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Existence of infinitely many solutions for a p-Kirchhoff problem in RN

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Abstract

We consider the existence of multiple solutions of the following singular nonlocal elliptic problem:

$$\begin{cases} -M(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p)\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)=h(x)|u|^{r-2}u+H(x)|u|^{q-2}u, \\ u(x)\to 0 \quad \text{as } |x|\to\infty, \end{cases}$$

where $x\in\mathbb{R}^N$, and $M(t)=\alpha+\beta t$. By the variational method we prove that the problem has infinitely many solutions when some conditions are fulfilled.

Keywords: Singular elliptic problem; Variational methods; Palais–Smale condition

1 Introduction and main results

In this paper, we consider the existence of multiple solutions for the following singular elliptic problem:

$$\begin{cases} -M(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p)\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) \\ \quad =h(x)|u|^{r-2}u+H(x)|u|^{q-2}u, \quad x\in\mathbb{R}^N, \\ u(x)\to 0 \quad \text{as } |x|\to\infty, \end{cases} \tag{1.1}$$

where $M(t)=\alpha+\beta t$ with parameters $\alpha,\beta>0$, $a<\frac{N-p}{p}$, $h(x)$ and $H(x)$ are nonnegative functions in $\mathbb{R}^N$, and further assumptions will be listed later. It is well known that problem like (1.1) originally comes from Kirchhoff's important work [1], and Kirchhoff-type equations received much attention only after the paper by Lions [2]; see [3–8] and the references therein. Zhang and Perera [6] considered the following problem:

$$\begin{cases} -(a+b\int_\Omega|\nabla u|^2\,dx)\Delta u=f(x,u), \quad x\in\Omega, \\ u(x)=0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^N$. By the variational method the authors proved that problem (1.2) has a positive solution, a negative solution, and a sign-changing solution.

We note that the function $f(x,t)$ in (1.2) is required to meet the condition

$$uf(x,u) \geq \nu F(x,u), \quad \nu > 4, \forall x \in \mathbb{R}^N, \forall u \in \mathbb{R}. \tag{1.3}$$

Condition like (1.3) also appears in [9]. In our paper, $f(x,u) = h(x)|u|^{r-2}u + H(x)|u|^{q-2}u$. If $r < p < q$ or $\min\{r,q\} < p$, then $f(x,u)$ does not satisfy (1.3) even for $p = 2$. Therefore the methods applied in [6] cannot be simply extended to p-Kirchhoff problem (1.1). In [10] the authors considered the following Kirchhoff-type equation in $\mathbb{R}^3$:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx)\Delta u + V(x)u = f(x,u), \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{1.4}$$

By applying the symmetric mountain pass theorem, the authors proved the existence of infinitely many high-energy solutions for (1.4). The authors in [11] studied the following superlinear Kirchhoff equation:

$$\begin{cases} -(a\int_{\mathbb{R}^N} |\nabla u|^2\,dx + b)\Delta u + \lambda V(x)u = f(x,u), \\ u(x) \in H^1(\mathbb{R}^N). \end{cases} \tag{1.5}$$

Under some new superlinear hypotheses on $f(x,u)$, the authors proved the existence and nonexistence of solutions for (1.5). The results show that the existence of solutions is closely related to the parameters λ and a. Particularly, the authors proved that one solution blows up as the nonlocal term vanishes.

In recent years, Kirchhoff-type equations with p-Laplacian operator have been an interesting topic; see [12–16]. When $a = 0$, in [16] the authors studied the following critical Kirchhoff problem with p-Laplacian on a bounded domain:

$$\begin{cases} -[M(\int_{\Omega} |\nabla u|^p\,dx)]^{p-1}\Delta_p u = f(x,u), \quad x \in \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

By the genus theorem the authors proved the existence of multiple solutions of (1.6). The function $f(x,u)$ in (1.6) satisfies the following condition:

