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Global existence and exponential decay of solutions for generalized coupled non-degenerate Kirchhoff system with a time varying delay term

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Abstract

The paper studies a system of nonlinear viscoelastic Kirchhoff system with a time varying delay and general coupling terms. We prove the global existence of solutions in a bounded domain using the energy and Faedo–Galerkin methods with respect to the condition on the parameters in the coupling terms together with the weight condition as regards the delay terms in the feedback and the delay speed. Furthermore, we construct some convex function properties, and we prove the uniform stability estimate.

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1 Introduction

The Kirchhoff equation belongs to the famous wave equation's models describing the transverse vibration of a string fixed in its ends. It has been introduced in 1876 by Kirchhoff [8] and it is more general than the D'Alembert equation. In one dimensional space it takes the following form:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{\rho h} + \frac{E}{2L\rho} \int_0^L \left| \frac{\partial u}{\partial x}(x,t) \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where the function $u(x,t)$ is the vertical displacement at the space coordinate x, varying in the segment $[0,L]$ and over time $t > 0$, ρ is the mass density, h is the area of the cross section of the string, P_0 is the initial tension on the string, L is the length of the string and E is the Young modulus of the material. The nonlinear coefficient

$$C(t) = \int_0^L \left| \frac{\partial u}{\partial x}(x,t) \right|^2 dx$$

is obtained by the variation of the tension during the deformation of the string. When we do not have an initial tension (i.e. $P_0 = 0$), we call that a degenerate case as opposed to the non-degenerate case.

In this paper, we are interested in studying, in $\mathcal{A} = \Omega \times (0,\infty)$, the following coupled viscoelastic Kirchhoff system:

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h_1(t-s)\Delta u(s)\,ds - \mu_1 \Delta u_t(x,t-\tau(t)) + f_1(u,v) \\ \quad = 0 \quad \text{in } \Omega \times]0,+\infty[, \\ |v_t|^l v_{tt} - M(\|\nabla v\|^2)\Delta v - \Delta v_{tt} + \int_0^t h_2(t-s)\Delta v(s)\,ds - \mu_2 \Delta v_t(x,t-\tau(t)) + f_2(u,v) \\ \quad = 0 \quad \text{in } \Omega \times]0,+\infty[, \\ u(x,t) = v(x,t) = 0 \quad \text{on } \Gamma \times]0,+\infty[, \\ (u(x,0),v(x,0)) = (u_0(x),v_0(x)), \qquad (u_t(x,0),v_t(x,0)) = (u_1(x),v_1(x)) \quad \text{in } \Omega, \\ (u_t(x,t-\tau(0)),v_t(x,t-\tau(0))) = (f_0(x,t-\tau(0)),g_0(x,t-\tau(0))) \quad \text{in } \Omega \times]0,\tau(0)[, \end{cases} \tag{1.2}$$

in which Ω is an n dimensional bounded domain of $\mathbb{R}^n$ and we have a smooth boundary Γ, $l > 0$, μ_1 and μ_2 are positive real constants, h_1 and h_2 are positive functions with exponential decay, and $\tau(t)$ is a positive time varying delay. In addition the initial condition $(u_0, v_0, u_1, v_1, f_0, g_0)$ will be specified in their function space later. M is a smooth function defined by

$$\begin{aligned} M: \quad & \mathbb{R}_+ \longrightarrow \mathbb{R}_{,+} \\ & r \longmapsto M(r) = a + br^{\gamma}, \end{aligned}$$

with $a, b > 0$, and $\gamma \geq 1$. f_1 and f_2 are two functions taking a particular form that we will make precise later.

The problem (1.2) is a description of axially moving viscoelastic strings composed of two different materials (like the wires of electricity) that are nonhomogeneous and which will be of influence on its moving, specially on the acceleration. From the mathematical point of view, this influence is represented by $|w_t|^l w''$, where $|w_t|^l$ is the material density, varying the velocity. A lot of work has been published with this term, for example see [11] and [14], where we find different results about the global existence and nonexistence of solutions and the decay of energy.

In recent years, the study of wave equations with delay has become an active area and with different forms of delay (constant [7], switching [5], varying in time [12], distributed [6]). The delay appears in modeling of a lot of domains, like the physical, chemical, biological and engineering domains. It is introduced when we have a time lag between an action on a system and a response of the system to this action. Furthermore, a delay can be small enough in feedback yet can destabilize a system [10], or improve the performance of the system [17].

In the absence of delay, Cavalcanti et al. [3] studied the following viscoelastic wave equations with strong damping:

$$|u_t|^l u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)\,ds - \mu \Delta u_t(x,t) = 0, \quad \text{in } \Omega \times]0,+\infty[.$$

They used the Fadeo–Galerkin method to prove the global existence of a solution; also an explicit decay rate of the energy has been given provided $m > 0$.

In the other hand, in the same case and for $l = 0$, Raslan et al. [16] and El-Sayed et al. [4] have studied coupled equal width wave equations with strong damping, as they were looking for the new exact solution.

The problem treated in [2] has the following form:

$$u_{tt} - \Delta u + \mu_1 \sigma(t) g_1\big(u_t(x,t)\big) + \mu_2 \sigma(t) g_2\big(u_t\big(x, t - \tau(t)\big)\big) = 0, \quad \text{in } \Omega \times]0, +\infty[.$$

Under the assumptions set on g_1, g_2, σ and τ, the authors have gotten the global existence of a solution and the decay rate of the energy.

Recently, Mezouar and Boulaaras [13] have studied the viscoelastic non-degenerate Kirchhoff equation with varying delay term in the internal feedback.

In the present paper, we extend our recently published paper in [13] for a coupled system (1.2). The famous technique of using the presence of a delay in the PDE problem is to set a new variable defined by a velocity dependent on the delay, which will give us a new problem equivalent to our studied problem; but the last one is a coupled system without delay. After this, we can prove the existence of global solutions in suitable Sobolev spaces by combining the energy method with the Fadeo–Galerkin procedure and under the choice of a suitable Lyapunov functional, we establish an exponential decay result.

The outline of the paper is as follows: In the second section, some hypotheses related to the problem are given and we state our main result. Then in the third section, the global existence of weak solutions is proven. Finally, in the fourth section, we give the uniform energy decay.

1.1 Preliminaries and assumptions

Similar to that [12], we present the new variables

$$z_1(x, \rho, t) = u_t\big(x, t - \rho\tau(t)\big), \quad x \in \Omega, \rho \in (0,1), t > 0,$$

and

$$z_2(x, \rho, t) = v_t\big(x, t - \rho\tau(t)\big), \quad x \in \Omega, \rho \in (0,1), t > 0.$$

Then we have

$$\tau(t) z_1'(x, \rho, t) + \big(1 - \rho\tau'(t)\big) \frac{\partial}{\partial \rho} z_1(x, \rho, t) = 0, \quad \text{in } \Omega \times (0,1) \times (0, +\infty). \tag{1.3}$$

In the same way, we have

$$\tau(t) z_2'(x, \rho, t) + \big(1 - \rho\tau'(t)\big) \frac{\partial}{\partial \rho} z_2(x, \rho, t) = 0, \quad \text{in } \Omega \times (0,1) \times (0, +\infty). \tag{1.4}$$

Therefore, problem (1.2) is equivalent to

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h_1(t-s)\Delta u(s)\,ds - \mu_1 \Delta z_1(x,1,t) + f_1(u,v) = 0 \\ \quad \text{in } \Omega \times]0,+\infty[, \\ |v_t|^l v_{tt} - M(\|\nabla v\|^2)\Delta v - \Delta v_{tt} + \int_0^t h_2(t-s)\Delta v(s)\,ds - \mu_2 \Delta z_2(x,1,t) + f_2(u,v) = 0 \\ \quad \text{in } \Omega \times]0,+\infty[, \\ \tau(t) z_1'(x,\rho,t) + (1-\rho\tau'(t))\frac{\partial}{\partial\rho} z_1(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \\ \tau(t) z_2'(x,\rho,t) + (1-\rho\tau'(t))\frac{\partial}{\partial\rho} z_2(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \\ u(x,t) = v(x,t) = 0, \quad \text{on } \partial\Omega \times [0,\infty[, \\ (z_1(x,0,t), z_2(x,0,t)) = (u_t(x,t), v_t(x,t)), \quad \text{on } \Omega \times]0,\infty[, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), \qquad (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)), \quad \text{in } \Omega, \\ (z_1(x,\rho,0), z_2(x,\rho,0)) = (f_0(x,-\rho\tau(0)), g_0(x,-\rho\tau(0))), \quad \text{in } \Omega \times]0,1[. \end{cases} \tag{1.5}$$

Throughout this work and for simplifying our formulas, we will adopt the notation z_i, u and v instead of $z_i(x,\rho,t)$, $u(x,t)$ and $v(x,t)$, except if that makes things inconvenient.

In order to demonstrate the main result in this paper, a few assumptions are needed.

(A-1) Consider that $0 < l \leq \gamma$ verifies

$$\begin{cases} \gamma \leq \frac{2}{n-2} & \text{in the case } n > 2, \\ \gamma < \infty & \text{in the case } n \leq 2. \end{cases}$$

(A-2) As regards the relaxation functions $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ we see that they are bounded C^1 functions such that

$$a - \int_0^\infty h_i(s)\,ds \geq k > 0.$$

We assume also that there exist some positive constants ζ_i verifying

$$h_i'(t) \leq -\zeta_i h_i(t)$$

for $i = 1, 2$.

(A-3) We have $\tau \in C^2([0,T],[\tau_0,\tau_1])$ a positive function, where

$$\tau'(t) \leq d < 1, \quad \forall t \in [0,T].$$

(A-4) $f_1(u,v) = \alpha v + b_1|v|^{q+1}|u|^{p-1}u$ and $f_2(u,v) = \alpha u + b_2|u|^{p+1}|v|^{q-1}v$ where $\alpha > 0$, $b_1 = (p+1)(p+q)$, $b_2 = (q+1)(p+q)$ such that p and q are conjugate (i.e. $\frac{1}{p} + \frac{1}{q} = 1$), $p, q < \gamma - \frac{1}{2}$ and satisfy

$$2 \leq p, q \leq \begin{cases} \sqrt{\frac{n}{2(n-2)}} & \text{if } n > 2, \\ +\infty & \text{if } n \leq 2. \end{cases}$$

The energy related to the system solution of (1.5) is defined as follows:

$$\begin{aligned} E(t) &= \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) + \frac{b}{2(\gamma+1)}\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) \\ &\quad + \frac{1}{2}\left(a - \int_0^t h_1(s)\,ds\right)\|\nabla u\|^2 + \frac{1}{2}\left(a - \int_0^t h_2(s)\,ds\right)\|\nabla v\|^2 \\ &\quad + \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\ &\quad + \frac{1}{2}(h_1 \, o \, \nabla u)(t) + \frac{1}{2}(h_2 \, o \, \nabla v)(t) + \xi\tau(t)\int_0^1 \left(\|\nabla z_1\|^2 + \|\nabla z_2\|^2\right) d\rho \\ &\quad + \alpha\int_\Omega uv\,dx + (p+q)\int_\Omega |u|^{p+1}|v|^{q+1}\,dx, \end{aligned} \tag{1.6}$$

where ξ is a positive constant such that

$$\frac{\max\{\mu_1,\mu_2\}}{2(1-d)} < \xi \tag{1.7}$$

and

$$(h_i \, o \, w)(t) = \int_0^t h_i(t-s)\left\|w(\cdot,t) - w(\cdot,s)\right\|^2 ds, \quad \text{for } i = 1,2.$$

Lemma 1.1 (Sobolev–Poincaré's inequality) *Let q be a number with*

$$2 \le q < +\infty \quad (n = 1,2) \quad \textit{or} \quad 2 \le q \le 2n/(n-2) \quad (n \ge 3).$$

Then there exists a constant $C_s = C_s(\Omega, q)$ such that

$$\|u\|_q \le C_s\|\nabla u\|_2 \quad \textit{for } u \in H_0^1(\Omega).$$

We present the following lemma.

Lemma 1.2 [15] *For h, φ C^1-real functions, we have*

$$\begin{aligned} &\frac{d}{dt}\left[(h \, o \, \varphi)(t) - \left(\int_0^t h(s)\,ds\right)\|\varphi\|^2\right] \\ &\quad = (h'^2 - 2\int_\Omega\int_0^t h(t-s)\varphi(s)\varphi_t(t)\,ds\,dx \quad \forall t \ge 0. \end{aligned} \tag{1.8}$$

Lemma 1.3 *Let (u, v, z_1, z_2) be a solution of the problem (1.5). Then the energy functional defined by (1.6) satisfies*

$$\begin{aligned} E'(t) &\le -\beta\left(\left\|\nabla z_1(x,1,t)\right\|^2 + \left\|\nabla z_2(x,1,t)\right\|^2\right) + \lambda\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\ &\quad + \frac{1}{2}\left[\left(h_1' o \nabla u\right)(t) + \left(h_2' o \nabla v\right)(t)\right], \end{aligned} \tag{1.9}$$

where $\lambda = \xi + \frac{\mu}{2}$, $\beta = \xi(1-d) - \frac{\mu}{2}$ and $\mu = \max\{\mu_1, \mu_2\}$ are positive.

