i, y_i \in X$, $i = 0, 1, \ldots, m$ and the inequality

$$E\bigl\|A^\beta F(t, x_0, x_1, \ldots, x_m)\bigr\|^2 \le M_2\Bigl(\max_{i=0,1,\ldots,m} E\|x_i\|^2 + 1\Bigr) \tag{2.2}$$

holds for $(t, x_0, x_1, \ldots, x_m) \in J \times X^{m+1}$.

$(H2)$ The function $G : J \times X^{k+1} \to L(K, X)$ satisfies the following conditions:

(i) for each $t \in J$, the function $G(t, \cdot) : X^{k+1} \to L(K, X)$ is continuous and for each $(x_0, x_1, \ldots, x_n) \in X^{n+1}$; the function $G(\cdot, x_0, x_1, \ldots, x_k) : J \to L(K, X)$ is Υ_t-measurable;

(ii) for each positive number $q \in N$, there is a positive function $h_q(\cdot) : (0, T] \to R^+$ such that

$$\sup_{\|x_0\|^2,\ldots,\|x_n\|^2 \le q} \int_0^t E\bigl\|G(s, x_0, x_1, \ldots, x_k)\bigr\|_Q^2\,ds \le h_q(t),$$

the function $s \to (t-s)^{\mu-1}h_q(s) \in L^1((0, T], R^+)$ and there exists a $\Lambda_1 > 0$ such that

$$\lim_{q\to\infty}\inf \frac{\int_0^t (t-s)^{\mu-1}h_q(s)\,ds}{q} = \Lambda_1 < \infty, \quad t \in (0, T].$$

$(H3)$ The function $\sigma : J \times X^{p+1} \to L_2^0(Y, X)$ satisfies the following conditions:

(i) for each $t \in J$, the function $\sigma(t, \cdot) : X^{p+1} \to L_2^0(Y, X)$ is continuous and for each $(x_0, x_1, \ldots, x_p) \in X^{p+1}$; the function $\sigma(\cdot, x_0, x_1, \ldots, x_p) : J \to L_2^0(Y, X)$ is Υ_t-measurable;

(ii) for each positive number $q \in N$, there is a positive function $\bar{h}_q(\cdot) : (0, T] \to R^+$ such that

$$\sup_{\|x_0\|^2,\ldots,\|x_n\|^2 \le q} E\bigl\|\sigma(t, x_0, x_1, \ldots, x_p)\bigr\|_{L_2^0}^2 \le \bar{h}_q(t),$$

the function $s \to (t-s)^{\mu-1}\bar{h}_q(s) \in L^1((0, T], R^+)$ and there exists a $\Lambda_2 > 0$ such that

$$\lim_{q\to\infty}\inf \frac{\int_0^t (t-s)^{\mu-1}\bar{h}_q(s)\,ds}{q} = \Lambda_2 < \infty, \quad t \in (0, T],$$

$(H4)$ The function $g_i : (t_i, s_i] \times X \to X$ is continuous and satisfies the following two conditions:

(i) There exists a constant $M_3 > 0$, such that

$$E\|g_i(t,x)\|^2 \leq M_3 E\|x\|^2, \quad \forall x \in X; t \in (t_i, s_i], i = 1, 2, \ldots, m.$$

(ii) There exists a constant $M_6 > 0$, such that

$$E\|g_i(t,x_1) - g_i(t,x_2)\|^2 \leq M_6 E\|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in X; t \in (t_i, s_i], i = 1, 2, \ldots, m.$$

$(H5)$ The function $\xi : C(J,X) \to X$ satisfies the following two conditions:

(i) There exist positive constants M_4 and M_5 such that

$$E\|\xi(x)\|^2 \leq M_4 E\|x\|^2 + M_5 \quad \forall x \in X;$$

(ii) There exists a constant $M_7 > 0$, such that

$$E\|\xi(x_1) - \xi(x_2)\|^2 \leq M_7 E\|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in X.$$

Theorem 2.2 *Let Φ be a condensing operator on a Banach space X, that is, Φ is continuous and takes bounded sets into bounded sets, and $\mu(\Phi(\bar{B})) \leq \mu(\bar{B})$ for every bounded set $\bar{B}$ of X with $\mu(\bar{B}) > 0$. If $\Phi(Z) \subset Z$ for a convex, closed and bounded set Z of X, then Φ has a fixed point in X (where $\mu(\cdot)$ denotes Kuratowski's measure of noncompactness).*

Definition 2.3 An Υ_t-adapted stochastic process $x(t) : J \to X$ is said to be a mild solution of problem (1.1) if $x_0 \in X$ for each $s \in [0,T)$ the function $AP_\mu(t-s)F(s,x(s),x(b_1(s)),\ldots,x(b_m(s)))$ is integrable and the following stochastic integral equation is verified:

$$\begin{aligned}
x(t) &= S_{\nu,\mu}(t)\big[x_0 - \xi(x) + F\big(0,x(0),x\big(b_1(0)\big),\ldots,x\big(b_m(0)\big)\big)\big] \\
&\quad - F\big(t,x(t),x\big(b_1(t)\big),\ldots,x\big(b_m(t)\big)\big) \\
&\quad - \int_0^t AP_\mu(t-s)F\big(s,x(s),x\big(b_1(s)\big),\ldots,x\big(b_m(s)\big)\big)\,ds + \int_0^t P_\mu(t-s)Bu(s)\,ds \\
&\quad + \int_0^t P_\mu(t-s)\int_0^s G\big(\tau,x(\tau),x\big(a_1(\tau)\big),\ldots,x\big(a_k(\tau)\big)\big)\,d\omega(\tau)\,ds \\
&\quad + \int_0^t P_\mu(t-s)\sigma\big(s,x(s),x\big(c_1(s)\big),\ldots,x\big(c_p(s)\big)\big)\,dB^H(s), \quad t \in (0,t_1], \\
x(t) &= g_i\big(t,x(t)\big), \quad t \in (t_i,s_i], i = 1,2,\ldots,m, \qquad (2.3) \\
x(t) &= S_{\nu,\mu}(t-s_i)g_i\big(s_i,x(s_i)\big) - F\big(t,x(t),x\big(b_1(t)\big),\ldots,x\big(b_m(t)\big)\big) \\
&\quad - \int_{s_i}^t AP_\mu(t-s)F\big(s,x(s),x\big(b_1(s)\big),\ldots,x\big(b_m(s)\big)\big)\,ds + \int_{s_i}^t P_\mu(t-s)Bu(s)\,ds \\
&\quad + \int_{s_i}^t P_\mu(t-s)\int_0^s G\big(\tau,x(\tau),x\big(a_1(\tau)\big),\ldots,x\big(a_k(\tau)\big)\big)\,d\omega(\tau)\,ds \\
&\quad + \int_{s_i}^t P_\mu(t-s)\sigma\big(s,x(s),x\big(c_1(s)\big),\ldots,x\big(c_p(s)\big)\big)\,dB^H(s), \\
&\quad t \in (s_i,t_{i+1}], i = 1,2,\ldots,m,
\end{aligned}$$

where

$$\begin{aligned} S_{\nu,\mu}(t) &= I_{0+}^{\nu(1-\mu)} P_\mu(t), \\ P_\mu(t) &= t^{\mu-1} T_\mu(t), \\ T_\mu(t) &= \int_0^\infty \mu\theta \Psi_\mu(\theta) S\bigl(t^\mu \theta\bigr)\, d\theta, \end{aligned} \tag{2.4}$$

where

$$\Psi_\mu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\mu)}, \quad 0 < \mu < 1, \theta \in (0,\infty),$$

is a function of Wright-type which satisfies the following inequality $\int_0^\infty \theta^\tau \Psi_\mu(\theta)\, d\theta = \frac{\Gamma(1+\tau)}{\Gamma(1+\mu\tau)}$ for $\theta \geq 0$.

Lemma 2.2 (see [32]) *The operator $S_{\nu,\mu}$ and P_μ have the following properties.*

(i) $\{P_\mu(t) : t > 0\}$ *is continuous in the uniform operator topology.*

(ii) *For any fixed $t > 0$, $S_{\nu,\mu}(t)$ and $P_\mu(t)$ are linear and bounded operators, and*

$$\bigl\|P_\mu(t)x\bigr\| \leq \frac{Mt^{\mu-1}}{\Gamma(\mu)}\|x\|, \qquad \bigl\|S_{\nu,\mu}(t)x\bigr\| \leq \frac{Mt^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)}\|x\|. \tag{2.5}$$

(iii) $\{P_\mu(t) : t > 0\}$ *and* $\{S_{\nu,\mu}(t) : t > 0\}$ *are strongly continuous.*

Lemma 2.3 *For any $x \in X$, $\beta \in (0,1)$ and $\delta \in (0,1]$, we have $AT_\mu(t)x = A^{1-\beta} T_\mu(t) A^\beta x$, $0 \leq t \leq T$ and*

$$\bigl\|A^\delta T_\mu(t)x\bigr\| \leq \frac{\mu C_\delta \Gamma(2-\delta)}{t^{\delta\mu}\Gamma(1+\mu(1-\delta))}\|x\|, \quad 0 < t \leq T.$$

In order to study the approximate controllability for the fractional control system (1.1), we introduce the following linear fractional differential system

$$\begin{cases} D_{0+}^{\nu,\mu} x(t) + Ax(t) = Bu(t), & t \in (0,T], \\ I_{0+}^{(1-\nu)(1-\mu)} x(0) = x_0. \end{cases} \tag{2.6}$$

It is convenient at this point to introduce the operators associated with (2.6) as

$$\Gamma_0^T = \int_0^T (T-s)^{\mu-1} T_\mu(T-s) BB^* T_\mu^*(T-s)\, ds,$$

and $R(T, \Gamma_0^T) = (TI + \Gamma_0^T)^{-1}$, $T > 0$, where B^* and T_μ^* denote the adjoint of B and T_μ, respectively.

Let $x(T; x_0, u)$ be the state value of (1.1) at terminal state T, corresponding to the control u and the initial value x_0. Denote by $R(T, x_0) = \{x(T; x_0, u) : u \in L_2(J, U)\}$ the reachable set of system (1.1) at terminal time T, its closure in X is denoted by $\overline{R(T, x_0)}$

Definition 2.4 The system (1.1) is said to be approximately controllable on the interval J if $\overline{R(T, x_0)} = L_2(\Omega, X)$.