(f_1) there exist constants Q_1 and Q_2 such that

$$Q_1 t^{q-1} \leq f(x,t) \leq Q_2 t^{q-1} \tag{1.7}$$

for all $t \geq 0$ and $x \in \overline{\Omega}$, where $q \in (p, p^* = Np/(N-p))$. In the present paper, however, it is not difficult to check that when $\max\{r,q\} < p$, the function $f(x,u)$ in (1.1) does not satisfy condition (1.7). Thus problem (1.6) does not include our problem (1.1). Chen and Chen [17] considered a class of more general Kirchhof-type equations

$$\begin{aligned} &\left(a + \mu\left(\int_{\mathbb{R}^N} \left(|\nabla u|^p + V(x)|u|^p\right)dx\right)^{\tau}\right)\left(-\Delta_p u + V(x)|u|^{p-2}u\right) \\ &\quad = f(x,u), \quad x \in \mathbb{R}^N, \end{aligned} \tag{1.8}$$

where $f(x,u) = \lambda_1 h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u$. Under appropriate assumptions, the authors proved that there exists $\lambda_0 > 0$ such that (1.8) has infinitely many high-energy solutions for $\lambda \in [0,\lambda_0)$. Particularly, when $\beta = 0$ and $\alpha = 1$, problems like (1.1) reduce to elliptic equation without nonlocal term. This class of problems have been investigated by many authors; we refer to [18–21] and the references therein.

Motivated by the references mentioned, we consider the existence of multiple solutions of singular problem (1.1) by variational methods and the genus theorem. To our best knowledge, there are few results on singular problem (1.1). We prove that problem (1.1) has infinitely many solutions when some certain conditions are fulfilled. Note that our problem (1.1) is considered in the whole space $\mathbb{R}^N$; the loss of compactness of the Sobolev embedding renders the variational technique more delicate.

The natural space in this paper is the weighted Sobolev space $X = D_a^{1,p}(\mathbb{R}^N)$, which is the completion of the space $C_0^\infty(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

The following weighted Sobolev–Hardy inequality is due to Caffarelli et al. [22], which is called the Caffarelli–Kohn–Nirenberg inequality: There exist constants $S_1, S_2 > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}\,dx\right)^{p/p^*} \le S_1 \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{1.9}$$

and

$$\int_{\mathbb{R}^N} |x|^{-(a+1)p}|u|^p\,dx \le S_2 \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{1.10}$$

where $-\infty < a < (N-p)/p$, $p^* = pN/(N-pd)$, $d = a+1-b$, and $a \le b < a+1$.

In this paper, we make the following assumptions:

(A_1) for $1 < r < p$, $h(x) \in L^\infty(\mathbb{R}^N) \cap L^{r_1}(\mathbb{R}^N, g_1)$ with $r_1 = \frac{p}{p-r}$, $g_1 = |x|^{(a+1)rr_1}$;

(A_2) for $1 < p \le r < 2p$, $h(x) \in L^\infty(\mathbb{R}^N) \cap L^{r_2}(\mathbb{R}^N, g_2)$ with $r_2 = \frac{p^*}{p^*-r}$, $g_2 = |x|^{brr_2}$;

(A_3) for $1 < p \le q < 2p$, $H(x) \in L^\infty(\mathbb{R}^N) \cap L^{q_1}(\mathbb{R}^N, f_1)$ with $q_1 = \frac{p^*}{p^*-q}$, $f_1 = |x|^{bqq_1}$;

(A_4) for $1 < q < p$, $H(x) \in L^\infty(\mathbb{R}^N) \cap L^{q_2}(\mathbb{R}^N, f_2)$ with $q_2 = \frac{p}{p-q}$, $f_2 = |x|^{(a+1)qq_2}$.

Here the space $L^\infty(\mathbb{R}^N)$ consists of all functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $|u|$ is bounded on $\mathbb{R}^N \setminus \Omega$ for some $\Omega \subset \mathbb{R}^N$ of Lebesgue measure zero with norm $\|u\|_\infty = \sup_{\mathbb{R}^N} |u|$, and the space $L^p(\mathbb{R}^N)$ with $1 \le p < \infty$ consists of all functions $u : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}^N} |u|^p\,dx < \infty$ and $\|u\|_{L^p(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |u|^p\,dx)^{1/p}$. Now we give the definition of weak solution for problem (1.1).