Proof After the multiplication of the first equation in (1.5) by u_t followed by integration of the result by parts over Ω, we get

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{l+2}\|u_t\|_{l+2}^{l+2}+\frac{b}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\frac{1}{2}a\|\nabla u\|^2+\frac{1}{2}\|\nabla u_t\|^2\right]\\&\quad-\int_\Omega\int_0^t h_1(t-s)\nabla u(s)\nabla u_t(t)\,ds\,dx\\&\quad+\mu_1\int_\Omega\nabla u_t\nabla z_1(x,1,t)\,dx+\alpha\int_\Omega vu_t\,dx+b_1\int_\Omega|v|^{q+1}|u|^{p-1}uu_t\,dx=0.\end{aligned}\tag{1.10}$$

Using (1.8) and (1.10) leads to

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{l+2}\|u_t\|_{l+2}^{l+2}+\frac{b}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\frac{1}{2}\left(a-\int_0^t h_1(s)\,ds\right)\|\nabla u\|^2\right.\\&\quad\left.+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}(h_1o\nabla u)(t)\right]\\&\quad+\frac{1}{2}h_1(t)\|\nabla u\|^2-\frac{1}{2}\left(h_1'o\nabla u\right)(t)+\mu_1\int_\Omega\nabla u_t\nabla z_1(x,1,t)\,dx\\&\quad+\alpha\int_\Omega vu_t\,dx+b_1\int_\Omega|v|^{q+1}|u|^{p-1}uu_t\,dx=0.\end{aligned}\tag{1.11}$$

Similarly by multiplying the second equation in (1.5) by v_t, integrating over Ω and using integration by parts, we get

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{l+2}\|v_t\|_{l+2}^{l+2}+\frac{b}{2(\gamma+1)}\|\nabla v\|^{2(\gamma+1)}+\frac{1}{2}\left(a-\int_0^t h_2(s)\,ds\right)\|\nabla v\|^2\right.\\&\quad\left.+\frac{1}{2}\|\nabla v_t\|^2+\frac{1}{2}(h_2o\nabla v)(t)\right]\\&\quad+\frac{1}{2}h_2(t)\|\nabla v\|^2-\frac{1}{2}\left(h_2'o\nabla v\right)(t)+\mu_2\int_\Omega\nabla v_t\nabla z_2(x,1,t)\,dx+\alpha\int_\Omega uv_t\,dx\\&\quad+b_2\int_\Omega|u|^{p+1}|v|^{q-1}vv_t\,dx=0.\end{aligned}\tag{1.12}$$

Multiplying the third equation in (1.5) by $\xi\Delta z_1$ and integrating the result over $\Omega\times(0,1)$, we obtain

$$\xi\tau(t)\int_\Omega\int_0^1 z_1'\Delta z_1\,d\rho\,dx=-\xi\int_{\Omega\times(0,1)}\left(1-\rho\tau'(t)\right)\frac{\partial}{\partial\rho}z_1\Delta z_1\,d\rho\,dx.$$

Consequently,

$$\begin{aligned}\frac{d}{dt}\left(\xi\tau(t)\int_0^1\|\nabla z_1\|^2\,d\rho\right)&=\xi\int_{\Omega\times(0,1)}\left[\tau'(t)|\nabla z_1|^2-\rho\tau'(t))\frac{\partial}{\partial\rho}|\nabla z_1|^2\right]d\rho\,dx\\&=-\xi\int_0^1\frac{\partial}{\partial\rho}\left(\left(1-\rho\tau'(t)\right)\|\nabla z_1\|^2\right)d\rho\\&=\xi\left[\|\nabla u_t\|^2-\xi\left(1-\tau'(t)\right)\left\|\nabla z_1(x,1,t)\right\|^2\right].\end{aligned}\tag{1.13}$$

Similarly we get

$$\frac{d}{dt}\left(\xi\tau(t)\int_0^1\|\nabla z_2\|^2\,d\rho\right)=\xi\left[\|\nabla v_t\|^2-\left(1-\tau'(t)\right)\left\|\nabla z_2(x,1,t)\right\|^2\right]. \tag{1.14}$$

Combining (1.11)–(1.14), taking the derivation of energy leads to

$$\begin{aligned}E'(t)={}&\xi\left[\|\nabla u_t\|^2+\|\nabla v_t\|^2-\left(1-\tau'(t)\right)\left(\left\|\nabla z_1(x,1,t)\right\|^2+\left\|\nabla z_2(x,1,t)\right\|^2\right)\right]\\&-\frac{1}{2}\left[h_1(t)\|\nabla u\|^2+h_2(t)\|\nabla v\|^2\right]\\&+\frac{1}{2}\left[\left(h_1'o\nabla u\right)(t)+\left(h_2'o\nabla v\right)(t)\right]-\mu_1\int_\Omega\nabla u_t(x,t)\nabla z_1(x,1,t)\,dx\\&-\mu_2\int_\Omega\nabla v_t(x,t)\nabla z_2(x,1,t)\,dx.\end{aligned}$$

From $(A3)$, we find the following bound:

$$\begin{aligned}E'(t)\le{}&-\left(\xi(1-d)-\frac{\mu_1}{2}\right)\int_\Omega\left|\nabla z_1(x,1,t)\right|^2dx-\left(\xi(1-d)-\frac{\mu_2}{2}\right)\int_\Omega\left|\nabla z_2(x,1,t)\right|^2dx\\&-\frac{1}{2}h_1(t)\left\|\nabla u_t(t)\right\|^2+\left(\xi+\frac{\mu_1}{2}\right)\left\|\nabla u_t(t)\right\|^2-\frac{1}{2}h_2(t)\left\|\nabla v(t)\right\|^2\\&+\left(\xi+\frac{\mu_2}{2}\right)\left\|\nabla v_t(t)\right\|^2+\frac{1}{2}\left[\left(h_1'o\nabla u\right)(t)+\left(h_2'o\nabla v\right)(t)\right].\end{aligned} \tag{1.15}$$

Using (1.7), we complete the proof of the lemma. □

2 Global existence

Theorem 2.1 *Let* $(u_0,v_0)\in(H^2(\Omega)\cap H_0^1(\Omega))^2$, $(u_1,v_1)\in(H_0^1(\Omega))^2$ *and* $(f_0,g_0)\in(H_0^1(\Omega,H^1(0,1)))^2$ *satisfy the compatibility condition*

$$\left(f_0(\cdot,0),g_0(\cdot,0)\right)=(u_1,v_1).$$

Assume that **(A1)–(A3)** *hold. Then the problem* (1.2) *admits a weak solution such that* $u,v\in L^\infty(0,\infty;H^2(\Omega)\cap H_0^1(\Omega))$, $u_t,v_t\in L^\infty(0,\infty;H_0^1(\Omega))$, *and* $u_{tt},v_{tt}\in L^2(0,\infty,H_0^1(\Omega))$.

Proof As in the previous assumptions in [2] for the initial conditions $u_0,v_0\in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1,v_1\in H_0^1(\Omega)$, $f_0,g_0\in H_0^1(\Omega,H^1(0,1))$ and the basic functions, we introduce the approximate solutions (u^k,v^k,z_1^k,z_2^k), $k=1,2,3,\ldots$, in the form

$$u^k(t)=\sum_{j=1}^k a^{jk}(t)w^j,\qquad v^k(t)=\sum_{j=1}^k b^{jk}(t)w^j,$$
$$z_1^k(t)=\sum_{j=1}^k c^{jk}(t)\phi^j,\qquad z_2^k(t)=\sum_{j=1}^k d^{jk}(t)\phi^j,$$

where a^{jk}, b^{jk}, c^{jk} and d^{jk} $(j = 1, 2, \dots, k)$ are determined by the following ordinary differential equations:

$$\begin{cases} (|u_t^k|^l u_{tt}^k, w^j) + M(\|\nabla u^k(t)\|^2)(\nabla u^k, \nabla w^j) + (\nabla u_{tt}^k, \nabla w^j) \\ \quad - \int_0^t h_1(t-s)(\nabla u^k(s), \nabla w^j)\,ds + \mu_1(\nabla z_1^k(\cdot,1)), \nabla w^j) + (f_1(u^k, v^k), w^j) = 0, \\ \quad 1 \le j \le k, \\ (|v_t^k|^l v_{tt}^k, w^j) + M(\|\nabla v^k(t)\|^2)(\nabla v^k, \nabla w^j) + (\nabla v_{tt}^k, \nabla w^j) \\ \quad - \int_0^t h_2(t-s)(\nabla v^k(s), \nabla w^j)\,ds + \mu_2(\nabla z_2^k(\cdot,1)), \nabla w^j) + (f_2(u^k, v^k), w^j) = 0, \\ \quad 1 \le j \le k, \\ z_1^k(x,0,t) = u_t^k(x,t), z_2^k(x,0,t) = v_t^k(x,t), \end{cases} \tag{2.1}$$

$$u^k(0) = u_0^k = \sum_{j=1}^k (u_0, w^j) w^j \to u_0, \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } k \to +\infty, \tag{2.2}$$

$$v^k(0) = v_0^k = \sum_{j=1}^k (v_0, w^j) w^j \to v_0, \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } k \to +\infty, \tag{2.3}$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^k (u_1, w^j) w^j \to u_1, \quad \text{in } H_0^1(\Omega) \text{ as } k \to +\infty, \tag{2.4}$$

$$v_t^k(0) = v_1^k = \sum_{j=1}^k (v_1, w^j) w^j \to v_1, \quad \text{in } H_0^1(\Omega) \text{ as } k \to +\infty. \tag{2.5}$$

Also

$$\begin{cases} (\tau(t)\frac{\partial}{\partial t} z_1^k + (1-\rho\tau'(t))\frac{\partial}{\partial \rho} z_1^k, \phi^j) = 0, & 1 \le j \le k, \\ (\tau(t)\frac{\partial}{\partial t} z_2^k + (1-\rho\tau'(t))\frac{\partial}{\partial \rho} z_2^k, \phi^j) = 0, & \le j \le k, \end{cases} \tag{2.6}$$

$$z_1^k(\rho,0) = \sum_{j=1}^k (f_0, \phi^j)\phi^j \to f_0, \quad \text{in } H_0^1(\Omega, H^1(0,1)) \text{ as } k \to +\infty. \tag{2.7}$$

$$z_2^k(\rho,0) = \sum_{j=1}^k (g_0, \phi^j)\phi^j \to g_0, \quad \text{in } H_0^1(\Omega, H^1(0,1)) \text{ as } k \to +\infty. \tag{2.8}$$

Noting that $\frac{l}{2(l+1)} + \frac{1}{2(l+1)} + \frac{1}{2} = 1$, by applying the generalized Hölder inequality, we find

$$(|u_t^k|^l u_{tt}^k, w_j) = \int_\Omega |u_t^k|^l u_{tt}^k w_j\,dx \le \left(\int_\Omega |u_t^k|^{2(l+1)}\,dx\right)^{\frac{l}{2(l+1)}} \|u_{tt}^k\|_{2(l+1)} \|w_j\|_2.$$

Since $(A1)$ holds, according to the Sobolev embedding the nonlinear terms $(|u_t^k|^l u_{tt}^k, w_j)$ and $(|v_t^k|^l v_{tt}^k, w_j)$ in (2.1) make sense (see [2]).

