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Blow-up analysis for two kinds of nonlinear wave equations

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Abstract

In this paper, we discuss the blow-up and lifespan phenomenon for the following wave equation with variable coefficient:

$$u_{tt}(t,x) - \mathbf{div}(a(x)\mathbf{grad}u(t,x)) = f(u, Du, D_x Du), \quad x \in \mathbf{R}^n, t > 0,$$

with small initial data, where $a(x) > 0$, $Du = (u_{x_0}, u_{x_1}, \dots, u_{x_n})$ and $D_x Du = (u_{x_k x_l}, k, l = 0, 1, \dots, n, k + l \geq 1)$.

Then we find a new phenomenon. The Cauchy problem

$$u_{tt}(t,x) - \Delta u(t,x) = u(t,x)e^{u(t,x)^2}, \quad x \in \mathbf{R}^2, t > 0,$$

is globally well-posed for small initial data, while for the combined nonlinearities

$$u_{tt}(t,x) - \Delta u(t,x) = u(t,x)(e^{u(t,x)^2} + e^{u_t(t,x)^2}), \quad x \in \mathbf{R}^2, t > 0$$

with small initial data will blow up in finite time. Moreover, we obtain the lifespan results for the above problems.

Keywords: Wave equation; Blow up; Lifespan

1 Introduction and main results

1.1 Introduction

The blow-up results concerning the semilinear wave equation

$$\partial_{tt} u - \sum_{i=1}^{n} \partial_i^2 u = |u|^p, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, n \geq 2, \tag{1}$$

were firstly studied by John [8] when $n = 3$. More precisely, he showed that the semilinear wave equation (1) has the global solutions if $p > 1 + \sqrt{2}$ and initial data are sufficiently small. Meanwhile, he proved the finite time blow-up of solutions if $p < 1 + \sqrt{2}$ and the initial data are not both identically zero. Then Strauss conjectured, when $n \geq 2$, the existence or nonexistence of global solutions to equation (1) for $p \in (p_c(n), \infty)$ or $p \in (1, p_c(n)]$, where $p_c(n)$ is the positive root of the quadratic equation

$$(n-1)p^2-(n+1)p-2=0.$$

After that, there has been much work concerning this conjecture. We give a brief summary here. To see the global existence of solutions to (3), one can refer to Glassey [3] for $n=2$, Lindblad and Sogge [14] for $n\le 8$ for $n\ge 4$; Georgiev, Lindblad, and Sogge [2] for $n\ge 4$ and $p_c<p\le\frac{n+3}{n-1}$. To see the finite time blow-up of solutions to (3), one can see Glassey [4] for $n=2$ and Sideris [15] for $n\ge 4$, Yordanov and Zhang [21] and Zhou [23] for $n\ge 4$, Takamura and Wakasa [18] and Zhou and Han [25] for $n\ge 2$ and the sharp upper bound of the lifespan of the solution by using a different method, respectively. The lifespan $T(\epsilon)$ of the solutions of (1) is the largest value such that solutions exist for $x\in\mathbb{R}^n$, $0\le t<T(\epsilon)$. To the best knowledge of the authors, there is little work concerning the analog of the Strauss conjecture on cosmological spacetimes except the work of Lindblad et al. [13]. They showed the global existence of solutions for the semilinear wave equation on Kerr black hole backgrounds. Zhou and Han [24] first obtained a blow-up result on semilinear wave equations with variable coefficients and boundary. Lai and Zhou [9, 10] obtained finite time blow-up result for nonlinear wave equations in exterior domains. Yan [19] verified this conjecture on blow-up result for semilinear wave equation in de Sitter spacetimes. After that, Li, Li, and Yan [11] gave the blow-up results of semilinear damped wave equation in de Sitter spacetimes. We refer the reader to [1, 5, 12, 22] for more related results. In this paper, one of our main results is to study the blow-up result on quasilinear wave equations with variable coefficients.

