Duan et al. *Boundary Value Problems* (2020) 2020:22
https://doi.org/10.1186/s13661-020-01331-6

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Ground states for a coupled Schrödinger system with general nonlinearities

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Abstract

We study a coupled Schrödinger system with general nonlinearities. By using variational methods, we prove the existence and asymptotic behaviour of ground state solution for the system with periodic couplings. Moreover, we prove the existence and nonexistence of ground state solution for the system with non-periodic couplings via Nehari manifold method. Especially, the ground state solution with both nontrivial components is obtained, and the sign of nontrivial components is considered.

MSC: 35J50; 35J60

Keywords: Coupled Schrödinger system; Nehari manifold; Ground state solutions; Variational methods

1 Introduction and main results

We study the existence, nonexistence and asymptotic behaviour of ground state solution of the coupled Schrödinger system

$$\begin{cases} (-\Delta)^s u + a_1 u = f_1(u) + b(x)|u|^{q-2}u|v|^q + \lambda v & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + a_2 v = f_2(v) + b(x)|u|^q|v|^{q-2}v + \lambda u & \text{in } \mathbb{R}^N, \\ u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $a_i > 0$, $i = 1,2$, $\lambda \in (-\sqrt{a_1 a_2}, 0) \cup (0, \sqrt{a_1 a_2})$, $0 < s < 1$, $N > 2s$, $2^* = \frac{2N}{N-2s}$ and $2 < 2q < p < 2^*$. $(-\Delta)^s$ stands for fractional Laplacian, see [1, 2]. The coupled Schrödinger system arises from Hartree–Fock theory in Bose–Einstein condensates and nonlinear optics, among other physical problems [3, 4].

Solutions with both nontrivial components (u,v), $u, v \neq 0$ are called nontrivial solutions. Solutions with both positive components are called positive solutions (u,v), $u, v > 0$. A nontrivial solution is called a ground state solution if its energy is minimum among all nontrivial solutions.

As is well known, there are nonlinear and linear forms of coupling terms for coupled Schrödinger systems. When $\lambda = 0$, Eqs. (1.1) reduce to a Schrödinger system with nonlinear couplings. In [5], the authors studied a Schrödinger system with nonlinear couplings

$$\begin{cases} (-\Delta)^s u + u = (|u|^{2p} + b|u|^{p-1}|v|^{p+1})u & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + a^{2s} v = (|v|^{2p} + b|v|^{p-1}|u|^{p+1})v & \text{in } \mathbb{R}^N, \\ \quad u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $a > 0$ and $2 < 2p + 2 < 2^*$. In the autonomous case, they proved that if $b > 0$ is large enough, Eqs. (1.2) have a positive ground state solution with both nontrivial components. Similar systems were also studied in [6–9]. When $b(x) = 0$, Eqs. (1.1) reduce to a Schrödinger system with linear couplings. In [10], the authors studied a Schrödinger system with linear couplings. Applying the classical Nehari manifold approach, they proved the existence of ground state solution and multiplicity results. For the other works about linearly coupled system, we refer the readers to [11, 12] and the references therein. When $\lambda b(x) \not\equiv 0$, Eqs. (1.1) are a Schrödinger system with linear and nonlinear couplings. There are few papers concerning this class of system. The authors in [13–15] proved the existence results of (1.1) with $f_1(u) = f_2(u) = u^3$, $q = 2$ and $b(x) = b$. To the best of our knowledge, there is almost no research concerning the system with general nonlinearities.

When $\lambda = 0$ and $b(x) = 0$, Eqs. (1.1) reduce to two scalar equations. The Schrödinger equation with different potentials and nonlinearities is actively studied, see for instance [16–21]. We just mention some results about asymptotic behaviour of ground state solution. Guo and Mederski in [16] studied a Schrödinger equation with sum of periodic and inverse-square potentials as follows:

$$-\Delta u + \left(V(x) - \frac{\mu}{|x|^2} \right) u = f(x, u),$$

where $V(x)$ is periodic. The superlinear and subcritical term f satisfies a weak monotonicity condition. They proved the existence of ground state solution and the asymptotic behaviour of ground state solution in the limit $\mu \to 0$. Later in [17, Theorem 1.3], Bieganowski studied the Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x, u) - K(x)|u|^{q-2}u,$$

where $2 < q < p < 2^*$ and the potential functions $V(x)$ and $K(x)$ are $\mathbb{Z}^N$-periodic. The author studied the asymptotic behaviour of ground state solution as $K(x) \to 0$ in $L^\infty(\mathbb{R}^N)$ by using variational methods.

In the presence of general nonlinearities, periodic potentials and nonlinear couplings, we study the asymptotic behaviour of ground state solutions of (1.1) in the limit $b(x) \to 0$ in $L^\infty(\mathbb{R}^N)$. We assume that

(B) $0 \le b(x) \in L^\infty(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic.

The nonlinearities f_i, $i = 1, 2$, satisfy:

(F_1) $f_i \in \mathcal{C}^1(\mathbb{R})$ and there exist $c_1, c_2 > 0$ such that

$$|f_i'(u)| \le c_1 \left(1 + |u|^{p-2}\right) \quad \text{and} \quad |f_i(u)| \le c_2 \left(1 + |u|^{p-1}\right) \quad \text{for all } u \in \mathbb{R}.$$

(F_2) $\lim_{|u| \to 0^+} \frac{f_i(u)}{|u|} = 0$, $f_i(-u) = -f_i(u)$ for all $u \in \mathbb{R}$.

(F_3) $\lim_{|u|\to+\infty} \frac{F_i(u)}{|u|^2} \to +\infty$, where $F_i(u) = \int_0^u f_i(s)\,ds$.

(F_4) $u \mapsto \frac{f_i(u)}{|u|^{2q-1}}$ is nondecreasing on $(-\infty, 0) \cup (0, +\infty)$.

To study asymptotic behaviour of ground state solution of (1.1), we introduce the following condition from [17]:

(F_5) There exist $d > 0$ and $2 < t \le p$ such that

$$f_i(u)u - 2F_i(u) \ge d|u|^t.$$

We state our main results in what follows.

Theorem 1.1 *Suppose that* (F_1)–(F_4) *and* (B) *are satisfied.*

(i) *Then Eqs.* (1.1) *have a ground state solution* ω, *where*

$$\omega = (u, v) \begin{cases} u > 0 \quad \textit{and} \quad v < 0 & \textit{as } -\sqrt{a_1 a_2} < \lambda < 0, \\ u > 0 \quad \textit{and} \quad v > 0 & \textit{as } 0 < \lambda < \sqrt{a_1 a_2}. \end{cases}$$

(ii) *Moreover,* (F_5) *holds, every function in the sequence* (b_n) *satisfies* (B) *and* $b_n \to 0$ *in* $L^\infty(\mathbb{R}^N)$ *as* $n \to +\infty$. *If* (u_n, v_n) *is a ground state solution of* (1.1) *with* $b(x) = b_n(x)$, *then there is a sequence* $(z_n) \subset \mathbb{Z}^N$ *such that*

$$\big(u_n(\cdot + z_n), v_n(\cdot + z_n)\big) \to (u, v) \quad \textit{strongly in } E,$$

where (u, v) *is a ground state solution of* (1.1) *with* $b(x) = 0$.

In Theorem 1.1, since we are concerned with (1.1) involving general nonlinearities and nonlinear couplings, moreover f_1 and f_2 are independent with each other, the problem becomes complicated in applying variational methods. To prove the existence of ground state solution of (1.1), we find a Palais–Smale sequence on Nehari manifold and use concentration compactness argument to deal with the lack of compactness of the sequence in $\mathbb{R}^N$. The proof of (ii) is mainly based on the Nehari manifold method and takes inspiration from [17]. By concentration compactness argument and periodicity of energy functional, we find that there exists a sequence $(z_n) \subset \mathbb{Z}^N$ such that the weak limit of $(u_n(\cdot + z_n), v_n(\cdot + z_n))$ is nontrivial and is a ground state solution of (1.1) with $b(x) = 0$. Then, a further evaluation of the least energy functional allows us to get the convergence in (ii).

We also study the existence and nonexistence of ground state solution of (1.1) in the presence of non-periodic couplings. In what follows $b(x)$ satisfies:

(B_1) $0 \le b(x) \in L^\infty(\mathbb{R}^N)$ and $b(x) = b_{\mathrm{per}}(x) + b_{\mathrm{loc}}(x)$, where $0 \le b_{\mathrm{per}}(x) \in L^\infty(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic and $b_{\mathrm{loc}}(x) \in L^\infty(\mathbb{R}^N) \cap L^{\frac{p}{p-2q}}(\mathbb{R}^N)$ satisfies $\lim_{|x|\to\infty} b_{\mathrm{loc}}(x) = 0$.

Theorem 1.2 *Suppose that* (F_1)–(F_4) *and* (B_1) *are satisfied.*

(i) *If* $b_{\mathrm{loc}}(x) \ge 0$ *for a.e.* $x \in \mathbb{R}^N$ *and* $b_{\mathrm{loc}}(x) > 0$ *on a positive measure set, then* (1.1) *has a ground state solution* ω, *where*

$$\omega = (u, v) \begin{cases} u > 0 \quad \textit{and} \quad v < 0 & \textit{as } -\sqrt{a_1 a_2} < \lambda < 0, \\ u > 0 \quad \textit{and} \quad v > 0 & \textit{as } 0 < \lambda < \sqrt{a_1 a_2}. \end{cases}$$

(ii) *If* $b_{\mathrm{loc}}(x) \le 0$ *for a.e.* $x \in \mathbb{R}^N$ *and* $b_{\mathrm{loc}}(x) < 0$ *on a positive measure set, then* (1.1) *has no ground state solution.*

In Theorem 1.2, $b(x)$ is non-periodic, which brings some difficulties to prove that the weak limit of the obtained PS sequence is nontrivial since the translation of energy functional is not invariant. By comparing its least energy with that in the periodic case, we can deduce that the weak limit is nontrivial. Finally, with concentration compactness argument and direct energy estimation, the existence and nonexistence results are proved under suitable assumptions on the sign of $b_{\mathrm{loc}}(x)$.

The paper is organized in the following way. In Sect. 2 we present several technical results which will be used throughout this paper. In Sect. 3 we study PS sequences on Nehari manifold. We prove Theorem 1.1 in Sect. 4 and Theorem 1.2 in Sect. 5.