A. First estimate.

Since the sequences u_0^k, v_0^k, u_1^k, v_1^k, $z_1^k(\rho,0)$ and $z_2^k(\rho,0)$ converge and from Lemma 1.3 with employing Gronwall's lemma, we find $C_1 > 0$ independent of k such that

$$E^k(t) + \beta \int_0^t \left(\|\nabla z_1^k(x,1,s))\|^2 + \|\nabla z_2^k(x,1,s))\|^2\right) ds \le C_1, \tag{2.9}$$

where

$$\begin{aligned} E^k(t) &= \frac{1}{l+2}\left(\|u_t^k\|_{l+2}^{l+2} + \|v_t^k\|_{l+2}^{l+2}\right) + \frac{b}{2(\gamma+1)}\left(\|\nabla u^k\|^{2(\gamma+1)} + \|\nabla v^k\|^{2(\gamma+1)}\right) \\ &+ \frac{1}{2}\left(a - \int_0^t h_1(s)\,ds\right)\|\nabla u^k\|^2 + \frac{1}{2}\left(a - \int_0^t h_2(s)\,ds\right)\|\nabla v^k\|^2 \\ &+ \frac{1}{2}\left(\|\nabla u_t^k\|^2 + \|\nabla v_t^k\|^2\right) \\ &+ \frac{1}{2}\left[\left(h_1\, o\, \nabla u^k\right)(t) + \left(h_2\, o\, \nabla v^k\right)(t)\right] \\ &+ \xi\tau(t)\int_0^1 \left(\|\nabla z_1^k(x,\rho,t))\|^2 + \|\nabla z_2^k(x,\rho,t))\|^2\right)d\rho \\ &+ \alpha\int_\Omega u^k v^k\,dx + (p+q)\int_\Omega |u^k|^{p+1}|v^k|^{q+1}\,dx. \end{aligned}$$

Noting $(A1)$ and the estimate (2.9) yields

$$u^k, v^k \quad \text{are bounded in } L^\infty_{\text{loc}}\left(0,\infty,H_0^1(\Omega)\right), \tag{2.10}$$

$$u_t^k, v_t^k \quad \text{are bounded in } L^\infty_{\text{loc}}\left(0,\infty,H_0^1(\Omega)\right), \tag{2.11}$$

$$z_1^k(x,\rho,t), z_2^k(x,\rho,t) \quad \text{are bounded in } L^\infty_{\text{loc}}\left(0,\infty,L^1\left(0,1,H_0^1(\Omega)\right)\right). \tag{2.12}$$

B. The second estimate.

By multiplying the first side of equation (respectively, the second equation) in (2.1) by d_{tt}^{jk} (respectively, by b_{tt}^{jk}), by summing j from 1 to k, then

$$\begin{cases} \int_\Omega |u_t^k|^l |u_{tt}^k|^2\,dx + \int_\Omega M(\|\nabla u^k\|^2)\nabla u^k \nabla u_{tt}^k\,dx + \|\nabla u_{tt}^k\|^2 \\ \quad = \int_0^t h_1(t-s)\int_\Omega \nabla u^k(s)\nabla u_{tt}^k(t)\,dx\,ds \\ \qquad - \mu_1\int_\Omega \nabla u_{tt}^k \nabla(z_1^k(x,1,t))\,dx - \int_\Omega f_1(u^k,v^k)u_{tt}^k(t)\,dx, \\ \int_\Omega |v_t^k|^l |v_{tt}^k|^2\,dx + \int_\Omega M(\|\nabla v^k\|^2)\nabla v^k \nabla v_{tt}^k\,dx + \|\nabla v_{tt}^k\|^2 \\ \quad = \int_0^t h_2(t-s)\int_\Omega \nabla v^k(s)\nabla v_{tt}^k(t)\,dx\,ds \\ \qquad - \mu_2\int_\Omega \nabla v_{tt}^k \nabla(z_2^k(x,1,t))\,dx - \int_\Omega f_2(u^k,v^k)v_{tt}^k(t)\,dx. \end{cases} \tag{2.13}$$

Differentiating (2.6) with respect to t, we get

$$\begin{cases} \left(\left(\frac{\tau(t)}{(1-\rho\tau'(t))}\right)'\frac{\partial}{\partial t}z_1^k + \frac{\tau(t)}{(1-\rho\tau'(t))}\frac{\partial^2}{\partial t^2}z_1^k + \frac{\partial^2}{\partial t\partial\rho}z_1^k, \phi^j\right) = 0, \\ \left(\left(\frac{\tau(t)}{(1-\rho\tau'(t))}\right)'\frac{\partial}{\partial t}z_2^k + \frac{\tau(t)}{(1-\rho\tau'(t))}\frac{\partial^2}{\partial t^2}z_2^k + \frac{\partial^2}{\partial t\partial\rho}z_2^k, \phi^j\right) = 0. \end{cases}$$

Multiplying the first equation by c_t^{jk} (respectively the second equation by d_t^{jk}), summing over j from 1 to k, we have

$$\begin{cases} \frac{1}{2}\left(\frac{\tau(t)}{(1-\rho\tau'(t))}\right)'\|\frac{\partial}{\partial t}z_1^k\|^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\tau(t)}{(1-\rho\tau'(t))}\|\frac{\partial}{\partial t}z_1^k\|^2\right) + \frac{1}{2}\frac{d}{d\rho}\|\frac{\partial}{\partial t}z_1^k\|^2 = 0, \\ \frac{1}{2}\left(\frac{\tau(t)}{(1-\rho\tau'(t))}\right)'\|\frac{\partial}{\partial t}z_2^k\|^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\tau(t)}{(1-\rho\tau'(t))}\|\frac{\partial}{\partial t}z_2^k\|^2\right) + \frac{1}{2}\frac{d}{d\rho}\|\frac{\partial}{\partial t}z_2^k\|^2 = 0. \end{cases}$$

Integrating over $(0,1)$ with respect to ρ, we obtain

$$\begin{cases} \frac{1}{2}\int_0^1 (\frac{\tau(t)}{(1-\rho\tau'(t))})' \| \frac{\partial}{\partial t} z_1^k \|^2 \, d\rho + \frac{1}{2}\frac{d}{dt}(\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))} \| \frac{\partial}{\partial t} z_1^k \|^2 \, d\rho) \\ \quad + \frac{1}{2}\| \frac{\partial}{\partial t} z_1^k(x,1,t)\|^2 - \frac{1}{2}\| u_{tt}^k(x,t)\|^2 = 0, \\ \frac{1}{2}\int_0^1 (\frac{\tau(t)}{(1-\rho\tau'(t))})' \| \frac{\partial}{\partial t} z_2^k \|^2 \, d\rho + \frac{1}{2}\frac{d}{dt}(\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))} \| \frac{\partial}{\partial t} z_2^k \|^2 \, d\rho) \\ \quad + \frac{1}{2}\| \frac{\partial}{\partial t} z_2^k(x,1,t)\|^2 - \frac{1}{2}\| v_{tt}^k(x,t)\|^2 = 0. \end{cases} \tag{2.14}$$

Summing (2.13), (2.14) and as $M(r) \geq a$, we get

$$\begin{cases} \int_\Omega |u_t^k|^l |u_{tt}^k|^2 \, dx + \|\nabla u_{tt}^k\|^2 + \frac{1}{2}\frac{d}{dt}(\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))} \| \frac{\partial}{\partial t} z_1^k \|^2 \, d\rho) + \frac{1}{2}\| \frac{\partial}{\partial t} z_1^k(x,1,t)\|^2 \\ \quad \leq -a \int_\Omega \nabla u^k \nabla u_{tt}^k \, dx + \frac{1}{2}\|u_{tt}^k(x,t)\|^2 - \frac{1}{2}\int_0^1 (\frac{\tau(t)}{(1-\rho\tau'(t))})' \| \frac{\partial}{\partial t} z_1^k \|^2 \, d\rho \\ \qquad + \int_0^t h_1(t-s) \int_\Omega \nabla u^k(s) \nabla u_{tt}^k(t) \, dx \, ds - \mu_1 \int_\Omega \nabla u_{tt}^k \nabla z_1^k(x,1,t) \, dx \\ \qquad - \int_\Omega f_1(u^k,v^k) u_{tt}^k(t) \, dx, \\ \int_\Omega |v_t^k|^l |v_{tt}^k|^2 \, dx + \|\nabla v_{tt}^k\|^2 + \frac{1}{2}\frac{d}{dt}(\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))} \| \frac{\partial}{\partial t} z_2^k \|^2 \, d\rho) + \frac{1}{2}\| \frac{\partial}{\partial t} z_2^k(x,1,t)\|^2 \\ \quad \leq -a \int_\Omega \nabla v^k \nabla v_{tt}^k \, dx + \frac{1}{2}\|v_{tt}^k(x,t)\|^2 - \frac{1}{2}\int_0^1 (\frac{\tau(t)}{(1-\rho\tau'(t))})' \| \frac{\partial}{\partial t} z_2^k \|^2 \, d\rho \\ \qquad + \int_0^t h_2(t-s) \int_\Omega \nabla v^k(s) \nabla v_{tt}^k(t) \, dx \, ds - \mu_2 \int_\Omega \nabla v_{tt}^k \nabla z_2^k(x,1,t) \, dx \\ \qquad - \int_\Omega f_2(u^k,v^k) v_{tt}^k(t) \, dx. \end{cases} \tag{2.15}$$

We estimate the right hand side of (2.15) as follows:

From the integration by parts, we have

$$-\int_\Omega f_1(u^k,v^k) u_{tt}^k(t) \, dx = \alpha \int_\Omega v_t^k u_t^k \, dx - b_1 \int_\Omega |v^k|^{q+1} |u^k|^{p-1} u^k u_{tt}^k \, dx.$$

Using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and Sobolev–Poincaré inequalities, we obtain

$$\alpha \int_\Omega v_t^k u_t^k \, dx \leq \frac{\alpha}{2}\left(\|v_t^k\|^2 + \|u_t^k\|^2\right) \leq \frac{C_s^2 \alpha}{2}\left(\|\nabla v_t^k\|^2 + \|\nabla u_t^k\|^2\right). \tag{2.16}$$

On the other hand, by recalling **(A-4)** and Lemma 1.1 and using Young's inequality, we get

$$\begin{aligned} &\left| \int_\Omega |v^k|^{q+1} |u^k|^{p-1} u^k u_{tt}^k \, dx \right| \\ &\quad \leq \frac{1}{2}\int_\Omega |v^k|^{2(q+1)} |u^k|^{2p} \, dx + \frac{1}{2}\|u_{tt}^k\|^2 \\ &\quad \leq \frac{\eta}{2}\int_\Omega |v^k|^{2(q+1)q} \, dx + \frac{1}{8\eta}\int_\Omega |u^k|^{2p^2} \, dx + \frac{1}{2}\|u_{tt}^k\|^2 \\ &\quad \leq \frac{\eta}{2}|\Omega|^{\frac{q-1}{2q}} \|v^k\|_{4q^2}^{4(q+1)q} + \frac{1}{8\eta}\|u^k\|_{2p^2}^{2p^2} + \frac{1}{2}\|u_{tt}^k\|^2 \\ &\quad \leq \frac{\eta}{2}|\Omega|^{\frac{q-1}{2q}} C_s^{4(q+1)q} \|\nabla v^k\|^{4(q+1)q} + \frac{C_s^{2p^2}}{8\eta}\|\nabla u^k\|^{2p^2} + \frac{C_s^2}{2}\|\nabla u_{tt}^k\|^2. \end{aligned} \tag{2.17}$$

Hence from summing (2.16) and (2.17) we deduce that

$$\begin{aligned}\left|-\int_\Omega f_1(u^k,v^k)u^k_{tt}(t)\,dx\right| &\le \frac{C_s^2\alpha}{2}\left(\|\nabla v^k_t\|^2+\|\nabla u^k_t\|^2\right)\\&\quad+\frac{b_1\eta}{2}|\Omega|^{\frac{q-1}{2q}}C_s^{4(q+1)q}\|\nabla v^k\|^{4(q+1)q}\\&\quad+\frac{b_1C_s^{2p^2}}{8\eta}\|\nabla u^k\|^{2p^2}+\frac{b_1C_s^2}{2}\|\nabla u^k_{tt}\|^2.\end{aligned}\tag{2.18}$$

Similarly

$$\begin{aligned}\left|-\int_\Omega f_2(u^k,v^k)v^k_{tt}(t)\,dx\right| &\le \frac{C_s^2\alpha}{2}\left(\|\nabla v^k_t\|^2+\|\nabla u^k_t\|^2\right)\\&\quad+\frac{b_2\eta}{2}|\Omega|^{\frac{p-1}{2p}}C_s^{4(p+1)p}\|\nabla u^k\|^{4(p+1)p}\\&\quad+\frac{b_2C_s^{2p^2}}{8\eta}\|\nabla v^k\|^{2q^2}+\frac{b_2C_s^2}{2}\|\nabla v^k_{tt}\|^2.\end{aligned}\tag{2.19}$$

Also by Young's inequality, we get

$$\begin{cases}|\int_\Omega a\nabla u^k\nabla u^k_{tt}\,dx|\le\eta\|\nabla u^k_{tt}\|^2+\frac{a^2}{4\eta}\|\nabla u^k\|^2,\\|\int_\Omega a\nabla v^k\nabla v^k_{tt}\,dx|\le\eta\|\nabla v^k_{tt}\|^2+\frac{a^2}{4\eta}\|\nabla v^k\|^2.\end{cases}\tag{2.20}$$

We have

$$\begin{aligned}&\left|\int_0^t h_1(t-s)\int_\Omega\nabla u^k(s)\nabla u^k_{tt}(t)\,dx\,ds\right|\\&\quad\le\eta\|\nabla u^k_{tt}\|^2+\frac{1}{4\eta}\int_\Omega\left(\int_0^t h_1(t-s)\nabla u^k(s)\,ds\right)^2dx\\&\quad\le\eta\|\nabla u^k_{tt}\|^2+\frac{1}{4\eta}\int_0^t h_1(s)\,ds\int_\Omega\int_0^t h_1(t-s)|\nabla u^k(s)|^2\,ds\,dx\\&\quad\le\eta\|\nabla u^k_{tt}\|^2+\frac{a-k}{4\eta}\int_0^t h_1(t-s)\|\nabla u^k(s)\|^2\,ds\\&\quad\le\eta\|\nabla u^k_{tt}\|^2+\frac{(a-k)h_1(0)}{4\eta}\int_0^t\|\nabla u^k(s)\|^2\,ds.\end{aligned}\tag{2.21}$$

Similarly

$$\begin{aligned}&\left|\int_0^t h_2(t-s)\int_\Omega\nabla v^k(s)\nabla v^k_{tt}(t)\,dx\,ds\right|\\&\quad\le\eta\|\nabla v^k_{tt}\|^2+\frac{(a-k)h_2(0)}{4\eta}\int_0^t\|\nabla v^k(s)\|^2\,ds\end{aligned}\tag{2.22}$$

and

$$\begin{cases}|\mu_1\int_\Omega\nabla u^k_{tt}\nabla z^k_1(x,1,t)\,dx|\le\eta\mu_1^2\|\nabla u^k_{tt}\|^2+\frac{1}{4\eta}\|\nabla z^k_1(x,1,t)\|^2,\\|\mu_2\int_\Omega\nabla v^k_{tt}\nabla z^k_2(x,1,t)\,dx|\le\eta\mu_2^2\|\nabla v^k_{tt}\|^2+\frac{1}{4\eta}\|\nabla z^k_2(x,1,t)\|^2.\end{cases}\tag{2.23}$$