Lemma 2.4 (see [20]) *The fractional linear control system* (2.6) *is approximately controllable on J if and only if $z(zI+\Gamma_0^T)^{-1} \to 0$ as $z \to 0^+$.*

Lemma 2.5 *For any $\bar{x}_T \in L_2(\Omega, X)$ there exist $\bar{\psi}$ and $\bar{\varphi} \in L_2(\Omega; L_2(J; L_2^0))$ such that*

$$\bar{x}_T = E\bar{x}_T + \int_0^T \bar{\psi}(s)\,d\omega(s) + \int_0^T \bar{\varphi}(s)\,dB^H(s).$$

Now for any $\delta > 0$ and $\bar{x}_T \in L_2(\Omega, X)$, we define the control function in the following form

$$\begin{aligned}
u^\delta(t) &= B^* T_\mu^*(T-t)\bigl(zI+\Gamma_0^T\bigr)^{-1} \\
&\quad \times \Bigl\{E\bar{x}_T - S_{\nu,\mu}(T)\bigl[x_0 - \xi(x) + F\bigl(0, x(0), x\bigl(b_1(0)\bigr), \dots, x\bigl(b_m(0)\bigr)\bigr)\bigr] \\
&\quad - F\bigl(T, x(T), x\bigl(b_1(T)\bigr), \dots, x\bigl(b_m(T)\bigr)\bigr) + \int_0^T \bar{\psi}(s)\,d\omega(s) + \int_0^T \bar{\varphi}(s)\,dB^H(s)\Bigr\} \\
&\quad - B^* T_\mu^*(T-t)\int_0^T \bigl(zI+\Gamma_0^T\bigr)^{-1} P_\mu(T-s) A F\bigl(s, x(s), x\bigl(b_1(s)\bigr), \dots, x\bigl(b_m(s)\bigr)\bigr)\,ds \\
&\quad - B^* T_\mu^*(T-t)\int_0^T \bigl(zI+\Gamma_0^T\bigr)^{-1} P_\mu(T-s) \\
&\quad \times \int_0^s G\bigl(\tau, x(\tau), x\bigl(a_1(\tau)\bigr), \dots, x\bigl(a_k(\tau)\bigr)\bigr)\,d\omega(\tau)\,ds \\
&\quad - B^* T_\mu^*(T-t)\int_0^T \bigl(zI+\Gamma_0^T\bigr)^{-1} P_\mu(T-s) \\
&\quad \times \sigma\bigl(s, x(s), x\bigl(c_1(s)\bigr), \dots, x\bigl(c_p(s)\bigr)\bigr)\,dB^H(s), \quad t \in (0, t_1], \\
u^\delta(t) &= B^* T_\mu^*(T-t)\bigl(zI+\Gamma_0^T\bigr)^{-1}\Bigl\{E\bar{x}_T - S_{\nu,\mu}(T-s_i) g_i\bigl(s_i, x(s_i)\bigr) \\
&\quad - F\bigl(T, x(T), x\bigl(b_1(T)\bigr), \dots, x\bigl(b_m(T)\bigr)\bigr) + \int_0^T \bar{\psi}(s)\,d\omega(s) + \int_0^T \bar{\varphi}(s)\,dB^H(s)\Bigr\} \\
&\quad - B^* T_\mu^*(T-t)\int_{s_i}^T \bigl(zI+\Gamma_0^T\bigr)^{-1} P_\mu(T-s) A F\bigl(s, x(s), x\bigl(b_1(s)\bigr), \dots, x\bigl(b_m(s)\bigr)\bigr)\,ds \\
&\quad - B^* T_\mu^*(T-t)\int_{s_i}^T \bigl(zI+\Gamma_0^T\bigr)^{-1} P_\mu(T-s) \\
&\quad \times \int_0^s G\bigl(\tau, x(\tau), x\bigl(a_1(\tau)\bigr), \dots, x\bigl(a_k(\tau)\bigr)\bigr)\,d\omega(\tau)\,ds \\
&\quad - B^* T_\mu^*(T-t)\int_{s_i}^T \bigl(zI+\Gamma_0^T\bigr)^{-1} P_\mu(T-s) \\
&\quad \times \sigma\bigl(s, x(s), x\bigl(c_1(s)\bigr), \dots, x\bigl(c_p(s)\bigr)\bigr)\,dB^H(s), \quad t \in (s_i, t_{i+1}].
\end{aligned} \tag{2.7}$$

3 Approximate controllability

In this section, we formulate sufficient conditions for the approximate controllability of the system (1.1). For this purpose, we first prove the existence of a mild solution for the system (1.1). Second we shall prove the system (1.1) is approximately controllable under certain assumptions.

Theorem 3.1 *If the assumptions* $(H1)$–$(H5)$ *are satisfied, then the system* (1.1) *has a mild solution on* J, *provided that*

$$\begin{aligned}&\left[1+\frac{M^4T^{2\mu}M_B^4}{z^2\mu^2\Gamma^4(\mu)}\right]\left\{\frac{36M^2(M_0^2M_2+M_3+M_4)}{\Gamma^2(\nu(1-\mu)+\mu)}\right.\\&\quad+36T^{2(1-\nu)(1-\mu)}\left[{M_0}^2M_2+\frac{M^2T^\mu\Lambda_1\operatorname{Tr}(Q)}{\mu\Gamma^2(\mu)}\right.\\&\quad\left.\left.+\frac{2HM^2\Lambda_2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}+\frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta}M_2}{\beta^2\Gamma^2(1+\mu\beta)}\right]\right\}+T^{2(1-\nu)(1-\mu)}M_3<1 \qquad (3.1)\end{aligned}$$

and

$$\begin{aligned}\gamma_1&=9\left[\frac{M^2({M_0}^2M_1+M_6+M_7)}{\Gamma^2(\nu(1-\mu)+\mu)}+T^{2(1-\nu)(1-\mu)}\left(M_6+{M_0}^2M_1\right)\right.\\&\quad\left.+\frac{M_1(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta+2(1-\nu)(1-\mu)}}{\beta^2\Gamma^2(1+\mu\beta)}\right] \qquad (3.2)\\&<1, \qquad (3.3)\end{aligned}$$

where $M_0=\|A^{-\beta}\|$ *and* $M_B=\|B\|$.

Proof For any $\delta>0$, consider the map Φ_δ on $\bar{C}$ defined by

$$\begin{aligned}(\Phi_\delta x)(t)&=S_{\nu,\mu}(t)\left[x_0-\xi(x)+F\left(0,x(0),x\left(b_1(0)\right),\ldots,x\left(b_m(0)\right)\right)\right]\\&\quad-F\left(t,x(t),x\left(b_1(t)\right),\ldots,x\left(b_m(t)\right)\right)\\&\quad-\int_0^tAP_\mu(t-s)F\left(s,x(s),x\left(b_1(s)\right),\ldots,x\left(b_m(s)\right)\right)ds+\int_0^tP_\mu(t-s)Bu^\delta(s)\,ds\\&\quad+\int_0^tP_\mu(t-s)\int_0^sG\left(\tau,x(\tau),x\left(a_1(\tau)\right),\ldots,x\left(a_k(\tau)\right)\right)d\omega(\tau)\,ds\\&\quad+\int_0^tP_\mu(t-s)\sigma\left(s,x(s),x\left(c_1(s)\right),\ldots,x\left(c_p(s)\right)\right)dB^H(s),\quad t\in(0,t_1],\\(\Phi_\delta x)(t)&=g_i\left(t,x(t)\right),t\in(t_i,s_i],\quad i=1,2,\ldots,m,\\(\Phi_\delta x)(t)&=S_{\nu,\mu}(t-s_i)g_i\left(s_i,x(s_i)\right)-F\left(t,x(t),x\left(b_1(t)\right),\ldots,x\left(b_m(t)\right)\right)\\&\quad-\int_{s_i}^tAP_\mu(t-s)F\left(s,x(s),x\left(b_1(s)\right),\ldots,x\left(b_m(s)\right)\right)ds+\int_{s_i}^tP_\mu(t-s)Bu^\delta(s)\,ds\\&\quad+\int_{s_i}^tP_\mu(t-s)\int_0^sG\left(\tau,x(\tau),x\left(a_1(\tau)\right),\ldots,x\left(a_k(\tau)\right)\right)d\omega(\tau)\,ds\\&\quad+\int_{s_i}^tP_\mu(t-s)\sigma\left(s,x(s),x\left(c_1(s)\right),\ldots,x\left(c_p(s)\right)\right)dB^H(s),\\&\quad t\in(s_i,t_{i+1}],i=1,2,\ldots,m.\end{aligned}$$

We shall show that the operator Φ_δ has a fixed point, which then is a solution of system (1.1).

For each positive integer q, set $B_q=\{x\in\bar{C},\|x\|_{\bar{C}}^2\le q\}$.

Then, for each q, $B_q \subset \bar{C}$ is clearly a bounded closed convex set in $\bar{C}$. From Lemma 2.2, Lemma 2.3 and (2.2) together with the Hölder inequality,

$$\begin{aligned}
&E\left\|\int_0^t AP_\mu(t-s)F\bigl(s,x(s),x\bigl(b_1(s)\bigr),\dots,x\bigl(b_m(s)\bigr)\bigr)\,ds\right\|^2\\
&\leq E\left[\int_0^t \bigl\|A^{1-\beta}P_\mu(t-s)A^\beta F\bigl(s,x(s),x\bigl(b_1(s)\bigr),\dots,x\bigl(b_m(s)\bigr)\bigr)\bigr\|\,ds\right]^2\\
&\leq \frac{\mu^2(C_{1-\beta})^2\Gamma^2(1+\beta)}{\Gamma^2(1+\mu\beta)}\int_0^t (t-s)^{\mu\beta-1}\,ds\\
&\quad\times\int_0^t (t-s)^{\mu\beta-1}E\bigl\|A^\beta F\bigl(s,x(s),x\bigl(b_1(s)\bigr),\dots,x\bigl(b_m(s)\bigr)\bigr)\bigr\|^2\,ds\\
&\leq \frac{\mu(C_{1-\beta})^2\Gamma^2(1+\beta)T^{\mu\beta}M_2}{\beta\Gamma^2(1+\mu\beta)}\int_0^t (t-s)^{\mu\beta-1}\Bigl(\max_{i=1,2,\dots,m}E\|x_i\|^2+1\Bigr)\,ds.
\end{aligned}\tag{3.4}$$

It follows that $AP_\mu(t-s)F(s,x(s),x(b_1(s)),\dots,x(b_m(s)))$ is integrable on J, by Bochner's theorem [33] so Φ_δ is well defined on B_q.