Another problem concerns the blow-up solution of nonlinear wave equation with exponential type nonlinearity. The global existence of initial value problem for the nonlinear wave equation with exponential type nonlinearity

$$u_{tt}(t,x)-\triangle u(t,x)=u(t,x)e^{\alpha u(t,x)^2},\quad (t,x)\in\bigl(\mathbf{R},\mathbf{R}^2\bigr) \tag{2}$$

was studied by Ibrahim, Majdoub, and Masmoudi [6]. They showed that if the initial energy is small, then the nonlinear wave equation with exponential type nonlinearity is globally well-posed. Here α is a positive constant in $(0,4\pi]$. A scattering problem in the energy space for Klein–Gordon equations with nonlinearity of exponential growth in two space dimensions was studied in [7]. Struwe [16] established the global well-posedness of solutions to the Cauchy problem for the wave equations with exponential nonlinearities in the super-critical regime of large energies for smooth and radially symmetric data. Then, he [17] showed that the Cauchy problem for wave equations with critical exponential nonlinearities in two space dimensions is globally well-posed for arbitrary smooth initial data.

1.2 Main results

In this paper, we consider the following Cauchy problem with small initial data in $n\ge 2$ space dimensions:

$$\begin{aligned} &u_{tt}(t,x)-\mathbf{div}\bigl(a(x)\mathbf{grad}u(t,x)\bigr)\\ &\quad=F\bigl(\bigl|u(t,x)\bigr|,\bigl|\nabla u(t,x)\bigr|,\bigl|\triangle u(t,x)\bigr|\bigr)+\lambda_0\bigl|u_t(t,x)\bigr|^{p_0},\\ &t=0:\qquad u=\epsilon f(x),\qquad u_t=\epsilon g(x), \end{aligned} \tag{3}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, $a(x)$ is a positive smooth function,

$$\begin{aligned} &F\big(|u(t,x)|, |\nabla u(t,x)|, |\triangle u(t,x)|\big) \\ &\quad = O\big(\lambda_1 |u(t,x)|^{p_1} + \lambda_2 |\nabla u(t,x)|^{p_2} + \lambda_3 |\triangle u(t,x)|^{p_3}\big), \\ &Du = (u_{x_0}, u_{x_1}, \dots, u_{x_n}), \quad x_0 = t, \\ &D_x Du = (u_{x_k x_l}, k, l = 0, 1, \dots, n, k + l \geq 1), \end{aligned} \tag{4}$$

$f(x)$, $g(x) \in \mathbb{C}_0^\infty(\mathbb{R}^n)$, ϵ is a small parameter, λ_k $(k = 0,1,2,3)$ are nonnegative constants, $p_k > 1$. Here, for simplicity of notations, we write $x_0 = t$.

We assume that compactly supported nonnegative data f and g satisfy

$$f(x), g(x) \geq 0, \qquad f(x) = g(x) = 0 \quad \text{for } |x| > 1. \tag{5}$$

Here we give one of our main results.

Theorem 1 *Let f, g be smooth functions with compact support $f, g \in \mathbb{C}_0^\infty$ and satisfy* (5), *space dimensions $n \geq 2$. Assume that problem* (3) *has a solution $(u, u_t) \in \mathbb{C}([0,T), \mathbb{H}^1(\mathbb{R}^n) \times \mathbb{L}^r(\mathbb{R}^n))$, $a(x) > 0$ and $\frac{\triangle a(x)}{a(x)} \in (0, C(1 + |x|^{2+\delta})^{-1})$ is local Hölder continuous, where $r = \max\{2, p_0\}$ such that*

$$\mathbf{supp}(u, u_t) \subset \big\{(t,x) : |x| \leq 1 + t\big\},$$

and the index $p_0 > 1$, $p_2 > 1$, $p_3 > 1$ and p_1 satisfies

$$\begin{aligned} &(1 + 2\delta)\big[(n+1)\big(1 - p_0^{-1}\big) + P_0^{-1}\big] - (2 + n)\big(1 - p_0^{-1}\big) \\ &\quad < p_1 < n\big(1 - p_0^{-1}\big) + (1 + 2\delta)\big[(n+1)\big(1 - p_0^{-1}\big) + P_0^{-1}\big]. \end{aligned}$$

Then the solution $u(t,x)$ will blow up in finite time, that is, $T < \infty$. Moreover, we have the following estimates for the lifespan $T(\epsilon)$ of solutions of (3): *there exists a positive constant C, which is independent of ϵ, such that*

$$T(\epsilon) \leq C_0 \epsilon^{-\frac{p_1(p_0-1)(p_1-1)}{p_0(p_1+1)-2}},$$

where C_0 is a positive constant which is independent of ϵ.

