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Attractors for the nonclassical reaction–diffusion equations on time-dependent spaces

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Abstract
In this paper, based on the notation of time-dependent attractors introduced by Conti, Pata and Temam in (J. Differ. Equ. 255:1254–1277, 2013), we prove the existence of time-dependent global attractors in $\mathcal{H}_t$ for a class of nonclassical reaction–diffusion equations with the forcing term $g(x) \in H^{-1}(\Omega)$ and the nonlinearity f satisfying the polynomial growth of arbitrary $p-1$ ($p \geq 2$) order, which generalizes the results obtained in (Appl. Anal. 94:1439–1449, 2015) and (Bound. Value Probl. 2016: 10, 2016).

MSC: 35R10; 35B40; 35B41

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 3$) with smooth boundary, we consider the long-time behavior of the solutions for the following nonclassical reaction–diffusion equation:

$$\begin{cases} u_t - \varepsilon(t)\Delta u_t - \Delta u + f(u) = g(x) & \text{in } \Omega \times (\tau, \infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x,\tau) = u_\tau, & x \in \Omega, \end{cases} \tag{1.1}$$

where $t > \tau$, $\tau \in \mathbb{R}$ is the initial time, $g(x) \in H^{-1}(\Omega)$ is an external force term, $\varepsilon(t) \in C^1(\mathbb{R})$ is a decreasing bounded function satisfying

$$\lim_{t \to +\infty} \varepsilon(t) = 0, \tag{1.2}$$

and there exists $L > 0$ such that

$$\sup_{t \in \mathbb{R}} \big(|\varepsilon(t)| + |\varepsilon'(t)| \big) \leq L. \tag{1.3}$$

For the nonlinear term $f \in C(\mathbb{R},\mathbb{R})$, similar to that in [3, 20, 24], we make the following classical assumptions:

$$f'(u) \geq -l, \quad \forall u \in \mathbb{R}, \tag{1.4}$$

and

$$-c_0 + c_1|u|^p \leq f(u)u \leq c_0 + c_2|u|^p, \quad p \geq 2, \tag{1.5}$$

for some positive constants c_0, c_1, c_2.

Let $\mathcal{F}(u) = \int_0^u f(r)\,dr$, then there are constants $\tilde{c}_i > 0$ $(i = 0,1,2)$ such that

$$-\tilde{c}_0 + \tilde{c}_1|u|^p \leq \mathcal{F}(u) \leq \tilde{c}_0 + \tilde{c}_2|u|^p, \quad \forall u \in \mathbb{R}. \tag{1.6}$$

For Eq. (1.1), when $\varepsilon(t) > 0$ is a constant, the existence and long-time behavior of solutions have been extensively studied by several authors; see, e.g., [1, 4, 5, 23, 25–27, 29, 30, 32]. In [4, 5, 29], the authors main considered the existence of solutions for this type of equations. In [1, 23, 25–27, 30], the authors main considered the existence of the global attractors (see [23, 25–27]) and the pullback (or the uniform) attractors (see [1, 23, 30]) in $H_0^1(\Omega)$ (or $H^1(\mathbb{R}^N)$). In particular, in [32], we obtained the existence of the pullback attractors in $C_{H_0^1(\Omega)}$ (rather than in $H_0^1(\Omega)$) for the nonclassical reaction–diffusion equations with delays.

When $\varepsilon(t) = 0$, Eq. (1.1) becomes the classical reaction–diffusion equation. The existence and the long-time behavior of solutions have also been extensively investigated by several authors; see, e.g., [2, 11, 12, 17, 21, 28, 31]. In [2, 11, 12, 28], the authors mainly considered the existence (or the blowup), uniqueness and the long-time decay of the solutions for the semilinear parabolic equation [11, 12], the nonlinear parabolic equation [2] and the coupled parabolic systems [28]. In [17, 21, 31], the authors have proved the existence of the global attractors in $L^p(\Omega)$, $H_0^1(\Omega)$, $L^{2p-2}(\Omega)$, $H^2(\Omega)$ (see [31]) and the existence of the pullback attractors in $L^p(\Omega)$ and $H_0^1(\Omega)$ (see [17] and [21], respectively).

When $\varepsilon(t) \in C^1(\mathbb{R})$ satisfies (1.2)–(1.3), the long-time behavior of solutions for Eq. (1.1) has been considered by some researchers; see, e.g., [16, 18]. In [16], the authors have proved the existence of the time-dependent global attractors in $\mathcal{H}_t$ with the nonlinearity f satisfying $|f''(u)| \leq c(1 + |u|)$ (see *Theorem* 3.4 in [16] for details). Furthermore, in [18], the authors have considered the case of the nonlinearity f satisfying the critical exponent growth and proved the existence of the time-dependent global attractors in $\mathcal{H}_t$ (see *Theorem* 3.3 in [18] for details).

In this paper, we consider Eq. (1.1) with the nonlinearity f satisfying polynomial growth of arbitrary $p - 1$ $(p \geq 2)$ order, which makes that the Sobolev compact embedding is no longer valid and brings more difficulty for verifying the corresponding asymptotic compactness of the solutions process $\{U(t,\tau)\}_{t\geq\tau}$. In order to overcome the difficulty mentioned above, we verify the existence of the time-dependent global attractors $\hat{\mathcal{A}}$ in $\mathcal{H}_t$ for the process $\{U(t,\tau)\}_{t\geq\tau}$ by applying the contractive function methods as in [6, 13, 14, 19, 22, 27] (see Theorem 3.8).

2 Preliminaries

In this section, we firstly review briefly some notations, basic definitions and results about processes on time-dependent spaces (see [7–9, 19] for details).

2.1 Notations

Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of normed spaces, we introduce the R-ball of X_t as

$$\mathbb{B}_t(R) = \bigl\{z \in X_t : \|z\|_{X_t} \le R\bigr\}.$$

For any given $\epsilon > 0$, the ϵ-neighborhood of a set $B \subset X_t$ is defined as

$$\mathcal{O}_t^{\epsilon}(B) = \bigcup_{x\in B}\bigl\{y \in X_t : \|x-y\|_{X_t} < \epsilon\bigr\} = \bigcup_{x\in B}\bigl\{x + \mathbb{B}_t(\epsilon)\bigr\}.$$

We denote the Hausdorff semidistance of two (nonempty) sets $B, C \subset X_t$ by

$$\delta_t(B,C) = \sup_{x\in B}\inf_{y\in C}\|x-y\|_{X_t}.$$

Moreover, we introduce the time-dependent space $\mathcal{H}_t$ endowed with the norms

$$\|u\|_{\mathcal{H}_t}^2 = \|u\|_2^2 + \varepsilon(t)\|\nabla u\|_2^2,$$

where $\|\cdot\|_2$ denotes the usual norm in $L^2(\Omega)$.