Definition 1.1 A function $u \in X$ is said to be a weak solution of problem (1.1) if for any $\varphi \in X$, we have

$$\begin{aligned}&\left(\alpha + \beta\|u\|_X^p\right)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx\\&\quad = \int_{\mathbb{R}^N} h(x)|u|^{r-2}u\varphi\,dx + \int_{\mathbb{R}^N} H(x)|u|^{q-2}u\varphi\,dx.\end{aligned} \tag{1.11}$$

Assumptions (A_1)–(A_4) mean that all the integrals in (1.11) are well defined and converge.

Our main result is the following:

Theorem 1 *Assume that one of the following cases holds*:

(i) (A_1) *and* (A_3);

(ii) (A_1) *and* (A_4);

(iii) (A_2) *and* (A_4).

Then problem (1.1) *has infinitely many solutions. Particularly, if* (A_2) *and* (A_3) *hold and* $\alpha > 0$ *is small enough, then problem* (1.1) *also has infinitely many solutions.*

This paper is organized as follows. In Sect. 2, we give some basic definitions and set up the variational framework. Particularly, we prove some compact embedding theorems. In Sect. 3, by the variational methods and the genus theorem we consider the multiplicity results and prove Theorem 1.

2 Preliminary results

It is clear that problem (1.1) has a variational structure. Let $J(u) : X \to \mathbb{R}^1$ be the corresponding Euler functional of problem (1.1) defined by

$$J(u) = \frac{1}{p}\hat{M}\left(\|u\|_X^p\right) - \frac{1}{r}\int_{\mathbb{R}^N} h(x)|u|^r\,dx - \frac{1}{q}\int_{\mathbb{R}^N} H(x)|u|^q\,dx, \tag{2.1}$$

where $\hat{M}(t) = \int_0^t M(s)\,ds$. Then $J(u) \in C^1(X, \mathbb{R}^1)$, and for any $\varphi \in X$, we have

$$\begin{aligned}\langle J'(u), \varphi\rangle = {} & M\left(\|u\|_X^p\right)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx - \int_{\mathbb{R}^N} h(x)|u|^{r-2}u\varphi\,dx \\ & - \int_{\mathbb{R}^N} H(x)|u|^{q-2}u\varphi\,dx.\end{aligned} \tag{2.2}$$

Particularly, we have

$$\langle J'(u), u\rangle = M\left(\|u\|_X^p\right)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx - \int_{\mathbb{R}^N} h(x)|u|^r\,dx - \int_{\mathbb{R}^N} H(x)|u|^q\,dx. \tag{2.3}$$

It is well known that the weak solution of problem (1.1) is the critical point of $J(u)$. Thus, to prove the existence of infinitely many weak solutions for problem (1.1), it is sufficient to show that $J(u)$ admits a sequence of critical points. Our proof is based on the variational method, and one important aspect of applying this method is showing that the functional $J(u)$ satisfies the condition $(PS)_c$, which is introduced in the following definition.

Definition 2.1 Let $c \in \mathbb{R}^1$, and let X be a Banach space. The functional $J(u) \in C^1(X, \mathbb{R})$ satisfies the condition $(PS)_c$ if any $\{u_n\} \subset X$ such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in } X^* \quad \text{as } n \to \infty \tag{2.4}$$

contains a convergent subsequence in X.

The following embedding theorem is an extension of the classical Rellich–Kondrachov compactness theorem, see [23].

Lemma 2.1 *Suppose $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, and let $1 < p < N$ and $a < (N-p)/p$. Then the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is continuous if $1 \le r \le Np/(N-p)$ and $0 \le \alpha \le (1+a)r + N(1-r/p)$ and is compact if $1 \le r < Np/(N-p)$ and $0 \le \alpha < (1+a)r + N(1-r/p)$.*

We further give some embedding theorems, which play an important role in the paper.