Taking into account (2.18)–(2.23) into (2.15) yields

$$\begin{cases} \int_\Omega |u_t^k|^l |u_{tt}^k|^2\,dx + \|\nabla u_{tt}^k\|^2 + \frac{1}{2}\frac{d}{dt}(\int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|\frac{\partial}{\partial t}z_1^k\|^2\,d\rho) + \frac{1}{2}\|\frac{\partial}{\partial t}z_1^k(x,1,t)\|^2 \\ \quad \le (\eta(\mu_1^2+2) + \frac{C_s^2}{2})\|\nabla u_{tt}^k\|^2 \\ \qquad + \frac{a^2}{4\eta}\|\nabla u^k\|^2 + \frac{C_s^2\alpha}{2}(\|\nabla v_t^k\|^2 + \|\nabla u_t^k\|^2) + \frac{b_1\eta}{2}|\Omega|^{\frac{q-1}{2q}} C_s^{4(q+1)q}\|\nabla v^k\|^{4(q+1)q} \\ \qquad + \frac{b_1 C_s^{2p^2}}{8\eta}\|\nabla u^k\|^{2p^2} + \frac{b_1 C_s^2}{2}\|\nabla u_{tt}^k\|^2 \\ \qquad + \frac{1}{4\eta}\|\nabla z_1^k(x,1,t)\|^2 + \frac{1}{4\eta}(a-k)h_1(0)\int_0^t \|\nabla u^k(s)\|^2\,ds - \frac{1}{2}\int_0^1 (\frac{\tau(t)}{1-\rho\tau'(t)})'\|\frac{\partial}{\partial t}z_1^k\|^2\,d\rho, \\ \int_\Omega |v_t^k|^l |v_{tt}^k|^2\,dx + \|\nabla v_{tt}^k\|^2 + \frac{1}{2}\frac{d}{dt}(\int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|\frac{\partial}{\partial t}z_2^k\|^2\,d\rho) + \frac{1}{2}\|\frac{\partial}{\partial t}z_2^k(x,1,t)\|^2 \\ \quad \le (\eta(\mu_2^2+2) + \frac{C_s^2}{2})\|\nabla v_{tt}^k\|^2 \\ \qquad + \frac{a^2}{4\eta}\|\nabla v^k\|^2 + \frac{C_s^2\alpha}{2}(\|\nabla v_t^k\|^2 + \|\nabla u_t^k\|^2) + \frac{b_2\eta}{2}|\Omega|^{\frac{p-1}{2p}} C_s^{4(p+1)p}\|\nabla u^k\|^{4(p+1)p} \\ \qquad + \frac{b_2 C_s^{2p^2}}{8\eta}\|\nabla v^k\|^{2q^2} + \frac{b_2 C_s^2}{2}\|\nabla v_{tt}^k\|^2 \\ \qquad + \frac{1}{4\eta}\|\nabla z_2^k(x,1,t)\|^2 + \frac{1}{4\eta}(a-k)h_2(0)\int_0^t \|\nabla v^k(s)\|^2\,ds \\ \qquad - \frac{1}{2}\int_0^1 (\frac{\tau(t)}{1-\rho\tau'(t)})'\|\frac{\partial}{\partial t}z_2^k\|^2\,d\rho. \end{cases}$$

By using **(A3)** and taking the first estimate (2.9) into account, we infer

$$\begin{cases} \int_\Omega |u_t^k|^l |u_{tt}^k|^2\,dx + (1 - (\eta(\mu_1^2+2) + \frac{(1+b_1)C_s^2}{2}))\|\nabla u_{tt}^k\|^2 \\ \quad + \frac{1}{2}\frac{d}{dt}(\int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|\frac{\partial}{\partial t}z_1^k\|^2\,d\rho) \\ \quad + \frac{1}{2}\|\frac{\partial}{\partial t}z_1^k(x,1,t)\|^2 \le C_2 + \frac{1}{4\eta}(a-k_1)h_1(0)C_1T, \\ \int_\Omega |v_t^k|^l |v_{tt}^k|^2\,dx + (1 - (\eta(\mu_2^2+2) + \frac{(1+b_2)C_s^2}{2}))\|\nabla v_{tt}^k\|^2 \\ \quad + \frac{1}{2}\frac{d}{dt}(\int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|\frac{\partial}{\partial t}z_2^k\|^2\,d\rho) \\ \quad + \frac{1}{2}\|\frac{\partial}{\partial t}z_2^k(x,1,t)\|^2 \le C_2 + \frac{1}{4\eta}(a-k_2)h_2(0)C_1T, \end{cases} \tag{2.24}$$

where C_2 is a positive constant that depends on $\eta, \alpha, a, C_s, |\Omega|, b_1, b_2, p, q, C_1$ for $i = 1, 2$.

Integrating (2.24) over (0,t) we obtain

$$\begin{cases} \int_0^t \int_\Omega |u_t^k|^l |u_{tt}^k|^2\,dx\,dt + (1 - (\eta(\mu_1^2+2) + \frac{(1+b_1)C_s^2}{2}))\int_0^t \|\nabla u_{tt}^k(s)\|^2\,ds \\ \quad + \int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|\frac{\partial}{\partial t}z_1^k\|^2\,d\rho + \frac{1}{2}\int_0^1 \|\frac{\partial}{\partial t}z_1^k(x,1,t)\|^2\,dt \le (C_2 + \frac{1}{4\eta}(a-k)h_1(0)C_1T)T, \\ \int_0^t \int_\Omega |v_t^k|^l |v_{tt}^k|^2\,dx\,dt + (1 - (\eta(\mu_2^2+2) + \frac{(1+b_2)C_s^2}{2}))\int_0^t \|\nabla v_{tt}^k(s)\|^2\,ds \\ \quad + \int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|\frac{\partial}{\partial t}z_2^k\|^2\,d\rho + \frac{1}{2}\int_0^1 \|\frac{\partial}{\partial t}z_2^k(x,1,t)\|^2\,dt \le (C_2 + \frac{1}{4\eta}(a-k)h_2(0)C_1T)T. \end{cases}$$

For a suitable $\eta > 0$ such that $1 - (\eta(\mu_i^2+2) + \frac{(1+b_i)C_s^2}{2}) > 0$ for $i = 1, 2$, we obtain the second estimate

$$\int_0^t \left(\|\nabla u_{tt}^k(s)\|^2 + \|\nabla v_{tt}^k(s)\|^2\right)ds + \int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\left(\left\|\frac{\partial}{\partial t}z_1^k\right\|^2 + \left\|\frac{\partial}{\partial t}z_2^k\right\|^2\right)d\rho \le C_3. \tag{2.25}$$

We observe from the estimate (2.9) and (2.25) that there exist subsequences (u^m) of (u^k) and (v^m) of (v^k) such that

$$(u^m, v^m) \rightharpoonup (u, v) \quad \text{weakly star in } L^\infty(0, T, H_0^1(\Omega)), \tag{2.26}$$

$$\left(u_t^m, v_t^m\right) \rightharpoonup (u_t, v_t) \quad \text{weakly star in } L^\infty\left(0, T, H_0^1(\Omega)\right), \tag{2.27}$$

$$\left(u_{tt}^m, v_{tt}^m\right) \rightharpoonup (u_{tt}, v_{tt}) \quad \text{weakly in} L^2\left(0, T, H_0^1(\Omega)\right), \tag{2.28}$$

$$\left(z_1^m, z_1^m\right) \rightharpoonup (z_1, z_2) \quad \text{weakly star in } L^\infty\left(0, T, H_0^1\left(\Omega, L^2(0,1)\right)\right), \tag{2.29}$$

$$\left(\frac{\partial}{\partial t} z_1^m, \frac{\partial}{\partial t} z_2^m\right) \rightharpoonup \left(\frac{\partial}{\partial t} z_1, \frac{\partial}{\partial t} z_2\right) \quad \text{weakly star in } L^\infty\left(0, T, L^2\left(\Omega \times (0,1)\right)\right). \tag{2.30}$$

In the following, we will treat the nonlinear term. From the first estimate (2.9) and Lemma 1.1, we deduce

$$\begin{aligned} \left\| |u_t^k|^l u_t^k \right\|_{L^2(0,T,L^2(\Omega))} &= \int_0^T \left\| u_t^k \right\|_{2(l+1)}^{2(l+1)} dt \\ &\le C_s^{2(l+1)} \int_0^T \left\| \nabla u_t^k \right\|_2^{2(l+1)} dt \le C_4, \end{aligned}$$

where C_4 depends only on C_s, C_1, T, l.

On the other hand, from the Aubin–Lions theorem (see Lions [9]), we deduce that there exists a subsequence of (u^m), still denoted by (u^m), such that

$$u_t^m \to u_t \quad \text{strongly in } L^2\left(0, T, L^2(\Omega)\right), \tag{2.31}$$

which implies

$$u_t^m \to u_t \quad \text{almost everywhere in } \mathcal{A}. \tag{2.32}$$

Hence

$$\left|u_t^m\right|^l u_t^m \to |u_t|^l u_t \quad \text{almost everywhere in } \mathcal{A}, \tag{2.33}$$

where $\mathcal{A} = \Omega \times (0, T)$. Thus, using (2.31), (2.33) and the Lions lemma, we derive

$$\left|u_t^m\right|^l u_t^m \rightharpoonup |u_t|^l u_t \quad \text{weakly in } L^2\left(0, T, L^2(\Omega)\right); \tag{2.34}$$

similarly

$$\left|v_t^m\right|^l v_t^m \rightharpoonup |v_t|^l v_t \quad \text{weakly in } L^2\left(0, T, L^2(\Omega)\right) \tag{2.35}$$

and

$$\left(z_1^m, z_2^m\right) \to (z_1, z_2) \quad \text{strongly in } L^2\left(0, T, L^2(\Omega)\right),$$

which implies $(z_1^m, z_2^m) \to (z_1, z_2)$ almost everywhere in $\mathcal{A}$.

The sequences (u^m) and (v^m) satisfy

$$f_1\left(u^m, v^m\right) \to f_1(u, v) \quad \text{strongly in } L^\infty\left(0, T, L^2(\Omega)\right) \tag{2.36}$$

and

$$f_2(u^m,v^m) \to f_2(u,v) \quad \text{strongly in } L^\infty(0,T,L^2(\Omega)); \tag{2.37}$$

we have

$$\|f_1(u^m,v^m)-f_1(u,v)\|^2 = \int_\Omega \big| |v^k|^{q+1}|u^k|^p u^k - |v|^{q+1}|u|^p u \big|^2\,dx.$$

As we add and subtract $|v^k|^{q+1}|u|^p u$ to the previous formula, we obtain

$$\begin{aligned}\|f_1(u^m,v^m)-f_1(u,v)\|^2 &\le \int_\Omega \big| |v^k|^{q+1}\big||u^k|^p u^k - |u|^p u\big| + |u|^{p+1}\big||v^k|^{q+1} - |v|^{q+1}\big|\big|^2\,dx \\ &\le 2\bigg[\int_\Omega |v^k|^{2(q+1)}\big||u^k|^p u^k - |u|^p u\big|^2\,dx \\ &\quad + \int_\Omega |u|^{2(p+1)}\big||v^k|^{q+1} - |v|^{q+1}\big|^2\,dx\bigg].\end{aligned} \tag{2.38}$$

We use the following elementary inequalities:

$$\begin{aligned}&\big||a|^k - |b|^k\big| \le C|a-b|\big(|a|^{k-1}+|b|^{k-1}\big),\\ &\big||a|^k a - |b|^k b\big| \le C|a-b|\big(|a|^k+|b|^k\big),\end{aligned}$$

and

$$(a+b)^2 \le 2(a^2+b^2),$$

for some constant C, $\forall k \ge 1$ and $\forall a,b \in \mathbb{R}$. Hence (2.38) becomes

$$\begin{aligned}\|f_1(u^m,v^m)-f_1(u,v)\|^2 &\le 4C\bigg[\int_\Omega |v^k|^{2(q+1)}|u^k-u|^2\big(|u^k|^{2p}+|u|^{2p}\big)\,dx \\ &\quad + \int_\Omega |u|^{2(p+1)}|v^k-v|^2\big(|v^k|^{2q}+|v|^{2q}\big)\,dx\bigg].\end{aligned} \tag{2.39}$$

The typical term in the above formula can be estimated as follows.