2 Preliminaries

We denote the Hilbert space $E := H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ endowed with the norm (see [1]) $\|\omega\|^2 := \|(u,v)\|^2 = \|u\|^2 + \|v\|^2$, where

$$\|u\|^2 := [u]_s^2 + a_i|u|_2^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} a_i|u|^2\,dx,$$

$|\cdot|_p$ stands for the norm of $L^p(\mathbb{R}^N)$ and $|(\cdot,\cdot)|_p = (|\cdot|_p^p + |\cdot|_p^p)^{\frac{1}{p}}$ stands for the norm of $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$. It is well known that weak solutions of (1.1) are critical points of functional $\mathcal{J}(\omega) = \mathcal{J}(u,v) : E \to \mathbb{R}$

$$\begin{aligned}\mathcal{J}(\omega) &:= \mathcal{J}(u,v)\\ &= \frac{1}{2}\|\omega\|^2 - \lambda\int_{\mathbb{R}^N} uv\,dx - \int_{\mathbb{R}^N} F_1(u)\,dx - \int_{\mathbb{R}^N} F_2(v)\,dx\\ &\quad - \frac{1}{q}\int_{\mathbb{R}^N} b(x)|u|^q|v|^q\,dx.\end{aligned} \tag{2.1}$$

Denote

$$\mathcal{I}(\omega) := \int_{\mathbb{R}^N} F_1(u)\,dx + \int_{\mathbb{R}^N} F_2(v)\,dx + \frac{1}{q}\int_{\mathbb{R}^N} b(x)|u|^q|v|^q\,dx,$$

and Nehari manifold

$$\begin{aligned}&\mathcal{N} := \{\omega \in E \setminus \{0,0\} : \mathcal{J}'(\omega)\omega = 0\},\\ &c := \inf\{\mathcal{J}(\omega) : \omega \in E \setminus \{0,0\}, \mathcal{J}'(\omega)\omega = 0\}.\end{aligned}$$

Assumptions (F_2) and (F_4) imply that

$$f_i(u)u = 2q\int_0^u \frac{f_i(u)}{u^{2q-1}}s^{2q-1}\,ds \ge 2q\int_0^u \frac{f_i(s)}{s^{2q-1}}s^{2q-1}\,ds = 2qF_i(u). \tag{2.2}$$

The following lemma is standard and follows from (F_1)–(F_2).

Lemma 2.1 *For $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\left|f_i(u)u\right| + \left|F_i(u)\right| \le \varepsilon|u|^2 + C_\varepsilon|u|^p.$$

We need several lemmas for our proof.

Lemma 2.2 *For $\lambda \in (-\sqrt{a_1a_2}, \sqrt{a_1a_2}) \setminus \{0\}$, there holds*

$$\left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega\|^2 \le \|\omega\|^2 - 2\lambda\int_{\mathbb{R}^N} uv\,dx \le \left(1 + \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega\|^2. \tag{2.3}$$

Proof Since $\lambda \in (-\sqrt{a_1a_2}, \sqrt{a_1a_2}) \setminus \{0\}$, then $0 < \frac{|\lambda|}{\sqrt{a_1a_2}} < 1$ and

$$\begin{aligned} -2\lambda\int_{\mathbb{R}^N} uv\,dx &\ge -2|\lambda|\int_{\mathbb{R}^N} |u||v|\,dx = -2\frac{|\lambda|}{\sqrt{a_1a_2}}\int_{\mathbb{R}^N}\sqrt{a_1a_2}|u||v|\,dx \\ &\ge -\frac{|\lambda|}{\sqrt{a_1a_2}}\int_{\mathbb{R}^N} a_1u^2 + a_2v^2\,dx \ge -\frac{|\lambda|}{\sqrt{a_1a_2}}\|\omega\|^2. \end{aligned} \tag{2.4}$$

It follows that $\|\omega\|^2 - 2\lambda\int_{\mathbb{R}^N} uv\,dx \ge (1 - \frac{|\lambda|}{\sqrt{a_1a_2}})\|\omega\|^2$. The proof of $\|\omega\|^2 - 2\lambda\int_{\mathbb{R}^N} uv\,dx \le (1 + \frac{|\lambda|}{\sqrt{a_1a_2}})\|\omega\|^2$ is analogous. □

Lemma 2.3 *Suppose that (F_1)–(F_4) are satisfied and a potential function $b(x)$ satisfies (B) or (B_1), one has $\beta := \inf_{\omega\in\mathcal{N}}\|\omega\| > 0$.*

Proof Let $\omega_n \in \mathcal{N}$ be such that $\|\omega_n\| \to 0$, then

$$\begin{aligned} \left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega_n\|^2 &\le \|\omega_n\|^2 - 2\lambda\int_{\mathbb{R}^N} u_nv_n\,dx \\ &= \int_{\mathbb{R}^N} f_1(u_n)u_n\,dx + \int_{\mathbb{R}^N} f_2(v_n)v_n\,dx + 2\int_{\mathbb{R}^N} b(x)|u_n|^q|v_n|^q\,dx, \end{aligned}$$

which implies that

$$\|\omega_n\|^2 \le C\left(\varepsilon\|u_n\|^2 + \varepsilon\|v_n\|^2 + C_\varepsilon\|u_n\|^p + C_\varepsilon\|v_n\|^p + \|u_n\|^{2q} + \|v_n\|^{2q}\right)$$

for a constant $C > 0$. Let $\varepsilon > 0$ be such that $1 - \varepsilon C > 0$, then

$$\begin{aligned} 1 - \varepsilon C &\le C\frac{C_\varepsilon(\|u_n\|^p + \|v_n\|^p) + \|u_n\|^{2q} + \|v_n\|^{2q}}{\|\omega_n\|^2} \\ &\le C\left[C_\varepsilon\left(\|u_n\|^{p-2} + \|v_n\|^{p-2}\right) + \|u_n\|^{2q-2} + \|v_n\|^{2q-2}\right] \to 0. \end{aligned}$$

It is a contradiction. Hence $\inf_{\omega\in\mathcal{N}}\|\omega\| > 0$. □

Lemma 2.4 *Suppose that (F_1)–(F_4) are satisfied, and a potential function $b(x)$ satisfies (B) or (B_1), then:*

- (A_1) *There exists $r > 0$ such that $a := \inf_{\|\omega\|=r}\mathcal{J}(\omega) > \mathcal{J}(0) = 0$;*
- (A_2) *For any $\omega \in E \setminus \{(0,0)\}$, there exists $t > 0$ such that $\mathcal{J}(t\omega) < 0$;*

(A_3) *For $t \in (0,\infty) \setminus \{1\}$ and $\omega \in \mathcal{N}$, there holds*

$$\varphi(t) := \frac{t^2-1}{2}\mathcal{I}'(\omega)\omega - \mathcal{I}(t\omega) + \mathcal{I}(\omega) < 0;$$

(A_4) *For any $\omega \in E \setminus \{(0,0)\}$, there exists a unique number $t > 0$ such that $t\omega \in \mathcal{N}$ and $\mathcal{J}(t\omega) = \max_{r\geq 0} \mathcal{J}(r\omega)$.*

Proof (A_1) Applying the fractional Sobolev embedding theorem [1] and Lemma 2.1, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} F_i(u)\,dx \leq C\big(\varepsilon\|u\|^2 + C_\varepsilon\|u\|^p\big).$$

Hölder's inequality implies that

$$\int_{\mathbb{R}^N} b(x)|u|^q|v|^q\,dx \leq |b|_\infty |u|_{2q}^q |v|_{2q}^q \leq C\big(\|u\|^{2q} + \|v\|^{2q}\big).$$

Let $r, C_1 > 0$ for $\|\omega\| \leq r$ and r be sufficiently small, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} F_1(u)\,dx + \int_{\mathbb{R}^N} F_2(v)\,dx + \frac{1}{q}\int_{\mathbb{R}^N} b(x)|u|^q|v|^q\,dx \\
&\quad\leq C(\varepsilon_1+\varepsilon_2)\|\omega\|^2 + C(C_{\varepsilon_1}+C_{\varepsilon_2})\|\omega\|^p + 2C\|\omega\|^{2q} \\
&\quad\leq \frac{1}{4}\left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega\|^2 \leq \frac{1}{4}\left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)r^2.
\end{aligned}$$

For $\|\omega\| = r$, it suffices to show that

$$\mathcal{J}(\omega) \geq \frac{1}{4}\left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)r^2 > 0.$$

(A_2) For any $\omega \in E \setminus \{(0,0)\}$ and $t > 0$, by using Fatou's lemma and (F_3), we have

$$\lim_{t\to+\infty}\int_{\mathbb{R}^3} \frac{F_1(tu)}{t^2u^2}u^2 + \frac{F_2(tv)}{t^2v^2}v^2\,dx \geq \int_{\mathbb{R}^3}\lim_{t\to+\infty}\left(\frac{F_1(tu)}{t^2u^2}u^2 + \frac{F_2(tv)}{t^2v^2}v^2\right)dx \to +\infty,$$

which implies that

$$\begin{aligned}
\mathcal{J}(t\omega)/t^2 = {}& \frac{1}{2}\left(\|\omega\|^2 - 2\lambda\int_{\mathbb{R}^N} uv\,dx\right) - \int_{\mathbb{R}^N}\frac{F_1(tu)+F_2(tv)}{t^2}\,dx \\
&- \frac{t^{2q-2}}{q}\int_{\mathbb{R}^N} b(x)|u|^q|v|^q\,dx \to -\infty \quad \text{as } t \to +\infty.
\end{aligned}$$

Hence $\mathcal{J}(t\omega) \to -\infty$ as $t \to +\infty$.

(A_3) For $\omega \in \mathcal{N}$ and $t > 0$, let

$$\varphi(t) = \frac{t^2-1}{2}\mathcal{I}'(\omega)\omega - \mathcal{I}(t\omega) + \mathcal{I}(\omega),$$

obviously, $\varphi'(t) = t\mathcal{I}'(\omega)\omega - \mathcal{I}'(t\omega)\omega$. It follows from Lemma 2.3 that

$$\mathcal{I}'(\omega)\omega = \|\omega\|^2 - 2\lambda \int_{\mathbb{R}^N} uv\,dx \geq \left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega\|^2 > 0.$$

We have

$$\begin{aligned} t\mathcal{I}'(\omega)\omega - \mathcal{I}'(t\omega)\omega &= t\left(\int_{\mathbb{R}^N} f_1(u)u\,dx + \int_{\mathbb{R}^N} f_2(v)v\,dx + 2\int_{\mathbb{R}^N} b(x)|u|^q|v|^q\,dx\right) \\ &\quad - \left(\int_{\mathbb{R}^N} f_1(tu)u\,dx + \int_{\mathbb{R}^N} f_2(tv)v\,dx + 2t^{2q-1}\int_{\mathbb{R}^N} b(x)|u|^q|v|^q\,dx\right). \end{aligned}$$

In view of (F_4), if $t < 1$, then

$$\begin{aligned} \varphi'(t) &= t\mathcal{I}'(\omega)\omega - \mathcal{I}'(t\omega)\omega \\ &> t^{2q-1}\left(\int_{\mathbb{R}^N} f_1(u)u\,dx + \int_{\mathbb{R}^N} f_2(v)v\,dx - \int_{\mathbb{R}^N} \frac{f_1(tu)u}{t^{2q-1}}\,dx - \int_{\mathbb{R}^N} \frac{f_2(tv)v}{t^{2q-1}}\right)dx \geq 0. \end{aligned}$$

While for $t > 1$, we have $\varphi'(t) = t\mathcal{I}'(\omega)\omega - \mathcal{I}'(t\omega)\omega < 0$. Hence $\varphi(t) < \varphi(1) = 0$ for $t \in (0, +\infty) \setminus \{1\}$.