2.2 Some concepts

In this subsection, we give some concepts about the time-dependent global attractors.

Definition 2.1 Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. A process is a two-parameter family of mappings $U(t,\tau) : X_\tau \to X_t$, $t \ge \tau$, $\tau \in \mathbb{R}$ with properties

(i) $U(\tau,\tau) = \mathrm{Id}$ is the identity operator on X_τ, $\tau \in \mathbb{R}$;

(ii) $U(t,s)U(s,\tau) = U(t,\tau)$, $\forall t \ge s \ge \tau$, $\tau \in \mathbb{R}$.

Definition 2.2 A family $\hat{C} = \{C_t\}_{t\in\mathbb{R}}$ of bounded sets $C_t \subset X_t$ is called uniformly bounded if there exists a constant $R > 0$ such that $C_t \subset \mathbb{B}_t(R)$ for all $t \in \mathbb{R}$.

Definition 2.3 A family $\hat{B} = \{B_t\}_{t\in\mathbb{R}}$ is called pullback absorbing if it is uniformly bounded and for every $R > 0$, there exists a constant $t_0 = t_0(t,R) \le t$ such that $U(t,\tau)\mathbb{B}_\tau(R) \subset B_t$ for all $\tau \le t_0$.

The process $\{U(t,\tau)\}_{t\ge\tau}$ is called dissipative whenever it admits a pullback absorbing family.

Definition 2.4 A time-dependent absorbing set for the process $\{U(t,\tau)\}_{t\ge\tau}$ is a uniformly bounded family $\hat{B} = \{B_t\}_{t\in\mathbb{R}}$ with the following property: for every $R \ge 0$ there exists a $t_0 = t_0(R) \ge 0$ such that

$$U(t,\tau)\mathbb{B}_\tau(R) \subset B_t \quad \text{for all } \tau \le t - t_0.$$

Definition 2.5 The process $\{U(t,\tau)\}_{t\geq\tau}$ is said to be pullback asymptotically compact if for any $t\in\mathbb{R}$, any bounded sequence $\{x_n\}_{n=1}^{\infty}\subset X_{\tau_n}$ and any sequence $\{\tau_n\}_{n=1}^{\infty}$ with $\tau_n\to-\infty$ as $n\to\infty$, the sequence $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ is precompact in $\{X_t\}_{t\in\mathbb{R}}$.

Definition 2.6 The time-dependent global attractor for the process $\{U(t,\tau)\}_{t\geq\tau}$ is the smallest family $\hat{\mathcal{A}}=\{\mathcal{A}_t\}_{t\in\mathbb{R}}$ such that

(i) $\mathcal{A}_t$ is compact in X_t;

(ii) $\hat{\mathcal{A}}$ is invariant, i.e., $U(t,\tau)\mathcal{A}_\tau=\mathcal{A}_t, \forall t\geq\tau$;

(iii) $\hat{\mathcal{A}}$ is pullback attracting, i.e., it is uniformly bounded and the limit

$$\lim_{\tau\to-\infty}\delta_t\big(U(t,\tau)C_\tau,\mathcal{A}_t\big)=0$$

holds for every uniformly bounded family $\hat{C}=\{C_t\}_{t\in\mathbb{R}}$ and every fixed $t\in\mathbb{R}$.

Remark 2.7 The attracting property can be equivalently stated in terms of pullback absorbing: a (uniformly bounded) family $\mathcal{K}=\{K_t\}_{t\in\mathbb{R}}$ is called pullback attracting if for any $\epsilon>0$ the family $\{\mathcal{O}_t^{\epsilon}(K_t)\}_{t\in\mathbb{R}}$ is pullback absorbing.

Similarly to Theorem 4.2 in [8], we have the following theorem.

Theorem 2.8 *The time-dependent global attractor $\hat{\mathcal{A}}$ exists and it is unique if and only if the process $\{U(t,\tau)\}_{t\geq\tau}$ is asymptotically compact, namely, the set*

$$\mathbb{K}=\big\{\mathcal{K}=\{K_t\}_{t\in\mathbb{R}}:K_t\subset X_t \textit{ is compact }, \mathbb{K} \textit{ is pullback attracting}\big\}$$

is not empty.

2.3 Some results

In order to obtain the time-dependent global attractors of Eq. (1.1), we need the following definitions and conclusions, which are similar to those in [6, 13, 14, 19, 22, 27].

Definition 2.9 Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces and $\hat{C}=\{C_t\}_{t\in\mathbb{R}}$ be a family of uniformly bounded subset of $\{X_t\}_{t\in\mathbb{R}}$. We call a function $\psi_\tau^t(\cdot,\cdot)$, defined on $\{X_t\}_{t\in\mathbb{R}}\times\{X_t\}_{t\in\mathbb{R}}$, a contractive function on $C_\tau\times C_\tau$ if for fixed $t\in\mathbb{R}$ and any sequence $\{x_n\}_{n=1}^{\infty}\subset C_\tau$, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}\subset\{x_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\psi_\tau^t(x_{n_k},x_{n_l})=0\quad\text{for all } t\geq\tau.$$

We denote the set of all contractive functions on $C_\tau\times C_\tau$ by $Contr(C_\tau)$.

Theorem 2.10 *Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process on Banach spaces $\{X_t\}_{t\in\mathbb{R}}$ and have a pullback absorbing set $\hat{B}=\{B_t\}_{t\in\mathbb{R}}$. Moreover, assume that, for any $\epsilon>0$, there exist $\tau_0=\tau_0(\epsilon)<t$ and $\psi_{\tau_0}^t(\cdot,\cdot)\in\hat{C}(B_{\tau_0})$ such that*

$$\big\|U(t,\tau_0)x-U(t,\tau_0)y\big\|_{X_t}\leq\epsilon+\psi_{\tau_0}^t(x,y),\quad\forall x,y\in B_{\tau_0},$$

for any $t\in\mathbb{R}$. Then $\{U(t,\tau)\}_{t\geq\tau}$ is pullback asymptotically compact in $\{X_t\}_{t\in\mathbb{R}}$.

