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The nonnegative weak solution of a degenerate parabolic equation with variable exponent growth order

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Abstract

A degenerate parabolic equation of the form

$$(|v|^{\beta-1}v)_t = \operatorname{div}(b(x,t)|\nabla v|^{p(x,t)-2}\nabla v) + \nabla\vec{g}\cdot\nabla\vec{\gamma}(v)$$

is considered, where $\vec{g} = \{g^i(x,t)\}$, $\vec{\gamma}(v) = \{\gamma_i(v)\}$. If the diffusion coefficient $b(x,t) \geq 0$ is degenerate on the boundary, by adding some restrictions on $b(x,t)$ and $\vec{g}$, the existence and uniqueness of weak solutions are proved. Based on the uniqueness, the stability of weak solutions can be proved without any boundary condition.

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1 Introduction and the main results

Consider the degenerate parabolic problem with exponent variable growth order

$$\left(|v|^{\beta-1}v\right)_t = \operatorname{div}\left(b(x,t)|\nabla v|^{p(x,t)-2}\nabla v\right) + \nabla\vec{g}\cdot\nabla\vec{\gamma}(v), \quad (x,t)\in Q_T = \Omega\times(0,T), \tag{1.1}$$

where $b(x,t)$ and $p(x,t)$ are $C(\overline{Q_T})$ nonnegative functions, $\vec{g} = \{g^i(x,t)\}$, $\vec{\gamma}(v) = \{\gamma_i(v)\}$, $\beta > 0$. We denote that $p^- = \min_{(x,t)\in\overline{Q_T}} p(x,t) > 1$ and $p^+ = \max_{(x,t)\in\overline{Q_T}} p(x,t)$ in this paper. The initial value matching up to equation (1.1) is

$$|v|^{\beta-1}v(x,0) = \left|v_0(x)\right|^{\beta-1}v_0(x), \quad x\in\Omega. \tag{1.2}$$

While the Dirichlet boundary value condition

$$v(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T) \tag{1.3}$$

is dispensable.

If $g = 0$, equation (1.1) arises from the branches of flows of electro-rheological or thermo-rheological fluids (see [1–3]), and the processing of digital images [4–15]. If the variable exponent $p(x,t)$ is replaced by a constant p, equation (1.1) becomes the well-

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known non-Newtonian polytropic filtration equation with orientated convection [16], as well as the convection-diffusion-reaction equation in which the variable can be interpreted as temperature for heat transfer problems, concentration for dispersion problems, etc. [17]. Now, let us give some details in part of the above references. Ye and Yin studied the propagation profile for the equation

$$u_t = \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) - \vec{\beta}(x)\cdot\nabla u^q,$$

in which the orientation of the convection was specified to be either the convection with counteracting diffusion or the convection with promoting diffusion, that is, $\vec{\beta}(x)\cdot(-x)\geq 0$ or $\vec{\beta}(x)\cdot x\geq 0$, respectively [16]. Guo, Li, and Gao considered the following evolutionary $p(x)$-Laplacian equation:

$$v_t = \operatorname{div}\left(|\nabla v|^{p(x)-2}\nabla v\right) + |v|^{r-2}v, \quad (x,t)\in Q_T,$$

subject to homogeneous Dirichlet boundary condition, where $r > 1$ is a constant. By using the energy estimate method, the regularity of weak solutions and blow-up in finite time were revealed in [7]. Antontsev and Shmarev have published a series of papers [8–13] on the homogeneous Dirichlet problem for the doubly nonlinear parabolic equation

$$v_t = \operatorname{div}\left(b(x,t,v)|v|^{\alpha(x,t)}|\nabla v|^{p(x,t)-2}\nabla v\right) + f(x,t), \quad (x,t)\in Q_T,$$

provided that $a(x,t,v)\geq a > 0$. They established conditions on the data that guarantee the comparison principle and uniqueness of bounded weak solutions in suitable Orlicz–Sobolev spaces subject to some additional restrictions [12]. Gao, Chu, and Gao in [14] studied the nonlinear diffusion equation

$$v_t = \operatorname{div}\left(|\nabla v|^{p(x,t)-2}\nabla v + b(x,t)\nabla v\right) + f(v), \quad (x,t)\in Q_T,$$

with the homogeneous Dirichlet boundary condition (1.3), where f is a continuous function satisfying

$$\left|f(v)\right| \leq a_0|v|^{\alpha-1},$$

with $a_0 > 0$ and $\alpha > 1$. They constructed suitable function spaces and used Galerkin's method to obtain the existence of weak solutions. It is worth pointing out that the requirement on $p_t(x,t)$ is only negative and integrable, which is a weaker condition than the corresponding conditions appearing in other papers. Recently, Liu and Dong [15] generalized [14]'s result to a more general equation

$$v_t = \operatorname{div}\left(\left|\nabla v^m\right|^{p(x,t)-2}\nabla v^m + b(x,t)\nabla v^m\right) + v^{q(x,t)}, \quad (x,t)\in Q_T$$

and gave a classification of the weak solutions. In addition, the equation arising from the double phase obstacle problems of the type

$$v_t = \operatorname{div}\left(a(x)|\nabla v|^{p-2}\nabla v + b(x)|\nabla v|^{q-2}\nabla v\right) + f(x,t,v,\nabla v), \quad (x,t)\in Q_T$$

has gained a wide attention [18, 19] etc., where $a(x) + b(x) > 0$.

In recent years, we have been interested in the well-posedness of weak solutions to the nonlinear equation

$$v_t = \operatorname{div}\big(b(x)|\nabla v|^{p(x)-2}\nabla v\big) + f(x,t,v,\nabla v), \quad (x,t) \in Q_T, \tag{1.4}$$

with some restrictions in $f(x,t,v,\nabla v)$. Different from other researchers' works [7–15], in which $b(x) = 1$ or $b(x) > b^- > 0$, where $b^- = \min_{x\in\overline{\Omega}} b(x)$, we only assumed that

$$b(x) > 0, \quad x \in \Omega, \qquad b(x) = 0, \quad x \in \partial\Omega,$$

and proved that the stability of weak solutions may be independent of the Dirichlet boundary value condition (1.3). One might refer to [20–24] for the details.

In this paper, for any $t \in [0,T]$, we assume

$$b(x,t) > 0, \quad x \in \Omega, \qquad b(x,t) = 0, \quad x \in \partial\Omega. \tag{1.5}$$

Comparing with equation (1.4), equation (1.1) is with the nonlinearity of $|v|^{\beta-1}v$, with the diffusion coefficient $b(x,t)$ and the variable exponent $p(x,t)$ depending on time variable t, and with a more complicate convection term $\nabla\vec{g}\cdot\nabla\vec{\gamma}(v)$. These nonlinearities not only bring some essential changes to the proof of the existence, but also add difficulties to proving the stability of weak solutions. The readers will see that, in order to overcome these difficulties, a new technique based on the mean value theorem is posed to prove the uniqueness of weak solution, another new technique based on the proof by contradiction is introduced. Both of them supply a new method to prove uniqueness of weak solution for the nonlinear degenerate parabolic equations.

