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Multiplicity and existence of solutions for generalized quasilinear Schrödinger equations with sign-changing potentials

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Abstract

We consider a class of generalized quasilinear Schrödinger equations

$$-\operatorname{div}(l^2(u)\nabla u) + l(u)l'(u)|\nabla u|^2 + V(x)u = f(u), \quad x \in \mathbb{R}^N,$$

where $l(t) : \mathbb{R} \to \mathbb{R}^+$ is a nondecreasing function with respect to $|t|$, the potential function V is allowed to be sign-changing so that the Schrödinger operator $-\Delta + V$ possesses a finite-dimensional negative space. We obtain existence and multiplicity results for the problem via the Symmetric Mountain Pass Theorem and Morse theory.

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Keywords: Quasilinear Schrödinger equations; Sign-changing potentials; Symmetric Mountain Pass Theorem; Morse theory

1 Introduction

In our article, we study the generalized quasilinear Schrödinger problem as follows:

$$-\operatorname{div}\big(l^2(u)\nabla u\big) + l(u)l'(u)|\nabla u|^2 + V(x)u = f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$, $f \in C(\mathbb{R},\mathbb{R})$ and the function l satisfies the following assumptions:

(l) $l \in C^2(\mathbb{R},\mathbb{R}^+)$, $l(t) = l(-t)$, $l(0) = 1$, $l'(t) \geq 0$ for all $t \geq 0$, $tl'(t) \leq l(t)$ for all $t \in \mathbb{R}$ and $l''(t) \geq 0$ is strict on a subset of positive measure in $\mathbb{R}$.

Solutions of (1.1) are related to the solitary wave solutions for the following quasilinear Schrödinger equations:

$$i\partial_t z = -\Delta z + W(x)z - \widetilde{f}(z) - \Delta g\big(|z|^2\big)g'\big(|z|^2\big)z, \tag{1.2}$$

where $z : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, and $\widetilde{f}$, g are real functions. The above problem (1.2) has been studied in several areas of physics corresponding to various types of g. For example, the case $g(t) = t$ was used in [9] for the superfluid film equation in plasma physics. If $g(t) = (1 + t)^{1/2}$, equation (1.2) models the self-channeling of a high-power ultrashort laser in matter (see [2] and [3]). Equation (1.2) also has relations with condensed matter theory (see [12]).

Taking $z(x, t) = \exp(-iEt)u(x)$ in (1.2), with $E > 0$, we are led to investigate the following elliptic equation:

$$-\Delta u + V(x)u - \Delta g\left(u^2\right)g'\left(u^2\right)u = f(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

with $V(x) = W(x) - E$. If we choose

$$l^2(u) = 1 + \frac{((g(u^2))')^2}{2},$$

then (1.3) turns into (1.1). In particular, if $g(t) = t$, we have

$$-\Delta u + V(x)u - \Delta\left(u^2\right)u = f(u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

For equation (1.4), to the best of our knowledge, the first results were proved by Poppenberg, Schmitt, and Wang in [13]. The idea in [13] is a constrained minimization argument. Subsequently, a general existence result for (1.4) was derived by Liu, Wang, and Wang [10]. The main existence results were obtained, through making a change of variable, reducing the quasilinear problem (1.4) to a semilinear one, and an Orlicz space framework was used to prove the existence of a positive solutions via Mountain Pass Theorem. The same method of variable change was also used by Colin and Jeanjean in [4], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was chosen as the working space. We refer the readers to [5, 6, 11, 15, 17, 18, 20, 22, 23] for more results.

In all these papers, it is required that the potential V satisfies the positivity condition

$$m := \inf_{x \in \mathbb{R}^N} V(x) > 0. \tag{1.5}$$

With this assumption and suitable conditions on the nonlinearity f, one can show that $u = 0$ is a local minimizer of the energy functional associated with (1.4), which would then verify the mountain pass geometry and so Mountain Pass Theorem can be applied to produce a solution. However, from $V(x) = W(x) - E$ we can see that, if the frequency E is large, then the potential $V(x)$ in (1.4) could not satisfy (1.5).

In the literature (see [7]), there are some existence results which allow the potential V to be negative somewhere. The strategy is to write $V = V^+ - V^-$ with $V^\pm = \max\{0, \pm V\}$. Then if V^- is in some sense small, the function

$$u \mapsto \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2 \right) dx \right)^{1/2} \tag{1.6}$$

is still a norm on the function space. This is the key to verify that **0** is a local minimizer of the corresponding energy functional.

Recently, Shi and Chen [19] studied problem (1.1) with a sign-changing potential V. Compared with [7], (1.6) fails to be a norm any more. To overcome this difficulty, the authors chose a constant $V_0 > 0$ satisfying

$$\widetilde{V}(x) = V(x) + V_0 > 0,$$

and considered the equivalent problem

$$-\operatorname{div}\big(l^2(u)\nabla u\big) + l(u)l'(u)|\nabla u|^2 + \widetilde{V}(x)u = \widetilde{f}(u), \quad x \in \mathbb{R}^N,$$

where $\widetilde{f}(u) = f(u) + V_0 u$. Unfortunately, from

$$\mathfrak{F}(t) = \frac{f(t)L(t)}{2l(t)} - F(t) \geq 0$$

in their condition (f_3), we cannot ensure

$$\widetilde{\mathfrak{F}}(t) = \frac{\widetilde{f}(t)L(t)}{2l(t)} - \widetilde{F}(t) = \mathfrak{F}(t) + \frac{V_0 tL(t)}{2l(t)} - \frac{V_0}{2}t^2 \geq 0, \tag{1.7}$$

since $\frac{tL(t)}{l(t)} - t^2 \leq 0$, where $L(t) = \int_0^t l(s)\,ds$, $F(t) = \int_0^t f(s)\,ds$, and $\widetilde{F}(t) = \int_0^t \widetilde{f}(s)\,ds$.