Proof We need to prove that, for any $\{x_n\}_{n=1}^{\infty} \subset B_{\tau_n}$ and any $\tau_n \to -\infty$ as $n \to \infty$,

$$\text{the sequence } \{U(t,\tau_n)x_n\}_{n=1}^{\infty} \text{ is precompact in } \{X_t\}_{t\in\mathbb{R}}.$$

In the following, we will show that $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ has a convergent subsequence via diagonal methods.

Taking $\epsilon_m > 0$ with $\epsilon_m \to 0$ as $m \to \infty$.

Then, for $\epsilon_1 > 0$, by the assumptions, there exist $\tau_0 = \tau_0(\epsilon_1) < t$ and $\psi_{\tau_0}^t(\cdot,\cdot) \in \hat{C}(B_{\tau_0})$ such that

$$\|U(t,\tau_0)x - U(t,\tau_0)y\|_{X_t} \le \epsilon_1 + \psi_{\tau_0}^t(x,y), \quad \forall x,y \in B_{\tau_0}, \tag{2.1}$$

for any $t \in \mathbb{R}$, where $\psi_{\tau_0}^t$ depends on τ_0.

Since $\tau_n \to -\infty$, without loss of generality, we assume that $\tau_n \le \tau_0$ such that $U(\tau_0,\tau_n)x_n \in B_{\tau_0}$ for each $n \in \mathbb{N}$. Set $y_n = U(\tau_0,\tau_n)x_n$, then from (2.1) we have

$$\begin{aligned}\|U(t,\tau_n)x_n - U(t,\tau_m)x_m\|_{X_t} &= \|U(t,\tau_0)U(\tau_0,\tau_n)x_n - U(t,\tau_0)U(\tau_0,\tau_m)x_m\|_{X_t} \\ &= \|U(t,\tau_0)y_n - U(t,\tau_0)y_m\|_{X_t} \\ &\le \epsilon_1 + \psi_{\tau_0}^t(y_n,y_m).\end{aligned} \tag{2.2}$$

By the definition of $\hat{C}(B_{\tau_0})$ and $\psi_{\tau_0}^t \in \hat{C}(B_{\tau_0})$, we know that $\{y_n\}_{n=1}^{\infty}$ have a subsequence $\{y_{n_k}^{(1)}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\psi_{\tau_0}^t\big(y_{n_k}^{(1)},y_{n_l}^{(1)}\big) \le \epsilon_1. \tag{2.3}$$

Similarly to [13, 22, 27], we have

$$\begin{aligned}&\lim_{k\to\infty}\sup_{q\in\mathbb{N}}\big\|U\big(t,\tau_{n_{k+q}}^{(1)}\big)x_{n_{k+q}}^{(1)} - U\big(t,\tau_{n_k}^{(1)}\big)x_{n_k}^{(1)}\big\|_{X_t} \\ &\quad\le \lim_{k\to\infty}\sup_{q\in\mathbb{N}}\limsup_{l\to\infty}\big\|U\big(t,\tau_{n_{k+q}}^{(1)}\big)x_{n_{k+q}}^{(1)} - U\big(t,\tau_{n_l}^{(1)}\big)x_{n_l}^{(1)}\big\|_{X_t} \\ &\qquad + \limsup_{k\to\infty}\limsup_{l\to\infty}\big\|U\big(t,\tau_{n_k}^{(1)}\big)x_{n_k}^{(1)} - U\big(t,\tau_{n_l}^{(1)}\big)x_{n_l}^{(1)}\big\|_{X_t} \\ &\quad\le \epsilon_1 + \lim_{k\to\infty}\sup_{q\in\mathbb{N}}\limsup_{l\to\infty}\psi_{\tau_0}^t\big(y_{n_{k+q}}^{(1)},y_{n_l}^{(1)}\big) + \epsilon_1 + \lim_{k\to\infty}\lim_{l\to\infty}\psi_{\tau_0}^t\big(y_{n_k}^{(1)},y_{n_l}^{(1)}\big),\end{aligned}$$

which, combining with (2.2) and (2.3), implies that

$$\lim_{k\to\infty}\sup_{q\in\mathbb{N}}\big\|U\big(t,\tau_{n_{k+q}}^{(1)}\big)x_{n_{k+q}}^{(1)} - U\big(t,\tau_{n_k}^{(1)}\big)x_{n_k}^{(1)}\big\|_{X_t} \le 4\epsilon_1.$$

Therefore, there exists a $K_1 \in \mathbb{N}$ such that

$$\lim_{k\to\infty}\sup_{q\in\mathbb{N}}\big\|U\big(t,\tau_{n_k}^{(1)}\big)x_{n_k}^{(1)} - U\big(t,\tau_{n_l}^{(1)}\big)x_{n_l}^{(1)}\big\|_{X_t} \le 5\epsilon_1, \quad \text{for all } k,l \ge K_1.$$

By induction, we can obtain that, for each $m \geq 1$, there exists a subsequence $\{U(t, \tau_{n_k}^{(m+1)})x_{n_k}^{(m+1)}\}_{k=1}^{\infty}$ of $\{U(t,\tau_{n_k}^{(m)})x_{n_k}^{(m)}\}_{k=1}^{\infty}$ and certain K_{m+1} such that

$$\lim_{k\to\infty}\sup_{q\in\mathbb{N}}\big\|U\big(t,\tau_{n_k}^{(m+1)}\big)x_{n_k}^{(m+1)}-U\big(t,\tau_{n_l}^{(m+1)}\big)x_{n_l}^{(m+1)}\big\|_{X_t}\leq 5\epsilon_{m+1},\quad \text{for all } k,l\geq K_{m+1}.$$

Now, we consider the diagonal subsequence $\{U(t,\tau_{n_k}^{(k)})x_{n_k}^{(k)}\}_{k=1}^{\infty}$. Since for each $m\in\mathbb{N}$, $\{U(t,\tau_{n_k}^{(k)})x_{n_k}^{(k)}\}_{k=m}^{\infty}$ is a subsequence of $\{U(t,\tau_{n_k}^{(k)})x_{n_k}^{(k)}\}_{k=1}^{\infty}$, then

$$\lim_{k\to\infty}\sup_{q\in\mathbb{N}}\big\|U\big(t,\tau_{n_k}^{(k)}\big)x_{n_k}^{(k)}-U\big(t,\tau_{n_l}^{(l)}\big)x_{n_l}^{(l)}\big\|_{X_t}\leq 6\epsilon_{m},\quad \text{for all } k,l\geq \max\{m,K_m\},$$

which combining with $\epsilon_m\to 0$ as $m\to\infty$, implies that $\{U(t,\tau_{n_k}^{(k)})x_{n_k}^{(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence in $\{X_t\}_{t\in\mathbb{R}}$. This shows that $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ is precompact in $\{X_t\}_{t\in\mathbb{R}}$. □

Similarly to Theorem 3.3 in [19], we have the following conclusion, which will be used to verify the existence of the time-dependent global attractor.