(A_4) In view of (A_1) and (A_2), for any $\omega \in E \setminus \{(0,0)\}$, there exists a maximum point $t_{\max}$ of $t \mapsto \mathcal{J}(t\omega)$ such that $\mathcal{J}'(t_{\max}\omega)\omega = 0$ and $t_{\max}\omega \in \mathcal{N}$.

For any $\omega \in \mathcal{N}$ and $t \in (0, +\infty) \setminus \{1\}$, we have

$$\mathcal{J}(t\omega) = \mathcal{J}(\omega) + \left(\mathcal{J}(t\omega) - \mathcal{J}(\omega) - \frac{t^2 - 1}{2}\mathcal{J}'(\omega)\omega\right) = \mathcal{J}(\omega) + \varphi(t) < \mathcal{J}(\omega). \qquad \square$$

Lemma 2.5 *$\mathcal{J}$ is coercive on $\mathcal{N}$, i.e. there is a sequence $(\omega_n) \subset \mathcal{N}$ such that $\mathcal{J}(\omega_n) \to +\infty$ as $\|\omega_n\| \to +\infty$.*

Proof Let $(\omega_n) \subset \mathcal{N}$ be a sequence such that $\|\omega_n\| \to +\infty$ as $n \to +\infty$. From (2.2), we find

$$\begin{aligned} \mathcal{J}(\omega_n) = \mathcal{J}(\omega_n) - \frac{1}{2q}\mathcal{J}'(\omega_n)\omega_n &\geq \left(\frac{1}{2} - \frac{1}{2q}\right)\left(\|\omega_n\|^2 - 2\lambda\int_{\mathbb{R}^N} u_nv_n\,dx\right) \\ &\geq \left(\frac{1}{2} - \frac{1}{2q}\right)\left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega_n\|^2 \to +\infty. \end{aligned} \qquad \square$$

The Nehari manifold $\mathcal{N}$ has the following properties.

Proposition 2.6

(i) *$\mathcal{N} \subset E$ is a $\mathcal{C}^1$-manifold;*

(ii) *ω is a nonzero free critical point of $\mathcal{J}$ if and only if ω is a critical point of $\mathcal{J}$ constrained on $\mathcal{N}$;*

(iii) *If (ω_n) is a (PS) sequence for $\mathcal{J}|_{\mathcal{N}}$, then ω_n is a (PS) sequence for $\mathcal{J}$.*

Proof (i) For $\omega \in \mathcal{N}$, we denote

$$\begin{aligned} \xi(u,v) &:= \mathcal{J}'(u,v)(u,v) \\ &= \|(u,v)\|^2 - 2\lambda\int uv\,dx - \int f_1(u)u\,dx \\ &\quad - \int f_2(v)v\,dx - 2\int b(x)|u|^q|v|^q\,dx. \end{aligned}$$

Let $\varphi_i(s) := \frac{f_i(s)}{s^{2q-1}}$ for $s > 0$. In view of (F_4), we have $\frac{d\varphi_i(s)}{ds} \geq 0$, i.e. $f_i'(s)s^{2q-1} - (2q-1)f_i(s)s^{2q-2} \geq 0$ for $s > 0$, which implies

$$f_i(s)s - f_i'(s)s^2 \leq -(2q-2)f_i(s)s \quad \text{for } s > 0.$$

Assume $s < 0$, then $-s > 0$ and $-f_i(-s)s - f_i'(-s)s^2 \leq (2q-2)f_i(-s)s$ for $s < 0$, in view of (F_2), we find

$$f_i(s)s - f_i'(s)s^2 \leq -(2q-2)f_i(s)s \quad \text{for } s < 0.$$

It is clear that

$$\begin{aligned}
\xi'(u,v)(u,v) &= 2\|(u,v)\|^2 - 4\lambda\int uv\,dx - \int f_1(u)u + f_1'(u)u^2\,dx \\
&\quad - \int f_2(v)v + f_2'(v)v^2\,dx - 4q\int b(x)|u|^q|v|^q\,dx \\
&= \int f_1(u)u - f_1'(u)u^2\,dx + \int f_2(v)v - f_2'(v)v^2\,dx \\
&\quad - (2q-2)2\int b(x)|u|^q|v|^q\,dx \\
&\leq -(2q-2)\left(\int f_1(u)u + f_2(v)v\,dx + 2\int b(x)|u|^q|v|^q\,dx\right) \\
&= -(2q-2)\left(\|(u,v)\|^2 - 2\lambda\int uv\,dx\right).
\end{aligned}$$

It follows from Lemma 2.3 that

$$\|(u,v)\|^2 - 2\lambda\int uv\,dx \geq \left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega\|^2 > 0,$$

then

$$\xi'(u,v)(u,v) \leq -(2q-2)\left(\|(u,v)\|^2 - 2\lambda\int uv\,dx\right) < 0. \tag{2.5}$$

Hence $\mathcal{N} \subset E$ is a $\mathcal{C}^1$-manifold.

(ii) If $\omega \neq (0,0)$ is a critical point of $\mathcal{J}$, then $\mathcal{J}'(\omega) = 0$ and $\omega \in \mathcal{N}$. If $\omega \in \mathcal{N}$ is a critical point of $\mathcal{J}$ on $\mathcal{N}$, by applying the Lagrange multiplier theorem, one has $\mathcal{J}'(\omega) = \delta\xi'(\omega)$ and $\mathcal{J}'(\omega)\omega = \delta\xi'(\omega)\omega$ for $\delta \in \mathbb{R}$. From (2.5) we deduce that $\delta = 0$ and $\mathcal{J}'(\omega) = 0$.

(iii) Let $(\omega_n) \subset \mathcal{N}$ be a (PS) sequence of $\mathcal{J}|_{\mathcal{N}}$, then

$$\mathcal{J}(\omega_n) \geq \left(\frac{1}{2} - \frac{1}{2q}\right)\left(\|\omega_n\|^2 - 2\lambda\int u_nv_n\,dx\right) \geq \left(\frac{1}{2} - \frac{1}{2q}\right)\left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega_n\|^2,$$

which implies (ω_n) is bounded in E. For some $\delta_n \in \mathbb{R}$, we have

$$\circ(1) = \mathcal{J}'|_{\mathcal{N}}(\omega_n) = \mathcal{J}'(\omega_n) - \delta_n\xi'(\omega_n), \tag{2.6}$$

thus $\delta_n\xi'(\omega_n)\omega_n + \circ(1) = \mathcal{J}'(\omega_n)\omega_n = 0$. From (2.5) we deduce that $\delta_n \to 0$. In view of (2.6), we get $\mathcal{J}'(\omega_n) \to 0$. □

3 Palais–Smale sequences on Nehari manifold

In this section, f_i satisfies (F_1)–(F_4), a potential function $b(x)$ satisfies (B) or (B_1).

Lemma 3.1 *There exists a bounded sequence $(u_n, v_n) \subset \mathcal{N}$ such that $\mathcal{J}(u_n, v_n) \to c$ and $\mathcal{J}'(u_n, v_n) \to 0$ as $n \to +\infty$.*

Proof It follows from Lemma 2.3 and Lemma 2.5 that $\mathcal{J}$ is bounded from below on $\mathcal{N}$. By using Ekeland's variational principle [22], there exists a sequence $(u_n, v_n) \subset \mathcal{N}$ such that

$$\begin{aligned} &\mathcal{J}(u_n, v_n) \le \inf_{\mathcal{N}} \mathcal{J}(u, v) + \frac{1}{n}, \\ &\mathcal{J}(u, v) \ge \mathcal{J}(u_n, v_n) - \frac{1}{n}\big\|(u_n - u, v_n - v)\big\| \quad \text{for any } (u, v) \in \mathcal{N}. \end{aligned} \tag{3.1}$$

Hence $\mathcal{J}(u_n, v_n) \to \inf_{\mathcal{N}} \mathcal{J}(u, v) = c$ as $n \to +\infty$. It follows that

$$\begin{aligned} c + \frac{1}{n} \ge \mathcal{J}(u_n, v_n) &\ge \left(\frac{1}{2} - \frac{1}{2q}\right)\left(\|\omega_n\|^2 - 2\lambda \int u_n v_n \, dx\right) \\ &\ge \left(\frac{1}{2} - \frac{1}{2q}\right)\left(1 - \frac{|\lambda|}{\sqrt{a_1 a_2}}\right)\|\omega_n\|^2, \end{aligned} \tag{3.2}$$

and

$$\|\omega_n\|^2 \le C. \tag{3.3}$$

For a fixed $(y, z) \in E$ and $\|(y, z)\| \le 1$, we denote

$$G_n(s, t) = \mathcal{J}'(u_n + sy + tu_n, v_n + sz + tv_n)(u_n + sy + tu_n, v_n + sz + tv_n). \tag{3.4}$$

Obviously, $G_n(0, 0) = \mathcal{J}'(u_n, v_n)(u_n, v_n) = 0$. In view of (2.5), we have

$$\frac{\partial G_n}{\partial t}(0, 0) = \xi'(u_n, v_n)(u_n, v_n) < 0.$$

By implicit function theorem, there exist C^1 functions $t_n(s) : (-\delta_n, \delta_n) \to \mathbb{R}$ such that $t_n(0) = 0$ and

$$G_n\big(s, t_n(s)\big) = 0 \quad \text{for } s \in (-\delta_n, \delta_n). \tag{3.5}$$

Differentiating $G_n(s, t_n(s))$ in s at $s = 0$, we have

$$\frac{\partial G_n}{\partial s}(0, 0) + \frac{\partial G_n}{\partial t}(0, 0)t_n'(0) = 0. \tag{3.6}$$