Here we give an example to indicate why (1.7) fails. Consider

$$l(t) = \sqrt{1+2t^2} \quad \text{and} \quad f(t) = l(t)\big|L(t)\big|^{p-2}L(t),$$

where $L(t) = \int_0^t l(s)\,ds = \frac{1}{2\sqrt{2}}\ln(\sqrt{2}t + \sqrt{1+2t^2}) + \frac{1}{2}t\sqrt{1+2t^2}$. Of course, we have $F(t) = \int_0^t f(s)\,ds = \frac{1}{p}|L(t)|^p$. It is easy to find that $f(t)$ and $l(t)$ satisfy conditions (f_1)–(f_4) (see [19]) and (l), respectively. Then we denote

$$\begin{aligned}
\widetilde{f}(t) &= f(t) + V_0 t = l(t)\big|L(t)\big|^{p-2}L(t) + V_0 t,\\
\widetilde{F}(t) &= F(t) + \frac{V_0}{2}t^2 = \frac{1}{p}\big|L(t)\big|^p + \frac{V_0}{2}t^2,\\
\widetilde{\mathfrak{F}}(t) &= \frac{\widetilde{f}(t)L(t)}{2l(t)} - \widetilde{F}(t) = \left(\frac{1}{2} - \frac{1}{p}\right)\big|L(t)\big|^p + \frac{V_0}{2}\left(\frac{L(t)}{l(t)} - t\right)t.
\end{aligned}$$

Due to

$$\lim_{t\to+\infty}\left(\frac{L(t)}{l(t)} - t\right) = \lim_{t\to+\infty}\left(-\frac{t^2}{2} + \frac{t\ln(\sqrt{2}t + \sqrt{1+2t^2})}{\sqrt{1+2t^2}}\right) = -\infty,$$

this implies that, for some $M \gg 1$, there exist T_1, T_2 $(1 < T_1 < T_2)$ such that

$$t\left(\frac{L(t)}{l(t)} - t\right) < -M, \quad t \in [T_1, T_2].$$

Since $L(t)$ is continuous on $[T_1, T_2]$, there exists $K > 0$ such that

$$\left(\frac{1}{2} - \frac{1}{p}\right)\big|L(t)\big|^p \leq K, \quad t \in [T_1, T_2].$$

Thus, for $V_0 \geq \frac{4K}{M}$, we have

$$\widetilde{\mathfrak{F}}(t) < -K, \quad t \in [T_1, T_2].$$

Therefore, unlike what the authors declared at the beginning of [19], this new nonlinearity $\widetilde{f}(t)$ does not satisfy their condition (f_3) any more. For this reason, their result may be valid for the case when the potential V is positive.

To the best of our knowledge, up to now in the literature there is only one research paper devoted to the situation that the quasilinear Schrödinger problems with "strongly" sign-changing potential. Recently, S. Liu et al. [11] proved the multiplicity of solutions of

$$-\Delta u + V(x)u - \Delta\left(u^2\right)u = f(u), \quad x \in \mathbb{R}^N,$$

where V is a sign-changing potential.

Our results extend and modify those obtained by S. Liu et al. [11] and H. Shi et al. [19]. Inspired by [11], we now present our hypotheses on the potential V and the nonlinearity f:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) > -\infty$;

(V_2) $\mu(V^{-1}(-\infty, M]) < \infty$ for all $M > 0$, where μ is the Lebesgue measure;

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist $C_1, C_2 > 0$ such that for all $t \in \mathbb{R}$, $p \in (2, 2^*)$,

$$\left|f(t)\right| \leq C_1 l(t)\left|L(t)\right| + C_2 l(t)\left|L(t)\right|^{p-1};$$

(f_2) there exists $\mu > 2$ such that for $t \neq 0$,

$$0 < \mu l(t)F(t) \leq L(t)f(t);$$

(f_3) $f(t) = o(t)$ as $t \to 0$;

(f_4) $f(t) = -f(-t)$.

We now summarize our main results:

Theorem 1.1 *Assume* (l), (V_1), (V_2), (f_1), (f_2), *and* (f_4) *hold. Then problem* (1.1) *has infinitely many solutions* $\{u_n\}$ *in* X *with* $I(u_n) \to \infty$ (X *and* $I(\cdot)$ *will be defined in Sect.* 2).

Remark 1.1 From (V_1), the potential $V(x)$ is allowed to be sign-changing.

Remark 1.2 Since l satisfies $l(t) = l(-t)$, $l'(t) \geq 0$ for all $t \geq 0$, and $tl'(t) \leq l(t)$ for all $t \in \mathbb{R}$, we can easily obtain

$$\left|l(t)\right| \leq C_3|t| + C_4 \quad \text{and} \quad l'(t) \leq C_5$$

for some constants $C_3, C_4, C_5 > 0$.

Theorem 1.2 *Assume* (l), (V_1), (V_2), *and* (f_1)–(f_3) *hold. If* 0 *is not an eigenvalue of* (2.2), *then problem* (1.1) *possesses at least one nontrivial solution.*

This paper is organized as follows. In Sect. 2, we describe the main preliminaries which we will use in this paper. Theorems 1.1 and 1.2 are proved in Sect. 3 and Sect. 4, respectively.

Notation. In this paper we use the following notations:

- $|u|_s = (\int_{\mathbb{R}^N} |u|^s\,dx)^{1/s}$ denotes the usual norm in L^s-space.
- $C, C_1, C_2, \ldots$ denote different positive constants.

- We denote the weak and strong convergence in X, as $n \to \infty$, by $u_n \rightharpoonup u$ and $u_n \to u$, respectively.

2 Preliminaries

Since $V(x)$ is bounded from below, there exists $V_0 > 0$ satisfying

$$\widetilde{V}(x) = V(x) + V_0 > 1 \quad \text{for all } x \in \mathbb{R}^N. \tag{2.1}$$

We now introduce the working space. Set

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\,dx < +\infty \right\},$$

which is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \left(\nabla u \nabla v + \widetilde{V}(x)uv\right) dx$$

and the corresponding norm

$$\|u\| = \langle u, u\rangle^{1/2}.$$

From condition (V_2), we have a compact embedding $X \hookrightarrow\hookrightarrow L^s(\mathbb{R}^N)$ for $s \in [2, 2^*)$ (see Bartsch–Wang [1]).