Theorem 2.11 *Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process on Banach space $\{X_t\}_{t\in\mathbb{R}}$, then $\{U(t,\tau)\}_{t\geq\tau}$ has a time-dependent global attractor in $\{X_t\}_{t\in\mathbb{R}}$ if the following conditions hold*:

(i) *$\{U(t,\tau)\}_{t\geq\tau}$ has a pullback absorbing set $\hat{B}=\{B_t\}_{t\in\mathbb{R}}$ in $\{X_t\}_{t\in\mathbb{R}}$;*
(ii) *$\{U(t,\tau)\}_{t\geq\tau}$ is pullback asymptotically compact in $\hat{B}=\{B_t\}_{t\in\mathbb{R}}$.*

3 Time-dependent global attractors

In this section, we will establish the existence of the time-dependent global attractors.

3.1 Existence and uniqueness of solutions

In this subsection, we consider the well-posedness of the solutions for Eq. (1.1) with (1.4)–(1.5). At first, we define the weak solutions as follows.

Definition 3.1 A weak solution of Eq. (1.1) is a function $u\in C([\tau,T];\mathcal{H}_t)\cap L^2(\tau,T;H_0^1(\Omega))\cap L^p(\tau,T;L^p(\Omega))$ for all $T>\tau$, with $u(\tau)=u_\tau$ and such that, for all $\varphi\in H_0^1(\Omega)$, it satisfies

$$\begin{aligned}&\frac{d}{dt}\big[\big(u(t),\varphi\big)+\varepsilon(t)\big(\nabla u(t),\nabla\varphi\big)\big]+\big(1-\varepsilon'(t)\big)\big(\nabla u(t),\nabla\varphi\big)+\big(f\big(u(t)\big),\varphi\big)\\&\quad=\big(g(x),\varphi\big),\quad \text{in } \mathcal{D}'(\tau,+\infty).\end{aligned}$$

Remark 3.2 We notice that, if $u(t)$ is a weak solution of Eq. (1.1), then it satisfies the energy equality

$$\begin{aligned}&\big\|u(t)\big\|_2^2+\varepsilon(t)\big\|\nabla u(t)\big\|_2^2+\int_s^t\big(2-\varepsilon'(r)\big)\big\|\nabla u(r)\big\|_2^2\,dr+2\int_s^t\big(f\big(u(r)\big),u(r)\big)\,dr\\&\quad=\big\|u(s)\big\|_2^2+\varepsilon(s)\big\|\nabla u(s)\big\|_2^2+2\int_s^t\big(g(r),u(r)\big)\,dr\quad \text{for all } \tau\leq s\leq t.\end{aligned}$$

The following theorem gives the existence of the weak solutions, which is similar to that in [10] and can be obtained by the Faedo–Galerkin methods.

Theorem 3.3 *Let f satisfy (1.4)–(1.5), $g \in H^{-1}(\Omega)$ and $u_\tau \in \mathcal{H}_\tau$. Then, for any $\tau \in \mathbb{R}$ and $t > \tau$, there exists a weak solution $u(t)$ to Eq. (1.1), which satisfies $u \in C([\tau,t];\mathcal{H}_t) \cap L^2(\tau,t;H_0^1(\Omega)) \cap L^p(\tau,t;L^p(\Omega))$, $u_t \in L^2(\tau,t;\mathcal{H}_t)$.*

Proof Let $\{w_j\}_{j\geq 1} \subset H_0^1(\Omega) \cap L^p(\Omega)$ be a Hilbert basis of $L^2(\Omega)$ such that $\operatorname{span}\{w_j\}_{j\geq 1}$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$. In order to establish the existence of the weak solutions, we need the approximate system for any $m \geq n$ seeking $\tilde{u}^m(t,x) = \Sigma_{j=1}^m \gamma_{mj}(t)\omega_j(x)$ that satisfies

$$\begin{cases} \frac{d}{dt}[(\tilde{u}^m(t),\omega_j) + \varepsilon(t)(\nabla\tilde{u}^m(t),\nabla\omega_j)] + (1-\varepsilon'(t))(\nabla\tilde{u}^m(t),\nabla\omega_j) + (f(\tilde{u}^m(t)),\omega_j) \\ \quad = (g(x),\omega_j), \\ \tilde{u}^m_\tau = u_\tau, \end{cases}$$

for a.e. $t > \tau, 1 \leq j \leq m$.

We will provide a priori estimates that show that these solutions are well-defined in the interval $[\tau,t]$ for any $t > \tau$.

Step 1: *First a priori estimates*. Multiplying each equation in the above system by $\gamma_{mj}(t)$, respectively, and summing from $j = 1$ to m, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|\tilde{u}^m(t)\|_2^2 + \varepsilon(t)\|\nabla\tilde{u}^m(t)\|_2^2\big) + \big(1-\varepsilon'(t)\big)\|\nabla\tilde{u}^m(t)\|_2^2 \\ &\quad + \big(f\big(\tilde{u}^m(t)\big),\tilde{u}^m(t)\big) = \big(g(x),\tilde{u}^m(t)\big) \leq \frac{1}{2}\|g\|_{H^{-1}}^2 + \frac{1}{2}\|\nabla\tilde{u}^m(t)\|_2^2, \quad \text{a.e. } t > \tau, \end{aligned}$$

where we have used the Hölder and Young inequalities.