Noting that $\frac{l}{2p}+\frac{1}{2q}+\frac{1}{2}=1$, by applying the generalized Hölder inequality, we find

$$\begin{aligned}&\int_\Omega |v^k|^{2(q+1)}|u^k-u|^2|u^k|^{2p}\,dx \\ &\quad\le \bigg(\int_\Omega |v^k|^{4(q+1)}\,dx\bigg)^{\frac{1}{2}}\bigg(\int_\Omega |u^k-u|^{4q}\,dx\bigg)^{\frac{1}{2q}}\bigg(\int_\Omega |u^k|^{4p^2}\,dx\bigg)^{\frac{1}{2p}}.\end{aligned} \tag{2.40}$$

Recalling $(A4)$, Lemma 1.1 and (2.9), we get

$$\int_\Omega |v^k|^{2(q+1)}|u^k-u|^2|u^k|^{2p}\,dx \le C\|\nabla(u^k-u)\|^2. \tag{2.41}$$

Hence (2.39) yields

$$\|f_1(u^m,v^m)-f_1(u,v)\|^2 \le C\big[\|\nabla(u^k-u)\|^2+\|\nabla(v^k-v)\|^2\big]. \tag{2.42}$$

As (u^m), (v^m) are Cauchy sequences in $L^\infty(0,T,H_0^1(\Omega))$ (we prove it as in [1]) then we deduce (2.36). Similarly we get the convergence (2.37).

By multiplying (2.1) and (2.6) by $\theta(t)\in\mathcal{D}(0,T)$ and by integrating over $(0,T)$, it follows that

$$\begin{cases} -\frac{1}{l+1}\int_0^T(|u_t^k(t)|^l u_t^k(t),w^j)\theta'(t)\,dt+\int_0^T M(\|\nabla u^k(t)\|^2)(\nabla u^k(t),\nabla w^j)\theta(t)\,dt \\ \quad +\int_0^T(\nabla u_{tt}^k,\nabla w^j)\theta(t)\,dt-\int_0^T\int_0^t h_1(t-s)(\nabla u^k(s),\nabla w^j)\theta(t)\,ds\,dt \\ \quad +\mu_1\int_0^T(\nabla z_1^k(\cdot,1)),\nabla w^j)\theta(t)\,dt+\int_0^T(f_1(u^k,v^k),w^j)\theta(t)\,dt=0, \\ -\frac{1}{l+1}\int_0^T(|v_t^k(t)|^l v_t^k(t),w^j)\theta'(t)\,dt+\int_0^T M(\|\nabla v^k(t)\|^2)(\nabla v^k(t),\nabla w^j)\theta(t)\,dt \\ \quad +\int_0^T(\nabla v_{tt}^k,\nabla w^j)\theta(t)\,dt-\int_0^T\int_0^t h_2(t-s)(\nabla v^k(s),\nabla w^j)\theta(t)\,ds\,dt \\ \quad +\mu_2\int_0^T(\nabla z_2^k(\cdot,1)),w^j)\theta(t)\,dt+\int_0^T(f_2(u^k,v^k),w^j)\theta(t)\,dt=0, \\ \int_0^T\int_0^1\int_\Omega(\tau(t)\frac{\partial}{\partial t}z_1^k+(1-\rho\tau'(t))\frac{\partial}{\partial\rho}z_1^k)\phi^j\theta(t)\,dx\,d\rho\,dt=0, \\ \int_0^T\int_0^1\int_\Omega(\tau(t)\frac{\partial}{\partial t}z_2^k+(1-\rho\tau'(t))\frac{\partial}{\partial\rho}z_2^k)\phi^j\theta(t)\,dx\,d\rho\,dt=0, \end{cases} \tag{2.43}$$

for all $j=1,\dots,k$.

The convergence of (2.26)–(2.30), (2.35), (2.34), (2.36) and (2.37) is sufficient to pass to the limit in (2.43). This completes the proof of the theorem. □

3 Exponential decay rate

In order to make precise the asymptotic behavior of our solutions, we introduce some functionality to determine a suitable Lyapunov functional equivalent to E.

Theorem 3.1 *Assume that* **(A1)**–**(A3)** *hold. Then for every* $t_0>0$ *there exist positive constants* K *and* c' *such that the energy defined by* (1.6) *obeys the following decay*:

$$E(t)\le Ke^{-c't},\quad \forall t\ge t_0. \tag{3.1}$$

Lemma 3.2 *Along a solution of the problem* (1.5) *the functional*

$$I(t)=\tau(t)\int_0^1 e^{-2\tau(t)\rho}\big(\|\nabla z_1\|^2+\|\nabla z_2\|^2\big)\,d\rho \tag{3.2}$$

satisfies the following estimates:

$$|I(t)|\le\frac{1}{\xi}E(t), \tag{3.3}$$

$$\begin{aligned} I'(t)\le{}& -2\tau(t)e^{-2\tau_1}\int_0^1\big(\|\nabla z_1\|^2+\|\nabla z_2\|^2\big)\,d\rho+\|\nabla u_t\|^2+\|\nabla v_t\|^2 \\ &-(1-d)e^{-2\tau_1}\big(\|\nabla z_1(x,1,t)\|^2+\|\nabla z_2(x,1,t)\|^2\big). \end{aligned} \tag{3.4}$$

Proof (ii) A direct derivation of (3.2) gives

$$I'(t)=\int_0^1\bigl[\tau'^{-2\tau(t)\rho}\bigl(\|\nabla z_1\|^2+\|\nabla z_2\|^2\bigr)+\tau(t)e^{-2\tau(t)\rho}\bigl(\|\nabla z_1'\|^2+\|\nabla z_2'\|^2\bigr)\bigr]\,d\rho.$$

Recalling (1.3)–(1.4)

$$\begin{aligned}I'(t)&=\int_0^1\biggl[\tau'^{-2\tau(t)\rho}\bigl(\|\nabla z_1\|^2+\|\nabla z_2\|^2\bigr)\\&\quad-\tau(t)e^{-2\tau(t)\rho}\bigl(1-\rho\tau'(t)\bigr)\frac{\partial}{\partial\rho}\bigl(\|\nabla z_1\|^2+\|\nabla z_2\|^2\bigr)\biggr]\,d\rho\\&=-\int_0^1\biggl[\frac{\partial}{\partial\rho}\bigl(e^{-2\tau(t)\rho}\bigl(1-\tau'(t)\rho\bigr)\bigl(\|\nabla z_1\|^2+\|\nabla z_2\|^2\bigr)\bigr)\\&\quad-2\tau(t)e^{-2\tau(t)\rho}\bigl(\|\nabla z_1\|^2+\|\nabla z_2\|^2\bigr)\biggr]\,d\rho\\&=\|\nabla u_t\|^2+\|\nabla v_t\|^2-e^{-2\tau(t)}\bigl(1-\tau'(t)\bigr)\bigl(\|\nabla z_1(x,1,t)\|^2+\|\nabla z_2(x,1,t)\|^2\bigr)-2I(t).\end{aligned}$$

Because the exponential function $e^{-2\rho\tau(t)}$ decreases on $(0,1)\times(\tau_0,\tau_1)$ and from $(A3)$, we get the results of this lemma. □

Lemma 3.3 *Along a solution of the problem* (1.5) *the functional*

$$\phi(t)=\frac{1}{l+1}\int_\Omega\bigl(|u_t|^lu_tu+|v_t|^lv_tv\bigr)\,dx+\int_\Omega\nabla u_t\nabla u\,dx+\int_\Omega\nabla v_t\nabla v\,dx$$

verifies the estimates

$$\begin{aligned}\bigl|\phi(t)\bigr|&\le\frac{1}{l+2}\bigl(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\bigr)+\biggl(\frac{(l+1)^{-1}}{l+2}c_s^{l+2}+\frac{c}{2}\biggr)\bigl(\|\nabla u\|^{l+2}+\|\nabla v\|^{l+2}\bigr)\\&\quad+\frac{1}{2}\bigl(\|\nabla u_t\|^2+\|\nabla v_t\|^2\bigr)\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\phi'(t)&\le\frac{1}{l+1}\bigl(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\bigr)\\&\quad+\biggl(\eta(a-k+1)-k+\biggl(\frac{b_1+b_2}{2}+\alpha\biggr)C_s^2\biggr)\bigl(\|\nabla u\|^2+\|\nabla v\|^2\bigr)\\&\quad+\frac{1}{4\eta}\bigl[(h_1o\nabla u)(t)+(h_2o\nabla v)(t)\bigr]+\frac{\mu_1^2}{4\eta}\bigl\|\nabla z_1(x,1,t)\bigr\|^2+\frac{\mu_2^2}{4\eta}\bigl\|\nabla z_2(x,1,t)\bigr\|^2\\&\quad+\|\nabla u_t\|^2+\|\nabla v_t\|^2.\end{aligned}\tag{3.6}$$

Proof (i) Applying Young's inequality, Sobolev–Poincaré's inequality and $L^{l+2}\hookrightarrow L^2$, we find

$$\begin{aligned}\bigl|\phi(t)\bigr|&\le\frac{1}{l+2}\|u_t\|_{l+2}^{l+2}+\frac{(l+1)^{-1}}{l+2}\|u\|_{l+2}^{l+2}+\frac{1}{l+2}\|v_t\|_{l+2}^{l+2}+\frac{(l+1)^{-1}}{l+2}\|v\|_{l+2}^{l+2}\\&\quad+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\|\nabla v_t\|^2+\frac{1}{2}\|\nabla v\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{l+2}\|u_t\|_{l+2}^{l+2}+\frac{(l+1)^{-1}}{l+2}c_s^{l+2}\|\nabla u\|^{l+2}+\frac{1}{l+2}\|v_t\|_{l+2}^{l+2}+\frac{(l+1)^{-1}}{l+2}c_s^{l+2}\|\nabla v\|^{l+2} \\
&\quad+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\|\nabla v_t\|^2+\frac{1}{2}\|\nabla v\|^2 \\
&\leq \frac{1}{l+2}\big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\big)+\left(\frac{(l+1)^{-1}}{l+2}c_s^{l+2}+\frac{c}{2}\right)\big(\|\nabla u\|^{l+2}+\|\nabla v\|^{l+2}\big) \\
&\quad+\frac{1}{2}\big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\big).
\end{aligned}
$$

(ii) Taking a direct derivation of (3.2) and replacing $|u_t|^l u_{tt}$, $|v_t|^l v_{tt}$ from the first and seconde equations of (1.5) give

$$
\begin{aligned}
\phi'(t) &= \frac{1}{l+1}\int_\Omega \big(|u_t|^l u_t\big)' u\,dx+\frac{1}{l+1}\int_\Omega |u_t|^{l+2}\,dx \\
&\quad+\frac{1}{l+1}\int_\Omega \big(|v_t|^l v_t\big)' v\,dx+\frac{1}{l+1}\int_\Omega |v_t|^{l+2}\,dx \\
&\quad+\int_\Omega \nabla u_{tt}\nabla u\,dx+\int_\Omega \nabla u_t\nabla u_t\,dx+\int_\Omega \nabla v_{tt}\nabla v\,dx+\int_\Omega \nabla v_t\nabla v_t\,dx \\
&= \int_\Omega \big[|u_t|^l u_{tt}\big]u\,dx+\frac{1}{l+1}\|u_t\|_{l+2}^{l+2}+\int_\Omega \big[|v_t|^l v_{tt}\big]v\,dx+\frac{1}{l+1}\|v_t\|_{l+2}^{l+2} \\
&\quad-\int_\Omega \Delta u_{tt}u\,dx+\|\nabla u_t\|^2-\int_\Omega \Delta v_{tt}u\,dx+\|\nabla v_t\|^2 \\
&= \frac{1}{l+1}\big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\big)+\int_\Omega \big[|u_t|^l u_{tt}-\Delta u_{tt}\big]u\,dx+\int_\Omega \big[|v_t|^l v_{tt} \\
&\quad-\Delta v_{tt}\big]v\,dx+\|\nabla u_t\|^2+\|\nabla v_t\|^2 \\
&= \frac{1}{l+1}\big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\big)+\int_\Omega \bigg[-f_1(u,v)+M\big(\|\nabla u\|^2\big)\Delta u \\
&\quad-\int_0^t h_1(t-s)\Delta u(s)\,ds+\mu_1\Delta z_1(x,1,t)\bigg]u\,dx \\
&\quad+\int_\Omega \bigg[-f_2(u,v)+M\big(\|\nabla v\|^2\big)\Delta v-\int_0^t h_2(t-s)\Delta v(s)\,ds+\mu_2\Delta z_2(x,1,t)\bigg]v\,dx \\
&\quad+\|\nabla u_t\|^2+\|\nabla v_t\|^2 \\
&= \frac{1}{l+1}\big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\big)-M\big(\|\nabla u\|^2\big)\|\nabla u\|^2 \\
&\quad+\int_\Omega \nabla u(t)\int_0^t h_1(t-s)\nabla u(s)\,ds\,dx-\mu_1\int_\Omega \nabla z_1(x,1,t)\nabla u\,dx \\
&\quad-M\big(\|\nabla v\|^2\big)\|\nabla v\|^2+\int_\Omega \nabla v(t)\int_0^t h_2(t-s)\nabla v(s)\,ds\,dx \\
&\quad-\mu_2\int_\Omega \nabla z_2(x,1,t)\nabla v\,dx+\|\nabla u_t\|^2+\|\nabla v_t\|^2 \\
&\quad-(b_1+b_2)\int_\Omega |v|^{q+1}|u|^{p+1}\,dx-2\alpha\int_\Omega uv\,dx.
\end{aligned}
\tag{3.7}
$$