From $(H2)(ii)$ together with Burkholder Gundy's inequality, we obtain

$$\begin{aligned}
&E\left\|\int_0^t P_\mu(t-s)\int_0^s G\bigl(\tau,x(\tau),x\bigl(a_1(\tau)\bigr),\dots,x\bigl(a_n(\tau)\bigr)\bigr)\,d\omega(\tau)\,ds\right\|^2\\
&\leq \operatorname{Tr}(Q)\frac{M^2T^\mu}{\mu\Gamma^2(\mu)}\int_0^t (t-s)^{\mu-1}\\
&\quad\times\Bigl(\sup_{\|x_0\|^2,\dots,\|x_n\|^2\leq q}\int_0^s E\bigl\|G\bigl(\tau,x(\tau),x\bigl(a_1(\tau)\bigr),\dots,x\bigl(a_k(\tau)\bigr)\bigr)\bigr\|_Q^2\,d\tau\Bigr)\,ds\\
&\leq \operatorname{Tr}(Q)\frac{M^2T^\mu}{\mu\Gamma^2(\mu)}\int_0^t (t-s)^{\mu-1}h_q(s)\,ds.
\end{aligned}\tag{3.5}$$

Similarly from $(H3)(ii)$ together with Burkholder Gundy's inequality, we obtain

$$\begin{aligned}
&E\left\|\int_0^t P_\mu(t-s)\sigma\bigl(s,x(s),x\bigl(c_1(s)\bigr),\dots,x\bigl(c_p(s)\bigr)\bigr)\,dB^H(s)\right\|^2\\
&\leq \frac{2HM^2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}\int_0^t (t-s)^{\mu-1}\\
&\quad\times\sup_{\|x_0\|^2,\dots,\|x_n\|^2\leq q}E\bigl\|\sigma\bigl(s,x(s),x\bigl(c_1(s)\bigr),\dots,x\bigl(c_p(s)\bigr)\bigr)\bigr\|_{L_2^0}^2\,ds\\
&\leq \frac{2HM^2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}\int_0^t (t-s)^{\mu-1}\bar{h}_q(s)\,ds.
\end{aligned}\tag{3.6}$$

Also, by using $(H1)$–$(H5)$ together with Hölder inequality, we obtain

$$\begin{aligned}
E\left\|\int_0^t P_\mu(t-s)Bu^\delta(s)\,ds\right\|^2 &= E\left\|\int_0^t (t-s)^{\mu-1}T_\mu(t-s)Bu^\delta(s)\,ds\right\|^2\\
&\leq \frac{M^2T^\mu M_B^2}{\mu\Gamma^2(\mu)}\int_0^t (t-s)^{\mu-1}E\bigl\|u^\delta(s)\bigr\|^2\,ds
\end{aligned}$$

where, for $t \in (0, t_1]$,

$$\begin{aligned}
&E\|u^\delta(s)\|^2 \\
&\quad\le \frac{M_B^2 M^2}{z^2\Gamma^2(\mu)}\biggl\{E\|\bar{x}_T\|^2 + \frac{M^2 T^{2(\nu-1)(1-\mu)}}{\Gamma^2(\nu(1-\mu)+\mu)}\bigl[E\|x(0)\|^2 + M_4 q + M_5 + M_0^2 M_2(q+1)\bigr] \\
&\qquad + M_0^2 M_2(q+1) + \operatorname{Tr}(Q)\int_0^T E\|\bar{\psi}(s)\|_Q^2\,ds + 2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|_{L_2^0}^2\,ds\biggr\} \\
&\qquad + \frac{M_B^2 M^2}{z^2\Gamma^2(\mu)}\biggl\{\frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta}M_2}{\beta^2\Gamma^2(1+\mu\beta)}(q+1) \\
&\qquad + \operatorname{Tr}(Q)\frac{M^2 T^\mu}{\mu\Gamma^2(\mu)}\int_0^T (T-s)^{\mu-1}h_q(s)\,ds + \frac{2HM^2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}\int_0^T (T-s)^{\mu-1}\bar{h}_q(s)\,ds\biggr\},
\end{aligned}$$

and for $t \in (s_i, t_{i+1}]$

$$\begin{aligned}
&E\|u^\delta(s)\|^2 \\
&\quad\le \frac{M_B^2 M^2}{z^2\Gamma^2(\mu)}\biggl\{E\|\bar{x}_T\|^2 + \frac{M^2 T^{2(\nu-1)(1-\mu)}}{\Gamma^2(\nu(1-\mu)+\mu)}M_3 q + M_0^2 M_2(q+1) \\
&\qquad + \operatorname{Tr}(Q)\int_0^T E\|\bar{\psi}(s)\|_Q^2\,ds + 2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|_{L_2^0}^2\,ds\biggr\} \\
&\qquad + \frac{M_B^2 M^2}{z^2\Gamma^2(\mu)}\biggl\{\frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta}M_2}{\beta^2\Gamma^2(1+\mu\beta)}(q+1) \\
&\qquad + \operatorname{Tr}(Q)\frac{M^2 T^\mu}{\mu\Gamma^2(\mu)}\int_{s_i}^T (T-s)^{\mu-1}h_q(s)\,ds \\
&\qquad + \frac{2HM^2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}\int_{s_i}^T (T-s)^{\mu-1}\bar{h}_q(s)\,ds\biggr\},
\end{aligned}$$

thus, we have

$$\begin{aligned}
&E\biggl\|\int_0^t P_\mu(t-s)Bu^\delta(s)\,ds\biggr\|^2 \\
&\quad\le \frac{M^4 T^{2\mu} M_B^4}{z^2\mu^2\Gamma^4(\mu)}\biggl\{E\|\bar{x}_T\|^2 + \frac{M^2 T^{2(\nu-1)(1-\mu)}}{\Gamma^2(\nu(1-\mu)+\mu)}\bigl[E\|x(0)\|^2 \\
&\qquad + M_4 q + M_5 + M_0^2 M_2(q+1)\bigr] + M_0^2 M_2(q+1) \\
&\qquad + \operatorname{Tr}(Q)\int_0^T E\|\bar{\psi}(s)\|_Q^2\,ds + 2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|_{L_2^0}^2\,ds \\
&\qquad + \frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta}M_2}{\beta^2\Gamma^2(1+\mu\beta)}(q+1) \\
&\qquad + \operatorname{Tr}(Q)\frac{M^2 T^\mu}{\mu\Gamma^2(\mu)}\int_0^T (T-s)^{\mu-1}h_q(s)\,ds \\
&\qquad + \frac{2HM^2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}\int_0^T (T-s)^{\mu-1}\bar{h}_q(s)\,ds\biggr\}, \quad t \in (0, t_1], \qquad (3.7)
\end{aligned}$$

$$
\begin{aligned}
&E\left\|\int_{s_i}^{t} P_\mu(t-s)Bu^\delta(s)\,ds\right\|^2\\
&\leq \frac{M^4T^{2\mu}M_B^4}{z^2\mu^2\Gamma^4(\mu)}\left\{E\|\bar{x}_T\|^2+\frac{M^2T^{2(\nu-1)(1-\mu)}}{\Gamma^2(\nu(1-\mu)+\mu)}M_3q\right.\\
&\quad+M_0^2M_2(q+1)+\operatorname{Tr}(Q)\int_0^T E\left\|\bar{\psi}(s)\right\|_Q^2\,ds+2HT^{2H-1}\int_0^T E\left\|\bar{\varphi}(s)\right\|_{L_2^0}^2\,ds\\
&\quad+\frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta}M_2}{\beta^2\Gamma^2(1+\mu\beta)}(q+1)+\operatorname{Tr}(Q)\frac{M^2T^\mu}{\mu\Gamma^2(\mu)}\int_{s_i}^T(T-s)^{\mu-1}h_q(s)\,ds\\
&\quad\left.+\frac{2HM^2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}\int_{s_i}^T(T-s)^{\mu-1}\bar{h}_q(s)\,ds\right\},\quad t\in(s_i,t_{i+1}].
\end{aligned}
$$

We claim that there exists a positive number q such that $\Phi_\delta(B_q)\subseteq B_q$. If it is not true, then, for each positive number q, there is a function $x_q(\cdot)\in B_q$, but $\Phi_\delta(x_q)\notin B_q$, that is $\|(\Phi_\delta x_q)(t)\|_C^2>q$ for some $t=t(q)\in J$, where $t(q)$ denotes that t is dependent of q. However, from $(H4)$–$(H5)$ and Eqs. (2.2), (3.4), (3.5), (3.6) and (3.7), we have for $t\in(0,t_1]$

$$
\begin{aligned}
&\|\Phi_\delta x_q\|_C^2\\
&\leq 36\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\Big\{E\left\|S_{\nu,\mu}(t)\big[x_0+\xi(x)+F\big(0,x(0),x\big(b_1(0)\big),\dots,x\big(b_m(0)\big)\big)\big]\right\|^2\\
&\quad+E\left\|F\big(t,x(t),x\big(b_1(t)\big),\dots,x\big(b_m(t)\big)\big)\right\|^2\\
&\quad+E\left\|\int_0^t AP_\mu(t-s)F\big(t,x(t),x\big(b_1(t)\big),\dots,x\big(b_m(t)\big)\big)\,ds\right\|^2\\
&\quad+E\left\|\int_0^t P_\mu(t-s)Bu^\delta(s)\,ds\right\|^2\\
&\quad+E\left\|\int_0^t P_\mu(t-s)\int_0^s G\big(\tau,x(\tau),x\big(a_1(\tau)\big),\dots,x\big(a_k(\tau)\big)\big)\,d\omega(\tau)\,ds\right\|^2\\
&\quad+E\left\|\int_0^t P_\mu(t-s)\sigma\big(s,x(s),x\big(c_1(s)\big),\dots,x\big(c_p(s)\big)\big)\,dB^H(s)\right\|^2\Big\}\\
&\leq 36\left\{\frac{M^2}{\Gamma^2(\nu(1-\mu)+\mu)}\big[E\left\|x(0)\right\|^2+M_4q+M_5+{M_0}^2M_2(q+1)\big]\right.\\
&\quad+T^{2(1-\nu)(1-\mu)}{M_0}^2M_2(q+1)\\
&\quad+\frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta+2(1-\nu)(1-\mu)}M_2(q+1)}{\beta^2\Gamma^2(1+\mu\beta)}\\
&\quad+\operatorname{Tr}(Q)\left(\frac{M^2T^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\right)q\frac{1}{q}\int_0^t(t-s)^{\mu-1}h_q(s)\,ds\\
&\quad+\frac{2HM^2T^{2H+\mu-1+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}q\frac{1}{q}\int_0^t(t-s)^{\mu-1}\bar{h}_q(s)\,ds\\
&\quad+\frac{M^4T^{2\mu}T^{2(1-\nu)(1-\mu)}M_B^4}{z^2\mu^2\Gamma^4(\mu)}\left\{E\|\bar{x}_T\|^2\right.\\
&\quad+\frac{M^2T^{2(\nu-1)(1-\mu)}}{\Gamma^2(\nu(1-\mu)+\mu)}\big[E\left\|x(0)\right\|^2+M_4q+M_5+M_0^2M_2(q+1)\big]+M_0^2M_2(q+1)
\end{aligned}
$$