Furthermore, by (1.5), we know that

$$\begin{aligned} &\frac{d}{dt}\big(\|\tilde{u}^m(t)\|_2^2 + \varepsilon(t)\|\nabla\tilde{u}^m(t)\|_2^2\big) + \big(1-2\varepsilon'(t)\big)\|\nabla\tilde{u}^m(t)\|_2^2 + 2c_1\|\tilde{u}^m(t)\|_p^p \\ &\quad \leq 2c_0|\Omega| + \|g\|_{H^{-1}}^2, \quad \text{a.e. } t > \tau. \end{aligned}$$

Integrating it in $[\tau,t]$, we have

$$\begin{aligned} &\|\tilde{u}^m(t)\|_2^2 + \varepsilon(t)\|\nabla\tilde{u}^m(t)\|_2^2 + \int_\tau^t \big(1-2\varepsilon'(s)\big)\|\nabla\tilde{u}^m(s)\|_2^2\,ds + 2c_1\int_\tau^t \|\tilde{u}^m(s)\|_p^p\,ds \\ &\quad \leq \|\tilde{u}^m(\tau)\|_2^2 + \varepsilon(\tau)\|\nabla\tilde{u}^m(\tau)\|_2^2 + \big(2c_0|\Omega| + \|g\|_{H^{-1}}^2\big)(t-\tau) \quad \text{for all } t \geq \tau. \end{aligned}$$

Hence,

$$\begin{aligned} &\|\tilde{u}^m(t)\|_2^2 + \varepsilon(t)\|\nabla\tilde{u}^m(t)\|_2^2 + \int_\tau^t \|\nabla\tilde{u}^m(s)\|_2^2\,ds + 2c_1\int_\tau^t \|\tilde{u}^m(s)\|_p^p\,ds \\ &\quad \leq \|\tilde{u}^m(\tau)\|_2^2 + \varepsilon(\tau)\|\nabla\tilde{u}^m(\tau)\|_2^2 + \big(2c_0|\Omega| + \|g\|_{H^{-1}}^2\big)(t-\tau) \quad \text{for all } t \geq \tau. \end{aligned} \tag{3.1}$$

So, from (3.1), we can get

$$\{\tilde{u}^m\}_{m\geq n} \quad \text{is bounded in } L^\infty(\tau,t;\mathcal{H}_t) \cap L^2\big(\tau,t;H_0^1(\Omega)\big) \cap L^p\big(\tau,t;L^p(\Omega)\big) \tag{3.2}$$

for all $t > \tau$.

Moreover, combining with (1.5) and (3.2), we obtain

$$\{f(\tilde{u}^m)\}_{m\geq n} \quad \text{is bounded in } L^q(\tau,t;L^q(\Omega)) \text{ for all } t>\tau,$$

where $q=p/(p-1)$.

Then there exist functions $\tilde{u}\in L^\infty(\tau,t;\mathcal{H}_t)\cap L^2(\tau,t;H_0^1(\Omega))\cap L^p(\tau,t;L^p(\Omega))$ and $\tilde{\chi}\in L^q(\tau,t;L^q(\Omega))$ for all $t>\tau$, and a subsequence such that

$$\begin{cases} \tilde{u}^m\to\tilde{u} & \text{weakly-star in } L^\infty(\tau,t;\mathcal{H}_t),\\ \tilde{u}^m\to\tilde{u} & \text{weakly in } L^2(\tau,t;H_0^1(\Omega)),\\ \tilde{u}^m\to\tilde{u} & \text{weakly in } L^p(\tau,t;L^p(\Omega)),\\ f(\tilde{u}^m)\to\tilde{\chi} & \text{weakly in } L^q(\tau,t;L^q(\Omega)). \end{cases} \tag{3.3}$$

Step 2: *Uniform estimate for the time derivatives*. Multiplying each equation of the approximate system by $\gamma'_{mj}(t)$ and summing from $j=1$ to m, we arrive at

$$\begin{aligned} &\|(\tilde{u}^m)'(t)\|_2^2+\varepsilon(t)\|(\nabla\tilde{u}^m)'(t)\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{u}^m(t)\|_2^2\\ &\quad+\big(f(\tilde{u}^m),(\tilde{u}^m)'(t)\big)=\big(g(x),(\tilde{u}^m)'(t)\big),\quad \text{a.e. } t>\tau. \end{aligned}$$

By the Hölder and Young inequalities, we have

$$\begin{aligned} &\|(\tilde{u}^m)'(t)\|_2^2+2\varepsilon(t)\|(\nabla\tilde{u}^m)'(t)\|_2^2+\frac{d}{dt}\|\nabla\tilde{u}^m(t)\|_2^2\\ &\quad+2\frac{d}{dt}\int_\Omega\mathcal{F}(\tilde{u}^m(t,x))\,dx\leq\|g\|_2^2,\quad \text{a.e. } t>\tau. \end{aligned}$$

Integrating it from τ to t, and from (1.6) we can get

$$\begin{aligned} &\|\nabla\tilde{u}^m(t)\|_2^2+2\tilde{c}_1\|\tilde{u}^m(t)\|_p^p+\int_\tau^t\big(\|(\tilde{u}^m)'(s)\|_2^2+\varepsilon(t)\|(\nabla\tilde{u}^m)'(s)\|_2^2\big)\,ds\\ &\quad\leq 4\tilde{c}_0|\Omega|+\|\nabla\tilde{u}^m(\tau)\|_2^2+2\tilde{c}_2\|\tilde{u}^m(\tau)\|_p^p+\|g\|_2^2(t-\tau) \end{aligned} \tag{3.4}$$

for all $t\geq\tau$ and any $m\geq n$.

Since $\tilde{u}^m_\tau=u_\tau$ for all $m\geq n$ and $\tilde{u}^m_\tau\in H_0^1(\Omega)\cap L^p(\Omega)$, by (3.4), we obtain

$$\{\tilde{u}^m(t)\}_{m\geq n} \quad \text{is bounded in } L^\infty(\tau,t;H_0^1(\Omega)\cap L^p(\Omega)) \tag{3.5}$$

and

$$\{(\tilde{u}^m)'(t)\}_{m\geq n} \quad \text{is bounded in } L^2(\tau,t;\mathcal{H}_t) \tag{3.6}$$

for all $t>\tau$. Then there exist functions $\tilde{u}\in L^\infty(\tau,t;H_0^1(\Omega)\cap L^p(\Omega))$ and $\tilde{u}_t\in L^2(\tau,t;\mathcal{H}_t)$ for all $t>\tau$, which improve the regularity of $\tilde{u}$ obtained in Step 1.

