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Existence of multiple positive solutions for nonhomogeneous fractional Laplace problems with critical growth

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Abstract

We prove the existence of multiple positive solutions of fractional Laplace problems with critical growth by using the method of monotonic iteration and variational methods.

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1 Introduction

Considerable attention has been devoted to fractional and non-local operators of elliptic type in recent years, both for their interesting theoretical structure and in view of concrete applications, like flame propagation, chemical reactions of liquids, population dynamics, geophysical fluid dynamics, and American options; see [3, 7, 19, 20] and the references therein.

In this paper we consider the following critical problem:

$$(P)_\gamma \qquad \begin{cases} (-\Delta)^s u = \lambda u + |u|^{p-2}u + \gamma g(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $s \in (0,1)$ is fixed and $(-\Delta)^s$ is the fractional Laplace operator, $\Omega \subset \mathbb{R}^N$ $(N > 2s)$ is a smooth bounded domain, $p = 2_s^* := \frac{2N}{N-2s}$, $g \in C_0(\Omega)$, $g(x) \geq 0$ a.e. in Ω and $g(x) \not\equiv 0$ in Ω, $\lambda \geq 0$, $\gamma > 0$ are some given constants.

We are interested in the existence of positive solutions of $(P)_\gamma$ since it exhibits many interesting existence phenomena which are related to some lack of compactness of the corresponding energy functional (see (1.2)). It is worth noting here that the problem $(P)_\gamma$, with $\lambda = 0$, $\gamma = 0$, has no positive solution whenever Ω is a star-shaped domain; see [6, 11]. This fact motivates the perturbation terms λu and $\gamma g(x)$, in our work. Servadei and Valdinoci [14, 15], and Tan [17] studied problem $(P)_\gamma$ with $\gamma = 0$ and obtained Brezis–Nirenberg type results. An interesting problem is whether the existence phenomena still remain true if we give $(P)_\gamma$ with $\gamma = 0$ a lower order homogeneous perturbation in the sense $\lim_{u\to 0} \frac{f(x,u)}{u^{p-1}} = 0$ and $f(x,0) = 0$. The existence results have been obtained in [14,

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15] for the fractional Laplace operator, and [8] for the fractional p-Laplace operator. We consider here the nonhomogeneous perturbation case. Note that problem $(P)_\gamma$ in the local case $s = 1$ has been investigated in [4, 18].

The fractional Laplace operator $(-\Delta)^s$ (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \big(u(x+y) + u(x-y) - 2u(x)\big) K(y)\,dy, \quad x \in \mathbb{R}^N,$$

where $K(x) = |x|^{-(N+2s)}$, $x \in \mathbb{R}^N$. We will denote by $H^s(\mathbb{R}^N)$ the usual fractional Sobolev space endowed with the so-called Gagliardo norm,

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \left(\int_{\mathbb{R}^{2N}} \big|u(x) - u(y)\big|^2 K(x-y)\,dx\,dy\right)^{1/2},$$

while X_0 is the function space defined as

$$X_0 = \big\{u \in H^s\big(\mathbb{R}^N\big) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\big\}.$$

We refer to [9, 12, 13] for a general definition of X_0 and its properties. The embedding $X_0 \hookrightarrow L^q(\Omega)$ is continuous for any $q \in [1, 2_s^*]$ and compact for any $q \in [1, 2_s^*)$. The space X_0 is endowed with the norm defined as

$$\|u\|_{X_0} = \left(\int_{\mathbb{R}^{2N}} \big|u(x) - u(y)\big|^2 K(x-y)\,dx\,dy\right)^{1/2}.$$

By Lemma 5.1 in [12] we have $C_0^2(\Omega) \subset X_0$. Thus X_0 is non-empty. Note that $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with scalar product

$$(u, v)_{X_0} = \int_{\mathbb{R}^{2N}} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\,dx\,dy.$$

We say that $u \in X_0$ is a weak solution of (1.1) if the identity

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y)\,dx\,dy \\ &\quad = \lambda \int_\Omega u\varphi\,dx + \int_\Omega |u|^{p-2} u\varphi\,dx + \gamma \int_\Omega g\varphi\,dx\end{aligned}$$

holds for all $\varphi \in X_0$.

We consider the energy functional associated with (1.1)

$$\begin{aligned}I(u) = {}& \frac{1}{2}\int_{\mathbb{R}^{2N}} \big|u(x) - u(y)\big|^2 K(x-y)\,dx\,dy - \frac{1}{2}\lambda \int_\Omega u^2\,dx \\ & - \frac{1}{p}\int_\Omega |u|^p\,dx - \gamma \int g u\,dx.\end{aligned} \tag{1.2}$$

The critical points of the functional I correspond to weak solutions of (1.1).

Let λ_1 be the first eigenvalue of $(-\Delta)^s$ on X_0. Our main results are as follows.

Theorem 1.1 *For $\lambda \in [0, \lambda_1)$ there exists a positive constant γ^* such that $(P)_\gamma$ admits a positive minimal solution for all $\gamma \in (0, \gamma^*]$ and admits no positive solution for $\gamma > \gamma^*$.*

We prove Theorem 1.1 by the method of monotonic iteration, also known as the super and subsolution method, which is a basic tool in nonlinear partial differential equations. In this paper, we discuss a fractional Laplace operator version of this method compared with second order linear or quasilinear elliptic operator. With respect to the classical case of the Laplacian, here some estimates are more delicate, due to the non-local nature of the operator $(-\Delta)^s$.