Combining Lemma 2.3 and (2.5), we get

$$\begin{aligned}\left|\frac{\partial G_n}{\partial t}(0,0)\right| &= \left|\xi'(u_n,v_n)(u_n,v_n)\right| \\ &\geq (2q-2)\left(\left\|(u_n,v_n)\right\|^2 - 2\lambda\int u_n v_n\,dx\right) \\ &\geq (2q-2)\left(1-\frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega_n\|^2 \geq (2q-2)\left(1-\frac{|\lambda|}{\sqrt{a_1a_2}}\right)\beta^2. \end{aligned} \tag{3.7}$$

It is clear that

$$\begin{aligned}\left|\frac{\partial G_n}{\partial s}(0,0)\right| &= \left|\xi'(u_n,v_n)(y,z)\right| \\ &\leq 2\left|\langle(u_n,v_n),(y,z)\rangle\right| + 2\left|\lambda\int u_n z\,dx\right| + 2\left|\lambda\int v_n y\,dx\right| \\ &\quad + \left|\int f_1(u_n)y\,dx\right| + \left|\int f_1'(u_n)u_n y\,dx\right| \\ &\quad + \left|\int f_2(v_n)z\,dx\right| + \left|\int f_2'(v_n)v_n z\,dx\right| \\ &\quad + 2q\int b(x)|u_n|^{q-1}|v_n|^q|y|\,dx + 2q\int b(x)|u_n|^q|v_n|^{q-1}|z|\,dx. \end{aligned} \tag{3.8}$$

By using Hölder's inequality, embedding theorem and (3.3), we find

$$2\left|\langle(u_n,v_n),(y,z)\rangle\right| + 2\left|\lambda\int u_n z\,dx\right| + 2\left|\lambda\int v_n y\,dx\right| \leq C_1. \tag{3.9}$$

In view of (F_1), Lemma 2.1 and (3.3), we have

$$\begin{aligned}&\left|\int f_1(u_n)y\,dx\right| + \left|\int f_1'(u_n)u_n y\,dx\right| \\ &\quad + \left|\int f_2(v_n)z\,dx\right| + \left|\int f_2'(v_n)v_n z\,dx\right| \leq C_2. \end{aligned} \tag{3.10}$$

Moreover, we deduce that

$$\begin{aligned}&2q\int b(x)|u_n|^{q-1}|v_n|^q|y|\,dx + 2q\int b(x)|u_n|^q|v_n|^{q-1}|z|\,dx \\ &\quad \leq 2q|b|_\infty|u_n|_{2q}^{q-1}|v_n|_{2q}^q|y|_{2q} + 2q|b|_\infty|u_n|_{2q}^q|v_n|_{2q}^{q-1}|z|_{2q} \leq C_3. \end{aligned} \tag{3.11}$$

It follows from (3.8)–(3.11) that

$$\left|\frac{\partial G_n}{\partial s}(0,0)\right| \leq C_4. \tag{3.12}$$

Combining (3.6), (3.7) and (3.12), we get

$$\left|t_n'(0)\right| \leq C_5. \tag{3.13}$$

Denote

$$(\bar{y},\bar{z})_{n,s} = s(y,z) + t_n(s)(u_n,v_n),$$
$$(y,z)_{n,s} = (u_n,v_n) + (\bar{y},\bar{z})_{n,s}. \tag{3.14}$$

From (3.4) and (3.5), we find $(y,z)_{n,s} \in \mathcal{N}$ for $s \in (-\delta_n,\delta_n)$. It follows from (3.1) that

$$\left|\mathcal{J}(y,z)_{n,s} - \mathcal{J}(u_n,v_n)\right| \le \frac{1}{n}\left\|(\bar{y},\bar{z})_{n,s}\right\|. \tag{3.15}$$

Applying Taylor's expansion on the left-hand side of (3.15), we get

$$\begin{aligned}\mathcal{J}(y,z)_{n,s} - \mathcal{J}(u_n,v_n) &= \mathcal{J}'(u_n,v_n)(\bar{y},\bar{z})_{n,s} + r(n,s)\\ &= s\mathcal{J}'(u_n,v_n)(y,z) + r(n,s),\end{aligned} \tag{3.16}$$

where $r(n,s) = o(\|(\bar{y},\bar{z})_{n,s}\|)$ as $|s| \to 0$. Combining (3.3), (3.13), (3.14) and $t_n(0) = 0$, we find

$$\limsup_{|s|\to 0}\frac{\|(\bar{y},\bar{z})_{n,s}\|}{|s|} \le C_6, \tag{3.17}$$

where C_6 is independent of n. It follows that $r(n,s) = o(|s|)$ as $|s| \to 0$. From (3.15), (3.16) and (3.17), we get

$$\left|\mathcal{J}'(u_n,v_n)(y,z)\right| \le \frac{C_6}{n}. \tag{3.18}$$

Hence $\mathcal{J}'(u_n,v_n) \to 0$ as $n \to +\infty$. □

From (iii) of Proposition 2.6 and Lemma 3.1, we get that (ω_n) is bounded in E and $\mathcal{J}'(\omega_n) \to 0$. Hence there exists a subsequence of (ω_n) such that $(u_n,v_n) \rightharpoonup (u_0,v_0)$ in E. Then we have the following result.

Lemma 3.2 *Suppose $\omega_n \rightharpoonup \omega_0$ in E and $\mathcal{J}'(\omega_n) \to 0$, then $\mathcal{J}'(\omega_0) = 0$.*

Proof For any $\phi = (\varphi,\psi)$, φ, $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\mathcal{J}'(\omega_n)\phi = {}&\langle(u_n,v_n),(\varphi,\psi)\rangle - \lambda\int_{\mathbb{R}^N} u_n\psi\,dx - \lambda\int_{\mathbb{R}^N} v_n\varphi\,dx\\ &- \int_{\mathbb{R}^N} f_1(u_n)\varphi\,dx - \int_{\mathbb{R}^N} f_2(v_n)\psi\,dx - \int_{\mathbb{R}^N} b(x)|u_n|^{q-2}u_n|v_n|^q\varphi\,dx\\ &- \int_{\mathbb{R}^N} b(x)|v_n|^{q-2}v_n|u_n|^q\psi\,dx.\end{aligned} \tag{3.19}$$

Up to a subsequence, we have

$$(u_n,v_n) \to (u_0,v_0) \quad \text{in } L^t_{\mathrm{loc}}(\mathbb{R}^N) \times L^t_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 1 \le t < 2^*,$$
$$(u_n,v_n) \to (u_0,v_0) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

The weak convergence $\omega_n \rightharpoonup \omega_0$ implies that $\langle (u_n, v_n), (\varphi, \psi)\rangle \to \langle (u_0, v_0), (\varphi, \psi)\rangle$, $\int_{\mathbb{R}^N} u_n \times \psi\, dx \to \int_{\mathbb{R}^N} u_0 \psi\, dx$ and $\int_{\mathbb{R}^N} v_n \varphi\, dx \to \int_{\mathbb{R}^N} v_0 \varphi\, dx$.

Let $K \subset \mathbb{R}^N$ be a compact set containing supports of φ, ψ, then $(u_n, v_n) \to (u_0, v_0)$ in $L^t(K) \times L^t(K)$ for $1 \le t < 2^*$. By [23, Theorem 4.9], there exist $l_K(x) \in L^{2q}(K)$ and $m_K(x) \in L^{2q}(K)$ such that $|u_n(x)| \le l_K(x)$ and $|v_n(x)| \le m_K(x)$ for a.e. $x \in K$. Let $h_K(x) := l_K(x) + m_K(x)$ for $x \in K$, then $h_K(x) \in L^{2q}(K)$ and

$$\big|u_n(x)\big|, \big|v_n(x)\big| \le h_K(x) \quad \text{for a.e. } x \in K.$$

Hence $b(x)|u_n|^{q-2}u_n|v_n|^q \varphi \le b(x) h_K^{2q-1}|\varphi|$ for a.e. $x \in K$, and

$$\int_K b(x) h_K^{2q-1} |\varphi|\, dx \le |b|_\infty |h_K \chi_K|_{2q}^{2q-1} |\varphi|_{2q}.$$

Applying Lebesgue's dominated convergence theorem, we deduce that

$$\int_K b(x)|u_n|^{q-2}u_n|v_n|^q \varphi\, dx \to \int_K b(x)|u_0|^{q-2}u_0|v_0|^q \varphi\, dx.$$

By similar arguments as above and Lemma 2.1, we deduce

$$\int_K f_1(u_n)\varphi\, dx \to \int_K f_1(u_0)\varphi\, dx \quad \text{and} \quad \int_K f_2(v_n)\psi\, dx \to \int_K f_2(v_0)\psi\, dx.$$

It follows from (3.19) that

$$\mathcal{J}'(u_n, v_n)(\varphi, \psi) \to \mathcal{J}'(u_0, v_0)(\varphi, \psi).$$

Hence $\mathcal{J}'(u_0, v_0) = 0$. □

We introduce the vanishing lemma from [24].

Lemma 3.3 ([24, Lemma 2.4]) *Assume that $\{u_k\}$ is a bounded sequence in $H^s(\mathbb{R}^N)$, which satisfies*

$$\sup_{z \in \mathbb{R}^N} \int_{B(z,1)} |u_k|^2\, dx \to 0. \tag{3.20}$$

Then $u_k \to 0$ strongly in $L^r(\mathbb{R}^N)$ for every $2 < r < \frac{2N}{N-2s}$.

Lemma 3.4 *Assume that $\{\omega_n\}$ is a PS sequence constrained on $\mathcal{N}$, which satisfies*

$$\sup_{z \in \mathbb{R}^N} \int_{B(z,1)} |\omega_n|^2\, dx \to 0, \tag{3.21}$$

then $\|\omega_n\| \to 0$.

Proof Combining Lemma 3.3 and (3.21), we get $u_n, v_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2^*$. For $\omega_n \subset \mathcal{N}$, we have

$$\left(1 - \frac{|\lambda|}{\sqrt{a_1 a_2}}\right)\|\omega_n\|^2 \le \|\omega_n\|^2 - 2\lambda \int_{\mathbb{R}^N} u_n v_n \, dx$$
$$= \int_{\mathbb{R}^N} f_1(u_n)u_n + f_2(v_n)v_n \, dx + 2\int_{\mathbb{R}^N} b(x)|u_n|^q |v_n|^q \, dx. \tag{3.22}$$

It is clear that

$$\left|\int_{\mathbb{R}^N} f_1(u_n)u_n \, dx\right| \le \varepsilon |u_n|_2^2 + C_\varepsilon |u_n|_p^p.$$

Let $\varepsilon \to 0$, we have $|\int_{\mathbb{R}^N} f_1(u_n)u_n \, dx| \to 0$. Moreover,

$$\int_{\mathbb{R}^N} b(x)|u_n|^q |v_n|^q \, dx \le |b|_\infty |u_n|_{2q}^q |v_n|_{2q}^q \to 0.$$

It follows from (3.22) that $\|\omega_n\| \to 0$. □

4 Ground states of a Schrödinger system with periodic couplings

We prove (i) and (ii) of Theorem 1.1 in Sects. 4.1–4.2, respectively.