Definition 1.1 A function $v(x,t)$ is said to be a weak solution of equation (1.1) with initial value (1.2), provided that $v(x,t)$ satisfies

$$\begin{aligned} &v \in L^\infty_{\mathrm{loc}}\big(0,T;W^{1,p(x,t)}(\Omega)\big), \qquad v \in W^{1,2}_{\mathrm{loc}}\big((0,T),L^2(\Omega)\big), \\ &b(x,t)|\nabla v|^{p(x,t)} \in L^1(Q_T), \end{aligned} \tag{1.6}$$

and for $\forall\phi(x,t) \in C_0^1(Q_T)$,

$$\begin{aligned} &\iint_{Q_T}\big(-|v|^{\beta-1}v\phi_t\big)\,dx\,dt + \iint_{Q_T} b(x,t)|\nabla v|^{p(x,t)-2}\nabla v\cdot\nabla\phi\,dx\,dt \\ &\quad + \sum_{i=1}^N \iint_{Q_T} g^i(x,t)\gamma_i(v)\phi_{x_i}\,dx\,dt \\ &= \sum_{i=1}^N \iint_{Q_T} \gamma_i(v)g^i(x,t)\phi(x,t)\,dx\,dt. \end{aligned} \tag{1.7}$$

Initial value (1.2) is true in the sense

$$\lim_{t\to 0}\int_\Omega |v|^{\beta-1}v(x,t)\varphi(x)\,dx = \int_\Omega |v|^{\beta-1}v_0(x)\varphi(x)\,dx, \quad \forall\varphi(x) \in C_0^\infty(\Omega). \tag{1.8}$$

The main results are the following theorems.

Theorem 1.2 *If $p^- \geq 2$, $b(x,t)$ satisfying (1.5), $v_0(x) \in L^\infty(\Omega)$ is nonnegative for $i \in \{1,2,\ldots,N\}$, $\gamma_i(s)$ is a C^1 function satisfying $|\gamma_i'(s)|^2|s|^{1-\beta} \leq c$ for $i = 1,2,\ldots,N$, $g^i(x,t)$ satisfies*

$$\int_\Omega \frac{p(x,t)-2}{p(x,t)} \left(\sum_{i=1}^N g^i(x,t) b(x,t)^{-\frac{2}{p(x,t)}} \right)^{\frac{p(x,t)}{p(x,t)-2}} dx \leq c(T), \tag{1.9}$$

then there is a nonnegative weak solution of equation (1.1) with initial value (1.2) in the sense of Definition 1.1.

Theorem 1.3 *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.1) with the different initial values $u_0(x)$ and $v_0(x)$ respectively, $0 < m \leq \|u\|_{L^\infty(Q_T)} \leq M$, $0 < m \leq \|v\|_{L^\infty(Q_T)} \leq M$. If $p^- > 1$, $\gamma_i(s)$ is a Lipschitz function, $b(x,t)$ satisfies (1.5), and*

$$\left| \sum_{i=1}^N \frac{\partial g^i(x,t)}{\partial x_i} \right| \leq c b(x,t)^{\frac{\alpha_1}{p(x,t)}}, \tag{1.10}$$

then there exists a constant $\alpha_1 \geq 2p^+$ such that

$$\begin{aligned} &\int_\Omega b(x,t)^{\frac{\alpha_1}{p(x,t)}} \left| |u|^{\beta-1}u(x,t) - |v|^{\beta-1}v(x,t) \right|^2 dx \\ &\quad \leq \int_\Omega b(x,0)^{\frac{\alpha_1}{p(x,0)}} \left| |u_0|^{\beta-1}u_0(x) - |v_0|^{\beta-1}v_0(x) \right|^2 dx, \quad \forall t \in [0,T). \end{aligned} \tag{1.11}$$

Theorem 1.4 *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.1) with the different initial values $u_0(x)$ and $v_0(x)$ respectively, $\gamma_i(s)$ be a Lipschitz function. Suppose that $g^i(x,t)$ satisfies (1.10) and*

$$\left| \sum_{i=1}^N g^i(x,t) \right| \leq c b(x,t)^{\frac{1}{p(x,t)}} \tag{1.12}$$

and one of the following conditions is true:

(i) $\beta \leq 1$.

(ii) *For $1 \leq i \leq N$, $\gamma_i(s)$ satisfies*

$$\left| \gamma_i(t_1) - \gamma_i(t_2) \right| \leq c \left| |t_1|^{\beta-1}t_1 - |t_2|^{\beta-1}t_2 \right|. \tag{1.13}$$

Then

$$\begin{aligned} &\int_\Omega \left| |u|^{\beta-1}u(x,t) - |v|^{\beta-1}v(x,t) \right| dx \\ &\quad \leq \int_\Omega \left| |u_0|^{\beta-1}u_0(x) - |v_0|^{\beta-1}v_0(x) \right| dx, \quad \forall t \in [0,T). \end{aligned} \tag{1.14}$$

Conditions (1.9), (1.10), and (1.12) all reflect the internal mutually dependent relationships between the diffusion coefficient $b(x,t)$ and the convective coefficients $g^i(x,t)$. Such an internal mutually dependent relationship that can affect the finite propagation has been studied in [16], while the internal mutually dependent relationships between the diffusion coefficient and the convection term arise in mathematics finance model for studying the agent's decision under the risk [25].

At the end of introduction, it might be advisable to summarize briefly. First, as a classical work on the well-posed results of the solution of a nonlinear parabolic equation, there are many papers devoted to this problem (one can refer to [26–28] and the references therein). Secondly, the model studied in this paper is a parabolic equation with variable exponential term; we would like to point out that more details on the structural characteristics and the physical background of the variable exponential term have been described in [29–33], etc. Thirdly, one can see that the new method to prove uniqueness of weak solution can be generalized to study the double phase obstacle problems.

2 The existence of weak solutions

Let us consider the approximate initial-boundary value problem

$$\big(|v|^{\beta-1}v\big)_t = \operatorname{div}\big(\big(b(x,t)+\varepsilon\big)|\nabla v|^{p(x,t)-2}\nabla v\big) + \nabla\vec{g}\cdot\nabla\vec{\gamma}(v), \tag{2.1}$$

$$|v|^{\beta-1}v(x,0) = |v_0|^{\beta-1}v_0(x), \quad x\in\Omega, \tag{2.2}$$

$$v(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T). \tag{2.3}$$

Definition 2.1 If $u(x,t)$ satisfies

$$v\in L^\infty_{\mathrm{loc}}\big(0,T;W_0^{1,p(x,t)}(\Omega)\big), \qquad v\in W^{1,2}_{\mathrm{loc}}\big(0,T;L^2(\Omega)\big), \tag{2.4}$$

and for any $\phi(x,t)\in C_0^1(Q_T)$, there holds

$$\begin{aligned} &-\iint_{Q_T}|v|^{\beta-1}v\phi_t\,dx\,dt + \iint_{Q_T}\big(b(x,t)+\varepsilon\big)|\nabla v|^{p(x,t)-2}\nabla v\cdot\nabla\phi\,dx\,dt \\ &\quad + \sum_{i=1}^N\iint_{Q_T}\gamma_i(v)\phi_{x_i}\,dx\,dt \\ &= \sum_{i=1}^N\iint_{Q_T}\gamma_i(v)g^i(x,t)\phi(x,t)\,dx\,dt. \end{aligned} \tag{2.5}$$

Then we say that $v(x,t)$ is said to be the weak solution of problem (2.1)–(2.3).