Applying the spectral theory of self-adjoint compact operators, let

$$-\infty < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

be the sequence of eigenvalues of

$$-\Delta u + V(x)u = \lambda u, \quad u \in X, \tag{2.2}$$

where each eigenvalue is repeated according to its multiplicity, and let $e_1, e_2, \ldots$ be the corresponding orthonormal eigenfunctions in $L^2(\mathbb{R}^N)$.

Problem (1.1) is the Euler–Lagrange equation of the following energy functional:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} l^2(u)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx.$$

But $I(u)$ may be ill-behaved in X. To overcome this difficulty, we make a change of variables introduced in Shen and Wang [16], as

$$v = L(u) = \int_0^u l(t)\,dt.$$

Firstly, we give some properties for L and L^{-1} which are defined in Sect. 1.

Lemma 2.1 ([19]) *The functions $L(t)$ and $L^{-1}(s)$ satisfy the following properties*:

(1) *L is odd, from class C^2 and invertible*;

(2) $\lim_{|s|\to 0} \frac{L^{-1}(s)}{s} = 1$;
(3)

$$\lim_{s\to+\infty} \frac{L^{-1}(s)}{s} = \begin{cases} \frac{1}{l(\infty)}, & \textit{if } l \textit{ is bounded}, \\ o(1), & \textit{if } l \textit{ is unbounded}; \end{cases}$$

(4) $L(t) \le l(t)t$, *for all* $t \ge 0$;
(5) $L^{-1}(s) \le s$, *for all* $s \ge 0$;
(6) $\frac{L^{-1}(s)}{s}$ *is nonincreasing, for all* $s \ge 0$;
(7) *if* l *is unbounded, then* $\lim_{s\to+\infty} \frac{|L^{-1}(s)|^2}{s} = \frac{2}{l'(\infty)}$;
(8) $0 \le \frac{t}{l(t)} l'(t) \le 1$, *for all* $t \in \mathbb{R}$;
(9) *there exists a positive constant* C_6 *such that*

$$\left|L^{-1}(s)\right| \ge \begin{cases} C_6|s|, & \textit{if } |s| \le 1, \\ C_6|s|^{1/2}, & \textit{if } |s| \ge 1; \end{cases}$$

(10) $|L^{-1}(\alpha s)|^2 \le C(\alpha)|L^{-1}(s)|^2$, *for all* $\alpha > 0$ *and* $C(\alpha) > 0$ *depends on* α.

Thus, after the change of variables, we obtain the functional $J(v)$ in the following form:

$$J(v) = I\bigl(L^{-1}(v)\bigr) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left|L^{-1}(v)\right|^2 dx - \int_{\mathbb{R}^N} F\bigl(L^{-1}(v)\bigr)\,dx$$

which is well defined in X. By Lemma 2.1, we know that $J \in C^1$, and the critical points of J are the weak solutions of our problem (1.1) (see [16]). Hence, to prove our main results, we should find critical points of the functional J.

Secondly, we set

$$\widetilde{f}(t) = f(t) + V_0 t, \qquad \widetilde{F}(t) = \int_0^t \widetilde{f}(s)\,ds = F(t) + \frac{V_0}{2}t^2$$

and rewrite J in the following form:

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} \widetilde{V}(x)\left|L^{-1}(v)\right|^2 dx - \int_{\mathbb{R}^N} \widetilde{F}\bigl(L^{-1}(v)\bigr)\,dx,$$

where $\widetilde{V}(x) = V(x) + V_0$. Note that by (f_2), the new nonlinearity $\widetilde{f}$ satisfies

$$l(t)\widetilde{F}(t) - \frac{1}{\mu}L(t)\widetilde{f}(t) \le \frac{V_0}{2}t^2 l(t) - \frac{V_0}{\mu}tL(t). \tag{2.3}$$

It is easy to see that the nonlinearity $\widetilde{f}(t)$ does not satisfy Ambrosetti–Rabinowitz condition any more, hence the boundedness of Palais–Smale sequences seems hard to verify. For this reason, we will show the functional J satisfies the Cerami condition.

Thirdly, recall that a $(C)_c$-sequence $\{v_n\}$ in X at the level c is such that

$$J(v_n) \to c \quad \text{and} \quad \bigl(1 + \|v_n\|\bigr)J'(v_n) \to 0.$$

Then J is said to satisfy the Cerami condition if any $(C)_c$-sequence has a convergent subsequence in X.

Lemma 2.2 *Under assumptions* (V_1), (V_2), (l), (f_1), *and* (f_2), J *satisfies the Cerami condition.*

Proof Let $\{v_n\}$ be a Cerami sequence of J, i.e.,

$$J(v_n) \to c \quad \text{and} \quad \bigl(1 + \|v_n\|\bigr)J'(v_n) \to 0 \tag{2.4}$$

for some $c \in \mathbb{R}$.

Step 1. We prove that

$$\rho_n := \left(\int_{\mathbb{R}^N} \bigl(|\nabla v_n|^2 + \widetilde{V}(x)\bigl|L^{-1}(v_n)\bigr|^2\bigr)\,dx\right)^{1/2} < +\infty.$$

If this conclusion is not true, we can suppose $\rho_n \to +\infty$. Consider the sequence $\{h_n\}$, defined by

$$h_n = \frac{L^{-1}(v_n)}{\rho_n}.$$

Since $l(t) \geq 1$, we obtain

$$\begin{aligned}\|h_n\|^2 &= \frac{1}{\rho_n^2}\int_{\mathbb{R}^N}\left(\frac{1}{l^2(L^{-1}(v_n))}|\nabla v_n|^2 + \widetilde{V}(x)\bigl|L^{-1}(v_n)\bigr|^2\right)dx \\ &\leq \frac{1}{\rho_n^2}\int_{\mathbb{R}^N}\bigl(|\nabla v_n|^2 + \widetilde{V}(x)\bigl|L^{-1}(v_n)\bigr|^2\bigr)\,dx = 1.\end{aligned} \tag{2.5}$$

Passing to a subsequence, we may assume that

$$\begin{aligned}&h_n \rightharpoonup h \quad \text{in } X; \\ &h_n \to h \quad \text{in } L^2\bigl(\mathbb{R}^N\bigr); \\ &h_n \to h \quad \text{a.e. on } \mathbb{R}^N.\end{aligned}$$