4.1 Existence

Step 1: *We find* $(u_0, v_0) \in E$ *such that* $\mathcal{J}'(u_0, v_0) = 0$.

In view of Lemma 3.1, there exists a bounded $(PS)_c$-sequence of $\mathcal{J}$ constrained on $\mathcal{N}$, i.e. a sequence $\omega_n \subset \mathcal{N}$ such that $\mathcal{J}(\omega_n) \to c$ and $(\mathcal{J}|_{\mathcal{N}})'(\omega_n) \to 0$. It follows from (iii) of Proposition 2.6 that $\mathcal{J}'(\omega_n) \to 0$. In view of Lemma 3.2, up to a subsequence, then

$$\begin{aligned}
&(u_n, v_n) \rightharpoonup (u_0, v_0) \quad \text{in } E,\\
&(u_n, v_n) \to (u_0, v_0) \quad \text{in } L^t_{\mathrm{loc}}(\mathbb{R}^N) \times L^t_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 1 \le t < 2^*,\\
&(u_n, v_n) \to (u_0, v_0) \quad \text{for a.e. } x \in \mathbb{R}^N,
\end{aligned}$$

and $\mathcal{J}'(u_0, v_0) = 0$.

Step 2: *We check whether* $(u_0, v_0) \neq (0, 0)$.

Suppose

$$\sup_{z \in \mathbb{R}^N} \int_{B(z,1)} |\omega_n|^2 \, dx \to 0.$$

It follows from Lemma 3.4 that $\|(u_n, v_n)\| \to 0$. We get a contradiction with respect to Lemma 2.3. By Lions' lemma [25] there exists $(y_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,1)} |u_n|^2 \, dx > \delta \quad \text{or} \quad \liminf_{n\to\infty} \int_{B(y_n,1)} |v_n|^2 \, dx > \delta.$$

We assume, without loss of generality, that

$$\liminf_{n\to\infty} \int_{B(y_n,1)} |u_n|^2 \, dx > \delta.$$

For each $y_n \in \mathbb{R}^N$, we will find $z_n \in \mathbb{Z}^N$ such that $B(y_n, 1) \subset B(z_n, 1 + \sqrt{N})$, then

$$\liminf_{n\to\infty} \int_{B(z_n,1+\sqrt{N})} |u_n|^2\,dx \geq \liminf_{n\to\infty} \int_{B(y_n,1)} |u_n|^2\,dx > \delta.$$

Since $\mathcal{J}$ and $\mathcal{N}$ are invariant under translations of the form $\omega \mapsto \omega(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that (z_n) is bounded in $\mathbb{Z}^N$. It is clear that $u_0 \not\equiv 0$ by $u_n \to u_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Hence $\omega_0 = (u_0, v_0) \not\equiv (0,0)$, $(u_0, v_0) \in \mathcal{N}$ and $\mathcal{J}(u_0, v_0) \geq c$.

Step 3: *We find* (u', v') *such that* $\mathcal{J}'(u', v') = 0$ *and* $\mathcal{J}(u', v') = c$, *where* $u' > 0$ *and* $v' < 0$ *as* $\lambda \in (-\sqrt{a_1 a_2}, 0)$, $u' > 0$ *and* $v' > 0$ *as* $\lambda \in (0, \sqrt{a_1 a_2})$.

Applying Fatou's lemma, we get

$$\begin{aligned} c &= \liminf_{n\to\infty} \mathcal{J}(u_n, v_n) \\ &= \liminf_{n\to\infty} \int \frac{1}{2} f_1(u_n)u_n - F_1(u_n) \\ &\quad + \frac{1}{2} f_2(v_n)v_n - F_2(v_n)\,dx + \left(1 - \frac{1}{q}\right) \int b(x)|u_n|^q |v_n|^q\,dx \\ &\geq \int \frac{1}{2} f_1(u_0)u_0 - F_1(u_0) + \frac{1}{2} f_2(v_0)v_0 - F_2(v_0)\,dx \\ &\quad + \left(1 - \frac{1}{q}\right) \int b(x)|u_0|^q |v_0|^q\,dx \\ &= \mathcal{J}(u_0, v_0). \end{aligned}$$

From the above computations, we find that $\mathcal{J}(u_0, v_0) = c$. Hence $(u_0, v_0) \not\equiv (0,0)$ is a ground state solution of (1.1).

Case 1. $\lambda \in (-\sqrt{a_1 a_2}, 0)$.

It is clear that $\|(|u_0|, -|v_0|)\| \leq \|(u_0, v_0)\|$. By (A_4) of Lemma 2.4, there exists $t > 0$ such that $(t|u_0|, -t|v_0|) \in \mathcal{N}$ and $\mathcal{J}(t|u_0|, -t|v_0|) \geq c$, then

$$c \leq \mathcal{J}\big(t|u_0|, -t|v_0|\big) \leq \mathcal{J}(tu_0, tv_0) \leq \mathcal{J}(u_0, v_0) = c.$$

Let $(u', v') := (t|u_0|, -t|v_0|)$, $u' \geq 0$ and $v' \leq 0$, we get that (u', v') is a ground state solution of (1.1) by (ii) of Proposition 2.6. It follows from (1.1) that

$$(-\Delta)^s u' + a_1 u' = f_1\big(u'\big) + b(x)\big|u'\big|^{q-2} u' \big|v'\big|^q + \lambda v' \geq 0$$

and

$$(-\Delta)^s v' + a_2 v' = f_2\big(v'\big) + b(x)\big|u'\big|^q \big|v'\big|^{q-2} v' + \lambda u' \leq 0.$$

In view of (1.1), if $u' = 0$, then $v' = 0$. Hence $u', v' \not\equiv 0$ by $(u', v') \not\equiv (0,0)$. Applying the strong maximum principle [26] to each equality of (1.1), we get that (u', v'), $u' > 0$ and $v' < 0$ is a ground state solution of (1.1).

Case 2. $\lambda \in (0, \sqrt{a_1 a_2})$.

There exists $t' > 0$ such that $(t'|u_0|, t'|v_0|) \in \mathcal{N}$ and $\mathcal{J}(t'|u_0|, t'|v_0|) \geq c$. We deduce that

$$c \leq \mathcal{J}\big(t'|u_0|, t'|v_0|\big) \leq \mathcal{J}\big(t'u_0, t'v_0\big) \leq \mathcal{J}(u_0, v_0) = c.$$

By similar arguments in Case 1, we get that (u',v'), $u'>0$ and $v'>0$ is a ground state solution of (1.1). This completes the proof of (i) of Theorem 1.1.

4.2 Asymptotic behaviour of ground states as $b_n \to 0$ in $L^\infty(\mathbb{R}^N)$

Denote

$$G_i(u) := \frac{1}{2} f_i(u)u - F_i(u), \quad i=1,2.$$

From (2.2), it suffices to show that $G_i(u) \geq 0$. The following version of Brezis–Lieb lemma [16] is crucial to proving the asymptotic behaviour of ground states.

Lemma 4.1 (Brezis–Lieb lemma) *Assume that* (F_1)–(F_4) *are satisfied, let* $\{u_n\}$ *be a bounded sequence such that* $u_n \rightharpoonup u$ *weakly in* $H^s(\mathbb{R}^N)$. *Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \bigl[G_i(u_n) - G_i(u_n - u)\bigr]\,dx = \int_{\mathbb{R}^N} G_i(u)\,dx, \quad i=1,2. \tag{4.1}$$

Proof It is clear that

$$\begin{aligned}\int_{\mathbb{R}^N} \bigl[G_i(u_n) - G_i(u_n-u)\bigr]\,dx &= \int_{\mathbb{R}^N} \int_0^1 \frac{d}{dt} G_i(u_n - u + tu)\,dt\,dx \\ &= \int_0^1 \int_{\mathbb{R}^N} g_i(u_n - u + tu)u\,dx\,dt,\end{aligned} \tag{4.2}$$

where $g_i(u) := \frac{d}{du} G_i(u)$, and $g_i(u) = \frac{1}{2} f_i'(u)u - \frac{1}{2} f_i(u)$. From (F_1), we find

$$\bigl|f_i'(u_n - u + tu)(u_n - u + tu)\bigr| \leq c_1\bigl(|u_n - u + tu| + |u_n - u + tu|^{p-1}\bigr).$$

Since $(u_n - u + tu)$ is bounded in $H^s(\mathbb{R}^N)$, by using Hölder's inequality, (F_1) and Lemma 2.1, we get $\int_{\mathbb{R}^N} g_i(u_n - u + tu)u\,dx$ is bounded. For every $\varepsilon > 0$, there is $\sigma > 0$ such that

$$\int_{\Omega} \bigl|g_i(u_n - u + tu)u\bigr|\,dx < \varepsilon$$

for any $n \in \mathbb{N}$ and every measurable subset $\Omega \subset \mathbb{R}^N$ such that $|\Omega| < \sigma$. Thus $(g_i(u_n - u + tu)u)$ is uniformly integrable. Moreover, for any $\varepsilon > 0$, there exists a measurable subset $\Omega \subset \mathbb{R}^N$ of finite measure $|\Omega| < +\infty$ such that, for any $n \geq 1$,

$$\int_{\mathbb{R}^N \setminus \Omega} \bigl|g_i(u_n - u + tu)u\bigr|\,dx < \varepsilon.$$

Hence $(g_i(u_n - u + tu)u)$ is tight over $\mathbb{R}^N$. Since $g_i(u_n - u + tu)u \to g_i(tu)u$ a.e. in $\mathbb{R}^N$, in view of the Vitali convergence theorem, $g_i(tu)u$ is integrable and

$$\int_{\mathbb{R}^N} g_i(u_n - u + tu)u\,dx \to \int_{\mathbb{R}^N} g_i(tu)u\,dx \quad \text{as } n \to +\infty.$$

From (4.2), we deduce

$$\int_{\mathbb{R}^N}\big[G_i(u_n)-G_i(u_n-u)\big]\,dx \to \int_0^1\int_{\mathbb{R}^N} g_i(tu)u\,dx\,dt$$
$$=\int_{\mathbb{R}^N} G_i(u)\,dx \quad \text{as } n\to+\infty.$$

This completes the proof of Lemma 4.1. □

We denote that $\mathcal{J}_n$ is the corresponding functional of (1.1) with $b(x)=b_n(x)$, $\mathcal{J}_0$ is the corresponding functional of (1.1) with $b(x)=0$. $\mathcal{N}_n$ and $\mathcal{N}_0$ are well defined in a similar way. Denote

$$c_n := \inf_{\mathcal{N}_n}\mathcal{J}_n \quad \text{and} \quad c_0 := \inf_{\mathcal{N}_0}\mathcal{J}_0.$$

From (i) of Theorem 1.1, there exist $\omega_n\in\mathcal{N}_n$ such that $\mathcal{J}_n(\omega_n)=c_n$ and $\omega_0\in\mathcal{N}_0$ such that $\mathcal{J}_0(\omega_0)=c_0$. We need several lemmas for the proof.