For any $k>0$, let

$$a_k = k^{2-\beta}, \qquad b_k = k^{1-\beta}\frac{3-\beta}{2}, \quad k=1,2,\ldots.$$

$\varphi_k(v)$ is an even function and is defined as

$$\varphi_k(v) = \begin{cases} \beta v^{\beta-1}, & v\ge k^{-1}, \\ \beta(a_kv^2+b_kv), & 0\le v<k^{-1}. \end{cases}$$

Then $\varphi_k(v) \in C^1(\mathbb{R})$, $\varphi_k(v) \to \beta v^{\beta-1}$ as $k \to \infty$. Instead of (2.1)–(2.3), we now consider the following problem:

$$\varphi_k(v)v_t = \operatorname{div}\left(\left(b(x,t)+\varepsilon\right)\left(|\nabla v|^2 + \frac{1}{k}\right)^{\frac{p(x,t)-2}{2}} \nabla v\right) + \nabla\vec{g}\cdot\nabla\vec{\gamma}(v), \tag{2.6}$$

$$v_k(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.7}$$

$$v_k(x,0) = v_{0k}(x), \quad x \in \Omega, \tag{2.8}$$

where $\|v_{0k}(x) - v_0(x)\|_{p^+(0)} \to 0$ as $k \to 0$, and $p^+(0) = \max_{x\in\overline{\Omega}} p(x,0)$. From [8–14], there is a unique solution $v_{k\varepsilon}$ of initial-boundary value problem (2.6)–(2.8). Let $k \to \infty$. If $v_0(x) \in L^\infty(\Omega)$ is nonnegative, similar to the process subject to the existence of weak solutions in [12](also[34]), one can prove that there is a nonnegative weak solution $v_\varepsilon \in L^1(0,T;W_0^{1,p(x,t)}(\Omega))$ to initial-boundary value problem (2.1)–(2.3) in the sense of Definition 2.1. Moreover,

$$\|v_{k\varepsilon}\|_{L^\infty(Q_T)} \le c, \qquad \|v_\varepsilon\|_{L^\infty(Q_T)} \le c. \tag{2.9}$$

Proof of Theorem 1.2 Let us choose v_ε as a test function. Then

$$\begin{aligned}
&\frac{\beta}{\beta+1}\int_\Omega v_\varepsilon^{\beta+1}\,dx + \iint_{Q_T}\left(b(x,t)+\varepsilon\right)|\nabla v_\varepsilon|^{p(x,t)}\,dx\,dt \\
&\quad - \iint_{Q_T} v_\varepsilon \nabla\vec{g}\cdot\nabla\vec{\gamma}(v_\varepsilon)\,dx\,dt \\
&\quad = \frac{\beta}{\beta+1}\int_\Omega v_0(x)^{\beta+1}\,dx.
\end{aligned} \tag{2.10}$$

Since

$$\begin{aligned}
&-\iint_{Q_T} v_\varepsilon\nabla\vec{g}\cdot\nabla\vec{\gamma}(v_\varepsilon)\,dx\,dt \\
&\quad = \sum_{i=1}^N \iint_{Q_T} v_\varepsilon g^i(x,t)\frac{\partial\gamma_i(v_\varepsilon)}{\partial x_i}\,dx\,dt \\
&\quad = \sum_{i=1}^N \left[\iint_{Q_T} v_\varepsilon \frac{\partial(\gamma_i(v_\varepsilon)g^i(x,t))}{\partial x_i}\,dx\,dt - \iint_{Q_T} v_\varepsilon a_i(v_\varepsilon)\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt\right] \\
&\quad = -\sum_{i=1}^N \iint_{Q_T} \frac{\partial v_\varepsilon}{\partial x_i} g^i(x,t)\gamma_i(v_\varepsilon)\,dx\,dt - \sum_{i=1}^N \iint_{Q_T} v_\varepsilon\gamma_i(v_\varepsilon)\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt \\
&\quad = -\sum_{i=1}^N \int_\Omega \frac{\partial}{\partial x_i}\int_0^{v_\varepsilon}\gamma_i(v_\varepsilon)\,ds\,dx - \sum_{i=1}^N \iint_{Q_T} v_\varepsilon\gamma_i(v_\varepsilon)\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt \\
&\quad = -\sum_{i=1}^N \iint_{Q_T} v_\varepsilon\gamma_i(v_\varepsilon)\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt,
\end{aligned} \tag{2.11}$$

we have

$$\frac{\beta}{\beta+1}\int_\Omega v_\varepsilon^{\beta+1}\,dx + \iint_{Q_T}\left(b(x,t)+\varepsilon\right)|\nabla v_\varepsilon|^{p(x,t)}\,dx\,dt \le c, \tag{2.12}$$

which implies

$$\iint_{Q_T} b(x,t)|\nabla v_\varepsilon|^{p(x,t)}\,dx\,dt \le c. \tag{2.13}$$

Moreover, let us multiply (2.1) with $v_{\varepsilon t}$, and obtain

$$\begin{aligned}
&\beta\int_\Omega v_\varepsilon^{\beta-1}(v_{\varepsilon t})^2\,dx \\
&\quad = \int_\Omega \operatorname{div}\big(\big(\rho(x,t)+\varepsilon\big)|\nabla v_\varepsilon|^{p(x,t)-2}\nabla v_\varepsilon\big)v_{\varepsilon t}\,dx + \int_\Omega \nabla\vec{g}\cdot\nabla\vec{\gamma}(v_\varepsilon)v_{\varepsilon t}\,dx.
\end{aligned} \tag{2.14}$$

Since

$$\begin{aligned}
&\int_\Omega \operatorname{div}\big(\big(b(x,t)+\varepsilon\big)|\nabla v_\varepsilon|^{p(x,t)-2}\nabla v_\varepsilon\big)v_{\varepsilon t}\,dx \\
&\quad = -\frac{1}{2}\int_\Omega \big(b(x,t)+\varepsilon\big)|\nabla v_\varepsilon|^{p(x,t)-2}|\nabla v_\varepsilon|_t^2\,dx \\
&\quad = -\frac{1}{2}\int_\Omega \big(b(x,t)+\varepsilon\big)\frac{d}{dt}\int_0^{|\nabla v_\varepsilon|^2} s^{\frac{p(x,t)-2}{2}}\,ds\,dx \\
&\qquad + \frac{1}{2}\int_\Omega \big(b(x,t)+\varepsilon\big)\int_0^{|\nabla v_\varepsilon|^2} \frac{d}{dt}s^{\frac{p(x,t)-2}{2}}\,ds\,dx,
\end{aligned} \tag{2.15}$$

and by $|\gamma_i'(s)|^2|s|^{1-\beta} \le c$, $p^- \ge 2$ and by (1.9), using the Young inequality, we have

$$\begin{aligned}
&\left|\int_\Omega \sum_{i=1}^N \frac{\partial\gamma_i(v_\varepsilon)}{\partial x_i}g^i(x,t)v_{\varepsilon t}\,dx\right| \\
&\quad \le \int_\Omega \sum_{i=1}^N \big|\gamma_i'(v_\varepsilon)\big|\big|g^i(x,t)v_{\varepsilon x_i}\big||v_{\varepsilon t}|\,dx \\
&\quad \le \frac{\beta}{2}\int_\Omega v_\varepsilon^{\beta-1}(v_{\varepsilon t})^2\,dx + \frac{2}{\beta}\int_\Omega \sum_{i=1}^N \big|\gamma_i'(v_\varepsilon)g^i(x,t)v_{\varepsilon x_i}\big|^2|v_\varepsilon|^{1-\beta}\,dx \\
&\quad \le \frac{\beta}{2}\int_\Omega v_\varepsilon^{\beta-1}(v_{\varepsilon t})^2\,dx \\
&\qquad + c\int_\Omega \left[\frac{p(x,t)-2}{p(x,t)}\left(\sum_{i=1}^N g^i(x,t)b(x,t)^{-\frac{2}{p(x,t)}}\right)^{\frac{p(x,t)}{p(x,t)-2}} + \frac{2}{p(x,t)}b(x,t)|\nabla v_\varepsilon|^{p(x,t)}\right]dx.
\end{aligned} \tag{2.16}$$

From (2.14)–(2.16), we extrapolate that

$$\begin{aligned}
&\frac{\beta}{2}\int_\Omega v^{\beta-1}(u_{\varepsilon t})^2\,dx + \frac{1}{2}\int_\Omega \big(b(x,t)+\varepsilon\big)\frac{d}{dt}\int_0^{|\nabla v_\varepsilon|^2} s^{\frac{p(x,t)-2}{2}}\,ds\,dx \\
&\quad \le c\int_\Omega \left[\frac{p(x,t)-2}{p(x,t)}\big(g^i(x,t)b(x,t)^{-\frac{2}{p(x,t)}}\big)^{\frac{p(x,t)}{p(x,t)-2}} + \frac{2}{p(x,t)}b(x,t)|\nabla v_\varepsilon|^{p(x,t)}\right]dx
\end{aligned}$$

$$+\left|\frac{1}{2}\int_{\Omega}\big(b(x,t)+\varepsilon\big)\int_0^{|\nabla v_\varepsilon|^2}\frac{d}{dt}s^{\frac{p(x,t)-2}{2}}\,ds\,dx\right|$$
$$\leq c.$$