For any fixed $t>\tau$, since

$$\|\tilde{u}^m(t_2)-\tilde{u}^m(t_1)\|_{\mathcal{H}_t}^2=\|\tilde{u}^m(t_2)-\tilde{u}^m(t_1)\|_2^2+\varepsilon(t)\|\nabla\tilde{u}^m(t_2)-\nabla\tilde{u}^m(t_1)\|_2^2$$

$$
\begin{aligned}
&= \left\| \int_{t_1}^{t_2} \big(\tilde{u}^m\big)'(s)\,ds \right\|_2^2 + \varepsilon(t) \left\| \int_{t_1}^{t_2} \big(\nabla \tilde{u}^m\big)'(s)\,ds \right\|_2^2 \\
&\leq \big(\big\| \big(\tilde{u}^m\big)' \big\|_{L^2(\tau,t;L^2(\Omega))}^2 + \varepsilon(t) \big\| \big(\nabla \tilde{u}^m\big)' \big\|_{L^2(\tau,t;L^2(\Omega))}^2 \big) |t_2 - t_1| \\
&= \big\| \big(\tilde{u}^m\big)' \big\|_{L^2(\tau,t;\mathcal{H}_t)}^2 |t_2 - t_1|, \qquad (3.7)
\end{aligned}
$$

for all $t_1, t_2 \in [\tau, t]$, from (3.5), (3.6) and (3.7), by the Ascoli–Arzelà Theorem, and taking into account the initial data for all the sequence, we deduce that there is a subsequence such that

$$
\tilde{u}^m \to \tilde{u} \quad \text{in } C\big([\tau,t];\mathcal{H}_t\big) \qquad (3.8)
$$

for all $t > \tau$ and a.e. in $\Omega \times (\tau, \infty)$.

Since $f \in C(\mathbb{R},\mathbb{R})$, we conclude that $f(\tilde{u}^m) \to f(\tilde{u})$*a.e.* in $\Omega \times (\tau,\infty)$. So, combining with (3.3) and [15] (Lemma 1.3, p. 12) we obtain $\tilde{\chi} = f(\tilde{u})$.

Thus, together with (3.3) and (3.8), by taking the limit in the equations satisfied by $\{\tilde{u}^m\}$ and, thanks to the fact that span$\{\omega_j\}_{j\geq 1}$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$, we conclude that $\tilde{u}$ is a weak solution of Eq. (1.1).

Step 3: *Proof of the general statement by density*. For each $n \in \mathbb{N}$, we define $u_\tau^n = \Sigma_{j=1}^n (u_\tau, \omega_j)\omega_j$. (Due to the fact that $\{\omega_j\}_{j\geq 1}$ is a Hilbert basis of $L^2(\Omega)$, it is easy to check that $u_\tau^n \to u_\tau$ in $\mathcal{H}_\tau$.)

Let also consider a sequence $\{g^n\}_{n=1}^\infty \subset L^2(\Omega)$ converging to $g \in H^{-1}(\Omega)$.

Denote by u^n the corresponding solution to Eq. (1.1) with g replaced by g^n and initial data u_τ^n.

Then, by the energy equality for each u^n, we have

$$
\begin{aligned}
&\big\|u^n(t)\big\|_2^2 + \varepsilon(t)\big\|\nabla u^n(t)\big\|_2^2 + 2\int_\tau^t \big\|\nabla u^n(s)\big\|_2^2\,ds + 2\int_\tau^t \big(f\big(u^n(s)\big), u^n(s)\big)\,ds \\
&\quad = \big\|u^n(\tau)\big\|_2^2 + \varepsilon(\tau)\big\|\nabla u^n(\tau)\big\|_2^2 + 2\int_\tau^t \big(g^n(x), u^n(s)\big)\,ds, \quad \forall t \geq \tau.
\end{aligned}
$$

Similar to the reasoning process in Step 1, we get

$$
\big\{u^n\big\} \quad \text{is bounded in } L^\infty(\tau,t;\mathcal{H}_t) \cap L^2\big(\tau,t;H_0^1(\Omega)\big) \cap L^p\big(\tau,t;L^p(\Omega)\big) \qquad (3.9)
$$

for all $t > \tau$.

Now, combining with (1.5) and (3.9), we see that $\{f(u^n)\}$ is bounded in $L^q(\tau,t;L^q(\Omega))$ for all $t > \tau$.

Therefore, there exist functions $u \in L^\infty(\tau,t;\mathcal{H}_t) \cap L^2(\tau,t;H_0^1(\Omega)) \cap L^p(\tau,t;L^p(\Omega))$ and $\chi \in L^q(\tau,t;L^q(\Omega))$ for all $t > \tau$, and a subsequence such that

$$
\begin{cases}
u^n \to u & \text{weakly-star in } L^\infty(\tau,t;\mathcal{H}_t), \\
u^n \to u & \text{weakly in } L^2(\tau,t;H_0^1(\Omega)), \\
u^n \to u & \text{weakly in } L^p(\tau,t;L^p(\Omega)), \\
f(u^n) \to \chi & \text{weakly in } L^q(\tau,t;L^q(\Omega)),
\end{cases} \qquad (3.10)
$$

for all $t > \tau$.

Moreover, we may improve some of the above convergence. Taking into account the energy equality for $u^n - u^m$, we have

$$\begin{aligned} &\|u^n(t) - u^m(t)\|_2^2 + \varepsilon(t)\|\nabla u^n(t) - \nabla u^m(t)\|_2^2 + \int_\tau^t \|\nabla u^n(s) - \nabla u^m(s)\|_2^2\,ds \\ &\quad\le \|u^n(\tau) - u^m(\tau)\|_2^2 + \varepsilon(\tau)\|\nabla u^n(\tau) - \nabla u^m(\tau)\|_2^2 + 2l\int_\tau^t \|u^n(s) - u^m(s)\|_2^2\,ds \\ &\qquad + \|g^n - g^m\|_{H^{-1}}^2(t-\tau), \quad \forall t \ge \tau. \end{aligned} \tag{3.11}$$

By (3.11), we know that

$$\{u^n\} \quad \text{is a Cauchy sequence in } C\big([\tau,t];\mathcal{H}_t\big) \cap L^2\big(\tau,t;H_0^1(\Omega)\big) \text{ for all } t > \tau.$$

Thus, we have $u^n \to u$ a.e. in $\Omega \times (\tau,\infty)$.

Therefore, as before, combining with (3.10) and [15] (Lemma 1.3, p. 12) we obtain $\chi = f(u)$; and from (3.10) we may take the limit in the equations satisfied by u^n and conclude that u is a weak solution of Eq. (1.1). □

For the solutions of Eq. (1.1), the following theorem shows the uniqueness and continuity with respect to initial data.

Theorem 3.4 *Let f satisfy (1.4)–(1.5), $g \in H^{-1}(\Omega)$ and $u_\tau \in \mathcal{H}_\tau$, then the weak solution of Eq. (1.1) is unique. Moreover, for every two solutions $u^1(t)$ and $u^2(t)$ (with different initial data), the following Lipschitz continuity holds:*

$$\|\omega(t)\|_2^2 + \varepsilon(t)\|\nabla\omega(t)\|_2^2 \le \big(\|\omega_\tau\|_2^2 + \varepsilon(\tau)\|\nabla\omega_\tau\|_2^2\big)e^{2l(t-\tau)}, \quad \forall t \ge \tau,$$

where $\omega(t) = u^1(t) - u^2(t)$.