Lemma 2.2 *Assume (A_3) or (A_4). Then the embedding $X \hookrightarrow L^q(\mathbb{R}^N, H)$ is compact.*

Proof We divide the proof into two cases.

Case 1. $1 < p \le q < p^*$.

Let

$$\mu_1 = \frac{p^*}{q}, \qquad \mu_2 = q_1 = \frac{p^*}{p^*-q}.$$

Thus $\mu_1, \mu_2 > 1$ and $\frac{1}{\mu_1} + \frac{1}{\mu_2} = 1$. From (1.9) and the Hölder inequality it follows that

$$\begin{aligned}
\|u\|^q_{L^q(\mathbb{R}^N, H(x))} &= \int_{\mathbb{R}^N} H(x)|x|^{bq}|u|^q|x|^{-bq}\,dx \\
&\le \left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}\,dx\right)^{\frac{q}{p^*}} \left(\int_{\mathbb{R}^N} H(x)^{q_1}|x|^{bqq_1}\,dx\right)^{\frac{1}{q_1}} \\
&\le S_1^{\frac{q}{p}} \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx\right)^{\frac{q}{p}} \left(\int_{\mathbb{R}^N} H\Bigl(x\Bigr) q_1 |x|^{bqq_1}\,dx\right)^{\frac{1}{q_1}} \\
&\le S_1^{\frac{q}{p}} \|u\|_X^q \bigl\|H(x)\bigr\|_{L^{q_1}(\mathbb{R}^N, f_1)},
\end{aligned} \tag{2.5}$$

where q_1 and f_1 are defined in (A_3). Then (2.5) yields that the embedding is continuous. Next, we will prove that the embedding is compact. Let B_R be the ball with center at the origin and radius $R > 0$. Denote $L^q(\Omega, H(x))$ by $Y(\Omega)$. Then $Y(\mathbb{R}^N) = L^q(\mathbb{R}^N, H(x))$. Let $\{u_n\}$ be a bounded sequence in X. Then $\{u_n\}$ is bounded in $X(B_R)$, where $X(B_R)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|\cdot\| = \left(\int_{B_R} |x|^{-ap}|\nabla \cdot|^p\,dx\right)^{\frac{1}{p}}.$$

We choose $\alpha = 0$ in Lemma 2.1. Then there exist $u \in Y(B_R)$ and a subsequence, still denoted by $\{u_n\}$, such that $\|u_n - u\|_{Y(B_R)} \to 0$ as $n \to \infty$. We claim that

$$\lim_{R\to\infty} \sup_{u \in X\setminus\{0\}} \frac{\|u\|_{Y(B_R^c)}}{\|u\|_X} = 0, \tag{2.6}$$

where $B_R^c = \mathbb{R}^N \setminus B_R$. In fact, from (2.5) we obtain that

$$\|u\|^q_{Y(B_R^c)} \le S_1^{\frac{q}{p}} \|u\|_X^q \|H\|_{L^{q_1}(B_R^c, f_1)}. \tag{2.7}$$

From (A_3) it follows that

$$\lim_{R\to\infty}\int_{B_R^c} H^{q_1} f_1\,dx = 0. \tag{2.8}$$

Then (2.7) and (2.8) imply (2.6). Since X is a separable Banach space and $\{u_n\}$ is bounded in X, we may assume, up to a subsequence, that

$$u_n \rightharpoonup u \quad \text{in } X. \tag{2.9}$$

In view of (2.6), we get that for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ large enough such that

$$\|u_n\|_{Y(B_{R_\varepsilon}^c)} \le \varepsilon \|u_n\|_X \quad (n = 1, 2, \ldots). \tag{2.10}$$