Then

$$\big\|\big(v_\varepsilon^{\frac{\beta+1}{2}}\big)_t\big\|_{L^2(Q_T)}=\frac{\beta+1}{2}\big\|v_\varepsilon^{\frac{\beta-1}{2}}v_{\varepsilon t}\big\|_{L^2(Q_T)}\leq c, \tag{2.17}$$

and

$$\iint_{Q_T}|v_{\varepsilon t}|^2\,dx\,dt\leq\int_0^T\int_\Omega v_\varepsilon^{\beta-1}|v_{\varepsilon t}|^2v_\varepsilon^{1-\beta}\,dx\,dt\leq\|v_\varepsilon\|_{L^\infty(Q_T)}^{1-\beta}\int_0^T\int_\Omega v_\varepsilon^{\beta-1}|v_{\varepsilon t}|^2\,dx\,dt$$
$$\leq c. \tag{2.18}$$

From (2.12), (2.18), we are able to extrapolate that $v_\varepsilon\to v$ a.e. in Q_T. Accordingly, $\gamma_i(v_\varepsilon)\to\gamma_i(v)$ a.e. in Q_T.

Let $\varepsilon\to 0$ in (2.10). Similar to that in [35], which is subject to the evolutionary p-Laplacian equation, it is not difficult to deduce that

$$\big(b(x,t)+\varepsilon\big)|\nabla v_\varepsilon|^{p(x,t)-2}\nabla v_\varepsilon\rightharpoonup b(x,t)|\nabla v|^{p(x,t)-2}\nabla v,\quad\text{in } L^1\big(0,T;L^{\frac{p(x,t)}{p(x,t)-1}}(\Omega)\big).$$

Also, we can show that initial value (1.2) is true in the sense of (1.8) as in [12]. Theorem 1.2 is proved. □

3 Proof of Theorem 1.3

Lemma 3.1 ([36, 37])

(i) *The space* $(L^{p(x)}(\Omega),\|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega),\|\cdot\|_{W^{1,p(x)}(\Omega)})$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) *Let* $p(x)$ *and* $q(x)$ *be two functions with* $\frac{1}{p(x)}+\frac{1}{q(x)}=1$ *. The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$*. For any* $u\in L^{p(x)}(\Omega)$ *and* $v\in L^{q(x)}(\Omega)$,

$$\left|\int_\Omega uv\,dx\right|\leq 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{q(x)}(\Omega)}.$$

(iii)

$$\textit{If } \|u\|_{L^{p(x)}(\Omega)}=1,\quad\textit{then }\int_\Omega|u|^{p(x)}\,dx=1.$$
$$\textit{If } \|u\|_{L^{p(x)}(\Omega)}>1,\quad\textit{then } |u|_{L^{p(x)}(\Omega)}^{p^-}\leq\int_\Omega|u|^{p(x)}\,dx\leq|u|_{L^{p(x)}(\Omega)}^{p^+}.$$
$$\textit{If } \|u\|_{L^{p(x)}(\Omega)}<1,\quad\textit{then } |u|_{L^{p(x)}(\Omega)}^{p^+}\leq\int_\Omega|u|^{p(x)}\,dx\leq|u|_{L^{p(x)}(\Omega)}^{p^-}.$$

Proof of Theorem 1.3 For any given $t\in(0,T)$ and small enough $\lambda>0$, we denote $\Omega_{\lambda t}=\{x\in\Omega: b(x,t)>\lambda\}$ and define

$$\xi_\lambda(x,t)=\big[b(x,t)-\lambda\big]_+^{\frac{\alpha_1}{p(x,t)}},$$

where $\alpha_1\geq 2p^+$.

We choose $\chi_{[\tau,s]}(t)[u(x,t)-v(x,t)]\xi_\lambda(x,t)$ as a test function, where $\chi_{[\tau,s]}$ is the characteristic function on $[\tau,s]\subset(0,t)$. Then

$$\begin{aligned}
&\iint_{Q_{\tau s}}(u-v)\xi_\lambda(x,t)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt\\
&=-\iint_{Q_{\tau s}}\bigl(b(x,t)|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\nabla\bigl[(u-v)\xi_\lambda(x,t)\bigr]\,dx\,dt\\
&\quad-\sum_{i=1}^{N}\iint_{Q_{\tau s}}g^i(x,t)\bigl[\gamma_i(u)-\gamma_i(v)\bigr]\bigl[u-v)\xi_\lambda(x,t)\bigr]_{x_i}\,dx\,dt\\
&\quad-\sum_{i=1}^{N}\iint_{Q_{\tau s}}\bigl[\gamma_i(u)-\gamma_i(v)\bigr]\bigl[(u-v)\xi_\lambda(x,t)\bigr]\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt,
\end{aligned}\tag{3.1}$$

where $Q_{\tau s}=\Omega\times[\tau,s]$ as usual.

In the first place,

$$\begin{aligned}
&\iint_{Q_{\tau s}}b(x,t)\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\nabla\bigl[(u-v)\xi_\lambda\bigr]\,dx\,dt\\
&=\iint_{Q_{\tau s}}b(x,t)\xi_\lambda\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\nabla(u-v)\,dx\,dt\\
&\quad+\iint_{Q_{\tau s}}b(x,t)\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)(u-v)\nabla\xi_\lambda\,dx\,dt,
\end{aligned}\tag{3.2}$$

we have

$$\iint_{Q_{\tau s}}b(x,t)\xi_\lambda\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\nabla(u-v)\,dx\,dt\ge 0\tag{3.3}$$

and

$$\begin{aligned}
&\left|\iint_{Q_{\tau s}}(u-v)b(x,t)\bigl(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\bigr)\nabla\xi_\lambda\,dx\,dt\right|\\
&\le c\left(\int_\tau^s\int_{\Omega_{\lambda t}}b(x,t)\bigl(|\nabla u|^{p(x,t)}+|\nabla v|^{p(x,t)}\bigr)\,dx\,dt\right)^{\frac{1}{q_1}}\\
&\quad\cdot\left(\int_\tau^s\int_{\Omega_{\lambda t}}b(x,t)|\nabla\xi_\lambda|^{p(x,t)}|u-v|^{p(x,t)}\,dx\,dt\right)^{\frac{1}{p_1}}\\
&\le c\left(\int_\tau^s\int_{\Omega}b(x,t)\bigl(|\nabla u|^{p(x,t)}+|\nabla v|^{p(x,t)}\bigr)\,dx\,dt\right)^{\frac{1}{q_1}}\\
&\quad\cdot\left(\int_\tau^s\int_{\Omega_{\lambda t}}b(x,t)\bigl[b(x,t)-\lambda\bigr]^{p(x,t)(\frac{\alpha_1}{p(x,t)}-1)}|\nabla b|^{p(x,t)}|u-v|^{p(x,t)}\,dx\,dt\right)^{\frac{1}{p_1}}\\
&\le c\left(\int_\tau^s\int_{\Omega_{\lambda t}}b(x,t)\bigl[b(x,t)-\lambda\bigr]^{p(x,t)(\frac{\alpha_1}{p(x,t)}-1)}|u-v|^{p(x,t)}\,dx\,dt\right)^{\frac{1}{p_1}}.
\end{aligned}\tag{3.4}$$

Here, $q(x,t)=\frac{p(x,t)}{p(x,t)-1}$, from (iii) of Lemma 3.1, $q_1=q^+$ or q^- according to

$$\int_\tau^s\int_{\Omega_{\lambda t}} b(x,t)\big(|\nabla u|^{p(x,t)}+|\nabla v|^{p(x,t)}\big)\,dx\,dt<1,$$

or

$$\int_\tau^s\int_{\Omega_{\lambda t}} b(x,t)\big(|\nabla u|^{p(x,t)}+|\nabla v|^{p(x,t)}\big)\,dx\,dt\geq 1,$$

p_1 has a similar meaning, and we have used the fact that $|\nabla b|\leq c$ in (3.4).