As $M(r)\geq a$ and making use of Young's inequality we obtain

$$
\phi'(t)\leq \frac{1}{l+1}\big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\big)-a\|\nabla u\|^2+\int_\Omega \nabla u(t)\int_0^t h_1(t-s)\nabla u(s)\,ds\,dx
$$

$$
\begin{aligned}
&+\frac{\mu_1^2}{4\eta}\left\|\nabla z_1(x,1,t)\right\|^2+\eta\|\nabla u\|^2\\
&-a\|\nabla v\|^2+\int_\Omega \nabla v(t)\int_0^t h_2(t-s)\nabla v(s)\,ds\,dx\\
&+\frac{\mu_2^2}{4\eta}\left\|\nabla z_2(x,1,t)\right\|^2+\eta\|\nabla v\|^2+\|\nabla u_t\|^2+\|\nabla v_t\|^2\\
&-(b_1+b_2)\int_\Omega |v|^{q+1}|u|^{p+1}\,dx-2\alpha\int_\Omega uv\,dx. \qquad (3.8)
\end{aligned}
$$

By use of Young's inequality, the third term in the right side is estimated as follows:

$$
\begin{aligned}
\int_\Omega \nabla u(t)\int_0^t h_1(t-s)\nabla u(s)\,ds\,dx &\le \int_0^t h(t-s)\int_\Omega \left|\nabla u(t)\left(\nabla u(s)-\nabla u(t)\right)\right|dx\,ds\\
&\quad+\left\|\nabla u(t)\right\|^2\int_0^t h_1(t-s)\,ds\\
&\le (1+\eta)\left\|\nabla u(t)\right\|^2\int_0^t h_1(s)\,ds\\
&\quad+\frac{1}{4\eta}\int_0^t h_1(t-s)\left\|\nabla u(s)-\nabla u(t))\right\|^2 ds\\
&\le (1+\eta)(a-k)\left\|\nabla u(t)\right\|^2+\frac{1}{4\eta}(h_1 o\nabla u)(t).
\end{aligned}
$$

Similarly

$$
\int_\Omega \nabla v(t)\int_0^t h_2(t-s)\nabla v(s)\,ds\,dx \le (1+\eta)(a-k)\left\|\nabla v(t)\right\|^2+\frac{1}{4\eta}(h_2 o\nabla v)(t),
$$

and from $(A4)$

$$
-(b_1+b_2)\int_\Omega |v|^{q+1}|u|^{p+1}\,dx-2\alpha\int_\Omega uv\,dx \le \left(\frac{b_1+b_2}{2}+\alpha\right)C_s^2\left(\|\nabla v\|^2+\|\nabla u\|^2\right).
$$

Thus, (3.6) is valid. □

Lemma 3.4 *Along a solution of the problem* (1.5) *the functional*

$$
\begin{aligned}
\psi(t)={}&\int_\Omega\left(\Delta u_t-\frac{1}{l+1}|u_t|^l u_t\right)\int_0^t h_1(t-s)\left(u(t)-u(s)\right)ds\,dx+\int_\Omega\bigg(\Delta v_t\\
&-\frac{1}{l+1}|v_t|^l v_t\bigg)\int_0^t h_2(t-s)\left(v(t)-v(s)\right)ds\,dx
\end{aligned}
$$

satisfies the estimates

$$
\begin{aligned}
\left|\psi(t)\right| \le{}& \frac{1}{2}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)+\frac{1}{2}(a-k)\bigg(1\\
&+\frac{(l+1)^{-1}}{(l+2)}(a-k)^l c_s^{l+2}\bigg)\left[(h_1 o\nabla u)(t)+(h_2 o\nabla v)(t)\right]\\
&+\frac{(l+1)^{-1}}{(l+2)}(a-k)^{l+2}c_s^{l+2}2^{2l+1}\left(\|\nabla u\|^{2(l+1)}+\|\nabla v\|^{2(l+1)}\right)
\end{aligned}
$$

$$+\frac{1}{l+2}\big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\big) \tag{3.9}$$

and

$$\begin{aligned}
\psi'(t) \le{}& \delta\bigg[(a-k)+\frac{(l+1)^{-1}}{(l+2)}\big(h_1(0)\big)^{l+2}c_s^{l+2}2^{2(l+1)}+b_2\frac{c_s^{4(p+1)}}{2}\\
&+\frac{c_s^{4p}}{2}b_1\bigg]M\big(\|\nabla u\|^2\big)\|\nabla u\|^2\\
&+\bigg(2\delta(a-k)^2+\frac{\alpha c_s^2}{2}\bigg)\|\nabla u\|^2+\bigg(\frac{M(\|\nabla u\|^2)}{4\delta}\\
&+\bigg(2\delta+\frac{1}{3\delta}+\frac{\alpha c_s^2}{2}\bigg)(a-k)\bigg)(h_1 o\nabla u)(t)\\
&-\frac{h_1(0)}{4\delta}\bigg(1+\frac{(l+1)^{-1}}{(l+2)}\big(h_1(0)\big)^l c_s^{l+2}\bigg)\big(h_1' o\nabla u\big)(t)\\
&+\bigg(\delta-\int_0^t h_1(s)\,ds\bigg)\|\nabla u_t\|^2+\mu_1^2\delta\big\|\nabla z_1(x,1,t)\big\|^2\\
&+\frac{1}{l+1}\bigg(1-\int_0^t h_1(s)\,ds\bigg)\|u_t\|_{l+2}^{l+2}\\
&+\delta\bigg[(a-k)+\frac{(l+1)^{-1}}{(l+2)}\big(h_2(0)\big)^{l+2}c_s^{l+2}2^{2(l+1)}+b_1\frac{c_s^{4(q+1)}}{2}\\
&+\frac{c_s^{4q}}{2}b_2\bigg]M\big(\|\nabla v\|^2\big)\|\nabla v\|^2\\
&+\bigg(2\delta(a-k)^2+\frac{\alpha c_s^2}{2}\bigg)\|\nabla v\|^2+\bigg(\frac{M(\|\nabla v\|^2)}{4\delta}\\
&+\bigg(2\delta+\frac{1}{3\delta}+\frac{\alpha c_s^2}{2}\bigg)(a-k)\bigg)(h_2 o\nabla v)(t)\\
&-\frac{h_2(0)}{4\delta}\bigg(1+\frac{(l+1)^{-1}}{(l+2)}\big(h_2(0)\big)^l c_s^{l+2}\bigg)\big(h_2' o\nabla v\big)(t)\\
&+\bigg(\delta-\int_0^t h_2(s)\,ds\bigg)\|\nabla v_t\|^2+\mu_2^2\delta\big\|\nabla z_2(x,1,t)\big\|^2\\
&+\frac{1}{l+1}\bigg(1-\int_0^t h_2(s)\,ds\bigg)\|v_t\|_{l+2}^{l+2},
\end{aligned} \tag{3.10}$$

where $\delta > 0$ *and* c_s *is the Sobolev embedding constant.*

Proof We have

$$\begin{aligned}
\psi(t) ={}& -\int_\Omega \nabla u_t\int_0^t h_1(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\\
&-\int_\Omega \frac{1}{l+1}|u_t|^l u_t\int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx
\end{aligned}$$

$$\begin{aligned}
&-\int_\Omega \nabla v_t \int_0^t h_2(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx \\
&-\int_\Omega \frac{1}{l+1}|v_t|^l v_t \int_0^t h_2(t-s)\big(v(t)-v(s)\big)\,ds\,dx.
\end{aligned}$$

We use Young's inequality with the conjugate exponents $p' = \frac{l+2}{l+1}$ and $q' = l+2$, then the second term in the right hand side can be estimated as

$$\begin{aligned}
&\left|-\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\quad\le \frac{1}{l+1}\left|\int_\Omega \big(|u_t|^l u_t\big)\left(\int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\right)dx\right| \\
&\quad\le \frac{1}{l+1}\left[\frac{1}{p'}\int_\Omega \big||u_t|^l u_t\big|^{p'}\,dx + \frac{1}{q'}\int_\Omega \left|\int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\right|^{q'} dx\right] \\
&\quad\le \frac{1}{l+1}\left[\frac{1}{p'}\int_\Omega \big(|u_t|^{l+1}\big)^{p'}\,dx + \frac{1}{q'}\int_\Omega \left(\int_0^t h_1(t-s)\big|u(t)-u(s)\big|\,ds\right)^{q'} dx\right] \\
&\quad\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} \\
&\qquad + \frac{(l+1)^{-1}}{l+2}\int_\Omega \left[\int_0^t \big(h_1(t-s)\big)^{\frac{l+1}{l+2}}\Big(\big(h_1(t-s)\big)^{\frac{1}{l+2}}\big|u(t)-u(s)\big|\Big)\,ds\right]^{l+2} dx.
\end{aligned} \tag{3.11}$$

We get by using Hölder's inequality

$$\begin{aligned}
&\int_\Omega \left[\int_0^t \big(h_1(t-s)\big)^{\frac{l+1}{l+2}}\Big(\big(h_1(t-s)\big)^{\frac{1}{l+2}}\big|u(t)-u(s)\big|\Big)\,ds\right]^{l+2} dx \\
&\quad\le \int_\Omega \left[\left(\int_0^t \Big(\big(h_1(t-s)\big)^{\frac{l+1}{l+2}}\Big)^{p'}\,ds\right)^{\frac{1}{p'}}\left(\int_0^t \Big(\big(h_1(t-s)\big)^{\frac{1}{l+2}}\big|u(t)-u(s)\big|\Big)^{q'}\,ds\right)^{\frac{1}{q'}}\right]^{l+2} dx \\
&\quad\le \int_\Omega \left[\left(\int_0^t h_1(t-s)\,ds\right)^{\frac{l+1}{l+2}}\left(\int_0^t h_1(t-s)\big|u(t)-u(s)\big|^{l+2}\,ds\right)^{\frac{1}{l+2}}\right]^{l+2} dx \\
&\quad\le \left(\int_0^t h_1(t-s)\,ds\right)^{l+1}\int_0^t h_1(t-s)\big\|u(t)-u(s)\big\|_{l+2}^{l+2}\,ds \\
&\quad\le (a-k)^{l+1}c_s^{l+2}\int_0^t \sqrt{h_1(t-s)}\sqrt{h_1(t-s)}\big\|\nabla u(t)-\nabla u(s)\big\|^{l+1}\big\|\nabla u(t)-\nabla u(s)\big\|\,ds \\
&\quad\le (a-k)^{l+1}c_s^{l+2}\left(\frac{1}{2}\int_0^t h_1(t-s)\big\|\nabla u(t)-\nabla u(s)\big\|^{2l+2}\,ds\right. \\
&\qquad\left. + \frac{1}{2}\int_0^t h_1(t-s)\big\|\nabla u(t)-\nabla u(s)\big\|^2\,ds\right) \\
&\quad\le (a-k)^{l+1}c_s^{l+2}\left(\frac{1}{2}\int_0^t h_1(t-s)\big\|2\nabla u(t)\big\|^{2l+2}\,ds + \frac{1}{2}(h_1 o \nabla u)(t)\right) \\
&\quad\le (a-k)^{l+1}c_s^{l+2}\left(2^{2l+1}(a-k)\big\|\nabla u(t)\big\|^{2(l+1)} + \frac{1}{2}(h_1 o \nabla u)(t)\right).
\end{aligned} \tag{3.12}$$

Combining (3.12) with (3.11) we obtain

$$
\begin{aligned}
&\left|-\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\quad\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} \\
&\qquad+\frac{(l+1)^{-1}}{l+2}\left[(a-k)^{l+1}c_s^{l+2}\left(2^{2l+1}(a-k)\big\|\nabla u(t)\big\|^{2(l+1)}+\frac{1}{2}(h_1 o\nabla u)(t)\right)\right]. \qquad (3.13)
\end{aligned}
$$

In the same way, we get

$$
\begin{aligned}
&\left|-\int_\Omega \nabla u_t \int_0^t h_1(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\right| \\
&\quad\le \frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\int_\Omega\left(\int_0^t h_1(t-s)\big|\nabla u(t)-\nabla u(s)\big|\,ds\right)^2 dx \\
&\quad\le \frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}(a-k)(h_1 o\nabla u)(t). \qquad (3.14)
\end{aligned}
$$

Similarly

$$
\begin{cases}
|-\int_\Omega \frac{1}{l+1}|v_t|^l v_t\int_0^t h_2(t-s)(v(t)-v(s))\,ds\,dx|\le \frac{1}{l+2}\|v_t\|^{l+2} \\
\quad+\frac{(l+1)^{-1}}{l+2}[(a-k)^{l+1}c_s^{l+2}(2^{2l+1}(a-k)\|\nabla v(t)\|^{2(l+1)}+\frac{1}{2}(h_2 o\nabla v)(t))] \\
|-\int_\Omega \nabla v_t\int_0^t h(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx|\le \frac{1}{2}\|\nabla v_t\|^2+\frac{1}{2}(a-k)(h_2 o\nabla v)(t).
\end{cases} \qquad (3.15)
$$

Combining (3.13),(3.14) and (3.15), we deduce (i).