Subsequently, by (2.3) and Lemma 2.1(4), we get

$$\begin{aligned}c + o_n(1) &= J(v_n) - \frac{1}{\mu}\bigl\langle J'(v_n), v_n\bigr\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}\widetilde{V}(x)\bigl|L^{-1}(v_n)\bigr|^2\,dx \\ &\quad - \frac{1}{\mu}\int_{\mathbb{R}^N}\widetilde{V}(x)\frac{L^{-1}(v_n)}{l(L^{-1}(v_n))}v_n dx + \int_{\mathbb{R}^N}\frac{\widetilde{f}(L^{-1}(v_n))}{\mu l(L^{-1}(v_n))}v_n - \widetilde{F}\bigl(L^{-1}(v_n)\bigr)\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N}\bigl(|\nabla v_n|^2 + \widetilde{V}(x)\bigl|L^{-1}(v_n)\bigr|^2\bigr)\,dx \\ &\quad + \frac{1}{\mu}\int_{\mathbb{R}^N}\widetilde{V}(x)\left(\bigl|L^{-1}(v_n)\bigr|^2 - \frac{L^{-1}(v_n)}{l(L^{-1}(v_n))}v_n\right)dx \\ &\quad + \int_{\mathbb{R}^N}\left(\frac{V_0}{\mu}\frac{L^{-1}(v_n)}{l(L^{-1}(v_n))}v_n - \frac{V_0}{2}\bigl|L^{-1}(v_n)\bigr|^2\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\rho_n^2 - \frac{V_0}{2}\int_{\mathbb{R}^N}\bigl|L^{-1}(v_n)\bigr|^2\,dx.\end{aligned}$$

After multiplying both sides of the above equation by ρ_n^{-2}, for large n, we have

$$\frac{V_0}{2}\frac{\int_{\mathbb{R}^N}|L^{-1}(v_n)|^2\,dx}{\rho_n^2}=\frac{V_0}{2}\int_{\mathbb{R}^N}|h_n|^2\,dx\geq\frac{1}{2}\left(\frac{1}{2}-\frac{1}{\mu}\right).$$

Since $h_n\to h$ in $L^2(\mathbb{R}^N)$ and $\mu>2$, it implies that $h\neq 0$. Thus the set $\Theta=\{x\in\mathbb{R}^N:h(x)\neq 0\}$ has a positive Lebesgue measure.

Due to our assumption (f_2), it implies that

$$\begin{aligned}\frac{\widetilde{F}(t)}{t^2}&=\frac{1}{t^2}\left(F(t)+\frac{1}{2}V_0t^2\right)\\&\geq\frac{1}{2}V_0+\frac{|L(t)|^\mu}{t^2}\\&\geq\frac{1}{2}V_0+|t|^{\mu-2}\to+\infty,\quad\text{as }|t|\to\infty.\end{aligned}$$

Noticing that $\widetilde{F}(t)\geq 0$ and by Fatou's lemma, we obtain

$$\int_{\mathbb{R}^N}\frac{\widetilde{F}(L^{-1}(v_n))}{\rho_n^2}\,dx\geq\int_\Theta\frac{\widetilde{F}(L^{-1}(v_n))}{|L^{-1}(v_n)|^2}h_n^2\,dx\to+\infty,\quad\text{as }n\to\infty.$$

Hence, we get

$$\begin{aligned}c+o_n(1)&=J(v_n)\\&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N}\widetilde{V}(x)\left|L^{-1}(v_n)\right|^2dx-\int_{\mathbb{R}^N}\widetilde{F}\left(L^{-1}(v_n)\right)dx\\&=\rho_n^2\left(\frac{1}{2}-\frac{\int_{\mathbb{R}^N}\widetilde{F}(L^{-1}(v_n))\,dx}{\rho_n^2}\right)\to-\infty,\end{aligned}$$

which is a contradiction. Thus, we obtain

$$\int_{\mathbb{R}^N}\left(|\nabla v_n|^2+\widetilde{V}(x)\left|L^{-1}(v_n)\right|^2\right)dx<+\infty.$$

Step 2. We prove that there exists a constant $C_7>0$ such that

$$\rho_n^2=\int_{\mathbb{R}^N}\left(|\nabla v_n|^2+\widetilde{V}(x)\left|L^{-1}(v_n)\right|^2\right)dx\geq C_7\|v_n\|^2.\tag{2.6}$$

Indeed, we may assume $v_n\not\equiv 0$ (otherwise, the conclusion is trivial). If (2.6) is incorrect, passing to a subsequence, we suppose

$$\frac{\rho_n^2}{\|v_n\|^2}\to 0,\quad\text{as }n\to\infty.\tag{2.7}$$

Setting

$$w_n=\frac{v_n}{\|v_n\|}\quad\text{and}\quad g_n=\frac{|L^{-1}(v_n)|^2}{\|v_n\|^2},$$

one has

$$\begin{aligned}\frac{\rho_n^2}{\|v_n\|^2} &= \frac{1}{\|v_n\|^2}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx + \frac{1}{\|v_n\|^2}\int_{\mathbb{R}^N}\widetilde{V}(x)\big|L^{-1}(v_n)\big|^2\,dx \\ &= \int_{\mathbb{R}^N}|\nabla w_n|^2\,dx + \int_{\mathbb{R}^N}\widetilde{V}(x)g_n\,dx \to 0.\end{aligned}$$