Then

$$\begin{aligned}&\lim_{\lambda\to 0}\left|\iint_{Q_{\tau s}}(u-v)b(x,t)\big(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\big)\nabla\xi_\lambda\,dx\,dt\right|\\&\quad\leq\lim_{\lambda\to 0}c\left(\int_\tau^s\int_{\Omega_{\lambda t}} b(x,t)\big(b(x,t)-\lambda\big)^{p(x,t)(\frac{\alpha_1}{p(x,t)}-1)}|u-v|^{p(x,t)}\,dx\,dt\right)^{\frac{1}{p_1}}\\&\quad\leq c\left(\int_\tau^s\int_{\Omega} b(x,t)^{1+p(x,t)(\frac{\alpha_1}{p(x,t)}-1)}|u-v|^{p(x,t)}\,dx\,dt\right)^{\frac{1}{p_1}}.\end{aligned}\tag{3.5}$$

If we denote that

$$\Omega_{1t}=\big\{x\in\Omega:p(x,t)\geq 2\big\},\qquad \Omega_{2t}=\big\{x\in\Omega:p(x,t)<2\big\},$$

by $u,v\in L^\infty$, we have

$$\begin{aligned}&\left(\int_\tau^s\int_{\Omega_{1t}} b(x,t)^{1+p(x,t)(\frac{\alpha_1}{p(x,t)}-1)}|u-v|^{p(x,t)}\,dx\,dt\right)^{\frac{1}{p_1}}\\&\quad\leq c\left(\iint_{Q_{\tau s}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}|u-v|^2\,dx\,dt\right)^{\frac{1}{p_1}},\end{aligned}\tag{3.6}$$

and using the Hölder inequality, we get

$$\begin{aligned}&\left(\int_\tau^s\int_{\Omega_{2t}} b(x,t)^{1+p(x,t)(\frac{\alpha_1}{p(x,t)}-1)}|u-v|^{p(x,t)}\,dx\,dt\right)^{\frac{1}{p_1}}\\&\quad\leq\left(\int_\tau^s\int_{\Omega_{2t}}\big[b(x,t)^{1+p(x,t)(\frac{\alpha_1}{p(x,t)}-1)-\frac{\alpha_1}{2}}\big]^{\frac{2}{2-p(x,t)}}\,dx\,dt\right)^{\frac{1}{p_{12}}\frac{1}{p_1}}\\&\qquad\cdot\left(\int_\tau^s\int_{\Omega_{2t}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}|u-v|^2\,dx\,dt\right)^{\frac{1}{2}}\\&\quad\leq c\left(\iint_{Q_{\tau s}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}|u-v|^2\,dx\,dt\right)^{\frac{1}{2}},\end{aligned}\tag{3.7}$$

where $p_{12}(x,t)=\frac{2}{2-p(x,t)}$, from (iii) of Lemma 3.1, $p_{12}=p_{12}^+$ or p_{12}^- according to

$$\int_\tau^s\int_{\Omega_{2t}}\big[b(x,t)^{1+p(x,t)(\frac{\alpha_1}{p(x,t)}-1)-\frac{\alpha_1}{2}}\big]^{\frac{2}{2-p(x,t)}}\,dx\,dt<1,$$

or

$$\int_{\tau}^{s}\int_{\Omega_{2t}}\big[b(x,t)^{1+p(x,t)(\frac{\alpha_1}{p(x,t)}-1)-\frac{\alpha_1}{2}}\big]^{\frac{2}{2-p(x,t)}}\,dx\,dt\ge 1.$$

In the second place,

$$\begin{aligned}
&\sum_{i=1}^{N}\iint_{Q_{\tau s}}\big[\gamma_i(u)-\gamma_i(v)\big]\big[(u-v)\xi_\lambda\big]_{x_i}\,dx\,dt\\
&\quad=\sum_{i=1}^{N}\iint_{Q_{\tau s}}\big[\gamma_i(u)-\gamma_i(v)\big](u-v)\xi_{\lambda x_i}\,dx\,dt\\
&\qquad+\sum_{i=1}^{N}\iint_{Q_{\tau s}}\big[\gamma_i(u)-\gamma_i(v)\big](u-v)_{x_i}\xi_\lambda\,dx\,dt.
\end{aligned}\tag{3.8}$$

Since $|\nabla b|\le c$, $\alpha_1\ge 2p^+$, there hold

$$\begin{aligned}
&\lim_{\lambda\to 0}\sum_{i=1}^{N}\iint_{Q_{\tau s}}\big[\gamma_i(u)-\gamma_i(v)\big](u-v)\xi_{\lambda x_i}\,dx\,dt\\
&\quad\le c\lim_{\lambda\to 0}\sum_{i=1}^{N}\int_{\tau}^{s}\int_{\Omega_{\lambda t}}\big[\gamma_i(u)-\gamma_i(v)\big](u-v)\big[b(x,t)-\lambda\big]^{\frac{\alpha_1}{p(x,t)}-1}|b_{x_i}|\,dx\\
&\quad\le c\bigg(\iint_{Q_{\tau s}}b(x,t)^{\frac{\alpha_1}{p(x,t)}}|u-v|^2\,dx\bigg)^{\frac{1}{2}}
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&\lim_{\lambda\to 0}\sum_{i=1}^{N}\iint_{Q_{\tau s}}\big[\gamma_i(u)-\gamma_i(v)\big](u-v)_{x_i}\xi_\lambda\,dx\,dt\\
&\quad=\lim_{\lambda\to 0}\sum_{i=1}^{N}\iint_{Q_{\tau s}}\big[\gamma_i(u)-\gamma_i(v)\big](u-v)_{x_i}\big[b(x,t)-\lambda\big]_+^{\frac{\alpha_1}{p(x,t)}}\,dx\,dt\\
&\quad\le\sum_{i=1}^{N}\bigg(\iint_{Q_{\tau s}}\big(b(x,t)^{\frac{\alpha_1-1}{p(x,t)}}\big)^{q(x,t)}\big|\gamma_i(u)-\gamma_i(v)\big|^{p'(x,t)}\,dx\,dt\bigg)^{\frac{1}{q_1}}\\
&\qquad\cdot\bigg(\iint_{Q_{\tau s}}b(x,t)\big(|\nabla u|^{p(x,t)}+|\nabla v|^{p(x,t)}\big)\,dx\,dt\bigg)^{\frac{1}{p_1}}\\
&\quad\le c\sum_{i=1}^{N}\bigg(\iint_{Q_{\tau s}}b(x,t)^{\frac{\alpha_1-1}{p(x,t)-1}}\big|\gamma_i(u)-\gamma_i(v)\big|^{q(x,t)}\,dx\,dt\bigg)^{\frac{1}{q_1}}.
\end{aligned}\tag{3.10}$$