Secondly, we consider the following problem:

$$\begin{aligned} &u_{tt}(t,x) - \triangle u(t,x) = u(t,x)\big(e^{u(t,x)^2} + e^{u_t(t,x)^2}\big), \quad x \in \mathbf{R}^2, t > 0, \\ &t = 0: \qquad u = \epsilon f(x), \qquad u_t = \epsilon g(x). \end{aligned} \tag{6}$$

We assume that compactly supported nonnegative data f and g satisfy

$$f(x) = 0, \qquad g(x) \geq 0, \qquad g(x) = 0 \quad \text{for } |x| > 1. \tag{7}$$

Theorem 2 *Let g be a smooth function with compact support $g \in \mathbb{C}_0^\infty(\mathbb{R}^2)$ and satisfy (7). Assume that problem (6) has a solution $(u, u_t) \in \mathbb{C}([0,T), \mathbb{H}^1(\mathbb{R}^2) \times \mathbb{L}^2(\mathbb{R}^2))$ such that*

$$\mathbf{supp}(u, u_t) \subset \{(t,x) : |x| \le 1 + t\}.$$

Then the solution $u(t,x)$ will blow up in finite time, that is, $T < \infty$. Moreover, we have the following estimates for the lifespan $T(\epsilon)$ of solutions of (6): there exists a positive constant C, which is independent of ϵ, such that

$$T(\epsilon) \le C_0 \epsilon^{-2}.$$

where C_0 is a positive constant which is independent of ϵ.

The organization of this paper is as follows. In Sect. 2, we recall some blow-up criteria on ODEs. Section 3 is devoted to proving the finite time blow-up of solutions for the quasilinear wave equation (3) with variable coefficients. In the last section, the proof of Theorem 2 is given.

2 Preliminaries

This section recalls some blow-up results for ordinary differential inequality. The first relevant result on ODE was established by Sideris [15]. The following blow-up result can be found in [18, 21] as Lemma 2.1.

Lemma 1 ([18]) *Let $p > 1$, $a > 0$, and $(p-1)a = q - 2$. Assume that $G \in \mathbb{C}^2([0,T))$ satisfies*

$$\begin{aligned} &G(t) \ge Kt^a \quad \text{for } t \ge T_0, \\ &G''(t) \ge A(t+R)^{-q} \left|G(t)\right|^p \quad \text{for } t > 0, \\ &G(0) > 0, \qquad G'(0) > 0, \end{aligned}$$

where K, T_0, A, and R denote positive constants with $T_0 \ge R$. Then T must satisfy $T \le 2T_1$ provided that $K \ge K_0$, where

$$K_0 = \left\{ \frac{1}{2^{\frac{q}{2}} a} \sqrt{\frac{B}{p+1}} \left(1 - \frac{1}{2^{a\delta}}\right) \right\}^{\frac{-2}{p-1}}, \qquad T_1 = \max\left\{ T_0, \frac{G(0)}{G'(0)} \right\},$$

with arbitrarily chosen δ satisfying $0 < \delta < \frac{p-1}{2}$ and a fixed positive constant B.

A more general blow-up result was given in [19]. One can see Lemma 2.2 in [19] for more details on the proof.

Lemma 2 ([19]) *Let $p > 1$. Assume that $G \in \mathbb{C}^2([0,T))$ satisfies*

$$G(t) \ge Ka(t) \quad \text{for } t \ge T_0, \tag{8}$$

$$G''(t) \ge Ab^{-1}(t+R) \left|G(t)\right|^p \quad \text{for } t > 0, \tag{9}$$

$$G(0) > 0, \qquad G'(0) > 0, \tag{10}$$

where K, T_0, A, and R denote positive constants with $T_0 \geq R$, $a(t)$ and $b(t)$ are positive strictly increasing smooth functions, and $b^{-\frac{1}{2}}(t+R)a^{\frac{p-1}{2}-\delta}(t)$ is a strictly decreasing smooth function for $t > 0$, and there exist fixed $t_0 \geq 2T_1$ and a positive constant $\tilde{K}$ such that

$$\tilde{K}a^{-\delta}(T_1) \leq \int_{T_1}^{t_0} b^{-\frac{1}{2}}(t+R)a^{\frac{p-1}{2}-\delta}(t)\,dt. \tag{11}$$