On the other hand, due to the compact embedding $X(B_{R_\varepsilon}) \hookrightarrow Y(B_{R_\varepsilon})$ in Lemma 2.1, we have that

$$\lim_{n\to\infty}\|u_n - u\|_{Y(B_{R_\varepsilon})} = 0. \tag{2.11}$$

Therefore there is $N_0 \in \mathbb{N}$ such that

$$\|u_n - u\|_{Y(B_{R_\varepsilon})} < \varepsilon \tag{2.12}$$

for $n > N_0$. Then from (2.10) and (2.12) it follows that

$$\|u_n - u\|_Y \le \|u_n - u\|_{Y(B_{R_\varepsilon})} + \|u_n\|_{Y(B_{R_\varepsilon}^c)} + \|u\|_{Y(B_{R_\varepsilon}^c)} \le \bigl(1 + \|u_n\|_X + \|u\|_X\bigr)\varepsilon, \tag{2.13}$$

which implies that $u_n \to u$ strongly in $Y(\mathbb{R}^N)$.

Case 2. $1 < q < p$.

By (1.10) and (A_4) we get that

$$\begin{aligned}
\|u\|_{L^q(\mathbb{R}^N,H)}^q &= \int_{\mathbb{R}^N} H(x)|u|^q\,dx \\
&\le \left(\int_{\mathbb{R}^N} |x|^{-(a+1)p}|u|^p\,dx\right)^{\frac{q}{p}} \left(\int_{\mathbb{R}^N} |x|^{-(a+1)qq_2} H^{q_2}(x)\,dx\right)^{\frac{1}{q_2}} \\
&\le S_2^{\frac{q}{p}} \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx\right)^{\frac{q}{p}} \|H\|_{L^{q_2}(\mathbb{R}^N,f_2)},
\end{aligned} \tag{2.14}$$

which implies that the embedding $X \hookrightarrow L^q(\mathbb{R}^N, H)$ is continuous. Furthermore, proceeding in a similar manner to Case 1, we can also prove that the embedding is compact. □

In an analogous manner, we can prove the following result.

Lemma 2.3 *Assume (A_1) or (A_2). Then the embedding $X \hookrightarrow L^r(\mathbb{R}^N, h)$ is compact.*

We now prove that $J(u)$ satisfies the condition $(PS)_c$.

Lemma 2.4 *Assume that the hypotheses in Theorem 1 hold. Then $J(u)$ satisfies the condition $(PS)_c$ for any $c \in \mathbb{R}$.*

Proof Let $\{u_n\} \subset X$ be a $(PS)_c$ sequence such that (2.4) holds. We divide the proof into four cases. We only prove Case 1,

Case 1. $1 < \max\{r, q\} < p$.

Firstly, we prove that $\{u_n\}$ is bounded in X. Choosing $\theta > 2p$, from (A_1), (A_4), and (2.4) it follows that for large n,

$$\begin{aligned}
&1 + c + \|u_n\|_X \\
&\quad\geq J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle = \left(\frac{1}{p} - \frac{1}{\theta}\right)\alpha\|u_n\|_X^p + \beta\left(\frac{1}{2p} - \frac{1}{\theta}\right)\|u_n\|_X^{2p} \\
&\qquad - \left(\frac{1}{r} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} h(x)|u_n|^r\,dx - \left(\frac{1}{q} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} H(x)|u_n|^q\,dx \\
&\quad\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\alpha\|u_n\|_X^p + \beta\left(\frac{1}{2p} - \frac{1}{\theta}\right)\|u_n\|_X^{2p} - \left(\frac{1}{r} - \frac{1}{\theta}\right)S_2^{\frac{r}{p}}\|u_n\|_X^r\|h\|_{L^{r_1}(\mathbb{R}^N, g_1)} \\
&\qquad - \left(\frac{1}{q} - \frac{1}{\theta}\right)S_2^{\frac{q}{p}}\|u_n\|_X^q\|H\|_{L^{q_2}(\mathbb{R}^N, f_2)},
\end{aligned} \tag{2.15}$$

which means that $\{\|u_n\|_X\}$ is bounded.