When $1<p(x,t)<2$, we know $q(x,t)>2$. Since $\alpha_1\ge p^+$, if $b(x,t)<1$, then $b(x,t)^{\frac{\alpha_1-1}{p(x,t)-1}}\le b(x,t)^{\frac{\alpha_1}{p(x,t)}}$. If $1\le b(x,t)\le D=\max_{(x,t)\in\overline{\Omega}\times[0,T]}b(x,t)$, then

$$b(x,t)^{\frac{\alpha_1-1}{p(x,t)-1}-\frac{\alpha_1}{p(x,t)}}=b(x,t)^{\frac{\alpha_1-p(x,t)}{p(x,t)(p(x,t)-1)}}\le D^{\frac{\alpha_1-p(x,t)}{p(x,t)(p(x,t)-1)}}\le c,$$

which implies that $b(x,t)^{\frac{\alpha_1-1}{p(x,t)-1}} \leq cb(x,t)^{\frac{\alpha_1}{p(x,t)}}$ is always true. Thus, we extrapolate that

$$\begin{aligned} &\sum_{i=1}^{N}\left(\int_{\tau}^{s}\int_{\Omega_{1t}} b(x,t)^{\frac{\alpha_1-1}{p(x,t)-1}}\left|\gamma_i(u)-\gamma_i(v)\right|^{q(x,t)}\,dx\,dt\right)^{\frac{1}{q_1}} \\ &\quad\leq c\left(\iint_{Q_{\tau s}} \rho(x,t)^{\frac{\alpha_1}{p(x,t)}}|u-v|^2\,dx\,dt\right)^{\frac{1}{q_1}}. \end{aligned} \tag{3.11}$$

When $p(x,t)\geq 2$, we know $q(x,t)<2$. By $\alpha_1\geq 2$, using the Hölder inequality, we have

$$\begin{aligned} &\sum_{i=1}^{N}\left(\int_{\tau}^{s}\int_{\Omega_{2t}} b(x,t)^{\frac{\alpha_1-1}{p(x,t)-1}}\left|\gamma_i(u)-\gamma_i(v)\right|^{p'(x,t)}\,dx\,dt\right)^{\frac{1}{q_1}} \\ &\quad\leq c\left(\iint_{Q_{\tau s}} \left[b(x,t)^{\frac{\alpha_1-1}{p(x,t)-1}-\frac{\alpha_1}{2(p(x,t)-1)}}\right]^{\frac{2}{2-q(x,t)}}\,dx\,dt\right)^{\frac{1}{q_{22}}\frac{1}{q_1}} \\ &\qquad\cdot\left(\iint_{Q_{\tau s}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}|u-v|^2\,dx\,dt\right)^{\frac{1}{2}} \\ &\quad\leq c\left(\iint_{Q_{\tau s}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}|u-v|^2\,dx\,dt\right)^{\frac{1}{2}}, \end{aligned} \tag{3.12}$$

where $q_{22}(x,t)=\frac{2-q(x,t)}{2}$, $q_{22}=q_{22}^{+}$, or q_{22}^{-}.

In the third place, since $|\sum_{i=1}^{N}\frac{\partial g^i(x,t)}{\partial x_i}|\leq cb(x,t)^{\frac{\alpha_1}{p(x,t)}}$

$$\begin{aligned} &\left|-\lim_{\lambda\to 0}\sum_{i=1}^{N}\iint_{Q_{\tau s}}\left[\gamma_i(u)-\gamma_i(v)\right]\left[(u-v)\xi_\lambda(x,t)\right]\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt\right| \\ &\quad=\left|-\sum_{i=1}^{N}\iint_{Q_{\tau s}}\left[\gamma_i(u)-\gamma_i(v)\right](u-v)\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt\right| \\ &\quad\leq c\iint_{Q_{\tau s}}|u-v|^2\left|\sum_{i=1}^{N}\frac{\partial g^i(x,t)}{\partial x_i}\right|\,dx\,dt \\ &\quad\leq c\iint_{Q_{\tau s}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}|u-v|^2\,dx\,dt. \end{aligned} \tag{3.13}$$

From (3.4)–(3.13), letting $\lambda\to 0$ in (3.1), we deduce that

$$\begin{aligned} &\iint_{Q_{\tau s}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}(u-v)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt \\ &\quad\leq c\left(\iint_{Q_{\tau s}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}\left|u(x,t)-v(x,t)\right|^2\,dx\,dt\right)^{l}, \end{aligned} \tag{3.14}$$

where $l\leq 1$.

Last but not least, by the mean value theorem,

$$\begin{aligned}
&\iint_{Q_{\tau s}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}(u-v)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt \\
&= \iint_{Q_{\tau s}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}\frac{u-v}{|u|^{\beta-1}u-|v|^{\beta-1}v}\big(|u|^{\beta-1}u-|v|^{\beta-1}v\big)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt \\
&= \frac{1}{2}\iint_{Q_{\tau s}} \frac{b(x,t)^{\frac{\alpha_1}{p(x,t)}}}{\beta|\zeta|^{\beta-1}}\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)^2}{\partial t}\,dx\,dt, \qquad (3.15)
\end{aligned}$$

where $\zeta \in (v,u)$.

One of the possibilities of (3.15) is that, for any $s \geq \tau$,

$$\frac{d}{dt}\left\| b(x,t)^{\frac{\alpha_1}{2p(x,t)}}\big(|u|^{\beta-1}u-|v|^{\beta-1}v\big)\right\|_{L^2(\Omega)} \leq 0, \quad t\in[\tau,s], \qquad (3.16)$$

is true, then

$$\begin{aligned}
&\int_\Omega b(x,t)^{\frac{\alpha_1}{p(x,t)}}\left||u|^{\beta-1}u(x,s)-|v|^{\beta-1}v(x,s)\right|^2 dx \\
&\leq \int_\Omega b(x,t)^{\frac{\alpha_1}{p(x,t)}}\left||u|^{\beta-1}u(x,\tau)-|v|^{\beta-1}v(x,\tau)\right|^2 dx
\end{aligned}$$

is clear.

Another possibility of (3.15) is that there is $s_0 \geq \tau$ such that

$$\frac{d}{dt}\left\| b(x,t)^{\frac{\alpha_1}{2p(x,t)}}\big(|u|^{\beta-1}u-|v|^{\beta-1}v\big)\right\|_{L^2(\Omega)} > 0, \quad t\in[\tau,s_0], \qquad (3.17)$$

then

$$\begin{aligned}
&\iint_{Q_{\tau s_0}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}(u-v)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt \\
&= \frac{1}{2}\iint_{Q_{\tau s_0}} \frac{b(x,t)^{\frac{\alpha_1}{p(x,t)}}}{\beta|\zeta|^{\beta-1}}\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)^2}{\partial t}\,dx\,dt \\
&\geq \frac{1}{2\beta M^{\beta-1}}\iint_{Q_{\tau s_0}} b(x,t)^{\frac{\alpha_1}{p(x,t)}}\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)^2}{\partial t}\,dx\,dt, \qquad (3.18)
\end{aligned}$$

where $\zeta \in (v,u)$, $M = \max\{\|u\|_{L^\infty(Q_T)}, \|v\|_{L^\infty(Q_T)}\}$.