From (2.7), as $n \to \infty$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla w_n|^2\,dx \to 0, \\ &\int_{\mathbb{R}^N}\widetilde{V}(x)g_n\,dx \to 0,\end{aligned}$$

and

$$\int_{\mathbb{R}^N}\widetilde{V}(x)w_n^2\,dx \to 1. \tag{2.8}$$

We claim that for each $\varepsilon > 0$, there exists a constant $C_8 > 0$ independent of n, such that $\operatorname{meas}(\Omega_n) < \varepsilon$, where $\Omega_n := \{x \in \mathbb{R}^N : |v_n(x)| \geq C_8\}$. If this claim is not true, there is an $\varepsilon_0 > 0$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that for each positive integer k, $\operatorname{meas}(\{x \in \mathbb{R}^N : |v_{n_k}(x)| \geq k\}) \geq \varepsilon_0 > 0$. Set $\Omega_{n_k} := \{x \in \mathbb{R}^N : |v_{n_k}(x)| \geq k\}$. By Lemma 2.1(9), we obtain

$$\begin{aligned}\rho_{n_k}^2 &\geq \int_{\mathbb{R}^N}\widetilde{V}(x)\big|L^{-1}(v_{n_k})\big|^2\,dx \\ &\geq \int_{\Omega_{n_k}}\widetilde{V}(x)\big|L^{-1}(v_{n_k})\big|^2\,dx \\ &\geq Ck\varepsilon_0 \to +\infty, \quad \text{as } k \to +\infty.\end{aligned}$$

On the other hand, if $|v_n(x)| < C_8$, by Lemma 2.1(9)–(10), one has

$$C_6^2\left(\frac{v_n(x)}{C_8}\right)^2 \leq \left|L^{-1}\left(\frac{v_n(x)}{C_8}\right)\right|^2 \leq C_9\big|L^{-1}\big(v_n(x)\big)\big|^2.$$

Therefore, there exists a constant $C_{10} > 0$ such that

$$\int_{\mathbb{R}^N\setminus\Omega_n}\widetilde{V}(x)w_n^2\,dx \leq C_{10}\int_{\mathbb{R}^N}\widetilde{V}(x)g_n dx \to 0, \quad \text{as } n \to +\infty. \tag{2.9}$$

By the absolutely continuity of the Lebesgue integral, there exist $\varepsilon > 0$ and $n_0 > 0$ for $n > n_0$ we have $\operatorname{meas}(\Omega_n) < \varepsilon$ and $\int_{\Omega_n}\widetilde{V}(x)w_n^2\,dx < \frac{1}{2}$. For this ε, taking $n \to +\infty$, we get

$$\begin{aligned}\int_{\mathbb{R}^N}\widetilde{V}(x)w_n^2\,dx &= \int_{\Omega_n}\widetilde{V}(x)w_n^2\,dx + \int_{\mathbb{R}^N\setminus\Omega_n}\widetilde{V}(x)w_n^2\,dx \\ &\leq \frac{1}{2} + \int_{\mathbb{R}^N\setminus\Omega_n}\widetilde{V}(x)w_n^2\,dx.\end{aligned} \tag{2.10}$$

From (2.8), (2.9), and (2.10), we get a contradictory inequality $1 \leq \frac{1}{2}$. Thus, summing up the above arguments, we prove that the Cerami sequence $\{v_n\}$ in (2.4) is bounded in X.

Step 3. We prove that $v_n \to v$ in X.

From the boundedness of the sequence $\{v_n\}$ and the compactness of the embedding $X \hookrightarrow\hookrightarrow L^s(\mathbb{R}^N)$, up to subsequence, we may assume

$$v_n \rightharpoonup v \quad \text{in } X \quad \text{and} \quad v_n \to v \quad \text{in } L^s\big(\mathbb{R}^N\big) \quad \text{for } s \in [2, 2^*).$$

By the growth condition (f_1), the properties of L^{-1} described in Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\frac{\widetilde{f}(L^{-1}(v_n))}{l(L^{-1}(v_n))} - \frac{\widetilde{f}(L^{-1}(v))}{l(L^{-1}(v))} \right)(v_n - v)\, dx \\
&\quad \le C_{11} \int_{\mathbb{R}^N} \big(|v_n| + |v_n|^{p-1} + |v| + |v|^{p-1}\big)(v_n - v)\, dx \\
&\quad \le C_{11}\big(|v_n|_2 + |v|_2\big)|v_n - v|_2 + C_{11}\big(|v_n|_p^{p-1} + |v|_p^{p-1}\big)|v_n - v|_p \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

On the other hand, we claim that there exists $C_{12} > 0$ such that

$$\int_{\mathbb{R}^N} \left(\big|\nabla(v_n - v)\big|^2 + \widetilde{V}(x)\left(\frac{L^{-1}(v_n)}{l(L^{-1}(v_n))} - \frac{L^{-1}(v)}{l(L^{-1}(v))} \right)(v_n - v) \right) dx \ge C_{12}\|v_n - v\|. \tag{2.11}$$

There is no harm in supposing $v_n \not\equiv v$ (otherwise, the conclusion is trivial). Denote

$$b_n = \frac{v_n - v}{\|v_n - v\|} \quad \text{and} \quad d_n = \frac{1}{v_n - v}\left(\frac{L^{-1}(v_n)}{l(L^{-1}(v_n))} - \frac{L^{-1}(v)}{l(L^{-1}(v))} \right).$$

If (2.11) is not true, we can assume that

$$\int_{\mathbb{R}^N} \big(|\nabla b_n|^2 + \widetilde{V}(x) d_n(x) b_n^2\big)\, dx \to 0.$$

By Lemma 2.1(8), this implies

$$\frac{d}{ds}\left(\frac{L^{-1}(s)}{l(L^{-1}(s))} \right) = \frac{l(L^{-1}(s)) - L^{-1}(s) l'(L^{-1}(s))}{l^3(L^{-1}(s))} > 0$$

and $\frac{d}{ds}(\frac{L^{-1}(s)}{l(L^{-1}(s))})\,|_{s=0} = 1$. Moreover, for each $\delta > 0$ there exists $C_\delta > 0$ such that

$$\frac{d}{ds}\left(\frac{L^{-1}(s)}{l(L^{-1}(s))} \right) > C_\delta, \quad \text{when } |s| < \delta.$$

Therefore, we deduce that $d_n(x)$ is positive and

$$\int_{\mathbb{R}^N} |\nabla b_n|^2\, dx \to 0, \qquad \int_{\mathbb{R}^N} \widetilde{V}(x) d_n(x) b_n^2\, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \widetilde{V}(x) b_n^2\, dx \to 1.$$

By the argument of proving Lemma 3.11 in [8], we can obtain a contradiction.