Secondly, we prove that $\{u_n\}$ converges strongly in X. Since X is a separable Banach space, there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ in X. The compact embeddings in Lemmas 2.2 and 2.3 give that

$$\begin{aligned}
&\int_{\mathbb{R}^N} h(x)|u_n|^r\,dx \to \int_{\mathbb{R}^N} h(x)|u_0|^r\,dx, \\
&\int_{\mathbb{R}^N} H(x)|u_n|^q\,dx \to \int_{\mathbb{R}^N} H(x)|u_0|^q\,dx.
\end{aligned} \tag{2.16}$$

Furthermore, the Brezis–Leib lemma shows that

$$\int_{\mathbb{R}^N} h(x)|u_n - u_0|^r\,dx \to 0, \qquad \int_{\mathbb{R}^N} H(x)|u_n - u_0|^q\,dx \to 0. \tag{2.17}$$

Then from the Hölder inequality and (2.17) it follows that

$$\int_{\mathbb{R}^N} h(x)|u_n|^{r-2}u_n|u_n - u_0|\,dx \to 0, \qquad \int_{\mathbb{R}^N} H(x)|u_n|^{q-2}u_n|u_n - u_0|\,dx \to 0. \tag{2.18}$$

Let $\varphi = u_n - u_0$ in (2.2). Then

$$\begin{aligned}
&\langle J'(u_n), u_n - u_0\rangle \\
&\quad= \left(\alpha + \beta\|u_n\|_X^p\right)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u_0)\,dx \\
&\qquad - \int_{\mathbb{R}^N} h(x)|u_n|^{r-2}u_n(u_n - u_0)\,dx - \int_{\mathbb{R}^N} H(x)|u_n|^{q-2}u_n(u_n - u_0)\,dx.
\end{aligned} \tag{2.19}$$

Note that $J'(u_n) \to 0$. Therefore from (2.18) and (2.19) we get that

$$\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u_0)\, dx \to 0 \quad \text{as } n \to +\infty. \tag{2.20}$$

On the other hand, from the weak convergence $u_n \rightharpoonup u_0$ it follows that

$$\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla (u_n - u_0)\, dx \to 0 \quad \text{as } n \to +\infty. \tag{2.21}$$

Consequently, relations (2.20) and (2.21) yield that

$$\int_{\mathbb{R}^N} |x|^{-ap} \bigl(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_0|^{p-2} \nabla u_0\bigr) \cdot \nabla (u_n - u_0)\, dx \to 0. \tag{2.22}$$

Furthermore, by the standard inequalities (see [24])

$$|\xi - \zeta|^p \le \begin{cases} c\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta\rangle & \text{for } p \ge 2, \\ c\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta\rangle^{p/2} (|\xi|^p + |\zeta|^p)^{(2-p)/2} & \text{for } 1 < p < 2, \end{cases}$$

we obtain that

$$\int_{\mathbb{R}^N} |x|^{-ap} \bigl|\nabla (u_n - u_0)\bigr|^p\, dx \to 0, \tag{2.23}$$

that is, $u_n \to u_0$ strongly in X.

Case 2. $1 < p < \min\{r, q\}$ and $\max\{r, q\} < 2p$.

We choose $\theta = 2p$. Since $r, q < 2p$, from (A_2) and (A_3) it follows that

$$\begin{aligned} 1 + c + \|u_n\|_X &\ge J(u_n) - \frac{1}{\theta}\bigl\langle J'(u_n), u_n\bigr\rangle \\ &= \left(\frac{1}{p} - \frac{1}{2p}\right)\alpha \|u_n\|_X^p - \left(\frac{1}{r} - \frac{1}{2p}\right)\int_{\mathbb{R}^N} h(x)|u_n|^r\, dx \\ &\quad - \left(\frac{1}{q} - \frac{1}{2p}\right)\int_{\mathbb{R}^N} H(x)|u_n|^q\, dx \\ &\ge \left(\frac{1}{p} - \frac{1}{2p}\right)\alpha \|u_n\|_X^p - \left(\frac{1}{r} - \frac{1}{2p}\right) S_2^{\frac{r}{p}} \|u_n\|_X^r \|h\|_{L^{r_1}(\mathbb{R}^N, g_1)} \\ &\quad - \left(\frac{1}{q} - \frac{1}{2p}\right) S_2^{\frac{q}{p}} \|u_n\|_X^q \|H\|_L^{q_2}(\mathbb{R}^N, f_2), \end{aligned} \tag{2.24}$$