Combining (3.14)–(3.15) with (3.18), we can extrapolate that

$$\begin{aligned}
&\int_\Omega b(x,s_0)^{\frac{\alpha_1}{p(x,s_0)}}\left||u|^{\beta-1}u(x,s_0)-|v|^{\beta-1}v(x,s_0)\right|^2 dx \\
&\leq \int_\Omega b(x,\tau)^{\frac{\alpha_1}{p(x,\tau)}}\left||u|^{\beta-1}u(x,\tau)-|v|^{\beta-1}v(x,\tau)\right|^2 dx \\
&\quad + \frac{2cM^{\beta-1}}{\beta m^{2(\beta-1)}}\left(\int_\tau^{s_0}\int_\Omega b(x,t)^{\frac{\alpha_1}{p(x,t)}}\left||u|^{\beta-1}u(x,t)-|v|^{\beta-1}v(x,t)\right|^2 dx\,dt\right)^l. \qquad (3.19)
\end{aligned}$$

Here $m = \min\{\|u\|_{L^\infty(Q_T)}, \|v\|_{L^\infty(Q_T)}\}$. From (3.19), we have

$$\int_\Omega b(x,s_0)^{\frac{\alpha_1}{p(x,s_0}} \left||u|^{\beta-1}u(x,s_0) - |v|^{\beta-1}v(x,s_0)\right|^2 dx \\ \leq \int_\Omega b(x,\tau)^{\frac{\alpha_1}{p(x,\tau)}} \left||u|^{\beta-1}u(x,\tau) - |v|^{\beta-1}v(x,\tau)\right|^2 dx,$$

which contradicts assumption (3.17). In other words, (3.17) is impossible. This fact implies that, for any $s,\tau \in [0,T)$, inequality (3.16) is always true. By the arbitrariness of τ, we have

$$\int_\Omega b(x,s)^{\frac{\alpha_1}{p(x,s)}} \left||u|^{\beta-1}u(x,s) - |v|^{\beta-1}v(x,s)\right|^2 dx \\ \leq \int_\Omega b(x,0)^{\frac{\alpha_1}{p(x,0)}} \left||u_0|^{\beta-1}u_0(x) - |v_0|^{\beta-1}v_0(x)\right|^2 dx,$$

Theorem 1.3 follows. □

4 The stability of weak solutions

Let $h_n(u)$ be an odd function defined as

$$h_n(u) = \begin{cases} 1, & u > \frac{1}{n}, \\ n^2u^2e^{1-n^2u^2}, & 0 \leq u \leq \frac{1}{n}. \end{cases}$$

Then

$$\lim_{n\to\infty} h_n(u) = \operatorname{sign}(u), \quad u \in (-\infty,+\infty), \tag{4.1}$$

$$0 \leq h_n'(u) \leq \frac{c}{u}, \qquad 0 < u < \frac{1}{n}, \qquad \lim_{n\to\infty} h_n'(u)u = 0. \tag{4.2}$$

Proof of Theorem 1.4 Since $g^i(x,t)$ satisfies (1.10), then from Theorem 1.3 we know that the weak solution of equation (1.1) with initial value (1.2) is unique. Let $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.1) with the different initial values $u_0(x)$ and $v_0(x)$ respectively. Since the weak solution of equation (1.1) with initial value (1.2) is unique, there are two asymptotic solutions of asymptotic problem (2.1)–(2.3), u_ε and v_ε, satisfying

$$\lim_{\varepsilon\to 0} u_\varepsilon = u, \qquad \lim_{\varepsilon\to 0} v_\varepsilon = v, \quad \text{a.e. } (x,t) \in Q_T, \tag{4.3}$$

and

$$\begin{aligned} &b(x,t)^{\frac{1}{p(x,t)}}\nabla u_\varepsilon \rightharpoonup b(x,t)^{\frac{1}{p(x,t)}}\nabla u, \\ &b(x,t)^{\frac{1}{p(x,t)}}\nabla v_\varepsilon \rightharpoonup b(x,t)^{\frac{1}{p(x,t)}}\nabla v, \quad \text{in } L^1\big(0,T;L^{p(x,t)}(\Omega)\big). \end{aligned} \tag{4.4}$$

We now choose $\chi_{[\tau,s]}(t)h_n(u_\varepsilon(x,t)-v_\varepsilon(x,t))$ as a test function, and so

$$\begin{aligned}
&\iint_{Q_{\tau s}} h_n(u_\varepsilon - v_\varepsilon)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt \\
&\quad + \iint_{Q_{\tau s}} b(x,t)\bigl(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\bigr)\cdot\nabla(u_\varepsilon - v_\varepsilon)h_n'(u_\varepsilon - v_\varepsilon)\,dx\,dt \\
&\quad + \sum_{i=1}^N \iint_{Q_{\tau s}} g^i(x,t)\bigl[\gamma_i(u)-\gamma_i(v)\bigr]\cdot(u_\varepsilon - v_\varepsilon)_{x_i}h_n'(u_\varepsilon - v_\varepsilon)\,dx\,dt \\
&= -\sum_{i=1}^N \iint_{Q_{\tau s}} \bigl[\gamma_i(u)-\gamma_i(v)\bigr]h_n(u_\varepsilon - v_\varepsilon)\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt.
\end{aligned} \tag{4.5}$$

In the first place, (4.4) yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} \bigl[b(x,t)^{\frac{p(x,t)-1}{p(x,t)}}\bigl(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\bigr)\bigr] \\
&\quad \cdot b(x,t)^{\frac{1}{p(x,t)}}\bigl[\nabla(u_\varepsilon - v_\varepsilon) - \nabla(u-v)\bigr]\,dx\,dt \\
&= 0.
\end{aligned} \tag{4.6}$$

In the second place, by (4.6) and the second mean value theorem, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} b(x,t)^{\frac{p(x,t)-1}{p(x,t)}}\bigl(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\bigr) \\
&\quad \cdot b(x,t)^{\frac{1}{p(x,t)}}\bigl[\nabla(u_\varepsilon - v_\varepsilon) - \nabla(u-v)\bigr]h_n'(u-v)\,dx\,dt \\
&= h_n'(\zeta)\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} b(x,t)^{\frac{p(x,t)-1}{p(x,t)}}\bigl(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\bigr) \\
&\quad \cdot b(x,t)^{\frac{1}{p(x,t)}}\bigl[\nabla(u_\varepsilon - v_\varepsilon) - \nabla(u-v)\bigr]\,dx\,dt \\
&= 0,
\end{aligned} \tag{4.7}$$

and since $(u_\varepsilon - v_\varepsilon)\to(u-v)$, *a.e.* in Ω,

$$\bigl|\bigl[h_n'(u_\varepsilon - v_\varepsilon) - h_n'(u-v)\bigr]\bigr| \le c(n),$$

by (4.6),

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega} b(x,t)^{\frac{p(x,t)-1}{p(x,t)}}\bigl(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\bigr) \\
&\quad \cdot b(x,t)^{\frac{1}{p(x,t)}}\bigl[\nabla(u_\varepsilon - v_\varepsilon) - \nabla(u-v)\bigr]\bigl[h_n'(u_\varepsilon - v_\varepsilon) - h_n'(u-v)\bigr]\,dx\,dt \\
&= 0.
\end{aligned} \tag{4.8}$$

By (4.7)–(4.8), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} b(x,t)^{\frac{p(x,t)-1}{p(x,t)}}\big(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\big)\\
&\quad\cdot b(x,t)^{\frac{1}{p(x,t)}}\big[\nabla(u_\varepsilon-v_\varepsilon)-\nabla(u-v)\big]h_n'(u_\varepsilon-v_\varepsilon)\,dx\,dt\\
&\quad=\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} b(x,t)^{\frac{p(x,t)-1}{p(x,t)}}\big(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\big)\\
&\quad\cdot b(x,t)^{\frac{1}{p(x,t)}}\big[\nabla(u_\varepsilon-v_\varepsilon)-\nabla(u-v)\big]h_n'(u-v)\,dx\,dt\\
&\quad+\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} b(x,t)^{\frac{p(x,t)-1}{p(x,t)}}(x)\big(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\big)\\
&\quad\cdot b(x,t)^{\frac{1}{p(x,t)}}\big[\nabla(u_\varepsilon-v_\varepsilon)-\nabla(u-v)\big]\big[h_n'(u_\varepsilon-v_\varepsilon)-h_n'(u-v)\big]\,dx\,dt\\
&\quad=\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} b(x,t)^{\frac{p(x,t)-1}{p(x,t)}}\big(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\big)\\
&\quad\cdot b(x,t)^{\frac{1}{p(x,t)}}\big[\nabla(u_\varepsilon-v_\varepsilon)-\nabla(u-v)\big]h_n'(u-v)\,dx\,dt\\
&\quad=0.
\end{aligned}\tag{4.9}$$