Consequently,

$$\begin{aligned} o_n(1) &= \langle J'(v_n) - J'(v), v_n - v \rangle \\ &= \int_{\mathbb{R}^N} \left|\nabla(v_n - v)\right|^2 dx + \int_{\mathbb{R}^N} \widetilde{V}(x) \left(\frac{L^{-1}(v_n)}{l(L^{-1}(v_n))} - \frac{L^{-1}(v)}{l(L^{-1}(v))} \right)(v_n - v)\, dx \\ &\quad - \int_{\mathbb{R}^N} \left(\frac{\widetilde{f}(L^{-1}(v_n))}{l(L^{-1}(v_n))} - \frac{\widetilde{f}(L^{-1}(v))}{l(L^{-1}(v))} \right)(v_n - v)\, dx \\ &\geq C_{12}\|v_n - v\|^2 + o_n(1). \end{aligned}$$

We deduce that $v_n \to v$ in X. □

3 Proof of Theorem 1.1

Since 0 is not an eigenvalue of

$$-\Delta u + V(x)u = \lambda u,$$

we can assume that there exists an integer $d \geq 0$ such that $0 \in (\lambda_d, \lambda_{d+1})$. For $d \geq 1$, we denote

$$X^- = \operatorname{span}\{e_1, \ldots, e_d\} \quad \text{and} \quad X^+ = \left(X^-\right)^{\perp}.$$

Specially, if $d = 0$, we set $X^- = \{0\}$ and $X^+ = X$. Then X^- and X^+ are the negative and positive spaces of the quadratic form

$$B(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)v^2\right) dx,$$

respectively. Furthermore, for some $\eta > 0$ we get

$$\pm B(v) \geq \eta\|v\|^2, \quad v \in X^{\pm}. \tag{3.1}$$

Since the principle part $Q(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|L^{-1}(v)|^2)\, dx$ of $J(v)$ is a C^2-functional on X with derivatives given by

$$\begin{aligned} \langle Q'(v), \phi \rangle &= \int_{\mathbb{R}^N} \left(\nabla v \cdot \nabla \phi + V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))} \phi \right) dx, \\ \langle Q''(v)\phi, \psi \rangle &= \int_{\mathbb{R}^N} \left(\nabla \phi \cdot \nabla \psi + V(x) \frac{l(L^{-1}(v)) - L^{-1}(v) l'(L^{-1}(v))}{l^3(L^{-1}(v))} \phi \psi \right) dx \end{aligned}$$

for all $v, \phi, \psi \in X$, in particular, since $L^{-1}(0) = 0$, $l(0) = 1$ and $|l'(t)| \leq C_5$, we have

$$\langle Q'(0), \phi \rangle = 0$$

and

$$\langle Q''(0)\phi, \psi \rangle = \int_{\mathbb{R}^N} \left(\nabla \phi \cdot \nabla \psi + V(x)\phi\psi\right) dx.$$

Applying Taylor's formula, we have

$$\begin{aligned} Q(v) &= Q(0) + \langle Q'(0), v\rangle + \frac{1}{2!}\langle Q''(0)v, v\rangle + o(\|v\|^2) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2)\,dx + o(\|v\|^2) \quad \text{for } \|v\| \text{ small.} \end{aligned} \tag{3.2}$$

To prove Theorem 1.1, we will apply the following Symmetric Mountain Pass Theorem due to Ambrosetti–Rabinowitz [14].

Proposition 3.1 *Let X be an infinite-dimensional Banach space, $X = Y \bigoplus Z$ with $\dim Y < +\infty$. If $J \in C^1(X, \mathbb{R})$ satisfies Cerami condition and*

(1) *$J(0) = 0$, $J(-u) = J(u)$ for all $u \in X$;*
(2) *there are constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho \cap Z} \geq \alpha$;*
(3) *for any finite-dimensional subspace $W \subset X$, there is an $R = R(W)$ such that $J \leq 0$ on $W \setminus B_{R(W)}$,*

then J has a sequence of critical values $c_j \to +\infty$.

Lemma 3.1 *Assume that (V_1), (V_2), (l), (f_1), (f_2) hold, and W is a finite-dimensional subspace of X. If $v \in W$, then*

$$J(v) \to -\infty, \quad \textit{as } \|v\| \to \infty.$$

Proof For any $\{v_n\} \subset W$ with $\|v_n\| \to +\infty$, consider

$$a_n = \frac{v_n}{\|v_n\|}.$$

Then $\{a_n\}$ is a bounded sequence in W. Since $\dim W < \infty$, there exists $a \in W \setminus \{0\}$ such that

$$\begin{aligned} &a_n \to a \quad \text{in } W, \\ &a_n \to a \quad \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

For $x \in \{a \neq 0\}$, we have

$$|v_n(x)| \to \infty,$$

and, using Lemma 2.1(9), obtain

$$|L^{-1}(v_n(x))| \to \infty.$$

Thus, from (f_2), for $x \in \{a \neq 0\}$,

$$\frac{\widetilde{F}(L^{-1}(v_n))}{\|v_n\|^2} = \frac{\widetilde{F}(L^{-1}(v_n))}{|v_n|^2}a_n^2 \geq \frac{|v_n|^\mu}{|v_n|^2}a_n^2 \to +\infty.$$

By Fatou's Lemma, we get

$$\int_{\mathbb{R}^N} \frac{\widetilde{F}(L^{-1}(v_n))}{\|v_n\|^2}\,dx \geq \int_{\{a\neq 0\}} \frac{\widetilde{F}(L^{-1}(v_n))}{\|v_n\|^2}\,dx \to +\infty, \quad \text{as } n \to \infty.$$

Furthermore,

$$
\begin{aligned}
J(v_n) &= \|v_n\|^2 \left(\frac{1}{2\|v_n\|^2} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + \widetilde{V}(x) \left| L^{-1}(v_n) \right|^2 \right) dx - \int_{\mathbb{R}^N} \frac{\widetilde{F}(L^{-1}(v_n))}{\|v_n\|^2} \, dx \right) \\
&\le \left(\frac{1}{2} - \int_{\mathbb{R}^N} \frac{\widetilde{F}(L^{-1}(v_n))}{\|v_n\|^2} \, dx \right) \|v_n\|^2 \to -\infty, \quad \text{as } n \to \infty. \qquad \square
\end{aligned}
$$

Lemma 3.2 *Under assumptions (V_1), (V_2), (l), and (f_1), one has*

$$
J|_{\partial B_\rho \cap Z_k} \ge \alpha,
$$

where $Z_k := \overline{\operatorname{span}}\{e_k, e_{k+1}, \dots\}$, ρ, k, α are positive constants, and $k > d$.