Proof Let $\omega(t) = u^1(t) - u^2(t)$, then $\omega(t)$ satisfies the following equation:

$$\begin{cases} \omega_t - \varepsilon(t)\Delta\omega_t - \Delta\omega = f(u^1) - f(u^2) & \text{in } \Omega \times (\tau,\infty), \\ \omega(x,t) = 0 & \text{on } \partial\Omega \times (\tau,\infty), \\ \omega(x,\tau) = u_\tau^1 - u_\tau^2, & x \in \Omega. \end{cases} \tag{3.12}$$

Taking the L^2-inner product between (3.12) and ω, and using (1.4), we have

$$\frac{d}{dt}\big(\|\omega\|_2^2 + \varepsilon(t)\|\nabla\omega\|_2^2\big) + \big(2 - \varepsilon'(t)\big)\|\nabla\omega\|_2^2 \le 2l\|\omega\|_2^2.$$

Then

$$\frac{d}{dt}\big(\|\omega\|_2^2 + \varepsilon(t)\|\nabla\omega\|_2^2\big) \le 2l\big(\|\omega\|_2^2 + \varepsilon(t)\|\nabla\omega\|_2^2\big).$$

By the Gronwall lemma, it yields

$$\|\omega(t)\|_2^2 + \varepsilon(t)\|\nabla\omega(t)\|_2^2 \le \big(\|\omega_\tau\|_2^2 + \varepsilon(\tau)\|\nabla\omega_\tau\|_2^2\big)e^{2l(t-\tau)},$$

and the uniqueness holds. □

Thus, we define the solution processes $\{U(t,\tau)\}_{t\geq\tau}$ in the spaces $\mathcal{H}_t$ as:

$$U(t,\tau):\mathcal{H}_\tau\to\mathcal{H}_t,\qquad U(t,\tau)u_\tau=u(t),\quad \forall t\geq\tau. \tag{3.13}$$

Moreover, Theorem 3.4 shows that the process $\{U(t,\tau)\}_{t\geq\tau}$ is Lipschitz in $\mathcal{H}_t$:

$$\|U(t,\tau)u_\tau^1-U(t,\tau)u_\tau^2\|_{\mathcal{H}_t}\leq\|u_\tau^1-u_\tau^2\|_{\mathcal{H}_\tau}e^{2l(t-\tau)},\quad \forall t\geq\tau.$$

3.2 Time-dependent global attractors

In this subsection, we will verify the existence of the time-dependent global attractors in $\mathcal{H}_t$ for the process $\{U(t,\tau)\}_{t\geq\tau}$ defined by (3.13).

3.2.1 Time-dependent absorbing sets

In the following, we will obtain the time-dependent global absorbing sets.

Lemma 3.5 *Let f satisfy (1.4)–(1.5), $g\in H^{-1}(\Omega)$ and $u_\tau\in\mathbb{B}_\tau(R)\subset\mathcal{H}_\tau$. Then there exists a $R_0>0$ such that the family $\hat{B}=\{B_t(R_0)\}_{t\in\mathbb{R}}$ is a time-dependent absorbing set for the process $\{U(t,\tau)\}_{t\geq\tau}$.*

Proof Multiplying (1.1) by $u(t)$ and integrating over $x\in\Omega$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_2^2+\varepsilon(t)\|\nabla u\|_2^2\right)+\left(1-\frac{1}{2}\varepsilon'(t)\right)\|\nabla u\|_2^2+\left(f(u),u\right)=\langle g(x),u\rangle.$$

Thanks to (1.5) and the Hölder inequality, we have

$$\frac{d}{dt}\left(\|u\|_2^2+\varepsilon(t)\|\nabla u\|_2^2\right)+\left(1-\varepsilon'(t)\right)\|\nabla u\|_2^2+2c_1\|u\|_p^p\leq 2c_0|\Omega|+\|g\|_{H^{-1}}^2.$$

Furthermore, by (1.3), we can get

$$\frac{d}{dt}\left(\|u\|_2^2+\varepsilon(t)\|\nabla u\|_2^2\right)+\frac{1}{1+L}\left(\lambda_1\|u\|_2^2+\varepsilon(t)\|\nabla u\|_2^2\right)\leq 2c_0|\Omega|+\|g\|_{H^{-1}}^2.$$

Setting $\lambda=\min\{\lambda_1,1\}$ and $\beta=\frac{\lambda}{1+L}$, we deduce that

$$\frac{d}{dt}\left(\|u\|_2^2+\varepsilon(t)\|\nabla u\|_2^2\right)+\beta\left(\|u\|_2^2+\varepsilon(t)\|\nabla u\|_2^2\right)\leq 2c_0|\Omega|+\|g\|_{H^{-1}}^2. \tag{3.14}$$

Multiplying (3.14) by $e^{\beta t}$ and integrating it in $[\tau,t]$, we obtain

$$\begin{aligned}&\left(\|u(t)\|_2^2+\varepsilon(t)\|\nabla u(t)\|_2^2\right)e^{\beta t}\\&\quad\leq\left(\|u(\tau)\|_2^2+\varepsilon(\tau)\|\nabla u(\tau)\|_2^2\right)e^{\beta\tau}+\left(2c_0|\Omega|+\|g\|_{H^{-1}}^2\right)\int_\tau^t e^{\beta s}\,ds,\quad\forall t\geq\tau.\end{aligned}$$

Therefore,

$$\left(\|u(t)\|_2^2+\varepsilon(t)\|\nabla u(t)\|_2^2\right)\leq\left(\|u(\tau)\|_2^2+\varepsilon(\tau)\|\nabla u(\tau)\|_2^2\right)e^{-\beta(t-\tau)}+\frac{1}{\beta}\left(2c_0|\Omega|+\|g\|_{H^{-1}}^2\right)$$

$$\leq 1 + \frac{1}{\beta}\left(2c_0|\Omega| + \|g\|_{H^{-1}}^2\right) = R_0,$$

provided that $t - \tau \geq t_0$ with $t_0 = \frac{1}{\beta}\ln(\|u_\tau\|_2^2 + \varepsilon(\tau)\|\nabla u_\tau\|_2^2)$, from which we obtain the existence of the time-dependent absorbing set. □

3.2.2 Time-dependent global attractors

At first, we have the following lemma, which is similar to that in [15].