In the third place,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} b(x,t)\big(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\big)\nabla(u_\varepsilon-v_\varepsilon)h_n'(u_\varepsilon-v_\varepsilon)\,dx\,dt\\
&\quad=\iint_{Q_{\tau s}} b(x,t)\big(|\nabla u|^{p(x,t)-2}\nabla u-|\nabla v|^{p(x,t)-2}\nabla v\big)\nabla(u-v)h_n'(u-v)\,dx\,dt\\
&\quad\ge 0.
\end{aligned}\tag{4.10}$$

In the fourth place, since

$$\begin{aligned}
&\int_\Omega\big|b(x,t)^{\frac{1}{p(x,t)}}(u_\varepsilon-v_\varepsilon)_{x_i}h_n'(u_\varepsilon-v_\varepsilon)\big|^{p(x,t)}\,dx\\
&\quad\le c(n)\int_\Omega\big|b(x,t)^{\frac{1}{p(x,t)}}(u_\varepsilon-v_\varepsilon)_{x_i}\big|^{p(x,t)}\,dx\le c(n),
\end{aligned}$$

as $\varepsilon\to 0$, we have

$$b(x,t)^{\frac{1}{p(x,t)}}h_n'(u_\varepsilon-v_\varepsilon)(u_\varepsilon-v_\varepsilon)_{x_i}\rightharpoonup b(x,t)^{\frac{1}{p(x,t)}}(u-v)_{x_i}h_n'(u-v),\quad \text{in } L^1\big(0,T;L^{p(x,t)}(\Omega)\big).$$

By (1.12), $|\sum_{i=1}^N g^i(x,t)|\le cb(x,t)^{\frac{1}{p(x,t)}}$, we extrapolate that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}}\sum_{i=1}^N g^i(x,t)\big[\gamma_i(u)-\gamma_i(v)\big](u_\varepsilon-v_\varepsilon)_{x_i}h_n'(u_\varepsilon-v_\varepsilon)\,dx\,dt\\
&\quad=\iint_{Q_{\tau s}}\sum_{i=1}^N g^i(x,t)\big[\gamma_i(u)-\gamma_i(v)\big]\cdot(u-v)_{x_i}h_n'(u-v)\,dx\,dt.
\end{aligned}$$

Moreover, since

$$\left|\left[\gamma_i(u)-\gamma_i(v)\right]h_n'(u-v)\right| \le c\left|(u-v)h_n'(u-v)\right| \le c,$$

if we denote $\Omega_{1n}=\{x\in\Omega:|u-v|<\frac{1}{n}\}$, we have

$$\begin{aligned}
&\left|\iint_{Q_{\tau s}}\sum_{i=1}^{N}g^i(x,t)\left[\gamma_i(u)-\gamma_i(v)\right]h_n'(u-v)(u-v)_{x_i}\,dx\,dt\right| \\
&\quad=\left|\int_\tau^s\int_{\Omega_{1n}}\sum_{i=1}^{N}g^i(x,t)\left[\gamma_i(u)-\gamma_i(v)\right]h_n'(u-v)(u-v)_{x_i}\,dx\,dt\right| \\
&\quad\le c\int_\tau^s\int_{\Omega_{1n}}\left|\sum_{i=1}^{N}g^i(x,t)(u-v)_{x_i}\right|dx\,dt \\
&\quad\le c\left[\int_\tau^s\int_{\Omega_{1n}}\left|b^{\frac{1}{p(x,t)}}\nabla(u-v)\right|^{p(x,t)}dx\,dt\right]^{\frac{1}{p_1}} \\
&\quad\le c.
\end{aligned}\tag{4.11}$$

If Ω_{1n} has 0 measure, from (4.11), letting $n\to\infty$, we have

$$\lim_{n\to\infty}\int_{\Omega_{1n}}\left|b^{\frac{1}{p(x,t)}}\nabla(u-v)\right|^{p(x,t)}dx=0.$$

While Ω_{1n} is with a positive measure, from (4.11), using the dominated convergence theorem, we directly have

$$\lim_{n\to\infty}\iint_{Q_{\tau s}}\sum_{i=1}^{N}g^i(x,t)\left[\gamma_i(u)-\gamma_i(v)\right]h_n(u-v)(u-v)_{x_i}\,dx\,dt=0.$$

Therefore, we have

$$\lim_{n\to\infty}\iint_{Q_{\tau s}}\sum_{i=1}^{N}g^i(x,t)\left[\gamma_i(u)-\gamma_i(v)\right]\cdot(u-v)_{x_i}h_n'(u-v)\,dx\,dt=0.\tag{4.12}$$

Once more,

$$\begin{aligned}
&-\lim_{\varepsilon\to0}\sum_{i=1}^{N}\iint_{Q_{\tau s}}\left[\gamma_i(u)-\gamma_i(v)\right]h_n(u_\varepsilon-v_\varepsilon)\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt \\
&\quad=-\sum_{i=1}^{N}\iint_{Q_{\tau s}}\left[\gamma_i(u)-\gamma_i(v)\right]h_n(u-v)\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt,
\end{aligned}$$

and by assumption (i) $\beta\le1$, or (ii)

$$\left|\gamma_i(t_1)-\gamma_i(t_2)\right|\le c\left||t_1|^{\beta-1}t_1-|t_2|^{\beta-1}t_2\right|,$$

we easily deduce that

$$\lim_{n\to\infty}\left|-\sum_{i=1}^{N}\iint_{Q_{\tau s}}\big[\gamma_i(u)-\gamma_i(v)\big]h_n(u-v)\frac{\partial g^i(x,t)}{\partial x_i}\,dx\,dt\right| \\ \leq c\iint_{Q_{\tau s}}\big||u|^{\beta-1}u-|v|^{\beta-1}v\big|\,dx\,dt. \tag{4.13}$$

Last but not least,

$$\lim_{n\to\infty}\lim_{\varepsilon\to0}\iint_{Q_{\tau s}}h_n(u_\varepsilon-v_\varepsilon)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt \\ =\lim_{n\to\infty}\iint_{Q_{\tau s}}h_n(u-v)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt \\ =\iint_{Q_{\tau s}}\operatorname{sign}(u-v)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt \\ =\iint_{Q_{\tau s}}\operatorname{sign}\big(|u|^{\beta-1}u-|v|^{\beta-1}v\big)\frac{\partial(|u|^{\beta-1}u-|v|^{\beta-1}v)}{\partial t}\,dx\,dt \\ =\int_\tau^s\frac{d}{dt}\big\||u|^{\beta-1}u-|v|^{\beta-1}v\big\|_{L^1(\Omega)}\,dt. \tag{4.14}$$

Then, by (4.5), (4.6), (4.7), (4.9), (4.10), (4.12), (4.14), we have

$$\int_\tau^s\frac{d}{dt}\big\||u|^{\beta-1}u-|v|^{\beta-1}v\big\|_{L^1(\Omega)}\,dt\leq c\iint_{Q_{\tau s}}\big||u|^{\beta-1}u-|v|^{\beta-1}v\big|\,dx\,dt.$$

By the Gronwall inequality,

$$\int_\Omega\big||u|^{\beta-1}u(x,s)-|v|^{\beta-1}v(x,s)\big|\,dx \\ \leq\int_\Omega\Big|\big|u(x,\tau)\big|^{\beta-1}u(x,\tau)-\big|v(x,\tau)\big|^{\beta-1}v(x,\tau)\Big|\,dx,\quad\forall t\in[0,T).$$

By the arbitrariness of τ, we extrapolate that

$$\int_\Omega\big||u|^{\beta-1}u(x,s)-|v|^{\beta-1}v(x,s)\big|\,dx\leq\int_\Omega\big||u_0|^{\beta-1}u_0-|v_0|^{\beta-1}v_0\big|\,dx,\quad\forall s\in[0,T).$$

The proof is complete. □

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