Proof By condition (f_1), there exist positive constants C_1 and C_2 such that

$$
\int_{\mathbb{R}^N} F\left(L^{-1}(t)\right) dx \le \int_{\mathbb{R}^N} \left(\frac{C_1}{2} |t|^2 + \frac{C_2}{p} |t|^p \right) dx.
$$

For $i \ge d$, denote $Z_i = \overline{\operatorname{span}}\{e_i, e_{i+1}, \dots\}$. Then, similar to Lemma 3.8 in [21], we have the following fact:

$$
\beta_i = \sup_{v \in Z_i, \|v\|=1} |v|_2 \to 0, \quad \text{as } i \to \infty.
$$

Let $Y = \operatorname{span}\{e_1, \dots, e_{k-1}\}$ and $Z_k = \overline{\operatorname{span}}\{e_k, e_{k+1}, \dots\}$, where $k > d$ and k will be determined, then $Z_k \subset X^+$. For $v \in Z_k$ and $\|v\|$ small enough, using (3.1) and Taylor's expansion, we have

$$
\begin{aligned}
J(v) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) \left| L^{-1}(v) \right|^2 \right) dx - \int_{\mathbb{R}^N} F\left(L^{-1}(v)\right) dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) v^2 \right) dx - \int_{\mathbb{R}^N} F\left(L^{-1}(v)\right) dx + o\left(\|v\|^2\right) \\
&\ge \eta \|v\|^2 - C_1 |v|_2^2 - C_2 |v|_p^p + o\left(\|v\|^2\right) \\
&\ge \left(\eta - C_1 \beta_k^2 \right) \|v\|^2 + o\left(\|v\|^2\right),
\end{aligned}
$$

where we used $p > 2$. Noticing that $\beta_k \to 0$ as $k \to \infty$, we then take k large enough such that $\eta - C_1 \beta_k^2 > 0$. □

Proof of Theorem 1.1 Obviously, using Lemmas 2.2, 3.1, 3.2 and (f_4), all conditions of Proposition 3.1 are satisfied. Therefore, J possesses a sequence of critical points $\{v_n\}$ with $J(v_n) \to +\infty$. Letting $u_n = L^{-1}(v_n)$, we obtain that $\{u_n\}$ is a sequence of weak solutions for problem (1.1) with $I(u_n) \to +\infty$. □

4 Proof of Theorem 1.2

In this section, by employing Morse theory, we prove the existence of one nontrivial solution for problem (1.1).

Let X be a real Banach space. For a given $J \in C^1(X)$, we use the following notation:

$$
J^c := \left\{ u \in X : J(u) \le c \right\}, \qquad \mathcal{K} = \left\{ u \in X : J'(u) = 0 \right\},
$$

U is a neighborhood of $u \in \mathcal{K}$, where u is an isolated critical point of J with $J(u) = c$. Then the qth critical group of J at an isolated critical point u is defined by

$$C_q(J,u) := H_q\bigl(J^c \cap U, J^c \cap U \setminus \{u\}\bigr), \quad q \in \mathbb{N},$$

where $H_q(\cdot,\cdot)$ is a qth singular relative homology group with integer coefficients. If J satisfies the Cerami condition and $a < \inf_{u\in\mathcal{K}} J(u)$, then the critical groups of J at infinity are defined by

$$C_q(J,\infty) := H_q\bigl(X, J^a\bigr).$$

In Morse theory, the functional J is always required to satisfy the so-called deformation condition.

Definition 4.1 The functional J satisfies deformation condition if for every $\varepsilon > 0$ small enough, $c \in \mathbb{R}$ and any neighborhood $\mathcal{N}$ of $\mathcal{K}_c$, there is a continuous deformation $\eta : [0,1] \times X \to X$ such that
(i) $\eta(t,v) = v$ for either $t = 0$ or $v \notin J^{-1}[c-\varepsilon, c+\varepsilon]$;
(ii) $J(\eta(t,v))$ is nonincreasing in t for any $v \in X$;
(iii) $\eta(J^{c+\varepsilon} \setminus \mathcal{N}) \subset J^{c-\varepsilon}$.

We note that if the functional J satisfies the (PS)-condition or the Cerami condition, then J satisfies the deformation condition.

Morse theory tells us that if J satisfies the Cerami condition, $v = 0$ is a critical point of J and $\mathcal{K} = \{0\}$, then $C_q(J,\infty) \cong C_q(J,0)$ for all $q \in \mathbb{N}$. It follows that if $C_q(J,\infty) \ncong C_q(J,0)$ for some $q \in \mathbb{N}$ then J must have a nontrivial critical point. So one has to compute these groups to get the nontrivial critical point.

4.1 Critical groups at zero

In this section, we will use the following proposition to compute the critical groups of J at zero.

Proposition 4.1 *Suppose $J \in C^1(X,\mathbb{R})$ has a local linking at zero with respect to the decomposition $X = Y \oplus Z$, i.e., for some $\varepsilon > 0$.*

$$\begin{aligned} &J(u) \le 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\ &J(u) > 0 \quad \text{for } u \in \bigl(Z \setminus \{0\}\bigr) \cap B_\varepsilon, \end{aligned} \tag{4.1}$$

where $B_\varepsilon = \{u \in X : \|u\| < \varepsilon\}$. If $d = \dim Y < \infty$, then $C_d(J,0) \neq 0$.