Then T must satisfy $T \leq 2T_1$ provided that $K \geq K_0$, where

$$K_0 = \left(\delta\tilde{K}\sqrt{\frac{p+1}{A}}\right)^{\frac{2}{1-p}}, \qquad T_1 = \max\left\{T_0, \frac{G(0)}{G'(0)}\right\} \tag{12}$$

with arbitrarily chosen δ satisfying $0 < \delta < \frac{p-1}{2}$.

Now we have a new blow-up result.

Lemma 3 *Let $p > 1$ and $b_1 - a_1(p-1) = 2$. Assume that $G \in \mathbb{C}^2([0,T))$ satisfies*

$$G(t) \geq Ke^{a_1 t} \quad \text{for } t \geq T_0, \tag{13}$$

$$G''(t) \geq Ae^{b_1(t+R)}\left|G(t)\right|^p \quad \text{for } t > 0, \tag{14}$$

$$G(0) > 0, \qquad G'(0) > 0, \tag{15}$$

where K, T_0, A, and R denote positive constants with $T_0 \geq R$.

Then T must satisfy $T \leq 2T_1$ provided that $K \geq K_0$, where

$$K_0 = \left(\tilde{K}^{\delta}\sqrt{\frac{p+1}{A}}\right)^{\frac{2}{1-p}}, \qquad T_1 = \max\left\{T_0, \frac{G(0)}{G'(0)}\right\} \tag{16}$$

for arbitrarily chosen δ satisfying $0 < \delta < \frac{p-1}{2}$, and a positive constant $\tilde{K} \geq a_1\delta e^{-\frac{1}{2}}$.

Proof We verify condition (11). Let $t_0 = 2T_1$. Then direct computation shows that

$$\begin{aligned}
&\int_{T_1}^{t_0} b^{-\frac{1}{2}}(t+R)a^{\frac{p-1}{2}-\delta}(t)\,dt \\
&\quad = \int_{T_1}^{2T_1} e^{-\frac{1}{2}b_1(t+R)+a_1(\frac{p-1}{2}-\delta)t}\,dt \\
&\quad = e^{-\frac{1}{2}b_1 R}\left(-\frac{1}{2}b_1 + a_1\left(\frac{p-1}{2}-\delta\right)\right)^{-1}\left(e^{(-\frac{1}{2}b_1+a_1(\frac{p-1}{2}-\delta))2T_1} - e^{(-\frac{1}{2}b_1+a_1(\frac{p-1}{2}-\delta))T_1}\right) \\
&\quad \geq \tilde{K}e^{-a\delta T_1},
\end{aligned}$$

with $\tilde{K} \geq a_1\delta e^{-\frac{1}{2}}$. This completes the proof. □

It follows from Yordanov and Zhang [20] that we introduce $\phi_0(x) \in \mathbb{C}^2(\mathbb{R}^n)$ and

$$\phi_1(x) = \int_{\mathbb{S}^{n-1}} e^{x\cdot\omega}\,d\omega \geq 0,$$

which are solutions of

$$\triangle\phi_0(x) + V(x)\phi_0(x) = 0,$$
$$\triangle\phi_1(x) + V(x)\phi_1(x) = \phi_1(x),$$

respectively. It is easy to see that $\phi_0(x) \neq \phi_1(x)$.

Then one can verify $\phi_0(x)$ and $\phi_1(x)$ (see [26]) such that

$$C^{-1} \le \phi_0(x) \le C, \tag{17}$$

$$0 < \phi_1(x) \le Ce^{|x|}\bigl(1 + |x|\bigr)^{-\frac{n-1}{2}}, \quad n \ge 2, \tag{18}$$

$$\phi_1(x) \sim C_n e^{|x|}|x|^{-\frac{n-1}{2}} \quad \text{as } |x| \longrightarrow \infty, \tag{19}$$

where C is a positive constant.