$$
\begin{aligned}
&+ \operatorname{Tr}(Q)\int_0^T E\big\|\bar{\psi}(s)\big\|_Q^2\,ds + 2HT^{2H-1}\int_0^T E\big\|\bar{\varphi}(s)\big\|_{L_2^0}^2\,ds \\
&+ \frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta}M_2}{\beta^2\Gamma^2(1+\mu\beta)}(q+1) \\
&+ \operatorname{Tr}(Q)\frac{M^2T^{\mu}}{\mu\Gamma^2(\mu)}q\frac{1}{q}\int_0^T (T-s)^{\mu-1}h_q(s)\,ds \\
&+ \frac{2HM^2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}q\frac{1}{q}\int_0^T (T-s)^{\mu-1}\bar{h}_q(s)\,ds\bigg\}\bigg\},
\end{aligned}
\tag{3.8}
$$

for $t\in(t_i,s_i]$

$$
\|\Phi_\delta x_q\|_C^2 \le \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\big\|g\big(t,x(t)\big)\big\|^2 \le T^{2(1-\nu)(1-\mu)}M_3q,
\tag{3.9}
$$

and for $t\in(s_i,t_{i+1}]$

$$
\begin{aligned}
&\|\Phi_\delta x_q\|_C^2 \\
&\le 36\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\bigg\{E\big\|S_{\nu,\mu}(t-s_i)g_i\big(s_i,x(s_i)\big)\big\|^2 \\
&+ E\big\|F\big(t,x(t),x\big(b_1(t)\big),\dots,x\big(b_m(t)\big)\big)\big\|^2 \\
&+ E\bigg\|\int_{s_i}^t AP_\mu(t-s)F\big(t,x(t),x\big(b_1(t)\big),\dots,x\big(b_m(t)\big)\big)\,ds\bigg\|^2 \\
&+ E\bigg\|\int_{s_i}^t P_\mu(t-s)Bu^\delta(s)\,ds\bigg\|^2 \\
&+ E\bigg\|\int_{s_i}^t P_\mu(t-s)\int_0^s G\big(\tau,x(\tau),x\big(a_1(\tau)\big),\dots,x\big(a_k(\tau)\big)\big)\,d\omega(\tau)\,ds\bigg\|^2 \\
&+ E\bigg\|\int_{s_i}^t P_\mu(t-s)\sigma\big(s,x(s),x\big(c_1(s)\big),\dots,x\big(c_p(s)\big)\big)\,dB^H(s)\bigg\|^2\bigg\} \\
&\le 36\bigg\{\frac{M^2}{\Gamma^2(\nu(1-\mu)+\mu)}M_3q + T^{2(1-\nu)(1-\mu)}{M_0}^2M_2(q+1) \\
&+ \frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta+2(1-\nu)(1-\mu)}M_2(q+1)}{\beta^2\Gamma^2(1+\mu\beta)} \\
&+ \operatorname{Tr}(Q)\bigg(\frac{M^2T^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\bigg)q\frac{1}{q}\int_{s_i}^t (t-s)^{\mu-1}h_q(s)\,ds \\
&+ \frac{2HM^2T^{2H+\mu-1+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}q\frac{1}{q}\int_{s_i}^t (t-s)^{\mu-1}\bar{h}_q(s)\,ds \\
&+ \frac{M^4T^{2\mu}T^{2(1-\nu)(1-\mu)}M_B^4}{z^2\mu^2\Gamma^4(\mu)}\bigg\{E\|\bar{x}_T\|^2 + \frac{M^2T^{2(\nu-1)(1-\mu)}}{\Gamma^2(\nu(1-\mu)+\mu)}M_3q + M_0^2M_2(q+1) \\
&+ \operatorname{Tr}(Q)\int_0^T E\big\|\bar{\psi}(s)\big\|_Q^2\,ds + 2HT^{2H-1}\int_0^T E\big\|\bar{\varphi}(s)\big\|_{L_2^0}^2\,ds \\
&+ \frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta}M_2}{\beta^2\Gamma^2(1+\mu\beta)}(q+1)
\end{aligned}
$$

$$+ \operatorname{Tr}(Q)\frac{M^2T^{\mu}}{\mu\Gamma^2(\mu)}q\frac{1}{q}\int_{s_i}^{T}(T-s)^{\mu-1}h_q(s)\,ds$$
$$+ \frac{2HM^2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}q\frac{1}{q}\int_{s_i}^{T}(T-s)^{\mu-1}\bar{h}_q(s)\,ds\biggr\}\biggr\}, \tag{3.10}$$

Combining (3.8), (3.9), (3.10) in the inequality $q \le \|(\Phi x_q)(t)\|_C^2$ then dividing both sides of the inequality by q and taking the lower limit $q \to +\infty$, we get

$$\biggl[1+\frac{M^4T^{2\mu}M_B^4}{z^2\mu^2\Gamma^4(\mu)}\biggr]\biggl\{\frac{36M^2(M_0^2M_2+M_3+M_4)}{\Gamma^2(\nu(1-\mu)+\mu)}$$
$$+ 36T^{2(1-\nu)(1-\mu)}\biggl[M_0{}^2M_2+\frac{M^2T^{\mu}\Lambda_1\operatorname{Tr}(Q)}{\mu\Gamma^2(\mu)}$$
$$+ \frac{2HM^2\Lambda_2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)}+\frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta}M_2}{\beta^2\Gamma^2(1+\mu\beta)}\biggr]\biggr\}+T^{2(1-\nu)(1-\mu)}M_3 \ge 1.$$

This contradicts (3.1). Hence for positive q, $\Phi_\delta(B_q) \subseteq B_q$.

Next we will show that the operator Φ_δ has a fixed point on B_q, which implies that Eq. (1.1) has a mild solution. We decompose Φ_δ as $\Phi_\delta = \Phi_1 + \Phi_2$, where the operators Φ_1 and Φ_2 are defined on B_q, respectively, by

$$(\Phi_1 x)(t) = \begin{cases} S_{\nu,\mu}(t)[x_0-\xi(x)+F(0,x(0),x(b_1(0)),\dots,x(b_m(0)))] \\ \quad - F(t,x(t),x(b_1(t)),\dots,x(b_m(t))) \\ \quad - \int_0^t AP_\mu(t-s)F(s,x(s),x(b_1(s)),\dots,x(b_m(s)))\,ds, \quad t\in(0,t_1], \\ g_i(t,x(t)), \quad t\in(t_i,s_i], i=1,2,\dots,m, \\ S_{\nu,\mu}(t-s_i)g_i(s_i,x(s_i)) - F(t,x(t),x(b_1(t)),\dots,x(b_m(t))) \\ \quad - \int_{s_i}^t AP_\mu(t-s)F(s,x(s),x(b_1(s)),\dots,x(b_m(s)))\,ds, \\ \quad t\in(s_i,t_{i+1}], i=1,2,\dots,m, \end{cases}$$

$$(\Phi_2 x)(t) = \begin{cases} \int_{s_i}^t P_\mu(t-s)Bu(s)\,ds \\ \quad + \int_{s_i}^t P_\mu(t-s)\int_0^s G(\tau,x(\tau),x(a_1(\tau)),\dots,x(a_k(\tau)))\,d\omega(\tau)\,ds \\ \quad + \int_{s_i}^t P_\mu(t-s)\sigma(s,x(s),x(c_1(s)),\dots,x(c_p(s)))\,dB^H(s), \\ \quad t\in(s_i,t_{i+1}], i=0,1,\dots,m, \\ 0, \quad \text{otherwise}, \end{cases}$$

for $t \in J$. We will show that Φ_1 verifies a contraction condition while Φ_2 is a compact operator.

To prove that Φ_1 satisfies a contraction condition, we take $x_1, x_2 \in B_q$, then, for each $t \in J$ and by condition $(H1)$, $(H4)$ and $(H5)$, we have for $t \in (0, t_1]$

$$E\bigl\|(\Phi_1x_1)(t)-(\Phi_1x_2)(t)\bigr\|^2$$
$$\le 9\Bigl\{E\bigl\|S_{\nu,\mu}(t)\bigl[\xi(x_1)-\xi(x_2)\bigr]\bigr\|^2$$
$$+ E\bigl\|S_{\nu,\mu}(t)\bigl[F\bigl(0,x_1(0),x_1\bigl(b_1(0)\bigr),\dots,x_1\bigl(b_m(0)\bigr)\bigr)$$
$$- F\bigl(0,x_2(0),x_2\bigl(b_1(0)\bigr),\dots,x_2\bigl(b_m(0)\bigr)\bigr)\bigr]\bigr\|^2$$

$$
\begin{aligned}
&+ E\big\| F\big(t, x_1(t), x_1\big(b_1(t)\big), \ldots, x_1\big(b_m(t)\big)\big) - F\big(t, x_2(t), x_2\big(b_1(t)\big), \ldots, x_2\big(b_m(t)\big)\big)\big\|^2 \\
&+ E\bigg\| \int_0^t AP_\mu(t-s)\big[F\big(t, x_1(t), x_1\big(b_1(t)\big), \ldots, x_1\big(b_m(t)\big)\big) \\
&- F\big(t, x_2(t), x_2\big(b_1(t)\big), \ldots, x_2\big(b_m(t)\big)\big)\big]\, ds\bigg\|^2 \bigg\} \\
\leq{}& 9\bigg[\frac{M^2 T^{2(\nu-1)(1-\mu)}(M_0{}^2 M_1 + M_7)}{\Gamma^2(\nu(1-\mu)+\mu)} \\
&+ M_0{}^2 M_1 + \frac{M_1 (C_{1-\beta})^2 \Gamma^2(1+\beta) T^{2\mu\beta}}{\beta^2 \Gamma^2(1+\mu\beta)}\bigg] E\big\|x_1(t) - x_2(t)\big\|^2,
\end{aligned} \tag{3.11}
$$