which implies that $\{\|u_n\|_X\}$ is bounded in X. The remaining proofs are similar to those of (2.16)–(2.23).

Case 3. $1 < r < p < q < 2p$.

Using (A_1) and (A_3), we can similarly prove that that $J(u)$ satisfies the conditions $(PS)_c$.

Case 4. $1 < q < p < r < 2p$.

By (A_2) and (A_4) we get that $J(u)$ satisfies the conditions $(PS)_c$. □

3 Existence of solutions

In this section, we use the minimax procedure to prove the existence of infinitely many solutions for problem (1.1). Let $\mathcal{A}$ denotes the class of $A \subset X\backslash\{0\}$ such that A is closed in X and symmetric with respect to the origin. For $A \in \mathcal{A}$, the genus $\gamma(A)$ is defined by

$$\gamma(A) = \min\bigl\{m \in N : \exists\phi \in C\bigl(A, R^m\backslash\{0\}\bigr), \phi(x) = \phi(-x)\bigr\}.$$

We say that $\gamma(A) = \infty$ if there no such mapping for any $m \in N$. Particularly, $\gamma(\emptyset) = 0$.

The following proposition gives some main properties of the genus; see [25, 26].

Proposition 3.1 *Let* $A, B \in \mathcal{A}$. *Then*:

(1) *If there exists an odd map* $g \in C(A, B)$, *then* $\gamma(A) \le \gamma(B)$.
(2) *If* $A \subset B$, *then* $\gamma(A) \le \gamma(B)$.
(3) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.
(4) *If* S *is a sphere centered at the origin in* $\mathbb{R}^N$, *then* $\gamma(S) = N$.
(5) *If* A *is compact, then* $\gamma(A) < \infty$, *and there exists* $\delta > 0$ *such that* $N_\delta(A) \in \mathcal{A}$ *and* $\gamma(N_\delta(A)) = \gamma(A)$, *where* $N_\delta(A) = \{x \in X : \|x - A\| \le \delta\}$.

Lemma 3.2 *There exists* $\varepsilon = \varepsilon(m)$ *such that*

$$\gamma\bigl\{u \in X : J(u) < -\varepsilon\bigr\} \ge m.$$

Proof Given $m \in \mathbb{N}^+$, let X_m be an m-dimensional subspace of X. Similarly to Lemma 2.4, we will proceed the discussion according to the relationship of p, q, r.

Case 1. $1 < r, q < p$.

Without loss of generality, we assume that $r < q$. Then by (2.1)

$$J(u) \le \frac{\alpha}{p}\|u\|_X^p + \frac{\beta}{2p}\|u\|_X^{2p} - \frac{1}{r}\|u\|_{L^r(\mathbb{R}^N,h)}^r. \tag{3.1}$$

Note that since X_m is a finite-dimensional space, all norms on this space are equivalent. Therefore for all $u \in X_m$,

$$J(u) \le \frac{\alpha}{p}\|u\|_X^p + \frac{\beta}{2p}\|u\|_X^{2p} - c\|u\|_X^r \tag{3.2}$$

for some constant $c > 0$. Then there exist small $\rho_1 > 0$ and $\varepsilon > 0$ such that $J(u) < -\varepsilon$ for $u \in X_m$ and $\|u\|_X = \rho_1$. Set

$$S_{\rho_1} = \bigl\{u \in X_m : \|u\|_{X_m} = \rho_1\bigr\}. \tag{3.3}$$

Then S_{ρ_1} is a sphere centered at the origin with radius of ρ_1, and

$$S_{\rho_1} \subset \bigl\{u \in X : J(u) \le -\varepsilon\bigr\} \triangleq J^{-\varepsilon}. \tag{3.4}$$

Therefore Proposition 3.1 shows that $\gamma(J^{-\varepsilon}) \ge \gamma(S_{\rho_1}) = m$.