Lemma 4.2 *Suppose that* (F_1)–(F_4) *and* (B) *are satisfied, then* ω_n *is bounded in* E. *Moreover, one has*

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}\inf_{\mathcal{N}_n}\mathcal{J}_n = \inf_{\mathcal{N}_0}\mathcal{J}_0 = c_0.$$

Proof Let $t_n>0$ be such that $t_n\omega_n\in\mathcal{N}_0$, we have

$$\begin{aligned} c_n &= \mathcal{J}_n(\omega_n)\ge\mathcal{J}_n(t_n\omega_n)=\mathcal{J}_0(t_n\omega_n)-\frac{t_n^{2q}}{q}\int_{\mathbb{R}^N} b_n(x)|u_n|^q|v_n|^q\,dx \\ &\ge c_0-\frac{t_n^{2q}}{q}\int_{\mathbb{R}^N} b_n(x)|u_n|^q|v_n|^q\,dx. \end{aligned} \tag{4.3}$$

Let $t_n'>0$ be such that $t_n'\omega_0\in\mathcal{N}_n$, then

$$\begin{aligned} c_0 &= \mathcal{J}_0(\omega_0)\ge\mathcal{J}_0\big(t_n'\omega_0\big)=\mathcal{J}_n\big(t_n'\omega_0\big)+\frac{t_n'^{2q}}{q}\int_{\mathbb{R}^N} b_n(x)|u_0|^q|v_0|^q\,dx \\ &\ge c_n+\frac{t_n'^{2q}}{q}\int_{\mathbb{R}^N} b_n(x)|u_0|^q|v_0|^q\,dx. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), we have

$$\begin{aligned} c_n &\le c_0-\frac{t_n'^{2q}}{q}\int_{\mathbb{R}^N} b_n(x)|u_0|^q|v_0|^q\,dx \\ &\le c_0\le c_n+\frac{t_n^{2q}}{q}\int_{\mathbb{R}^N} b_n(x)|u_n|^q|v_n|^q\,dx. \end{aligned} \tag{4.5}$$

Since $t_n\omega_n \in \mathcal{N}_0$, we have

$$\|t_n\omega_n\|^2 - 2\lambda \int_{\mathbb{R}^N} t_n^2 u_n v_n\,dx - \int_{\mathbb{R}^N} f_1(t_nu_n)t_nu_n + f_2(t_nv_n)t_nv_n\,dx$$
$$- 2\int_{\mathbb{R}^N} b_n(x)|t_nu_n|^q|t_nv_n|^q\,dx = 0.$$

Suppose $t_n \to +\infty$, in view of (F_3) and (2.2), we get that

$$\begin{aligned}
0 &= \|\omega_n\|^2 - 2\lambda \int_{\mathbb{R}^N} u_nv_n\,dx - \int_{\mathbb{R}^N} \frac{f_1(t_nu_n)t_nu_n + f_2(t_nv_n)t_nv_n}{t_n^2}\,dx \\
&\quad - 2t_n^{2q-2}\int_{\mathbb{R}^N} b_n(x)|u_n|^q|v_n|^q\,dx \\
&\le \left(1 + \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega_n\|^2 - \int_{\mathbb{R}^N} \frac{2qF_1(t_nu_n) + 2qF_2(t_nv_n)}{t_n^2}\,dx \\
&\quad - 2t_n^{2q-2}\int_{\mathbb{R}^N} b_n(x)|u_n|^q|v_n|^q\,dx \to -\infty.
\end{aligned}$$

It is a contradiction. Hence (t_n) is bounded. It follows from (2.2) and (4.4) that

$$\begin{aligned}
c_0 \ge c_n &= \mathcal{J}_n(\omega_n) = \mathcal{J}_n(\omega_n) - \frac{1}{2q}\mathcal{J}_n'(\omega_n)\omega_n \\
&\ge \left(\frac{1}{2} - \frac{1}{2q}\right)\left(\|\omega_n\|^2 - 2\lambda\int_{\mathbb{R}^N} u_nv_n\,dx\right) \ge \left(\frac{1}{2} - \frac{1}{2q}\right)\left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega_n\|^2. \qquad (4.6)
\end{aligned}$$

Hence ω_n is bounded in E, then

$$\int_{\mathbb{R}^N} b_n(x)|u_n|^q|v_n|^q\,dx \le |b_n|_\infty|u_n|_{2q}^q|v_n|_{2q}^q \to 0 \quad \text{as } n \to +\infty.$$

In view of (4.5), we deduce that $c_n \to c_0$ as $n \to +\infty$. This completes the proof of Lemma 4.2. □

Lemma 4.3 *For each ground state solution ω_n of $\mathcal{J}_n$, there exist $\omega \neq (0,0)$ and $(z_n) \subset \mathbb{Z}^N$ such that $\omega_n(\cdot + z_n) \rightharpoonup \omega$ in E. Moreover, ω is a ground state solution of $\mathcal{J}_0$, i.e. $\mathcal{J}_0'(\omega) = 0$ and $\mathcal{J}_0(\omega) = c_0$.*

Proof In view of Lemma 4.2, ω_n is bounded in E. Suppose

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |\omega_n|^2\,dx \to 0.$$

Applying similar arguments in Lemma 3.4, we get $\|\omega_n\| \to 0$. Since

$$\|\omega_n\|^2 - 2\lambda\int_{\mathbb{R}^N} u_nv_n\,dx \le \left(1 + \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\omega_n\|^2 \to 0,$$

we have

$$\begin{aligned}\limsup_{n\to+\infty}\mathcal{J}_n(\omega_n) &= \limsup_{n\to+\infty}\biggl[\frac{1}{2}\biggl(\|\omega_n\|^2-2\lambda\int_{\mathbb{R}^N}u_nv_n\,dx\biggr)\\ &\quad -\int_{\mathbb{R}^N}F_1(u_n)+F_2(v_n)\,dx-\frac{1}{q}\int_{\mathbb{R}^N}b_n(x)|u_n|^q|v_n|^q\,dx\biggr]\\ &= \limsup_{n\to+\infty}\biggl[-\int_{\mathbb{R}^N}F_1(u_n)+F_2(v_n)\,dx\biggr]\le 0.\end{aligned}$$

On the other hand, from (A_1) and (A_4) in Lemma 2.4, we have

$$\mathcal{J}_n(\omega_n)\ge\mathcal{J}_n\biggl(r\cdot\frac{\omega_n}{\|\omega_n\|}\biggr)\ge a>0.$$

It is a contradiction. By Lions' lemma [25] there exists $(y_n)\subset\mathbb{R}^N$ such that

$$\liminf_{n\to\infty}\int_{B(y_n,1)}|u_n|^2\,dx>\delta\quad\text{or}\quad\liminf_{n\to\infty}\int_{B(y_n,1)}|v_n|^2\,dx>\delta.$$

We assume, without loss of generality, that

$$\liminf_{n\to\infty}\int_{B(y_n,1)}|u_n|^2\,dx>\delta.$$

For each $y_n\in\mathbb{R}^N$, we will find $z_n\in\mathbb{Z}^N$ such that $B(y_n,1)\subset B(z_n,1+\sqrt{N})$, then

$$\liminf_{n\to\infty}\int_{B(z_n,1+\sqrt{N})}|u_n|^2\,dx\ge\liminf_{n\to\infty}\int_{B(y_n,1)}|u_n|^2\,dx>\delta.$$

Let $\bar{\omega}_n:=\omega_n(\cdot+z_n)$, $\bar{u}_n:=u_n(\cdot+z_n)$ and $\bar{v}_n:=v_n(\cdot+z_n)$, up to a subsequence, there exists $\omega\in E$ such that

$$\begin{aligned}&\bar{\omega}_n\rightharpoonup\omega\quad\text{in }E,\\ &\bar{\omega}_n\to\omega\quad\text{in }L^t_{\mathrm{loc}}\bigl(\mathbb{R}^N\bigr)\times L^t_{\mathrm{loc}}\bigl(\mathbb{R}^N\bigr)\text{ for }t\in\bigl[1,2^*\bigr),\\ &\bar{\omega}_n\to\omega\quad\text{a.e. }x\in\mathbb{R}^N.\end{aligned}$$

We have

$$\liminf_{n\to\infty}\int_{B(0,1+\sqrt{N})}|\bar{u}_n|^2\,dx>\delta.$$

Hence $u\ne 0$ and $\omega\ne(0,0)$.

For any $\phi=(\varphi,\psi)$, $\varphi,\psi\in C_0^\infty(\mathbb{R}^N)$, it is clear that

$$\begin{aligned}0&=\mathcal{J}_n'(\omega_n)\phi(\cdot-z_n)\\ &=\mathcal{J}_0'(\bar{\omega}_n)\phi-\int_{\mathbb{R}^N}b_n(x+z_n)|\bar{u}_n|^{q-2}\bar{u}_n|\bar{v}_n|^q\varphi\,dx\\ &\quad-\int_{\mathbb{R}^N}b_n(x+z_n)|\bar{u}_n|^q|\bar{v}_n|^{q-2}\bar{v}_n\psi\,dx.\end{aligned}\tag{4.7}$$

By using Hölder's inequality, we deduce that

$$\int_{\mathbb{R}^N} b_n(x+z_n)|\bar{u}_n|^{q-2}\bar{u}_n|\bar{v}_n|^q\varphi\,dx \le |b_n|_\infty|\bar{u}_n|_{2q}^{q-1}|\bar{v}_n|_{2q}^q|\varphi|_{2q} \to 0. \tag{4.8}$$

Combining (4.7) and (4.8), we find that $\mathcal{J}_0'(\bar{\omega}_n)\phi \to 0$. It follows from Lemma 3.2 that $\mathcal{J}_0'(\bar{\omega}_n)\phi \to \mathcal{J}_0'(\omega)\phi$ and $\mathcal{J}_0'(\omega) = 0$.