for $t \in (t_i, s_i]$

$$
\begin{aligned}
&E\big\|(\Phi_1 x_1)(t) - (\Phi_1 x_2)(t)\big\|^2 \\
&\quad\leq E\big\|g_i\big(t, x_1(t)\big) - g_i\big(t, x_2(t)\big)\big\|^2 \\
&\quad\leq M_6 E\big\|x_1(t) - x_2(t)\big\|^2,
\end{aligned} \tag{3.12}
$$

and for $t \in (s_i, t_{i+1}]$

$$
\begin{aligned}
&E\big\|(\Phi_1 x_1)(t) - (\Phi_1 x_2)(t)\big\|^2 \\
&\quad\leq 9\Big\{E\big\|S_{\nu,\mu}(t-s_i)\big(g_i\big(s_i, x_1(s_i)\big) - g_i\big(s_i, x_2(s_i)\big)\big)\big\|^2 \\
&\qquad+ E\big\| F\big(t, x_1(t), x_1\big(b_1(t)\big), \ldots, x_1\big(b_m(t)\big)\big) - F\big(t, x_2(t), x_2\big(b_1(t)\big), \ldots, x_2\big(b_m(t)\big)\big)\big\|^2 \\
&\qquad+ E\bigg\| \int_{s_i}^t AP_\mu(t-s)\big[F\big(t, x_1(t), x_1\big(b_1(t)\big), \ldots, x_1\big(b_m(t)\big)\big) \\
&\qquad- F\big(t, x_2(t), x_2\big(b_1(t)\big), \ldots, x_2\big(b_m(t)\big)\big)\big]\, ds\bigg\|^2 \bigg\} \\
&\quad\leq 9\bigg[\frac{M^2 T^{2(\nu-1)(1-\mu)}}{\Gamma^2(\nu(1-\mu)+\mu)} M_6 + M_0{}^2 M_1 \\
&\qquad+ \frac{M_1 (C_{1-\beta})^2 \Gamma^2(1+\beta) T^{2\mu\beta}}{\beta^2 \Gamma^2(1+\mu\beta)}\bigg] E\big\|x_1(t) - x_2(t)\big\|^2.
\end{aligned} \tag{3.13}
$$

Combining (3.11), (3.12), (3.13) and taking $\sup_{t\in J} t^{2(1-\nu)(1-\mu)}$ for both sides of the inequality, we get

$$
\begin{aligned}
&\sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\big\|(\Phi_1 x_1)(t) - (\Phi_1 x_2)(t)\big\|^2 \\
&\quad\leq 9\bigg[\frac{M^2(M_0{}^2 M_1 + M_6 + M_7)}{\Gamma^2(\nu(1-\mu)+\mu)} + T^{2(1-\nu)(1-\mu)}\big(M_6 + M_0{}^2 M_1\big) \\
&\qquad+ \frac{M_1 (C_{1-\beta})^2 \Gamma^2(1+\beta) T^{2\mu\beta + 2(1-\nu)(1-\mu)}}{\beta^2 \Gamma^2(1+\mu\beta)}\bigg] \\
&\qquad\times \sup_{t\in J} E\big\|x_1(t) - x_2(t)\big\|^2
\end{aligned}
$$

hence, from the definition of $\bar{C}$ and (3.3) we get

$$\|\Phi_1 x_1 - \Phi_1 x_2\|_{\bar{C}}^2 \le \gamma_1 \|x_1 - x_2\|_{\bar{C}}^2.$$

Thus, Φ_1 is a contraction.

To prove that Φ_2 is compact, first we prove that Φ_2 is continuous on B_q.

Let $\{x_n\} \subseteq B_q$ with $x_n \to x$ in B_q and rewrite $u^\delta(t) = u^\delta(t,x)$, the control function defined above. Then, for each $s \in J$, $x_n(s) \to x(s)$, and by $H2(i)$ and $H3(i)$, we have $G(s,x_n(s),x_n(a_1(s)),\dots,x_n(a_k(s))) \to G(s,x(s),x(a_1(s)),\dots,x(a_k(s)))$, as $n \to \infty$, and $\sigma(s,x_n(s),x_n(c_1(s)),\dots,x_n(c_p(s))) \to \sigma(s,x(s),x(c_1(s)),\dots,x(c_p(s)))$, as $n \to \infty$.

By the dominated convergence theorem, we have

$$\begin{aligned}
\|\Phi_2 x_n - \Phi_2 x\|_{\bar{C}}^2 = \sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\bigg\| &\int_{s_i}^t P_\mu(t-s) B\big(u^\delta(s,x_n) - u^\delta(s,x)\big)\,ds \\
&+ \int_{s_i}^t P_\mu(t-s) \int_0^s \big(G\big(\tau,x_n(\tau),x_n\big(a_1(\tau)\big),\dots,x_n\big(a_k(\tau)\big)\big) \\
&- G\big(\tau,x(\tau),x\big(a_1(\tau)\big),\dots,x\big(a_k(\tau)\big)\big)\big)\,d\omega(\tau)\,ds \\
&+ \int_{s_i}^t P_\mu(t-s)\big(\sigma\big(s,x_n(s),x_n\big(c_1(s)\big),\dots,x_n\big(c_p(s)\big)\big) \\
&- \sigma\big(s,x(s),x\big(c_1(s)\big),\dots,x\big(c_p(s)\big)\big)\big)\,dB^H(s)\bigg\|^2 \\
\to 0,
\end{aligned}$$

as $n \to \infty$, which is continuous.

Next we prove that the family $\{\Phi_2 x : x \in B_q\}$ is an equicontinuous family of functions.

To do this, let $\epsilon > 0$ be small, $s_i < t_\alpha < t_\beta \le t_{i+1}$, then

$$\begin{aligned}
&E\big\|(\Phi_2 x)(t_\beta) - (\Phi_2 x)(t_\alpha)\big\|^2 \\
&\quad\le E\bigg\|\int_{t_\alpha}^{t_\beta} P_\mu(t_\beta - s) B u^\delta(s)\,ds\bigg\|^2 \\
&\qquad + E\bigg\|\int_{s_i}^{t_\alpha-\epsilon} \big(P_\mu(t_\beta - s) - P_\mu(t_\alpha - s)\big) B u^\delta(s)\,ds\bigg\|^2 \\
&\qquad + E\bigg\|\int_{t_\alpha-\epsilon}^{t_\alpha} \big(P_\mu(t_\beta - s) - P_\mu(t_\alpha - s)\big) B u^\delta(s)\,ds\bigg\|^2 \\
&\qquad + E\bigg\|\int_{s_i}^{t_\alpha-\epsilon} \big(P_\mu(t_\beta - s) - P_\mu(t_\alpha - s)\big) \\
&\qquad \times \int_0^s G\big(\tau,x(\tau),x\big(a_1(\tau)\big),\dots,x\big(a_k(\tau)\big)\big)\,d\omega(\tau)\,ds\bigg\|^2 \\
&\qquad + E\bigg\|\int_{t_\alpha-\epsilon}^{t_\alpha} \big(P_\mu(t_\beta - s) - P_\mu(t_\alpha - s)\big) \\
&\qquad \times \int_0^s G\big(\tau,x(\tau),x\big(a_1(\tau)\big),\dots,x\big(a_k(\tau)\big)\big)\,d\omega(\tau)\,ds\bigg\|^2
\end{aligned}$$

$$
\begin{aligned}
&+E\left\|\int_{t_\alpha}^{t_\beta} P_\mu(t_\beta-s)\int_0^s G\bigl(\tau,x(\tau),x\bigl(a_1(\tau)\bigr),\dots,x\bigl(a_k(\tau)\bigr)\bigr)\,d\omega(\tau)\,ds\right\|^2\\
&+E\left\|\int_{s_i}^{t_\alpha-\epsilon}\bigl(P_\mu(t_\beta-s)-P_\mu(t_\alpha-s)\bigr)\sigma\bigl(s,x(s),x\bigl(c_1(s)\bigr),\dots,x\bigl(c_p(s)\bigr)\bigr)\,dB^H(s)\right\|^2\\
&+E\left\|\int_{t_\alpha-\epsilon}^{t_\alpha}\bigl(P_\mu(t_\beta-s)-P_\mu(t_\alpha-s)\bigr)\sigma\bigl(s,x(s),x\bigl(c_1(s)\bigr),\dots,x\bigl(c_p(s)\bigr)\bigr)\,dB^H(s)\right\|^2\\
&+E\left\|\int_{t_\alpha}^{t_\beta} P_\mu(t_\beta-s)\sigma\bigl(s,x(s),x\bigl(c_1(s)\bigr),\dots,x\bigl(c_p(s)\bigr)\bigr)\,dB^H(s)\right\|^2.
\end{aligned}
$$

We see that $E\|(\Phi_2 x)(t_\beta)-(\Phi_2 x)(t_\alpha)\|^2$ tends to zero independently of $x\in B_q$ as $t_\beta\to t_\alpha$, with ϵ sufficiently small since the compactness of $S_{\nu,\mu}(t)$ for $t>0$ (see [28]) implies the continuity in the uniform operator topology. Similarly, we can prove that the function $\Phi_2 x$, $x\in B_q$ are equicontinuous at $t=0$. Hence Φ_2 maps B_q into a family of equicontinuous functions.

It remains to prove that $V(t)=\{(\Phi_2 x)(t):x\in B_q\}$ is relatively compact in B_q. Obviously, by condition $(H3)$, $V(0)$ is relatively compact in B_q.