Moreover, we introduce a test function

$$\psi_1(t,x) = e^{-t}\phi_1(x).$$

It is easy to see

$$\triangle\psi_1(t,x) = \psi_1(t,x).$$

One can see [20, 21, 26] for more details.

3 Proof of Theorem 1

Rewrite the variable wave equation (3) as

$$\begin{aligned} &u_{tt}(t,x) - \triangle\bigl(a(x)u(t,x)\bigr) + \bigl(\triangle a(x)\bigr)u(t,x) \\ &\quad = F\bigl(\bigl|u(t,x)\bigr|, \bigl|\nabla u(t,x)\bigr|, \bigl|\triangle u(t,x)\bigr|\bigr) + \lambda_0\bigl|u_t(t,x)\bigr|^{p_0} \end{aligned} \tag{20}$$

with the initial data (u_0, u_1) satisfying (5), where F takes the form of (4).

Define

$$G(t) = \int_{\mathbb{R}^n} u(t,x)\phi_0(x).$$

Since $0 < \frac{\triangle a(x)}{a(x)} < C(1 + |x|^{2+\delta})^{-1}$ is local Hölder continuous, we derive

$$\begin{aligned} &\int_{\mathbb{R}^n} \bigl(\triangle\bigl(a(x)u(t,x)\bigr) - \bigl(\triangle a(x)\bigr)u(t,x)\bigr)\phi_0(x)\,dx \\ &\quad = \int_{\mathbb{R}^n} \bigl(a(x)u(t,x)\triangle\phi_0(x) - \bigl(\triangle a(x)\bigr)u(t,x)\phi_0(x)\bigr)\,dx \\ &\quad = \int_{\mathbb{R}^n} a(x)u(t,x)\biggl(\triangle\phi_0(x) - \frac{\triangle a(x)}{a(x)}\phi_0(x)\biggr)\,dx \\ &\quad = 0. \end{aligned} \tag{21}$$

So multiplying (20) both sides by $\phi_0(x)$ and using (21), we have

$$\begin{aligned} G''(t) &= \int_{\mathbb{R}^n} \partial_{tt} u(t,x)\phi_0(x)\,dx \\ &= \int_{\mathbb{R}^n} \phi_0(x)\big(F\big(\big|u(t,x)\big|, \big|\nabla u(t,x)\big|, \big|\triangle u(t,x)\big|\big) + \lambda_0\big|u_t(t,x)\big|^{p_0}\big)\,dx \\ &\geq \lambda_0 \int_{\mathbb{R}^n} \phi_0(x)\big|u_t(t,x)\big|^{p_0}\,dx, \end{aligned} \tag{22}$$

where the last inequality is derived by noticing $F(|u(t,x)|, |\nabla u(t,x)|, |\triangle u(t,x)|) > 0$ from (4).

Similarly, by (4) and (17), we obtain

$$\begin{aligned} G''(t) &= \int_{\mathbb{R}^n} \phi_0(x)\big(F\big(\big|u(t,x)\big|, \big|\nabla u(t,x)\big|, \big|\triangle u(t,x)\big|\big) + \lambda_0\big|u_t(t,x)\big|^{p_0}\big)\,dx \\ &\geq C\lambda_1 \int_{\mathbb{R}^n} \big|u(t,x)\big|^{p_1}\,dx. \end{aligned} \tag{23}$$

By the Hölder inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \phi_0(x)\big|u_t(t,x)\big|^{p_0}\,dx &\geq \left|\int_{\mathbb{R}^n} \phi_0(x)u_t(t,x)\,dx\right|^{p_0} \left(\int_{|x|\leq t+R} \phi_0(x)\,dx\right)^{p_0-1} \\ &\geq C\big(\mathrm{vol}\big(\mathbf{B}^n\big)\big)^{-1}(t+R)^{-n(p_0-1)}\left|\int_{\mathbb{R}^n} \phi_0(x)u_t(t,x)\,dx\right|^{p_0}. \end{aligned}$$