Since $c_n \to c_0$, using Fatou's lemma, we have

$$\begin{aligned}
c_0 &= \liminf_{n\to+\infty} \mathcal{J}_n(\omega_n) \\
&= \liminf_{n\to+\infty}\biggl[\int_{\mathbb{R}^N} \frac{1}{2}f_1(\bar{u}_n)\bar{u}_n - F_1(\bar{u}_n) \\
&\quad + \frac{1}{2}f_2(\bar{v}_n)\bar{v}_n - F_2(\bar{v}_n)\,dx + \left(1-\frac{1}{q}\right)\int_{\mathbb{R}^N} b_n(x)|\bar{u}_n|^q|\bar{v}_n|^q\,dx\biggr] \\
&\ge \liminf_{n\to+\infty}\biggl[\int_{\mathbb{R}^N} \frac{1}{2}f_1(\bar{u}_n)\bar{u}_n - F_1(\bar{u}_n) + \frac{1}{2}f_2(\bar{v}_n)\bar{v}_n - F_2(\bar{v}_n)\,dx\biggr] \\
&\ge \int_{\mathbb{R}^N} \frac{1}{2}f_1(u)u - F_1(u) + \frac{1}{2}f_2(v)v - F_2(v)\,dx \\
&= \mathcal{J}_0(\omega) \ge c_0.
\end{aligned} \tag{4.9}$$

Thus ω is a ground state solution of $\mathcal{J}_0$. This completes the proof of Lemma 4.3. □

Proof of (ii) *of Theorem* 1.1 We find that

$$\begin{aligned}
&\left(1-\frac{|\lambda|}{\sqrt{a_1a_2}}\right)\|\bar{\omega}_n-\omega\|^2 \\
&\quad\le \|\bar{\omega}_n-\omega\|^2 - 2\lambda\int_{\mathbb{R}^N}(\bar{u}_n-u)(\bar{v}_n-v)\,dx \\
&\quad= \mathcal{J}_n'(\bar{\omega}_n)(\bar{\omega}_n-\omega) - \langle\omega,\bar{\omega}_n-\omega\rangle + \lambda\int_{\mathbb{R}^N} u(\bar{v}_n-v)\,dx \\
&\qquad + \lambda\int_{\mathbb{R}^N} v(\bar{u}_n-u)\,dx + \int_{\mathbb{R}^N} f_1(\bar{u}_n)(\bar{u}_n-u) + f_2(\bar{v}_n)(\bar{v}_n-v)\,dx \\
&\qquad + \int_{\mathbb{R}^N} b_n(x)|\bar{u}_n|^{q-2}\bar{u}_n|\bar{v}_n|^q(\bar{u}_n-u)\,dx \\
&\qquad + \int_{\mathbb{R}^N} b_n(x)|\bar{v}_n|^{q-2}\bar{v}_n|\bar{u}_n|^q(\bar{v}_n-v)\,dx.
\end{aligned} \tag{4.10}$$

Since $\bar{\omega}_n \rightharpoonup \omega$ in E, we have $\langle\omega,\bar{\omega}_n-\omega\rangle \to 0$, $\lambda\int_{\mathbb{R}^N} u(\bar{v}_n-v)\,dx \to 0$ and $\lambda\int_{\mathbb{R}^N} v(\bar{u}_n-u)\,dx \to 0$. It is suffices to show that

$$\mathcal{J}_n'(\bar{\omega}_n)(\bar{\omega}_n-\omega) = \mathcal{J}_n'(\omega_n)\omega_n - \mathcal{J}_n'(\omega_n)\omega(\cdot-z_n) = 0.$$

From Lemma 4.1, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\bigl[G_1(\bar{u}_n)-G_1(\bar{u}_n-u)+G_2(\bar{v}_n)-G_2(\bar{v}_n-v)\bigr]\,dx = \int_{\mathbb{R}^N} G_1(u)+G_2(v)\,dx. \tag{4.11}$$

In view of (4.9), we have

$$c_0 = \lim_{n\to\infty}\int_{\mathbb{R}^N} G_1(\bar{u}_n) + G_2(\bar{v}_n)\,dx = \int_{\mathbb{R}^N} G_1(u) + G_2(v)\,dx. \tag{4.12}$$

It follows from (4.11) and (4.12) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} G_1(\bar{u}_n - u) + G_2(\bar{v}_n - v)\,dx = 0.$$

Since $G_i \geq 0$, $i = 1, 2$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} G_1(\bar{u}_n - u)\,dx = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_{\mathbb{R}^N} G_2(\bar{v}_n - v)\,dx = 0.$$

From (F_5), we deduce that

$$|\bar{u}_n - u|_t^t = \int_{\mathbb{R}^N} |\bar{u}_n - u|^t\,dx \leq \frac{2}{d}\int_{\mathbb{R}^N} G_1(\bar{u}_n - u)\,dx \to 0, \tag{4.13}$$

and $|\bar{v}_n - v|_t^t \to 0$. By fractional embedding theorem [1], we get that $\bar{u}_n$ and $\bar{v}_n$ are bounded in $L^r(\mathbb{R}^N)$ for $2 \leq r \leq 2^*$. Using (4.13) and the interpolation inequality, we get $\bar{u}_n \to u$ and $\bar{v}_n \to v$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2^*$. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|\int_{\mathbb{R}^N} f_1(\bar{u}_n)(\bar{u}_n - u)\,dx\right| \leq \varepsilon|\bar{u}_n|_2|\bar{u}_n - u|_2 + C_\varepsilon|\bar{u}_n|_p^{p-1}|\bar{u}_n - u|_p.$$

Let $\varepsilon \to 0$, we have

$$\left|\int_{\mathbb{R}^N} f_1(\bar{u}_n)(\bar{u}_n - u)\,dx\right| \to 0.$$

Moreover,

$$\left|\int_{\mathbb{R}^N} b_n(x)|\bar{u}_n|^{q-2}\bar{u}_n|\bar{v}_n|^q(\bar{u}_n - u)\,dx\right| \leq |b_n|_\infty|\bar{u}_n|_{2q}^{q-1}|\bar{v}_n|_{2q}^q|\bar{u}_n - u|_{2q} \to 0.$$

It follows from (4.10) that $\|\bar{\omega}_n - \omega\| \to 0$. This completes the proof of (ii) of Theorem 1.1. □

5 Ground states of a Schrödinger system with non-periodic couplings

We prove (i) and (ii) of Theorem 1.2 in Sects. 5.1–5.2, respectively. We denote that $\mathcal{J}_{\text{per}}$ is the corresponding functional of (1.1) with $b(x) = b_{\text{per}}(x)$. $\mathcal{N}_{\text{per}}$ and c_{per} are well defined in a similar way.

5.1 Existence

We need the following lemma.

Lemma 5.1 *Assume that $b_{\text{loc}}(x) \geq 0$ for a.e. $x \in \mathbb{R}^N$ and $b_{\text{loc}}(x) > 0$ on a positive measure set, then $c < c_{\text{per}}$.*

Proof From (i) of Theorem 1.1, we find a critical point ω' of $\mathcal{J}_{\text{per}}$, where

$$\omega' = (u', v') \begin{cases} u' > 0 \quad \text{and} \quad v' < 0 \quad \text{as } -\sqrt{a_1 a_2} < \lambda < 0, \\ u' > 0 \quad \text{and} \quad v' > 0 \quad \text{as } 0 < \lambda < \sqrt{a_1 a_2}, \end{cases}$$

$\mathcal{J}_{\text{per}}(u', v') = c_{\text{per}}$ and $\mathcal{J}'_{\text{per}}(u', v') = 0$. We get that $\int b_{\text{loc}}(x)|u'|^q|v'|^q\,dx > 0$. Let $t > 0$ be such that $t(u', v') \in \mathcal{N}$, then

$$\begin{aligned} c &\le \mathcal{J}(tu', tv') = \mathcal{J}_{\text{per}}(tu', tv') - \frac{t^{2q}}{q} \int b_{\text{loc}}(x)|u'|^q|v'|^q\,dx \\ &< \mathcal{J}_{\text{per}}(tu', tv') \le \mathcal{J}_{\text{per}}(u', v') = c_{\text{per}}. \end{aligned} \tag{5.1}$$

□

Proof of (i) *of Theorem* 1.2 We divide the proof into three steps. *Step* 1 and *Step* 3 are similar with those in Sect. 4.1, we omit them here. By similar arguments as *Step* 1 in Sect. 4.1, we find $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in E and $\mathcal{J}'(u_0, v_0) = 0$.

Step 2: *We check whether* $(u_0, v_0) \neq (0, 0)$.

Similarly, from Step 2 in Sect. 4.1, there exists $z_n \in \mathbb{Z}^N$ such that $B(y_n, 1) \subset B(z_n, 1 + \sqrt{N})$ and

$$\liminf_{n\to\infty} \int_{B(z_n,1+\sqrt{N})} |u_n|^2\,dx \ge \liminf_{n\to\infty} \int_{B(y_n,1)} |u_n|^2\,dx > \delta. \tag{5.2}$$

We claim that (z_n) is bounded, and hence $u_0 \neq 0$, $(u_0, v_0) \in \mathcal{N}$ and $\mathcal{J}(u_0, v_0) \ge c$.