Let $s_i<t\le t_{i+1}$ be fixed, $s_i<\epsilon<t$, arbitrary $\rho>0$, for $x\in B_q$, we define

$$
\begin{aligned}
&\bigl(\Phi_2^{\epsilon,\rho}x\bigr)(t)\\
&\quad=\mu\int_{s_i}^{t-\epsilon}\int_\rho^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)Bu^\delta(s)\,d\theta\,ds\\
&\qquad+\mu\int_{s_i}^{t-\epsilon}\int_\rho^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)\\
&\qquad\times\int_0^s G(\tau,x(\tau),x\bigl(a_1(\tau)\bigr),\dots,x\bigl(a_k(\tau)\bigr)\,d\omega(\tau)\,d\theta\,ds\\
&\qquad+\mu\int_{s_i}^{t-\epsilon}\int_\rho^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)\\
&\qquad\times\sigma\bigl(s,x(s),x\bigl(c_1(s)\bigr),\dots,x\bigl(c_p(s)\bigr)\bigr)\,d\theta\,dB^H(s)\\
&\quad=\mu S\bigl(\epsilon^\mu\rho\bigr)\int_{s_i}^{t-\epsilon}\int_\rho^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta-\epsilon^\mu\rho\bigr)Bu^\delta(s)\,d\theta\,ds\\
&\qquad+\mu S\bigl(\epsilon^\mu\rho\bigr)\int_{s_i}^{t-\epsilon}\int_\rho^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta-\epsilon^\mu\rho\bigr)\\
&\qquad\times\int_0^s G(\tau,x(\tau),x\bigl(a_1(\tau)\bigr),\dots,x\bigl(a_k(\tau)\bigr)\,d\omega(\tau)\,d\theta\,ds\\
&\qquad+\mu S\bigl(\epsilon^\mu\rho\bigr)\int_{s_i}^{t-\epsilon}\int_\rho^\infty \theta(t-s)^{\mu-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta-\epsilon^\mu\rho\bigr)\\
&\qquad\times\sigma\bigl(s,x(s),x\bigl(c_1(s)\bigr),\dots,x\bigl(c_p(s)\bigr)\bigr)\,d\theta\,dB^H(s).
\end{aligned}
$$

Since $S(\epsilon^\mu\rho)$, $\epsilon^\mu\rho>0$ is a compact operator, the set $V^{\epsilon,\rho}(t)=\{(\Phi_2^{\epsilon,\rho}x)(t):x\in B_q\}$ is relatively compact in X for every ϵ, $s_i<\epsilon<t$ and for all $\rho>0$.

Moreover, for every $x \in B_q$, we have

$$
\begin{aligned}
&\left\| \Phi_2 x - \Phi_2^{\epsilon,\rho} x \right\|_{\bar{C}}^2 \\
&\quad \le 9 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} \biggl\{ \mu^2 E \left\| \int_{s_i}^t \int_0^\rho \theta (t-s)^{\mu-1} \Psi_\mu(\theta) S\bigl((t-s)^\mu \theta\bigr) B u^\delta(s)\, d\theta\, ds \right\|^2 \\
&\qquad + \mu^2 E \left\| \int_{t-\epsilon}^t \int_\rho^\infty \theta (t-s)^{\mu-1} \Psi_\mu(\theta) S\bigl((t-s)^\mu \theta\bigr) B u^\delta(s)\, d\theta\, ds \right\|^2 \\
&\qquad + \mu^2 E \biggl\| \int_{s_i}^t \int_0^\rho \theta (t-s)^{\mu-1} \Psi_\mu(\theta) S\bigl((t-s)^\mu \theta\bigr) \\
&\qquad \times \int_0^s G\bigl(\tau, x(\tau), x\bigl(a_1(\tau)\bigr), \ldots, x\bigl(a_k(\tau)\bigr)\bigr)\, d\omega(\tau)\, d\theta\, ds \biggr\|^2 \\
&\qquad + \mu^2 E \biggl\| \int_{t-\epsilon}^t \int_\rho^\infty \theta (t-s)^{\mu-1} \Psi_\mu(\theta) S\bigl((t-s)^\mu \theta\bigr) \\
&\qquad \times \int_0^s G\bigl(\tau, x(\tau), x\bigl(a_1(\tau)\bigr), \ldots, x\bigl(a_k(\tau)\bigr)\bigr)\, d\omega(\tau)\, d\theta\, ds \biggr\|^2 \\
&\qquad + \mu^2 E \biggl\| \int_{s_i}^t \int_0^\rho \theta (t-s)^{\mu-1} \Psi_\mu(\theta) S\bigl((t-s)^\mu \theta\bigr) \\
&\qquad \times \sigma\bigl(s, x(s), x\bigl(c_1(s)\bigr), \ldots, x\bigl(c_p(s)\bigr)\bigr)\, d\theta\, dB^H(s) \biggr\|^2 \\
&\qquad + \mu^2 E \biggl\| \int_{t-\epsilon}^t \int_0^\rho \theta (t-s)^{\mu-1} \Psi_\mu(\theta) S\bigl((t-s)^\mu \theta\bigr) \\
&\qquad \times \sigma\bigl(s, x(s), x\bigl(c_1(s)\bigr), \ldots, x\bigl(c_p(s)\bigr)\bigr)\, d\theta\, dB^H(s) \biggr\|^2 \biggr\} \\
&\quad \le 9 \biggl\{ T^{\mu+2(1-\nu)(1-\mu)} \mu M^2 M_B^2 \int_{s_i}^t (t-s)^{\mu-1} E \bigl\| u^\delta(s) \bigr\|^2 ds \left(\int_0^\rho \theta \Psi_\mu(\theta)\, d\theta \right)^2 \\
&\qquad + T^{2(1-\nu)(1-\mu)} \mu M^2 M_B^2 \epsilon^\mu \int_{t-\epsilon}^t (t-s)^{\mu-1} E \bigl\| u^\delta(s) \bigr\|^2 ds \left(\int_\rho^\infty \theta \Psi_\mu(\theta)\, d\theta \right)^2 \\
&\qquad + T^{\mu+2(1-\nu)(1-\mu)} \mu M^2 \operatorname{Tr}(Q) \int_{s_i}^t (t-s)^{\mu-1} \\
&\qquad \times \int_0^s E \bigl\| G\bigl(\tau, x(\tau), x\bigl(a_1(\tau)\bigr), \ldots, x\bigl(a_k(\tau)\bigr)\bigr) \bigr\|_Q^2\, d\tau\, ds \left(\int_0^\rho \theta \Psi_\mu(\theta)\, d\theta \right)^2 \\
&\qquad + T^{\mu+2(1-\nu)(1-\mu)} \mu M^2 \operatorname{Tr}(Q) \epsilon^\mu \int_{t-\epsilon}^t (t-s)^{\mu-1} \\
&\qquad \times \int_0^s E \bigl\| G\bigl(\tau, x(\tau), x\bigl(a_1(\tau)\bigr), \ldots, x\bigl(a_k(\tau)\bigr)\bigr) \bigr\|_Q^2\, d\tau\, ds \left(\int_\rho^\infty \theta \Psi_\mu(\theta)\, d\theta \right)^2 \\
&\qquad + 2T^{2H+\mu-1+2(1-\nu)(1-\mu)} \mu M^2 \\
&\qquad \times \int_{s_i}^t (t-s)^{\mu-1} E \bigl\| \sigma\bigl(s, x(s), x\bigl(c_1(s)\bigr), \ldots, x\bigl(c_p(s)\bigr)\bigr) \bigr\|_{L_2^0}^2\, ds \left(\int_0^\rho \theta \Psi_\mu(\theta)\, d\theta \right)^2 \\
&\qquad + 2T^{2H-1+2(1-\nu)(1-\mu)} \mu M^2 \epsilon^\mu \\
&\qquad \times \int_{t-\epsilon}^t (t-s)^{\mu-1} E \bigl\| \sigma\bigl(s, x(s), x\bigl(c_1(s)\bigr), \ldots, x\bigl(c_p(s)\bigr)\bigr) \bigr\|_{L_2^0}^2\, ds \left(\int_\rho^\infty \theta \Psi_\mu(\theta)\, d\theta \right)^2 \biggr\}.
\end{aligned}
$$

We see that, for each $x \in B_q$, $\|\Phi_2 x - \Phi_2^{\epsilon,\rho}\|_C^2 \to 0$ as $\epsilon \to 0^+$, $\rho \to 0^+$. Therefore, there are relative compact sets arbitrarily close to the set $V(t) = \{(\Phi_2 x)(t) : x \in B_q\}$, hence the set $V(t)$ is also relatively compact in B_q.

Thus, by the Ascoli–Arzela theorem Φ_2 is a compact operator. These arguments enable us to conclude that $\Phi_\delta = \Phi_1 + \Phi_2$ is a condensing map on B_q, and by the fixed point theorem of Sadovskii there exists a fixed point $x(\cdot)$ for Φ_δ on B_q. Therefore the system (1.1) has a mild solution. □

Theorem 3.2 *Assume that* $(H1)$–$(H5)$ *are satisfied. Furthermore, if the functions* F, G, *and* σ *are uniformly bounded, then the system* (1.1) *is approximately controllable on* J.

Proof Let x_δ be a fixed point of Φ_δ. By using the stochastic Fubini theorem, it can be easily seen that

$$\begin{aligned} x_\delta(T) = {}& \bar{x}_T - z\bigl(zI + \Gamma_0^T\bigr)^{-1}\biggl\{E\bar{x}_T - g_m\bigl(T, x(T)\bigr) \\ & + F\bigl(T, x(T), x\bigl(b_1(T)\bigr), \ldots, x\bigl(b_m(T)\bigr)\bigr) + \int_0^T \bar{\psi}(s)\,d\omega(s) + \int_0^T \bar{\varphi}(s)\,dB^H(s)\biggr\} \\ & - z\int_{s_m}^T \bigl(zI + \Gamma_0^T\bigr)^{-1} P_\mu(T-s) AF\bigl(s, x(s), x\bigl(b_1(s)\bigr), \ldots, x\bigl(b_m(s)\bigr)\bigr)\,ds \\ & + z\int_{s_m}^T \bigl(zI + \Gamma_0^T\bigr)^{-1} P_\mu(T-s) \int_0^s G\bigl(\tau, x(\tau), x\bigl(a_1(\tau)\bigr), \ldots, x\bigl(a_k(\tau)\bigr)\bigr)\,d\omega(\tau)\,ds \\ & + z\int_{s_m}^T \bigl(zI + \Gamma_0^T\bigr)^{-1} P_\mu(T-s)\sigma\bigl(s, x(s), x\bigl(c_1(s)\bigr), \ldots, x\bigl(c_p(s)\bigr)\bigr)\,dB^H(s). \end{aligned}$$