Lemma 3.6 *Let f satisfy (1.4)–(1.5), $g \in H^{-1}(\Omega)$, $u_\tau \in \mathcal{H}_\tau$ and $\{u^n(t)\}_{n=1}^\infty$ be a sequence of solutions for Eq. (1.1) with initial data $u_\tau^n \in \mathcal{H}_\tau$ ($n = 1, 2, \ldots$), then there exists a subsequence of $\{u^n(t)\}_{n=1}^\infty$ that converges strongly in $L^2(\tau, t; L^2(\Omega))$.*

Proof By (1.5) and Theorem 3.3, we know that there exists a sequence $\{u^n(t)\}_{n=1}^\infty \subset L^2(\tau, T; H_0^1(\Omega))$, $\{f(u^n(t))\}_{n=1}^\infty \subset L^q(\tau, T; L^q(\Omega))$. Then, from Eq. (1.1), we obtain $\partial_t u^n - \varepsilon(t)\partial_t \Delta u^n = \Delta u^n - f(u^n) + g(x) \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega)) \subset L^2(\tau, T; H^{-2}(\Omega))$. By the regularization theory for elliptic equations, we know that $\partial_t u^n \in L^2(\tau, T; L^2(\Omega))$. As in [15], there exists a subsequence of $\{u^n(t)\}_{n=1}^\infty$ (still denoted by $\{u^n(t)\}_{n=1}^\infty$) that converges strongly in $L^2(\tau, T; L^2(\Omega))$. □

Then we have the following theorem, which will obtain the pullback asymptotic compactness for the process $\{U(t,\tau)\}_{t\geq\tau}$ defined by (3.13).

Theorem 3.7 *Let f satisfy (1.4)–(1.5), $g \in H^{-1}(\Omega)$ and $u_\tau \in \mathbb{B}_\tau(R) \subset \mathcal{H}_\tau$, then $\{U(t,\tau)\}_{t\geq\tau}$ is pullback asymptotically compact in $\mathcal{H}_t$.*

Proof Let $u^i(t)$ ($i = 1, 2$) be the solutions corresponding to initial data $u_\tau^i \in \mathbb{B}_\tau(R) \subset \mathcal{H}_\tau$, that is, $u^i(t)$ satisfies the following equation:

$$u_t - \varepsilon(t)\Delta u_t - \Delta u + f(u) = g(x), \quad \text{in } \Omega \times (\tau, \infty),$$

with initial data

$$u^i(x,\tau) = u_\tau^i, \quad x \in \Omega.$$

Denoting $\omega(t) = u^1(t) - u^2(t)$, then $\omega(t)$ satisfies the following equation:

$$\omega_t - \varepsilon(t)\Delta\omega_t - \Delta\omega + f\left(u^1\right) - f\left(u^2\right) = 0, \quad \text{in } \Omega \times (\tau, \infty), \tag{3.15}$$

with initial data

$$\omega(x,\tau) = u_\tau^1 - u_\tau^2, \quad x \in \Omega.$$

Multiplying (3.15) by $\omega(t)$ and integrating it in Ω, then, by (1.4), we obtain

$$\frac{d}{dt}\left(\|\omega\|_2^2 + \varepsilon(t)\|\nabla\omega\|_2^2\right) + \left(2 - \varepsilon'(t)\right)\|\nabla\omega\|_2^2 \leq 2l\|\omega\|_2^2.$$

By the Poincaré inequality, we have

$$\frac{d}{dt}\left(\|\omega\|_2^2+\varepsilon(t)\|\nabla\omega\|_2^2\right)+\beta_1\left(\|\omega\|_2^2+\varepsilon(t)\|\nabla\omega\|_2^2\right)\le 2l\|\omega\|_2^2,$$

where $\beta_1=2\beta$, β is given by (3.14).

Thanks to the Gronwall lemma, we get

$$\begin{aligned}&\|\omega(t)\|_2^2+\varepsilon(t)\|\nabla\omega(t)\|_2^2\\&\quad\le\left(\|\omega(\tau)\|_2^2+\varepsilon(\tau)\|\nabla\omega(\tau)\|_2^2\right)e^{-\beta_1(t-\tau)}+2l\int_\tau^t\|\omega(s)\|_2^2\,ds,\quad\forall t\ge\tau.\end{aligned}$$

Setting

$$\psi_\tau^t(u_\tau^1,u_\tau^2)=2l\int_\tau^t\|\omega(s)\|_2^2\,ds,$$

combining with Definition 2.9 and Lemma 3.6, we know that $\psi_\tau^t(\cdot,\cdot)$ is a contractive function. Then, for any $\epsilon>0$ and any fixed $t\in\mathbb{R}$, let $\tau_0=t-\frac{1}{\beta_1}\ln\frac{\|w_\tau\|_2^2+\varepsilon(\tau)\|\nabla w_\tau\|_2^2}{\epsilon}$, we easily see that $\{U(t,\tau)\}_{t\ge\tau}$ is pullback asymptotically compact in $\mathcal{H}_t$ by Theorem 2.10. □

Combining with Lemma 3.5 and Theorem 3.7, we have the main result of this paper.

Theorem 3.8 *Let f satisfy* (1.4)–(1.5), $g\in H^{-1}(\Omega)$ *and* $u_\tau\in\mathbb{B}_\tau(R)\subset\mathcal{H}_\tau$, *then* $\{U(t,\tau)\}_{t\ge\tau}$ *possesses a time-dependent global attractor* $\hat{\mathcal{A}}=\{\mathcal{A}_t\}_{t\in\mathbb{R}}$ *in* $\mathcal{H}_t$; *that is,* $\mathcal{A}_t$ *is compact,* $\hat{\mathcal{A}}$ *is nonempty, invariant in* $\mathcal{H}_t$ *and pullback attracts every bounded subset of* $\mathcal{H}_t$ *with respect to the* $\mathcal{H}_t$*-norm.*

Remark 3.9 In Theorem 3.8, we have obtained the time-dependent global attractor $\hat{\mathcal{A}}=\{\mathcal{A}_t\}_{t\in\mathbb{R}}$ in $\mathcal{H}_t$. From (1.2) we know that $\varepsilon(t)\to 0$ as $t\to+\infty$, then Eq. (1.1) becomes the classical reaction–diffusion equation $u_t-\Delta u+f(u)=g(x)$. An interesting question is about the limitation of $\mathcal{A}_t$ as $t\to+\infty$, that is, how to describe $\lim_{t\to+\infty}\mathcal{A}_t$? We will consider this problem in our next work.

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