Lemma 4.1 *Under assumptions (V_1), (V_2), (l), (f_1), and (f_3), functional J has a local linking at zero with respect to decomposition $X = X^- \oplus X^+$, where X^-, X^+ are defined in Sect. 3 and $d = \dim X^-$.*

Proof From (f_1) and (f_3), for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$\bigl|F\bigl(L^{-1}(t)\bigr)\bigr| \le \varepsilon t^2 + C_\varepsilon |t|^p. \tag{4.2}$$

Hence, we get

$$\int_{\mathbb{R}^N} F\big(L^{-1}(v)\big)\,dx \le o\big(\|v\|^2\big) \quad \text{as } \|v\| \to 0.$$

Using this and (3.2), we obtain

$$\begin{aligned} J(v) &= Q(v) - \int_{\mathbb{R}^N} F\big(L^{-1}(v)\big)\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla v|^2 + V(x)v^2\big)\,dx + o\big(\|v\|^2\big). \end{aligned} \tag{4.3}$$

From this and (3.1), one obtains that J has a local linking property at zero. Then it follows from Proposition 4.1 that $C_d(J,0) \ncong 0$. □

4.2 Critical groups at infinity

Lemma 4.2 *Suppose that* (f_1)–(f_3), (V_1), (V_2), *and* (l) *hold. For any* $q \in \mathbb{N}$, $C_q(J,\infty) \cong 0$.

Proof Let S^∞ be the unit sphere in X. Firstly, we will establish the following fact:

$$J(sw) \to -\infty \quad \text{as } s \to +\infty \text{ for any } w \in S^\infty. \tag{4.4}$$

Due to $|L^{-1}(sw)| \le |sw|$ and (f_2), we deduce

$$\begin{aligned} J(sw) &\le \frac{1}{2}\int_{\mathbb{R}^N} \big(\big|\nabla(sw)\big|^2 + \widetilde{V}(x)(sw)^2\big)\,dx - \int_{\mathbb{R}^N} F\big(L^{-1}(sw)\big)\,dx \\ &\le s^2\left(\frac{1}{2} - s^{\mu-2}\int_{\mathbb{R}^N} |w|^\mu\,dx\right) \to -\infty. \end{aligned}$$

Secondly, we will show that the following claim is true.

Claim *There exists* $A > 0$ *such that if* $J(v) \le -A$ *then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=1} J(tv) < 0.$$

If this claim is false, there exists a sequence $\{v_n\} \subset X$ such that

$$J(v_n) \le -n \quad \text{and} \quad \big\langle J'(v_n), v_n\big\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=1} J(tv_n) \ge 0.$$

By the same argument as in Lemma 2.2, we deduce

$$\begin{aligned} 0 &\ge J(v_n) - \frac{1}{\mu}\big\langle J'(v_n), v_n\big\rangle \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \big(|\nabla v_n|^2 + \widetilde{V}(x)\big|L^{-1}(v_n)\big|^2\big)\,dx - \frac{V_0}{2}\int_{\mathbb{R}^N} \big|L^{-1}(v_n)\big|^2\,dx. \end{aligned} \tag{4.5}$$

Set

$$\rho_n^2 = \int_{\mathbb{R}^N} \big(|\nabla v_n|^2 + \widetilde{V}(x)\big|L^{-1}(v_n)\big|^2\big)\,dx \quad \text{and} \quad h_n = \frac{L^{-1}(v_n)}{\rho_n}.$$

By the similar arguments of Lemma 2.2, we obtain

$$\rho_n \to +\infty \quad \text{and} \quad \{h_n\} \text{ is a bounded sequence in } X.$$

Then

$$\begin{aligned} &h_n \rightharpoonup h \quad \text{in } X; \\ &h_n \to h \quad \text{in } L^2(\mathbb{R}^N); \\ &h_n \to h \quad \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

Multiplying both sides of (4.5) by ρ_n^{-2}, we obtain $h \neq 0$. By the assumption $\langle J'(v_n), v_n\rangle \geq 0$, we have

$$\begin{aligned} 0 &\leq \langle J'(v_n), v_n\rangle \\ &\leq \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + \widetilde{V}(x)\left|L^{-1}(v_n)\right|^2\right) dx - \int_{\mathbb{R}^N} \frac{\widetilde{f}(L^{-1}(v_n))v_n}{l(L^{-1}(v_n))}\, dx \\ &\leq \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + \widetilde{V}(x)\left|L^{-1}(v_n)\right|^2\right) dx - \mu \int_{\mathbb{R}^N} \frac{l(L^{-1}(v_n))F(L^{-1}(v_n))}{l(L^{-1}(v_n))}\, dx \\ &= \rho_n^2 \left(1 - \frac{\mu \int_{\mathbb{R}^N} F(L^{-1}(v_n))}{\rho_n^2}\right) \\ &\leq \rho_n^2 \left(1 - \mu \int_{\mathbb{R}^N} \frac{F(L^{-1}(v_n))}{|L^{-1}(v_n)|^2} h_n^2\, dx\right) \to -\infty, \quad \text{as } n \to \infty. \end{aligned} \tag{4.6}$$

This is impossible, thus the conclusion of the claim must be true.

Hence, by this claim and (4.4), for any fixed $a > A$ and $v \in S^\infty$, there exists a unique $T := T(v) > 0$ such that

$$J\big(T(v)v\big) = -a.$$

By the implicit function theorem, this implies

$$T \text{ is a continuous function from } S^\infty \text{ to } \mathbb{R}.$$

Therefore the deformation retract $\eta : [0,1] \times (X \setminus B^\infty) \to X$ defined by

$$\eta(s,v) = (1-s)v + sT(v)v$$

satisfies $\eta(0,v) = v$, $\eta(1,v) \in J^{-a}$ for a large enough, where $B^\infty = \{v \in X : \|v\| \leq 1\}$. It follows that

$$C_q(J,\infty) = H^q\big(X, J^{-a}\big) \cong H^q\big(X, X \setminus B^\infty\big) \cong 0 \quad \text{for all } q \in \mathbb{N}.$$

Proof of Theorem 1.2 We have verified that J satisfies the Cerami condition. By Lemma 4.1, J has a local linking at zero with respect to the decomposition $X = X^- \oplus X^+$, hence, by Proposition 4.1, for $d = \dim X^-$, we have

$$C_d(J,0) \neq 0.$$

On the other hand, Lemma 4.2 says that for all $q \in \mathbb{N}$, $C_q(J,\infty) = 0$. Hence, J has a nontrivial critical point v. Now $u = L^{-1}(v)$ is a nontrivial solution of problem (1.1). □

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