Case 2. $1 < p < \min\{r, q\}$ and $\max\{r, q\} < 2p$.
By the equivalence of norms on the finite-dimensional space X_m we get from (2.1) that

$$\begin{aligned} J(u) &= \frac{\alpha}{p}\|u\|_X^p + \frac{\beta}{2p}\|u\|_X^{2p} - c_1\|u\|_X^r - c_2\|u\|_X^q \\ &= \|u\|_X^p\left(\frac{\alpha}{p} + \frac{\beta}{2p}\|u\|_X^p - c_1\|u\|_X^{r-p} - c_2\|u\|_X^{q-p}\right), \end{aligned} \tag{3.5}$$

where c_1, c_2 are positive constants. If $\alpha > 0$ is sufficiently small, then there exists small ρ_3 such that $J(u) < -\varepsilon$ with $\|u\|_{X_m} = \rho_3$.

Case 3. $1 < r < p < q < 2p$.
We can similarly get that

$$J(u) \le \frac{\alpha}{p}\|u\|_X^p + \frac{\beta}{2p}\|u\|_X^{2p} - c\|u\|_X^r$$

for $u \in X_m$ and some constant $c > 0$. There exist small $\rho_4 > 0$ and $\varepsilon > 0$ such that $J(u) < -\varepsilon$ with $\|u\|_{X_m} = \rho_4$. Then $S_{\rho_4} \subset J^{-\varepsilon}$ and $\gamma(J^{-\varepsilon}) \ge \gamma(S_{\rho_4}) = m$. The sphere S_{ρ_4} with radius ρ_4 is defined as in (3.3).

Case 4. $1 < q < p < r < 2p$.
It is not difficult to check that

$$J(u) \le \frac{\alpha}{p}\|u\|_X^p + \frac{\beta}{2p}\|u\|_X^{2p} - c\|u\|_X^q$$

for $u \in X_m$ and $c > 0$. Similarly, there exist small $\rho_5 > 0$ and $\varepsilon > 0$ such that $J(u) < -\varepsilon$ with $\|u\|_{X_m} = \rho_5$ and $\gamma(J^{-\varepsilon}) \ge \gamma(S_{\rho_5}) = m$. Therefore we complete the proof of Lemma 3.2. □

Let $\mathcal{A}_m = \{A \in \mathcal{A} : \gamma(A) \ge m\}$. Then $\mathcal{A}_{m+1} \subset \mathcal{A}_m$ $(m = 1, 2, \ldots)$. Define

$$c_m = \inf_{A \in \mathcal{A}_m} \sup_{u \in A} J(u).$$

Since $J(u)$ is coercive, $J(u)$ is bounded from below. It is not difficult to find that

$$c_1 \le c_2 \le \cdots \le c_m \le \cdots$$

and $c_m > -\infty$ for any $m \in \mathbb{N}$. Furthermore, set

$$K_c = \{u \in X : J(u) = c, J'(u) = 0\}.$$

Then K_c is compact, and the following lemma can be similarly proved as Lemma 6 in [21]; see also [26].

Lemma 3.3 *All c_m are critical values of $J(u)$. Moreover, if $c = c_m = c_{m+1} = \cdots = c_{m+\tau}$, then $\gamma(K_c) \ge 1 + \tau$.*

Now we can prove Theorem 1.

Proof Lemma 2.4 shows that $J(u)$ satisfies the conditions $(PS)_c$ in X. Then by the standard argument in [25–27] we obtain from Lemma 3.3 that $J(u)$ has infinitely many critical points, that is, problem (1.1) has infinitely many weak solutions in X. Therefore we complete the proof of Theorem 1. □

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