It follows from the assumption on F, G and σ that there exists $D > 0$ such that

$$\begin{aligned} & \bigl\|F\bigl(s, x_\delta(s), x_\delta\bigl(b_1(s)\bigr), \ldots, x_\delta\bigl(b_m(s)\bigr)\bigr)\bigr\|^2 \le D, \\ & \bigl\|G\bigl(s, x_\delta(s), x_\delta\bigl(a_1(s)\bigr), \ldots, x_\delta\bigl(a_k(s)\bigr)\bigr)\bigr\|^2 \le D, \\ & \bigl\|\sigma\bigl(s, x_\delta(s), x_\delta\bigl(c_1(s)\bigr), \ldots, x_\delta\bigl(c_p(s)\bigr)\bigr)\bigr\|^2 \le D. \end{aligned}$$

Consequently, the sequences $\{F(s, x_\delta(s), x_\delta(b_1(s)), \ldots, x_\delta(b_m(s)))\}$, $\{G(s, x_\delta(s), x_\delta(a_1(s)), \ldots, x_\delta(a_k(s)))\}$, $\{\sigma(s, x_\delta(s), x_\delta(c_1(s)), \ldots, x_\delta(c_p(s)))\}$ are weakly compact in $L_2(J, X)$, $L_2(L_Q(K, X))$ and $L_2(L_2^0(Y, X))$, so, there are subsequences, still denoted by $\{F(s, x_\delta(s), x_\delta(b_1(s)), \ldots, x_\delta(b_m(s)))\}$, $\{G(s, x_\delta(s), x_\delta(a_1(s)), \ldots, x_\delta(a_k(s)))\}$, $\{\sigma(s, x_\delta(s), x_\delta(c_1(s)), \ldots, x_\delta(c_p(s)))\}$, that are weakly converge to $\{F(s)\}$, $\{G(s)\}$, $\{\sigma(s)\}$ in $L_2(J, X)$, $L_2(L_Q(K, X))$ and $L_2(L_2^0(Y, X))$.

From the last equation, we have

$$\begin{aligned} & E\bigl\|x_\delta(T) - \bar{x}_T\bigr\|^2 \\ & \quad \le 9E\bigl\|z\bigl(zI + \Gamma_0^T\bigr)^{-1}\bigl\{E\bar{x}_T - g_m\bigl(T, x(T)\bigr)\bigr\}\bigr\|^2 \\ & \qquad + 9E\bigl\|z\bigl(zI + \Gamma_0^T\bigr)^{-1} F\bigl(T, x(T), x\bigl(b_1(T)\bigr), \ldots, x\bigl(b_m(T)\bigr)\bigr)\bigr\|^2 \\ & \qquad + 9E\biggl\|\int_0^T z\bigl(zI + \Gamma_0^T\bigr)^{-1}\bar{\psi}(s)\,d\omega(s)\biggr\|^2 \end{aligned}$$

$$
\begin{aligned}
&+ 9E\left\| \int_0^T z\big(zI + \Gamma_0^T\big)^{-1} \bar{\varphi}(s)\, dB^H(s) \right\|^2 \\
&+ 9E\left\| \int_{s_m}^T z\big(zI + \Gamma_0^T\big)^{-1} P_\mu(T-s) A\big(F\big(s, x(s), x\big(b_1(s)\big), \dots, x\big(b_m(s)\big)\big) - F(s)\big)\, ds \right\|^2 \\
&+ 9E\left\| \int_{s_m}^T z\big(zI + \Gamma_0^T\big)^{-1} P_\mu(T-s) A F(s)\, ds \right\|^2 \\
&+ 9E\left\| \int_{s_m}^T z\big(zI + \Gamma_0^T\big)^{-1} P_\mu(T-s) \right. \\
&\left. \times \int_0^s G\big(\tau, x(\tau), x\big(a_1(\tau)\big), \dots, x\big(a_k(\tau)\big)\big) - G(\tau)\, d\omega(\tau)\, ds \right\|^2 \\
&+ 9E\left\| \int_{s_m}^T z\big(zI + \Gamma_0^T\big)^{-1} P_\mu(T-s) \int_0^s G(\tau)\, d\omega(\tau)\, ds \right\|^2 \\
&+ 9E\left\| \int_{s_m}^T z\big(zI + \Gamma_0^T\big)^{-1} P_\mu(T-s) \sigma\big(s, x(s), x\big(c_1(s)\big), \dots, x\big(c_p(s)\big)\big) - \sigma(s)\, dB^H(s) \right\|^2 \\
&+ 9E\left\| \int_{s_m}^T z\big(zI + \Gamma_0^T\big)^{-1} P_\mu(T-s) \sigma(s)\, dB^H(s) \right\|^2.
\end{aligned}
$$

On the other hand, by Lemma 2.4, the operator $z(zI + \Gamma_0^T)^{-1} \to 0$ strongly as $z \to 0^+$ for all $s_m < s \le T$, and, moreover, $\|z(zI + \Gamma_0^T)^{-1}\| \le 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $P_\mu(t)$ implies that $E\|x_\delta(T) - \bar{x}_T\|^2 \to 0$ as $z \to 0^+$. This proves the approximate controllability of (1.1). □

4 Application

In this section, we present an example to illustrate our main result.

Let us consider the following stochastic control Hilfer fractional partial differential equation with fractional Brownian motion:

$$
\begin{cases}
D_{0+}^{\frac{1}{3},\frac{3}{5}}\big[x(t,z) + \int_0^\pi a(y,z)x(t,y)\, dy\big] \\
\quad = \frac{\partial^2}{\partial z^2} x(t,z) + \eta(t,z) + \int_0^t 3^{-s} x(s,z)\, d\omega(s) \\
\qquad + \frac{\sin t}{1+\sin t} x(t,z) \frac{dB^H(t)}{dt}, \quad t \in (0, \frac{1}{3}] \cup (\frac{2}{3}, 1], 0 \le z \le \pi, \\
x(t,0) = x(t,\pi) = 0, \quad t \in (0,1], \\
x(t,z) = \frac{1}{5} e^{-(t-\frac{1}{3})} \frac{\|x(t,z)\|}{1+\|x(t,z)\|}, \quad t \in (\frac{1}{3}, \frac{2}{3}], 0 \le z \le \pi, \\
I_{0+}^{\frac{4}{15}}(x(0,z)) + \sum_{i=1}^2 c_i x(t_i, z) = x_0(z), \quad 0 \le z \le \pi,
\end{cases}
\tag{4.1}
$$

where $D_{0+}^{\frac{1}{3},\frac{3}{5}}$ is Hilfer fractional derivative of order $\nu = \frac{1}{3}$, $\mu = \frac{3}{5}$, ω is a Wiener process and B^H is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$.

Let $X = Y = K = U = L_2([0,\pi])$ and A be defined by $Ay = -(\frac{\partial^2}{\partial z^2})y$ with domain $D(A) = \{y \in X : y, \frac{dy}{dz}$ are absolutely continuous, and $(\frac{d^2}{dz^2})y \in X, y(0) = y(\pi) = 0\}$.

Then $-A$ generates a strongly continuous semigroup $S(\cdot)$ which is compact, analytic, and self-adjoint. Furthermore, A has a discrete spectrum with eigenvalues n^2, $n \in N$ and the corresponding normalized eigenfunctions are given by

$$
e_n = \sqrt{\frac{2}{\pi}} \sin nx, \quad n = 1, 2, \dots.
$$

In addition $(e_n)_{n\in N}$ is a complete orthonormal basis in X. Then

$$-Ay = \sum_{n=1}^{\infty} n^2 \langle y, e_n\rangle e_n, \quad y \in D(A).$$

Furthermore, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operator, $\{S(t)\}_{t\geq 0}$ on X and is given by

$$S(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}\langle y, e_n\rangle_{e_n}, \quad y \in X, t \geq 0,$$

with $\|S(t)\| \leq e^{-t} \leq 1$.

Moreover, the two operators $S_{\frac{1}{3},\frac{3}{5}}(t)$ and $P_{\frac{3}{5}}(t)$ can be defined by

$$S_{\frac{1}{3},\frac{3}{5}}(t)x = \frac{3}{5\Gamma(\frac{2}{15})}\int_0^t \int_0^{\infty} \theta(t-s)^{\frac{-13}{15}} s^{\frac{-2}{5}} \Psi_{\frac{3}{5}}(\theta) S\bigl(s^{\frac{3}{5}}\theta\bigr)x \, d\theta \, ds,$$

$$P_{\frac{3}{5}}(t)x = \frac{3}{5}\int_0^{\infty} \theta t^{\frac{-2}{5}} \Psi_{\frac{3}{5}}(\theta) S\bigl(s^{\frac{3}{5}}\theta\bigr)x \, d\theta.$$

Clearly,

$$\bigl\|P_{\frac{3}{5}}(t)\bigr\| \leq \frac{t^{\frac{-2}{5}}}{\Gamma(\frac{3}{5})}, \qquad \bigl\|S_{\frac{1}{3},\frac{3}{5}}(t)\bigr\| \leq \frac{t^{\frac{-4}{15}}}{\Gamma(\frac{11}{15})}.$$

In order to define the operator $Q : Y \to Y$, we choose a sequence $\{\lambda_n\}_{n\in N} \subset R^+$, set $Qe_n = \lambda_n e_n$, and assume that

$$\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.$$

Define the fractional Brownian motion in Y by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta^H(t)e_n$$

where $H \in (\frac{1}{2}, 1)$ and $\{\beta_n^H\}_{n\in N}$ is a sequence of one-dimensional fractional Brownian motions mutually independent.

We also use the following properties:

(a) If $y \in D(A)$, then $Ay = \sum_{n=1}^{\infty} n^2\langle y, x_n\rangle x_n$.

(b) For each $y \in X$, $A^{-1/2}y = \sum_{n=1}^{\infty} \frac{1}{n}\langle y, x_n\rangle x_n$. In particular, $\|A^{-1/2}\|^2 = 1$.

(c) The operator $A^{1/2}$ is given by $A^{1/2}y = \sum_{n=1}^{\infty} n\langle y, x_n\rangle x_n$ on the space $D[A^{1/2}] = \{y(\cdot) \in X, \sum_{n=1}^{\infty} n\langle y, x_n\rangle x_n \in X\}$.

We assume the following conditions hold:

(i) The function a is measurable and

$$\int_0^{\pi}\int_0^{\pi} a^2(y,z)\, dy\, dz < \infty.$$

(*ii*) The function $\frac{\partial}{\partial z}a(y,z)$ is measurable, $a(y,0) = a(y,\pi) = 0$, and let

$$N_1 = \left[\int_0^\pi \int_0^\pi \left(\frac{\partial}{\partial z}a(y,z)\right)^2 dy\,dz\right]^{1/2} < \infty.$$

Define the bounded operator $B : U \to X$ by $Bu(t)(z) = \eta(t,z)$, $0 \le z \le \pi$, $u \in U$.