(ii) We use the Leibnitz formula and the first and second equations of (1.5) to find

$$
\begin{aligned}
\psi'(t) &= \int_\Omega \big(\Delta u_{tt}-|u_t|^l u_{tt}\big)\int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad+\int_\Omega\left(\Delta u_t-\frac{1}{l+1}|u_t|^l u_t\right)\left(\int_0^t\big(h_1'(t-s)\big(u(t)-u(s)\big)+h_1(t-s)u_t(t)\big)\,ds\right)dx \\
&\quad+\int_\Omega \big(\Delta v_{tt}-|v_t|^l v_{tt}\big)\int_0^t h_2(t-s)\big(v(t)-v(s)\big)\,ds\,dx \\
&\quad+\int_\Omega\left(\Delta v_t-\frac{1}{l+1}|v_t|^l v_t\right)\left(\int_0^t\big(h_2'(t-s)\big(v(t)-v(s)\big)+h_2(t-s)v_t(t)\big)\,ds\right)dx \\
&= \int_\Omega f_1(u,v)\int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad+\int_\Omega M\big(\|\nabla u\|^2\big)\nabla u(t)\int_0^t h_1(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\
&\quad-\int_\Omega\int_0^t h_1(t-s)\nabla u(s)\,ds\int_0^t h_1(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\
&\quad+\mu_1\int_\Omega \nabla z_1(x,1,t)\int_0^t h_1(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\
&\quad-\int_\Omega \nabla u_t\int_0^t h_1'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx
\end{aligned}
$$

$$\begin{aligned}
&- \frac{1}{l+1}\int_{\Omega}|u_t|^l u_t\int_0^t h_1'(t-s)\bigl(u(t)-u(s)\bigr)\,ds\,dx\\
&- \|\nabla u_t\|^2\int_0^t h_1(s)\,ds - \frac{1}{l+1}\|u_t\|_{l+2}^{l+2}\int_0^t h_1(s)\,ds\\
&+ \int_{\Omega} f_2(u,v)\int_0^t h_2(t-s)\bigl(u(t)-u(s)\bigr)\,ds\,dx\\
&+ \int_{\Omega} M\bigl(\|\nabla v\|^2\bigr)\nabla v(t)\int_0^t h_2(t-s)\bigl(\nabla v(t)-\nabla v(s)\bigr)\,ds\,dx\\
&- \int_{\Omega}\int_0^t h_2(t-s)\nabla v(s)\,ds\int_0^t h_2(t-s)\bigl(\nabla v(t)-\nabla v(s)\bigr)\,ds\,dx\\
&+ \mu_2\int_{\Omega}\nabla z_2(x,1,t)\int_0^t h_2(t-s)\bigl(\nabla v(t)-\nabla v(s)\bigr)\,ds\,dx\\
&- \int_{\Omega}\nabla v_t\int_0^t h_2'(t-s)\bigl(\nabla v(t)-\nabla v(s)\bigr)\,ds\,dx\\
&- \frac{1}{l+1}\int_{\Omega}|v_t|^l v_t\int_0^t h_2'(t-s)\bigl(v(t)-v(s)\bigr)\,ds\,dx\\
&- \|\nabla v_t\|^2\int_0^t h_2(s)\,ds - \frac{1}{l+1}\|v_t\|_{l+2}^{l+2}\int_0^t h_2(s)\,ds\\
={}& I_1+I_2+I_3+I_4+I_5+I_6 - \|\nabla u_t\|^2\int_0^t h_1(s)\,ds - \frac{1}{l+1}\|u_t\|^{l+2}\int_0^t h_1(s)\,ds\\
&- \|\nabla v_t\|^2\int_0^t h_2(s)\,ds - \frac{1}{l+1}\|v_t\|^{l+2}\int_0^t h_2(s)\,ds, \qquad (3.16)
\end{aligned}$$

where

$$\begin{cases}
I_1 = \int_{\Omega} M(\|\nabla u\|^2)\nabla u(t)\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
\qquad + \int_{\Omega} M(\|\nabla v\|^2)\nabla v(t)\int_0^t h_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx,\\
I_2 = -\int_{\Omega}\int_0^t h_1(t-s)\nabla u(s)\,ds\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
\qquad - \int_{\Omega}\int_0^t h_2(t-s)\nabla v(s)\,ds\int_0^t h_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx,\\
I_3 = \mu_1\int_{\Omega}\nabla z_1(x,1,t)\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
\qquad + \mu_2\int_{\Omega}\nabla z_2(x,1,t)\int_0^t h_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx,
\end{cases}$$

and

$$\begin{cases}
I_4 = -\int_{\Omega}\nabla u_t\int_0^t h_1'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
\qquad - \int_{\Omega}\nabla v_t\int_0^t h_2'(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx,\\
I_5 = -\frac{1}{l+1}\int_{\Omega}|u_t|^l u_t\int_0^t h_1'(t-s)(u(t)-u(s))\,ds\,dx\\
\qquad - \frac{1}{l+1}\int_{\Omega}|v_t|^l v_t\int_0^t h_2'(t-s)(v(t)-v(s))\,ds\,dx,\\
I_6 = \int_{\Omega} f_1(u,v)\int_0^t h_1(t-s)(u(t)-u(s))\,ds\,dx\\
\qquad + \int_{\Omega} f_2(u,v)\int_0^t h_2(t-s)(u(t)-u(s))\,ds\,dx.
\end{cases}$$

Next we will estimate $I_1,\dots,I_6$.

For I_1, by applying Hölder's and Young's inequalities, we obtain

$$\begin{aligned}
|I_1| &\leq M\bigl(\|\nabla u\|^2\bigr)\int_\Omega \bigl|\nabla u(t)\bigr|\biggl(\int_0^t h_1(s)\,ds\biggr)^{\frac{1}{2}}\biggl(\int_0^t h_1(t-s)\bigl|\nabla u(t)-\nabla u(s)\bigr|^2\,ds\biggr)^{\frac{1}{2}}\,dx \\
&\quad + M\bigl(\|\nabla v\|^2\bigr)\int_\Omega \bigl|\nabla v(t)\bigr|\biggl(\int_0^t h_2(s)\,ds\biggr)^{\frac{1}{2}}\biggl(\int_0^t h_2(t-s)\bigl|\nabla v(t)-\nabla v(s)\bigr|^2\,ds\biggr)^{\frac{1}{2}}\,dx \\
&\leq M\bigl(\|\nabla u\|^2\bigr)\biggl[\delta\int_\Omega \bigl|\nabla u(t)\bigr|^2\int_0^t h_1(s)\,ds\,dx \\
&\quad + \frac{1}{4\delta}\int_\Omega\int_0^t h_1(t-s)\bigl|\nabla u(t)-\nabla u(s)\bigr|^2\,ds\,dx\biggr] \\
&\quad + M\bigl(\|\nabla v\|^2\bigr)\biggl[\delta\int_\Omega \bigl|\nabla v(t)\bigr|^2\int_0^t h_2(s)\,ds\,dx \\
&\quad + \frac{1}{4\delta}\int_\Omega\int_0^t h_2(t-s)\bigl|\nabla v(t)-\nabla v(s)\bigr|^2\,ds\,dx\biggr] \\
&\leq M\bigl(\|\nabla u\|^2\bigr)\biggl(\delta(a-k)\bigl\|\nabla u(t)\bigr\|^2 + \frac{1}{4\delta}(h_1 o\nabla u)(t)\biggr) \\
&\quad + M\bigl(\|\nabla v\|^2\bigr)\biggl(\delta(a-k)\bigl\|\nabla v(t)\bigr\|^2 + \frac{1}{4\delta}(h_2 o\nabla v)(t)\biggr).
\end{aligned} \tag{3.17}$$

Similarly,

$$\begin{aligned}
|I_2| &\leq \delta\int_\Omega\biggl(\int_0^t h_1(t-s)\bigl|\nabla u(s)\bigr|\,ds\biggr)^2\,dx + \frac{1}{4\delta}\int_\Omega\biggl(\int_0^t h_1(t-s)\bigl|\nabla u(t)-\nabla u(s)\bigr|\,ds\biggr)^2\,dx \\
&\quad + \delta\int_\Omega\biggl(\int_0^t h_2(t-s)\bigl|\nabla v(s)\bigr|\,ds\biggr)^2\,dx \\
&\quad + \frac{1}{4\delta}\int_\Omega\biggl(\int_0^t h_2(t-s)\bigl|\nabla v(t)-\nabla v(s)\bigr|\,ds\biggr)^2\,dx \\
&\leq \delta\int_\Omega\biggl(\int_0^t h_1(t-s)\bigl(\bigl|\nabla u(s)-\nabla u(t)\bigr| + \bigl|\nabla u(t)\bigr|\bigr)\,ds\biggr)^2\,dx \\
&\quad + \frac{1}{4\delta}\biggl(\int_0^t h_1(s)\,ds\biggr)(h_1 o\nabla u)(t) \\
&\quad + \delta\int_\Omega\biggl(\int_0^t h_2(t-s)\bigl(\bigl|\nabla v(s)-\nabla v(t)\bigr| + \bigl|\nabla v(t)\bigr|\bigr)\,ds\biggr)^2\,dx \\
&\quad + \frac{1}{4\delta}\biggl(\int_0^t h_2(s)\,ds\biggr)(h_2 o\nabla v)(t) \\
&\leq 2\delta\bigl\|\nabla u(t)\bigr\|^2\biggl(\int_0^t h_1(t)\,ds\biggr)^2\,dx + \biggl(2\delta+\frac{1}{4\delta}\biggr)\biggl(\int_0^t h_1(s)\,ds\biggr)(h_1 o\nabla u)(t) \\
&\quad + 2\delta\bigl\|\nabla v(t)\bigr\|^2\biggl(\int_0^t h_2(t)\,ds\biggr)^2\,dx + \biggl(2\delta+\frac{1}{4\delta}\biggr)\biggl(\int_0^t h_2(s)\,ds\biggr)(h_2 o\nabla v)(t) \\
&\leq 2\delta\bigl\|\nabla u(t)\bigr\|^2(a-k)^2 + \biggl(2\delta+\frac{1}{4\delta}\biggr)(a-k)(h_1 o\nabla u)(t) + 2\delta\bigl\|\nabla v(t)\bigr\|^2(a-k)^2 \\
&\quad + \biggl(2\delta+\frac{1}{4\delta}\biggr)(a-k)(h_2 o\nabla v)(t),
\end{aligned} \tag{3.18}$$

$$
\begin{aligned}
|I_3| &\le \delta\left(\mu_1^2\left\|\nabla z_1(x,1,t)\right\|^2+\mu_2^2\left\|\nabla z_2(x,1,t)\right\|^2\right)\\
&\quad+\frac{(a-k)}{4\delta}(h_1o\nabla u)(t)+\frac{(a-k)}{4\delta}(h_2o\nabla v)(t), \qquad (3.19)\\
|I_4| &\le \delta\int_\Omega|\nabla u_t|^2\,dx+\frac{1}{4\delta}\int_\Omega\left(\int_0^t\left|h_1'(t-s)\right|\left|\nabla u(t)-\nabla u(s)\right|ds\right)^2dx\\
&\quad+\delta\int_\Omega|\nabla v_t|^2\,dx+\frac{1}{4\delta}\int_\Omega\left(\int_0^t\left|h_2'(t-s)\right|\left|\nabla v(t)-\nabla v(s)\right|ds\right)^2dx\\
&\le \delta\|\nabla u_t\|^2+\frac{1}{4\delta}\int_0^t\left(-h_1'(t-s)\right)ds\int_\Omega\int_0^t(-h_1'^2\,ds\,dx\\
&\quad+\delta\|\nabla v_t\|^2+\frac{1}{4\delta}\int_0^t\left(-h_2'(t-s)\right)ds\int_\Omega\int_0^t(-h_2'^2\,ds\,dx\\
&\le \delta\|\nabla u_t\|^2-\frac{h_1(0)}{4\delta}\left(h_1'o\nabla u\right)(t)+\delta\|\nabla v_t\|^2-\frac{h_2(0)}{4\delta}\left(h_2'o\nabla u\right)(t),
\end{aligned}
$$

and using the fact that $l \le \gamma$

$$
\begin{aligned}
|I_5| &\le \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\right)+\frac{(l+1)^{-1}}{l+2}\left[\left(h_1(0)\right)^{l+1}\int_0^t\left(-h_1'(t-s)\right)\left\|u(t)-u(s)\right\|_{l+2}^{l+2}ds\right.\\
&\quad\left.+\left(h_2(0)\right)^{l+1}\int_0^t\left(-h_2'(t-s)\right)\left\|v(t)-v(s)\right\|_{l+2}^{l+2}ds\right]\\
&\le \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2}+\|u_t\|_{l+2}^{l+2}\right)\\
&\quad+\frac{(l+1)^{-1}}{l+2}c_s^{l+2}\left[\left(h_1(0)\right)^{l+1}\int_0^t\left(-h_1'(t-s)\right)\left\|\nabla u(t)-\nabla u(s)\right\|^{l+2}ds\right.\\
&\quad\left.+\left(h_2(0)\right)^{l+1}\int_0^t\left(-h_2'(t-s)\right)\left\|\nabla v(t)-\nabla v(s)\right\|^{l+2}ds\right]\\
&\le \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2}+\|u_t\|_{l+2}^{l+2}\right)\\
&\quad+\frac{(l+1)^{-1}}{l+2}c_s^{l+2}\left(h_1(0)\right)^{l+1}\left[\delta 2^{2(l+1)}h_1(0)\left\|\nabla u(t)\right\|^{2(l+1)}-\frac{1}{4\delta}\left(h_1'o\nabla u\right)(t)\right]\\
&\quad+\frac{(l+1)^{-1}}{l+2}c_s^{l+2}\left(h_2(0)\right)^{l+1}\left[\delta 2^{2(l+1)}h_2(0)\left\|\nabla v(t)\right\|^{2(l+1)}-\frac{1}{4\delta}\left(h_2'o\nabla v\right)(t)\right]\\
&\le \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2}+\|u_t\|_{l+2}^{l+2}\right)\\
&\quad+\frac{(l+1)^{-1}}{l+2}c_s^{l+2}\left(h_1(0)\right)^{l+1}\left[\delta 2^{2(l+1)}h_1(0)M\left(\left\|\nabla u(t)\right\|^2\right)\left\|\nabla u(t)\right\|^2\right.\\
&\quad\left.-\frac{1}{4\delta}\left(h_1'o\nabla u\right)(t)\right]\\
&\quad+\frac{(l+1)^{-1}}{l+2}c_s^{l+2}\left(h_2(0)\right)^{l+1}\left[\delta 2^{2(l+1)}h_2(0)M\left(\left\|\nabla v(t)\right\|^2\right)\left\|\nabla v(t)\right\|^2\right.\\
&\quad\left.-\frac{1}{4\delta}\left(h_2'o\nabla v\right)(t)\right]. \qquad (3.20)
\end{aligned}
$$