We check the claim. Suppose that (z_n) is unbounded, then we can choose a subsequence of (z_n) such that $|z_n| \to \infty$ as $n \to \infty$. Let $\bar{u}_n := u_n(\cdot + z_n)$, $\bar{v}_n := v_n(\cdot + z_n)$, up to a subsequence, then

$$\begin{aligned} &(\bar{u}_n, \bar{v}_n) \rightharpoonup (\bar{u}, \bar{v}) \quad \text{in } E, \\ &(\bar{u}_n, \bar{v}_n) \to (\bar{u}, \bar{v}) \quad \text{in } L^t_{\text{loc}}(\mathbb{R}^N) \times L^t_{\text{loc}}(\mathbb{R}^N) \text{ for } 1 \le t < 2^*, \\ &(\bar{u}_n, \bar{v}_n) \to (\bar{u}, \bar{v}) \quad \text{for a.e. } x \in \mathbb{R}^N. \end{aligned}$$

We deduce that

$$\liminf_{n\to\infty} \int_{B(0,1+\sqrt{N})} |\bar{u}_n|^2\,dx > \delta$$

by (5.2). We find that $\bar{u} \neq 0$ by $\bar{u}_n \to \bar{u}$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, thus $\bar{\omega} = (\bar{u}, \bar{v}) \neq (0, 0)$. For any $\phi = (\varphi, \psi)$, $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} 0 &\leftarrow \mathcal{J}'(\omega_n)\phi(\cdot - z_n) \\ &= \mathcal{J}'_{\text{per}}(\bar{\omega}_n)\phi - \int_{\mathbb{R}^N} b_{\text{loc}}(x + z_n)|\bar{u}_n|^{q-2}\bar{u}_n|\bar{v}_n|^q\varphi\,dx \\ &\quad - \int_{\mathbb{R}^N} b_{\text{loc}}(x + z_n)|\bar{u}_n|^q|\bar{v}_n|^{q-2}\bar{v}_n\psi\,dx. \end{aligned} \tag{5.3}$$

Let $K \subset \mathbb{R}^N$ be a compact set containing supports of φ, ψ, then

$$\begin{aligned}
&\int_K b_{\mathrm{loc}}(x+z_n)|\bar{u}_n|^{q-2}\bar{u}_n|\bar{v}_n|^q\varphi\,dx \\
&\quad\leq |\bar{u}_n|_p^{q-1}|\bar{v}_n|_p^q|\varphi|_p\left(\int_K \left|b_{\mathrm{loc}}(x+z_n)\right|^{\frac{p}{p-2q}}\,dx\right)^{\frac{p-2q}{p}} \to 0 \qquad (5.4)
\end{aligned}$$

as $|z_n| \to \infty$. Combining (5.3) and (5.4), we get $\mathcal{J}'_{\mathrm{per}}(\bar{\omega}_n)\phi \to 0$. It follows from Lemma 3.2 that $\mathcal{J}'_{\mathrm{per}}(\bar{\omega}_n)\phi \to \mathcal{J}'_{\mathrm{per}}(\bar{\omega})\phi$ and $\mathcal{J}'_{\mathrm{per}}(\bar{\omega}) = 0$. Hence $(\bar{u},\bar{v}) \in \mathcal{N}_{\mathrm{per}}$ and $\mathcal{J}_{\mathrm{per}}(\bar{u},\bar{v}) \geq c_{\mathrm{per}}$. It is clear that

$$\begin{aligned}
a_1\bar{u}_n^2 + a_2\bar{v}_n^2 - 2\lambda\bar{u}_n\bar{v}_n &\geq a_1\bar{u}_n^2 + a_2\bar{v}_n^2 - 2|\lambda||\bar{u}_n||\bar{v}_n| \\
&\geq a_1\bar{u}_n^2 + a_2\bar{v}_n^2 - \frac{|\lambda|}{\sqrt{a_1a_2}}\left(a_1\bar{u}_n^2 + a_2\bar{v}_n^2\right) \\
&= \left(1 - \frac{|\lambda|}{\sqrt{a_1a_2}}\right)\left(a_1\bar{u}_n^2 + a_2\bar{v}_n^2\right) \geq 0.
\end{aligned}$$

Applying Fatou's lemma, we find

$$\begin{aligned}
c &= \liminf_{n\to\infty} \mathcal{J}(\omega_n) \\
&= \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{2q}\right)\left([u_n]_s^2 + [v_n]_s^2 + \int a_1|u_n|^2 + a_2|v_n|^2 - 2\lambda u_nv_n\,dx\right)\right. \\
&\quad \left. + \int \frac{1}{2q}f_1(u_n)u_n - F_1(u_n) + \frac{1}{2q}f_2(v_n)v_n - F_2(v_n)\,dx\right] \\
&= \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{2q}\right)\left([\bar{u}_n]_s^2 + [\bar{v}_n]_s^2 + \int a_1|\bar{u}_n|^2 + a_2|\bar{v}_n|^2 - 2\lambda\bar{u}_n\bar{v}_n\,dx\right)\right. \\
&\quad \left. + \int \frac{1}{2q}f_1(\bar{u}_n)\bar{u}_n - F_1(\bar{u}_n) + \frac{1}{2q}f_2(\bar{v}_n)\bar{v}_n - F_2(\bar{v}_n)\,dx\right] \\
&\geq \left(\frac{1}{2} - \frac{1}{2q}\right)\left([\bar{u}]_s^2 + [\bar{v}]_s^2 + \int a_1|\bar{u}|^2 + a_2|\bar{v}|^2 - 2\lambda\bar{u}\bar{v}\,dx\right) \\
&\quad + \int \frac{1}{2q}f_1(\bar{u})\bar{u} - F_1(\bar{u}) + \frac{1}{2q}f_2(\bar{v})\bar{v} - F_2(\bar{v})\,dx \\
&= \mathcal{J}_{\mathrm{per}}(\bar{u},\bar{v}) \geq c_{\mathrm{per}}. \qquad (5.5)
\end{aligned}$$

We get a contradiction with Lemma 5.1. Hence (z_n) is bounded. □

5.2 Nonexistence

Suppose by contradiction that there exists a ground state solution of (1.1), i.e. $\omega_0 = (u_0,v_0) \neq (0,0)$ such that $\mathcal{J}(u_0,v_0) = c$ and $\mathcal{J}'(u_0,v_0) = 0$. By using similar arguments as Step 3 in Sect. 4.1, we find a critical point of $\mathcal{J}$, where

$$\omega' = \left(u',v'\right)\begin{cases} u' > 0 \quad \text{and} \quad v' < 0 \quad \text{as } -\sqrt{a_1a_2} < \lambda < 0, \\ u' > 0 \quad \text{and} \quad v' > 0 \quad \text{as } 0 < \lambda < \sqrt{a_1a_2}, \end{cases}$$

$\mathcal{J}(u',v') = c$ and $\mathcal{J}'(u',v') = 0$.

Lemma 5.2 *Assume that $b_{\mathrm{loc}}(x) \le 0$ for a.e. $x \in \mathbb{R}^N$ and $b_{\mathrm{loc}}(x) < 0$ on a positive measure set, then $c > c_{\mathrm{per}}$.*

Proof It is clear that $\int b_{\mathrm{loc}}(x)|u'|^q|v'|^q\,dx < 0$. Let $t > 0$ be such that $t(u',v') \in \mathcal{N}_{\mathrm{per}}$, then

$$\begin{aligned} c_{\mathrm{per}} &\le \mathcal{J}_{\mathrm{per}}(tu',tv') = \mathcal{J}(tu',tv') + \frac{t^{2q}}{q}\int b_{\mathrm{loc}}(x)|u'|^q|v'|^q\,dx \\ &< \mathcal{J}(tu',tv') \le \mathcal{J}(u',v') = c. \end{aligned} \tag{5.6}$$

□

Let $\omega \in \mathcal{N}_{\mathrm{per}}$ be a ground state solution of $\mathcal{J}_{\mathrm{per}}$, i.e. $\mathcal{J}_{\mathrm{per}}(u,v) = c_{\mathrm{per}}$ and $\mathcal{J}'_{\mathrm{per}}(u,v) = 0$. Denote that $\bar{\omega} := \omega(\cdot - y)$ for $y \in \mathbb{Z}^N$, we find that $\bar{\omega} \in \mathcal{N}_{\mathrm{per}}$. There exists $t > 0$ such that $t\bar{\omega} \in \mathcal{N}$. For any $y \in \mathbb{Z}^N$, we have

$$\begin{aligned} c_{\mathrm{per}} &= \mathcal{J}_{\mathrm{per}}(\omega) = \mathcal{J}_{\mathrm{per}}(\bar{\omega}) \ge \mathcal{J}_{\mathrm{per}}(t\bar{\omega}) = \mathcal{J}(t\bar{\omega}) + \frac{t^{2q}}{q}\int_{\mathbb{R}^N} b_{\mathrm{loc}}(x)|\bar{u}|^q|\bar{v}|^q\,dx \\ &\ge c + \frac{t^{2q}}{q}\int_{\mathbb{R}^N} b_{\mathrm{loc}}(x)|\bar{u}|^q|\bar{v}|^q\,dx. \end{aligned} \tag{5.7}$$

Obviously, $\int_{\mathbb{R}^N} b_{\mathrm{loc}}(x)t^{2q}|\bar{u}|^q|\bar{v}|^q\,dx = \int_{\mathbb{R}^N} b_{\mathrm{loc}}(x+y)t^{2q}|u|^q|v|^q\,dx$. Since $\mathcal{J}_{\mathrm{per}}$ is coercive on $\mathcal{N}_{\mathrm{per}}$ and $\mathcal{J}_{\mathrm{per}}(t\omega) = \mathcal{J}_{\mathrm{per}}(t\bar{\omega}) \le c_{\mathrm{per}}$, we find that $t\omega$ is bounded in E. Furthermore, u, v are bounded in $L^{2q}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ by embedding theorem. For any $0 < \varepsilon < 1$, we choose $R(\varepsilon) > 0$ such that $\int_{B_R^c}|u|^{2q}\,dx < \varepsilon^2$, and choose $y(\varepsilon) > 0$ such that $\int_{B_R}|b_{\mathrm{loc}}(x+y)|^{\frac{p}{p-2q}}\,dx < \varepsilon^{\frac{p}{p-2q}}$, then there exist $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} b_{\mathrm{loc}}(x+y)t^{2q}|u|^q|v|^q\,dx \\ &\quad\le \left|\int_{B_R} b_{\mathrm{loc}}(x+y)t^{2q}|u|^q|v|^q\,dx\right| + \left|\int_{B_R^c} b_{\mathrm{loc}}(x+y)t^{2q}|u|^q|v|^q\,dx\right| \\ &\quad\le t^{2q}|u|_p^q|v|_p^q\left(\int_{B_R}\left|b_{\mathrm{loc}}(x+y)\right|^{\frac{p}{p-2q}}\,dx\right)^{\frac{p-2q}{p}} + t^{2q}|b_{\mathrm{loc}}|_\infty|v|_{2q}^q\left(\int_{B_R^c}|u|^{2q}\,dx\right)^{\frac{1}{2}} \\ &\quad\le C_1\varepsilon + C_2\varepsilon \le C_3\varepsilon, \end{aligned} \tag{5.8}$$

where ε is arbitrary. In view of (5.7), let $|y|$ be sufficiently large, we get $c_{\mathrm{per}} \ge c$. It is contradictory to Lemma 5.2. This completes the proof of (ii) of Theorem 1.2.

Acknowledgements
The authors would like to thank the referees for their pertinent comments and valuable suggestions.

Funding
This research is supported by the National Natural Science Foundation of China (No. 11671354).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that this study was independently finished. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 2 September 2019 Accepted: 19 January 2020 Published online: 30 January 2020

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