Thus it follows from (22) that

$$\begin{aligned} G''(t) &\geq \lambda_0 C\big(\mathrm{vol}\big(\mathbf{B}^n\big)\big)^{-1}(t+R)^{-n(p_0-1)}\left|\int_{\mathbb{R}^n} \phi_0(x)u_t(t,x)\,dx\right|^{p_0} \\ &= \lambda_0 C\big(\mathrm{vol}\big(\mathbf{B}^n\big)\big)^{-1}(t+R)^{-n(p_0-1)}\big|G'(t)\big|^{p_0}, \end{aligned}$$

which implies that

$$\frac{d}{dt}\big|G'(t)\big|^{1-p_0} \leq (1-p_0)\lambda_0 C^{-1}\big(\mathrm{vol}\big(\mathbf{B}^n\big)\big)^{-1}(t+R)^{-n(p_0-1)}. \tag{24}$$

Integrating (24) over $[0,t]$, we get

$$\big|G'(t)\big| \geq \left(\frac{\lambda_0(p_0-1)}{n-1}C^{-1}\big(\mathrm{vol}\big(\mathbf{B}^n\big)\big)^{-1}\right)^{\frac{-1}{p_0-1}}(t+R)^{n+\frac{1}{p_0-1}}. \tag{25}$$

On the other hand, it follows from (22) that $G'(t) = \int_{\mathbb{R}^n} \partial_t u(t,x)\phi_0(x)\,dx$ is an increasing function for $t \geq 0$. Since $g(x) \geq 0$ in (5), $G(t)$ is also an increasing function for $t \geq 0$. By $f(x) \geq 0$ in (5), we know that $G(t) > 0$. Thus it follows from (25) that

$$G(t) \geq \left(\frac{\lambda_0(p_0-1)}{n-1}C^{-1}\big(\mathrm{vol}\big(\mathbf{B}^n\big)\big)^{-1}\right)^{\frac{-1}{p_0-1}}\left(\frac{n-1}{p_0-1}+1\right)(t+R)^{n+\frac{1}{p_0-1}+1}. \tag{26}$$

On the other hand, by the Hölder inequality, we derive

$$\begin{aligned} G(t) &= \int_{\mathbb{R}^n} u(t,x)\phi_0(x)\,dx \\ &\le \left(\int_{\mathbb{R}^n} \left|u(t,x)\phi_0(x)\right|^{p_1}\right)^{\frac{1}{p_1}} \left(\int_{|x|\le 1+t} dx\right)^{1-\frac{1}{p_1}} \\ &\le C(1+t)^{n(1-\frac{1}{p_1})}\left(\int_{\mathbb{R}^n} \left|u(t,x)\psi_0(x)\right|^{p_1}\right)^{\frac{1}{p_1}}, \end{aligned}$$

which combining with (17) gives that

$$C\int_{\mathbb{R}^n} \left|u(t,x)\right|^{p_1}dx \ge \left(\int_{\mathbb{R}^n} \left|u(t,x)\psi_0(x)\right|^{p_1}\right)dx \ge (1+t)^{-n(p_1-1)}F_0^{p_1}(t).$$

Then by (23) we obtain

$$G''(t) \ge C_{\lambda_1}(1+t)^{-n(p_1-1)}G^{p_1}(t), \quad \forall p_1 > 1, \tag{27}$$

where C_λ is a positive constant which depends on λ_1.

Let

$$a_1 = n + \frac{1}{p_0-1} + 1, \qquad b_1 = n(p_1-1). \tag{28}$$

Next we apply Lemma 2 to prove our result. Let $0 < \delta < \min\{\frac{1}{n}, \frac{p_1-1}{2}\}$. It follows from (26)–(27) that (8)–(10) hold. The rest is to verify conditions (11) and (12). Substituting (28) into (11), direct computation shows that if we take $p_0 > 1$ and

$$\begin{aligned} &(1+2\delta)\left[(n+1)\left(1-p_0^{-1}\right)+P_0^{-1}\right]-(2+n)\left(1-p_0^{-1}\right) \\ &\quad < p_1 < n\left(1-p_0^{-1}\right)+(1+2\delta)\left[(n+1)\left(1-p_0^{-1}\right)+P_0^{-1}\right], \end{aligned}$$

then (11) holds.