For I_6, we have

$$\begin{aligned} I_6 &= \alpha \int_\Omega v(t) \int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx + \alpha \int_\Omega u(t) \int_0^t h_2(t-s)\big(v(t)-v(s)\big)\,ds\,dx \\ &\quad + b_1 \int_\Omega |v|^{q+1}|u|^{p-1}u \int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &\quad + b_2 \int_\Omega |u|^{p+1}|v|^{q-1}v \int_0^t h_2(t-s)\big(v(t)-v(s)\big)\,ds\,dx \\ &= I_6^0 + b_1 I_6^1 + b_2 I_6^2, \end{aligned} \tag{3.21}$$

$$\big|I_6^0\big| \le \frac{\alpha c_s^2}{2}\big(\big\|\nabla u(t)\big\|^2 + \big\|\nabla v(t)\big\|^2 + (a-k)\big[(h_1 o \nabla u)(t) + (h_2 o \nabla v)(t)\big]\big), \tag{3.22}$$

and

$$\big|I_6^1\big| \le \frac{1}{2}\int_\Omega \big(|v|^{2(q+1)} + |u|^{2p}\big)\int_0^t \big|h_1(t-s)(u(t)-u(s)\big|\,ds\,dx = I_6^{11} + I_6^{12}. \tag{3.23}$$

By using the Young and Hölder inequalities and Lemma 1.1, we find

$$\begin{aligned} I_6^{11} &= \frac{1}{2}\int_\Omega |v|^{2(q+1)} \int_0^t \big|h_1(t-s)\big(u(t)-u(s)\big)\big|\,ds\,dx \\ &\le \frac{\delta}{2}\int_\Omega |v|^{4(q+1)}\,dx + \frac{1}{8\delta}\int_\Omega \bigg[\int_0^t h_1(t-s)\big|u(t)-u(s)\big|\,ds\bigg]^2 dx \\ &\le \frac{c_s^{4(q+1)}\delta}{2}\|\nabla v\|^{4(q+1)} + \frac{1}{8\delta}(a-k)(h_1 o \nabla u)(t) \\ &\le \frac{c_s^{4(q+1)}\delta}{2} M\big(\|\nabla v\|^2\big)\|\nabla v\|^2 + \frac{1}{8\delta}(a-k)(h_1 o \nabla u)(t). \end{aligned} \tag{3.24}$$

Also by following a similar technique to above, we get

$$\big|I_6^{12}\big| \le \frac{c_s^{4p}\delta}{2} M\big(\|\nabla u\|^2\big)\|\nabla u\|^2 + \frac{1}{8\delta}(a-k)(h_1 o \nabla u)(t). \tag{3.25}$$

Hence

$$\begin{aligned} \big|I_6^1\big| &\le \frac{c_s^{4(q+1)}\delta}{2} M\big(\|\nabla v\|^2\big)\|\nabla v\|^2 + \frac{c_s^{4p}\delta}{2} M\big(\|\nabla u\|^2\big)\|\nabla u\|^2 \\ &\quad + \frac{1}{4\delta}(a-k)(h_1 o \nabla u)(t). \end{aligned} \tag{3.26}$$

Similarly

$$\begin{aligned} \big|I_6^2\big| &\le \frac{c_s^{4(p+1)}\delta}{2} M\big(\|\nabla u\|^2\big)\|\nabla u\|^2 + \frac{c_s^{4q}\delta}{2} M\big(\|\nabla v\|^2\big)\|\nabla v\|^2 \\ &\quad + \frac{1}{4\delta}(a-k)(h_2 o \nabla v)(t). \end{aligned} \tag{3.27}$$

Summing (3.22), (3.26) and (3.27), we get

$$\begin{aligned} I_6 &\le \left(b_2\frac{c_s^{4(p+1)}\delta}{2}+\frac{c_s^{4p}\delta}{2}b_1\right)M\big(\|\nabla u\|^2\big)\|\nabla u\|^2 \\ &\quad+\left(b_2\frac{c_s^{4q}\delta}{2}+b_1\frac{c_s^{4(q+1)}\delta}{2}\right)M\big(\|\nabla v\|^2\big)\|\nabla v\|^2 \\ &\quad+\left(\frac{\alpha c_s^2}{2}+\frac{1}{4\delta}\right)(a-k)\big((h_1o\nabla u)(t)+(h_2o\nabla v)(t)\big)+\frac{\alpha c_s^2}{2}\big(\|\nabla u\|^2+\|\nabla v\|^2\big). \end{aligned} \tag{3.28}$$

Combining (3.16) and (3.17)–(3.28), we complete the proof. □

Proof of Theorem 3.1 Now, for $M,\varepsilon_1>0$, we introduce the following functional:

$$F(t)=ME(t)+I(t)+\psi(t)+\varepsilon_1\phi(t). \tag{3.29}$$

Firstly we prove that $F(t)$ is equivalent to $E(t)$; for this we show that $F(t)$ verified the following boundedness:

$$\kappa_1E(t)\le F(t)\le\kappa_2E(t) \tag{3.30}$$

for some positive constants κ_2, κ_2.

We recall (3.3), (3.5), and (3.9) and, using the fact that $l\le\gamma$, we get

$$\begin{aligned} &\big|I(t)+\psi(t)+\varepsilon_1\phi(t)\big| \\ &\quad\le\frac{\varepsilon_1+1}{l+2}\big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\big)+\frac{\varepsilon_1+1}{2}\big(\|\nabla u_t\|^2)+\|\nabla v_t\|^2)\big) \\ &\qquad+\left(\frac{\varepsilon_1 c}{2}+\frac{(l+1)^{-1}}{(l+2)}c_s^{l+2}\big(\varepsilon_1+2^{2l+1}(a-k)^{l+2}\big)\right)\big(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\big) \\ &\qquad+\frac{a-k}{2}\left(1+\frac{(l+1)^{-1}}{(l+2)}(a-k)^lc_s^{l+2}\right)\big((h_1o\nabla u)(t)+(h_2o\nabla v)(t)\big)+\frac{1}{\xi}E(t) \\ &\quad\le\kappa E(t), \end{aligned}$$

where $\kappa>0$ depending the ε_1, a, b, l, c, c_s, k, ξ. For the choice of $M=\kappa+\epsilon$ with $\epsilon>0$, we get $F(t)\sim E(t)$.

By recalling (1.9), (3.4), (3.6), (3.10) and **(A2)**, we deduce that

$$\begin{aligned} F'(t)&\le\left(\mu^2\delta+\varepsilon_1\frac{\mu^2}{4\eta}-(1-d)e^{-2\tau_1}-M\beta\right)\big(\big\|\nabla z_1(x,1,t)\big)\big\|^2+\big\|\nabla z_2(x,1,t)\big)\big\|^2\big) \\ &\quad-2\tau(t)e^{-2\tau_1}\int_0^1\big(\|\nabla z_1\|^2+\|\nabla z_2\|^2\big)\,d\rho \\ &\quad-(\varepsilon_1\left[k-\eta(a-k+1)-\left(\frac{b_1+b_2}{2}+\alpha\right)c_s^2\right] \\ &\quad-\left(2\delta(a-k)^2-\delta\left[(a-k)+\frac{(l+1)^{-1}}{(l+2)}(h_2c_s)^{l+2}2^{2(l+1)}+\omega\right]M_0\right)\big(\|\nabla u\|^2+\|\nabla v\|^2\big) \\ &\quad-\frac{1}{l+1}(h_0-1-\varepsilon_1)\big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\big) \end{aligned}$$

$$\begin{aligned}&-(h_0-\delta-M\lambda-1-\varepsilon_1)\big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\big)\\&+\left[\left(\frac{M}{2}-\frac{h_1}{4\delta}\left(1+\frac{(l+1)^{-1}}{(l+2)}h_1^l c_s^{l+2}\right)-\frac{1}{\zeta}\left(\frac{\varepsilon_1}{4\eta}+\frac{M_0}{4\delta}\right.\right.\right.\\&\left.\left.\left.+\left(2\delta+\frac{1}{3\delta}+\frac{\alpha c_s^2}{2}\right)(a-k)\right)\right]\big(\big(h_1' o\nabla u\big)(t)+\big(h_2' o\nabla v\big)(t)\big),\quad \forall t\geq t_0>0,\end{aligned}$$

where $M_0=\max\{M\|\nabla u\|^2, M\|\nabla v\|^2\}$, $h_0=\min\{\int_0^{t_0}h_1(s)\,ds, \int_0^{t_0}h_2(s)\,ds\}$, $h_1=\min\{h_1(0), h_2(0)\}$, $h_2=\max\{h_1(0),h_2(0)\}$, $\omega=\max\{b_1\frac{c_s^{4(q+1)}}{2}+\frac{c_s^{4q}}{2}b_2, b_2\frac{c_s^{4(p+1)}}{2}+\frac{c_s^{4p}}{2}b_1\}$ and $\zeta=\max\{\zeta_1, \zeta_1\}$.

Let $\epsilon>0$ be sufficiently small so M is fixed, we take $h_0-M\lambda-1>\varepsilon_1$ and δ small enough such that

$$a_3=h_0-1-\varepsilon_1>0\quad\text{and}\quad a_4=h_0-\delta-M\lambda-1-\varepsilon_1.$$

Further, we choose η small enough such that

$$\begin{aligned}a_1&=\mu_1^2\delta+\varepsilon_1\frac{\mu^2}{4\eta}-(1-d)e^{-2\tau_1}-M\beta>0,\\a_2&=\varepsilon_1\left[k-\eta(a-k+1)-\left(\frac{b_1+b_2}{2}+\alpha\right)c_s^2\right]-2\delta(a-k)^2\\&\quad-\delta\left((a-k)+\frac{(l+1)^{-1}}{(l+2)}(h_2c_s)^{l+2}2^{2(l+1)}+\omega\right)M_0>0,\end{aligned}$$

and

$$a_5=\frac{M}{2}-\frac{h_1}{4\delta}\left(1+\frac{(l+1)^{-1}}{(l+2)}h_1^l c_s^{l+2}\right)-\frac{1}{\zeta}\left(\frac{\varepsilon_1}{4\eta}+\frac{M_0}{4\delta}+\left(2\delta+\frac{1}{3\delta}+\frac{\alpha c_s^2}{2}\right)(a-k)\right)<0.$$

Thus

$$\begin{aligned}F'(t)&\leq -a_3\frac{1}{l+2}\big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\big)-a_2\big(\|\nabla u\|^2+\|\nabla v\|^2\big)\\&\quad-2\tau(t)e^{-2\tau_1}\int_0^1\big(\|\nabla z_1\|^2+\|\nabla z_2\|^2\big)\,d\rho\\&\quad+a_1\big(\big\|\nabla z_1(x,1,t)\big\|^2+\big\|\nabla z_2(x,1,t)\big\|^2\big)-a_4\big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\big)\\&\quad+a_5\big[\big(h_1' o\nabla u\big)(t)+\big(h_2' o\nabla v\big)(t)\big]\\&\leq -mE(t)-cE'(t),\end{aligned}\tag{3.31}$$

where $m=\min\{\frac{2e^{-2\tau_1}}{\xi}, 2\frac{a_2}{a}, a_3\}$ and $c=\min\{\frac{a_1}{\beta}, \frac{a_4}{\lambda}, -2a_5\}$.

Let $L(t)=F(t)+cE(t)\sim E(t)$. From (3.31), we get

$$L'(t)\leq -c'L(t),\quad \forall t\geq t_0,\tag{3.32}$$

for some $c'>0$. A simple integration over (t_0,t) yields

$$L(t)\leq L(t_0)e^{-c'(t-t_0)},\quad \forall t\geq t_0.\tag{3.33}$$

Thanks to the equivalence between L and E, we obtain (3.1). □

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