We define $F : (0,1] \times X \to X$, $G : (0,1] \times X \to L(K,X)$, $\sigma : (0,1] \times X \to L_2^0(Y,X)$, $g_i : (t_i,s_i] \times X \to X$ and $\xi : C((0,1],X) \to X$ by $F(t,x) = Z_1(x)$, $G(s,x)(z) = 3^{-s}x(s,z)$, $\sigma(t,x)(z) = \frac{\sin t}{1+\sin t}x(t,z)$, $g_1 = \frac{1}{5}e^{-(t-\frac{1}{3})}\frac{\|x(t,z)\|}{1+\|x(t,z)\|}$ and $\xi = \sum_{i=1}^{2} c_i x(t_i,z)$, respectively, where

$$Z_1(x)(z) = \int_0^\pi a(y,z)x(y)\,dy.$$

Then G, σ, g_1 and ξ satisfy $(H2)$–$(H5)$. From (i) it is clear that Z_1 is a bounded linear operator on X. Furthermore, $Z_1(x) \in D[A^{1/2}]$, and $\|A^{1/2}Z_1\|^2 \le N_1$. In fact, from the definition of Z_1 and (ii) it follows that

$$\langle Z_1(x), x_n\rangle = \int_0^\pi x_n(z)\left[\int_0^\pi a(y,z)x(y)\,dy\right]dz = \frac{1}{n}\left(\frac{2}{\pi}\right)^{1/2}\langle Z_2(x), \cos(nx)\rangle,$$

where Z_2 is defined by

$$Z_2(x)(z) = \int_0^\pi \frac{\partial}{\partial z}a(y,z)x(y)\,dy.$$

From (ii) we know that $Z_2 : X \to X$ is a bounded linear operator with $\|Z_2\|^2 \le N_1$.

Hence $\|A^{1/2}Z_1(x)\|^2 = \|Z_2(x)\|^2$.

If $u \in L_2((0,1],U)$, then $B = I$, $B^* = I$.

Therefore, with the above choice, the system (4.1) can be written in the abstract form of (1.1). On the other hand the linear system corresponding to (4.1) is approximately controllable. Therefore, all the hypotheses of Theorem 3.1 and Theorem 3.2 are satisfied and

$$\begin{aligned}
&\left[1 + \frac{M^4T^{2\mu}M_B^4}{z^2\mu^2\Gamma^4(\mu)}\right]\left\{\frac{36M^2(M_0^2M_2 + M_3 + M_4)}{\Gamma^2(\nu(1-\mu)+\mu)}\right. \\
&\quad + 36T^{2(1-\nu)(1-\mu)}\left[{M_0}^2M_2 + \frac{M^2T^\mu\Lambda_1\operatorname{Tr}(Q)}{\mu\Gamma^2(\mu)}\right. \\
&\quad \left.\left. + \frac{2HM^2\Lambda_2T^{2H+\mu-1}}{\mu\Gamma^2(\mu)} + \frac{(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta}M_2}{\beta^2\Gamma^2(1+\mu\beta)}\right]\right\} + T^{2(1-\nu)(1-\mu)}M_3 < 1
\end{aligned}$$

and

$$\begin{aligned}
\gamma_1 = 9&\left[\frac{M^2({M_0}^2M_1 + M_6 + M_7)}{\Gamma^2(\nu(1-\mu)+\mu)} + T^{2(1-\nu)(1-\mu)}\left(M_6 + {M_0}^2M_1\right)\right. \\
&\left. + \frac{M_1(C_{1-\beta})^2\Gamma^2(1+\beta)T^{2\mu\beta+2(1-\nu)(1-\mu)}}{\beta^2\Gamma^2(1+\mu\beta)}\right] \\
&< 1.
\end{aligned}$$

Thus, we can conclude that the Hilfer fractional stochastic partial differential equation with fractional Brownian motion and nonlocal conditions (4.1) is approximately controllable on $(0, 1]$.

5 Conclusion

In this paper, by using fractional calculus, stochastic analysis, the fractional power of operators and the Sadovskii fixed point theorem, we obtained sufficient conditions for the approximate controllability of a class of noninstantaneous impulsive Hilfer fractional stochastic integrodifferential equations with fractional Brownian motion and nonlocal conditions. Also, we provided an example to illustrate our results. In the future we aim to study the existence of mild solutions and controllability of a class of Sobolev-type nonlinear Hilfer fractional stochastic differential inclusions with noninstantaneous impulsive in Hilbert space.

Acknowledgements
We would like to thank the referees and the editor for their important comments and suggestions, which have significantly improved the paper.

Funding
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the manuscript.

Author details
[1]Higher Institute of Engineering, El-Shorouk Academy, El-Shorouk City, Cairo, Egypt. [2]Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt. [3]Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt. [4]Department of Mathematics, Faculty of Science, Islamic University in Madinah, Medina, Saudi Arabia.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 27 June 2019 Accepted: 22 June 2020 Published online: 06 July 2020

References

1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
2. Wang, J.R., Feckan, M., Zhou, Y.: A survey on impulsive fractional differential equations. Fract. Calc. Appl. Anal. **19**(4), 806–831 (2016)
3. Polidoro, S., Ragusa, M.A.: Harnack inequality for hypoelliptic ultraparabolic equations with a singular lower order term. Rev. Mat. Iberoam. **24**(3), 1011–1046 (2008)
4. Boufoussi, B., Hajji, S.: Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. Stat. Probab. Lett. **82**, 1549–1558 (2012)
5. Diop, M.A., Ezzinbi, K., Mbaye, M.M.: Existence and global attractiveness of a pseudo almost periodic solution in p-th mean sense for stochastic evolution equation driven by a fractional Brownian motion. Stochastics **87**, 1061–1093 (2015)
6. Arthi, G., Park, J.H., Jung, H.Y.: Existence and exponential stability for neutral stochastic integrodifferential equations with impulses driven by a fractional Brownian motion. Commun. Nonlinear Sci. Numer. Simul. **32**, 145–157 (2016)
7. Boudaoui, A., Caraballo, T., Ouahab, A.: Impulsive neutral functional differential equations driven by a fractional Brownian motion with unbounded delay. Appl. Anal. **95**, 2039–2062 (2016)
8. Tamilalagan, P., Balasubramaniam, P.: Moment stability via resolvent operators of fractional stochastic differential inclusions driven by fractional Brownian motion. Appl. Math. Comput. **305**, 299–307 (2017)
9. Ren, Y., Wang, J., Hu, L.: Multi-valued stochastic differential equations driven by G-Brownian motion and related stochastic control problems. Int. J. Control **90**, 1132–1154 (2017)
10. Luan, N.N.: Chung's law of the iterated logarithm for subfractional Brownian motion. Acta Math. Sin. Engl. Ser. **33**(6), 839–850 (2017)
11. Ballinger, G., Liu, X.: Boundedness for impulsive delay differential equations and applications in populations growth models. Nonlinear Anal. **53**, 1041–1062 (2003)

12. Hernández, E., O'Regan, D.: On a new class of abstract impulsive differential equations. Proc. Am. Math. Soc. **141**, 1641–1649 (2013)
13. Pierri, M., O'Regan, D., Rolnik, V.: Existence of solutions for semi-linear abstract differential equations with non instantaneous impulses. Appl. Math. Comput. **219**, 6743–6749 (2013)
14. Pierri, M., Henriquez, H.R., Prokopczyk, A.: Global solutions for abstract differential equations with non-instantaneous impulses. Mediterr. J. Math. **13**, 1685–1708 (2016)
15. Ahmed, H.M.: Controllability of impulsive neutral stochastic differential equations with fractional Brownian motion. IMA J. Math. Control Inf. **32**, 781–794 (2015)
16. Kumar, A., Muslim, M., Sakthivel, R.: Controllability of the second-order nonlinear differential equations with non-instantaneous impulses. J. Dyn. Control Syst. **24**, 325–342 (2018)
17. Balachandran, K., Sakthivel, R.: Controllability of integrodifferential systems in Banach spaces. Appl. Math. Comput. **118**, 63–71 (2001)
18. Ahmed, H.M., El-Borai, M.M., El-Owaidy, H.M., Ghanem, A.S.: Impulsive Hilfer fractional differential equations. Adv. Differ. Equ. **2018**, 226, 1–20 (2018)
19. Sakthivel, R., Ganesh, R., Ren, Y., Anthoni, S.M.: Approximate controllability of nonlinear fractional dynamical systems. Commun. Nonlinear Sci. Numer. Simul. **18**, 3498–3508 (2013)
20. Sakthivel, R., Ganesh, R., Anthoni, S.M.: Approximate controllability of fractional nonlinear differential inclusions. Appl. Math. Comput. **225**, 708–717 (2013)
21. Sakthivel, R., Ren, Y., Debbouche, A., Mahmudov, N.I.: Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions. Appl. Anal. **95**, 2361–2382 (2016)
22. Debbouche, A., Torres, D.F.M.: Approximate controllability of fractional delay dynamic inclusions with nonlocal control conditions. Appl. Math. Comput. **243**, 161–175 (2014)
23. Ahmed, H.M.: Approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space. Adv. Differ. Equ. **2014**, 113, 1–11 (2014)
24. Muthukumar, P., Rajivganthi, C.: Approximate controllability of second-order neutral stochastic differential equations with infinite delay and Poisson jumps. J. Syst. Sci. Complex. **28**, 1033–1048 (2015)
25. Yan, Z., Jia, X.: Approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-dependent delay. Collect. Math. **66**, 93–124 (2015)
26. Yana, Z., Lu, F.: Approximate controllability of a multi-valued fractional impulsive stochastic partial integro-differential equation with infinite delay. Appl. Math. Comput. **292**, 425–447 (2017)
27. Debbouche, A., Antonov, V.: Approximate controllability of semilinear Hilfer fractional differential inclusions with impulsive control inclusion conditions in Banach spaces. Chaos Solitons Fractals **102**, 140–148 (2017)
28. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
29. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
30. Mandelbrot, B.B., Ness, J.W.V.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. **10**, 422–437 (1968)
31. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, vol. 44. Springer, New York (1983)
32. Gu, H., Trujillo, J.J.: Existence of mild solution for evolution equation with Hilfer fractional derivative. Appl. Math. Comput. **257**, 344–354 (2015)
33. Marle, C.M.: Measures et Probabilités. Hermann, Paris (1974)