Taking a fixed positive constant $\tilde{K}$ such that

$$\tilde{K} \ge \left(\frac{\lambda_0(p_0-1)}{n-1}C^{-1}\left(\operatorname{vol}\left(\mathbf{B}^n\right)\right)^{-1}\right)^{\frac{-p_1+1}{2\delta(p_0-1)}}\left(\frac{n-1}{p_0-1}+1\right)^{\frac{p_1-1}{2\delta}}\left(\frac{p_1+1}{A}\right)^{\frac{1}{2\delta}} > 0,$$

then (12) holds.

Thus $G(t)$ will blow up in finite time, then the solutions to problem (3) will blow up in finite time. At last, we estimate the lifespan result. Since $G''(t) \ge 0$ and $G'(0) \ge 0$, $G(t)$ is an increasing smooth function. So it holds

$$G(t) = \int_{\mathbb{R}^n} u(t,x)\phi_0(x)\,dx \ge \epsilon\int_{\mathbb{R}^n} f(x)\phi_0(x)\,dx \ge C\epsilon.$$

It follows from (27) that

$$G''(t) \ge (1+t)^{-n(p_1-1)}\epsilon^{p_1}, \quad \forall p_1 > 1.$$

Furthermore, we have

$$\begin{aligned} T(\epsilon) &\le C_0 \epsilon^{-\frac{p_1(p_1-1)}{(p_1-1)a_1-b_1+2}} \\ &\le C_0 \epsilon^{-\frac{p_1(p_0-1)(p_1-1)}{p_0(p_1+1)-2}}, \end{aligned}$$

where C_0 is a positive constant which is independent of ϵ. We complete the proof of Theorem 1.

4 Proof of Theorem 2

Then we consider the approximation equation of (6)

$$\begin{aligned} &u_{tt}^{(m)}(t,x) - \triangle u^{(m)}(t,x) = u(t,x)\big(e^{(u^{(m)}(t,x))^2} + e^{(u_t^{(m)}(t,x))^2}\big), \\ &t = 0: \qquad u^{(m)} = 0, \qquad u_t^{(m)} = \epsilon g(x), \end{aligned} \tag{29}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbf{R}^2$ and $m \in \mathbb{N}$.

By the local existence of classical solutions, the solution to Cauchy problem (29) can be approximated by Picard iteration. Set $u^{(0)} \equiv 0$. Then $u_t^{(0)} \equiv 0$. So by the positivity of the fundamental solution of the wave operator in two space dimensions, we can prove that $u^{(m)}(t,x)$ is a series of approximate solutions to (29) by induction. Let $m \longrightarrow \infty$, we conclude that $u(t,x)$ is a nonnegative solution to (6).

Let $r = |x|$ and $\mathcal{G}(r) = \frac{1}{2} r^{\frac{1}{2}} g(r)$, $x \in \mathbb{R}^2$. The radial symmetric form of equation (6) is

$$(\partial_{tt} - \partial_{rr})\big(r^{\frac{1}{2}} u\big) = \frac{1}{4} r^{-\frac{3}{2}} u + r^{\frac{1}{2}} u\big(e^{u^2} + e^{u_t^2}\big) \tag{30}$$

with initial data

$$t = 0: \qquad r^{\frac{1}{2}} u = 0, \qquad r^{\frac{1}{2}} u_t = \epsilon r^{\frac{1}{2}} g(r). \tag{31}$$

Using (31) and D'Alembert's formula, for $r > t$, we have

$$\begin{aligned} r^{\frac{1}{2}} u(t,r) = {}& \epsilon \int_{r-t}^{r+t} \mathcal{G}(\xi)\,d\xi + \frac{1}{8}\int_0^t \int_{r-t+\tau}^{r+t-\tau} \xi^{-\frac{3}{2}} u(\tau,\xi)\,d\tau\,d\xi \\ &+ \frac{1}{2}\int_0^t \int_{r-t+\tau}^{r+t-\tau} \xi^{\frac{1}{2}} u(\tau,\xi)\big(e^{(u(\tau,\xi))^2} + e^{(u_t(\tau,\xi))^2}\big)\,d\tau\,d\xi. \end{aligned} \tag{32}$$

Differentiating (32) with respect to t, we get

$$\begin{aligned} r^{\frac{1}{2}} u_t(t,r) = {}& \epsilon\big(\mathcal{G}(r+t) + \mathcal{G}(r-t)\big) \\ &+ \frac{1}{8}\int_0^t \big(\xi^{-\frac{3}{2}} u(\tau,\xi)|_{\xi=r+t-\tau} + \xi^{-\frac{3}{2}} u(\tau,\xi)|_{\xi=r-t+\tau}\big)\,d\tau \\ &+ \frac{1}{2}\int_0^t \big(\xi^{\frac{1}{2}} u(\tau,\xi)\big(e^{(u(\tau,\xi))^2} + e^{(u_t(\tau,\xi))^2}\big)|_{\xi=r-t+\tau} \\ &+ \xi^{\frac{1}{2}} u(\tau,\xi)\big(e^{(u(\tau,\xi))^2} + e^{(u_t(\tau,\xi))^2}\big)|_{\xi=r+t-\tau}\big)\,d\tau. \end{aligned} \tag{33}$$

For $t \geq \frac{1}{2}$ and $\frac{1}{4} < r - t \leq \frac{3}{4}$, by the form of $\mathcal{G}$, it follows from (33) that

$$u(t,r), u_t(t,r) \geq C\epsilon r^{-\frac{1}{2}}. \tag{34}$$

Let

$$G(t) = \int_{\mathbb{R}^2} u(t,x)\,dx.$$

Note that $e^u > u$ for $u > 0$. By (34), direct computation shows that

$$\begin{aligned} G''(t) &\geq \int_{\mathbb{R}^2} u(t,x)e^{u^2(t,x)} \\ &\geq \int_{\mathbb{R}^2} u^3(t,x)\,dx, \end{aligned} \tag{35}$$

and

$$\begin{aligned} G''(t) &\geq \int_{\mathbb{R}^2} u(t,x)e^{u_t^2(t,x)} \\ &\geq \int_{\mathbb{R}^2} u(t,x)u_t^2(t,x)\,dx \\ &\geq \int C\epsilon^3 \int_{t+\frac{1}{4}}^{t+\frac{3}{4}} r^{-\frac{3}{2}} \cdot r\,dr \\ &\geq C\epsilon^3(1+t)^{-\frac{1}{2}}, \end{aligned}$$

which implies that

$$G(t) \geq C\epsilon^3(1+t)^{\frac{3}{2}}, \quad t \geq 1. \tag{36}$$

Since ue^{u^2} is a positive function, by the Hölder inequality, we derive

$$\begin{aligned} G(t) &\leq \left(\int_{\mathbb{R}^2} u^3(t,x)\,dx\right)^{\frac{1}{3}} \left(\int_{|x|<1+t} dx\right)^{\frac{2}{3}} \\ &\leq C(1+t)^{\frac{4}{3}} \left(\int_{\mathbb{R}^2} u^3(t,x)\,dx\right)^{\frac{1}{3}}, \end{aligned}$$

which combining with (35) gives that

$$G''(t) \geq C(1+t)^{-4}G^3(t). \tag{37}$$

Let

$$a_1 = \frac{3}{2}, \qquad b_1 = 4, \qquad p = 3.$$

It is easy to see that (16) holds for $0 < \delta < \frac{1}{3}$. Applying Lemma 1 to $G(t)$, we know that $G(t)$ will blow up in finite time, then the solutions to problem (6) will blow up in finite time. Furthermore, we have $T(\epsilon) \leq C_0\epsilon^{-2}$, where $C_0 > 0$ is independent of ϵ. We complete the proof of Theorem 2.

5 Conclusion

In our paper, we show blow-up phenomena for two kinds of nonlinear wave equations, i.e., nonlinear wave equation with with variable coefficient and wave equation with exponential type nonlinearity. The importance of our results is that if the null condition or weak null condition cannot be satisfied, a perturbation of quasilinear term can destroy the global well-posedness.

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