Theorem 1.2 *For $\lambda \in [0, \lambda_1)$, $\gamma \in (0, \gamma^*)$, where γ^* is the one in Theorem 1.1, problem $(P)_\gamma$ admits at least two positive solutions.*

In order to prove Theorem 1.2, we adapt the variational approach used in [1] to the non-local framework (see also [15]).

This paper is organized as follows. In Sect. 2 we prove the existence of the first solution of $(P)_\gamma$ by the method of monotonic iteration. In Sect. 3 we prove the existence of the second solution of $(P)_\gamma$ by variational methods. We denote by $|\cdot|_p$ the $L^p(\Omega)$-norm for any $p > 1$, respectively.

2 Existence of the first positive solution

In this section we prove existence of the first solution of $(P)_\gamma$ by the method of monotonic iteration.

Definition 1 We say that $\overline{u} \in X_0$ is a weak supersolution of problem $(P)_\gamma$ if

$$\int_{\mathbb{R}^{2N}} \big(\overline{u}(x) - \overline{u}(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \\ \geq \lambda \int_\Omega \overline{u}\varphi\,dx + \int_\Omega |\overline{u}|^{p-2}\overline{u}\varphi\,dx + \gamma \int_\Omega g\varphi\,dx$$

for any $\varphi \in X_0$, $\varphi \geq 0$ a.e. in Ω.

Definition 2 We say that $\underline{u} \in X_0$ is a weak subsolution of problem $(P)_\gamma$ if

$$\int_{\mathbb{R}^{2N}} \big(\underline{u}(x) - \underline{u}(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \\ \leq \lambda \int_\Omega \underline{u}\varphi\,dx + \int_\Omega |\underline{u}|^{p-2}\underline{u}\varphi\,dx + \gamma \int_\Omega g\varphi\,dx$$

for any $\varphi \in X_0$, $\varphi \geq 0$ a.e. in Ω.

Let λ_1 be the first eigenvalue of $(-\Delta)^s$ on X_0 with $\phi_1 \geq 0$ the corresponding normalized eigenfunction; see Proposition 9 in [13]. We show $\phi_1 > 0$ in Ω. By Proposition 4 in [14], $\phi_1 \in L^\infty(\Omega)$. Furthermore, by Proposition 1.1 in [10], $\phi_1 \in C^s(\mathbb{R}^N)$. Assume by contradiction that there exists $x_0 \in \Omega$ such that $\phi_1(x_0) = 0$. It follows from the definition of the fractional Laplace $(-\Delta)^s$ that

$$0 > -\int_{\mathbb{R}^N} \big(\phi_1(x_0 + y) + \phi_1(x_0 - y) - 2\phi_1(x_0)\big)K(y)\,dy = \lambda_1\phi_1(x_0) = 0,$$

we get a contradiction. Thus, $\phi_1 > 0$ in Ω.

Lemma 2.1 *For $\lambda \in [0, \lambda_1)$ there exists a constant $\widehat{\gamma} > 0$ such that $(P)_\gamma$ has no positive solution for $\gamma > \widehat{\gamma}$.*

Proof Taking $C_1 = \min_{t\geq 0}[t^{p-1} - (\lambda_1 - \lambda)t]$ we get

$$t^{p-1} \geq [\lambda_1 - \lambda]t + C_1, \quad \forall t \geq 0. \tag{2.1}$$

Multiplying (1.1) by ϕ_1 and integrating on Ω we get

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \bigl(u(x) - u(y)\bigr)\bigl(\phi_1(x) - \phi_1(y)\bigr)K(x-y)\,dx\,dy \\ &\quad = \lambda \int_\Omega u\phi_1\,dx + \int_\Omega u^{p-1}\phi_1\,dx + \gamma \int_\Omega g\phi_1\,dx.\end{aligned}$$

Consequently,

$$\lambda_1 \int_\Omega u\phi_1\,dx = \lambda \int_\Omega u\phi_1\,dx + \int_\Omega u^{p-1}\phi_1\,dx + \gamma \int_\Omega g\phi_1\,dx.$$

Hence from (2.1) we have

$$\gamma \leq \widehat{\gamma} := \frac{-C_1 \int_\Omega \phi_1\,dx}{\int_\Omega g\phi_1\,dx}.$$

□

Lemma 2.2 *Let $u_1, u_2 \in X_0$ be supersolutions of (P_γ). Then $u_1 \wedge u_2 := \min\{u_1, u_2\}$ is a supersolution of (P_γ). Similarly, if $v_1, v_2 \in X_0$ are subsolutions of (P_γ), then so is $v_1 \vee v_2 :=$* $\max\{v_1, v_2\}$.

Proof By density results for X_0, there exists a sequence $\{w_n\} \subset C^\infty(\Omega)$ such that $w_n \to w := u_1 - u_2$ in X_0. It follows that $w_n(x) \to w(x)$ for a.e. $x \in \Omega$.

Let $\eta \in C^\infty(\mathbb{R})$ be a nondecreasing function such that (i) $0 \leq \eta(t) \leq 1$; (ii) $\eta(t) = 0$ for $t \leq 0$, $\eta(t) = 1$ for $t \geq 1$. Set $\eta_n(t) = \eta(nt)$. Then $\eta_n(t) = 0$ for $t \leq 0$, $\eta_n(t) = 1$ for $t \geq \frac{1}{n}$.

Now for any nonnegative function $\varphi \in C_0^\infty(\Omega)$ we define

$$\psi_{1,n} = (1 - \eta_n \circ w_n)\varphi, \qquad \psi_{2,n} = (\eta_n \circ w_n)\varphi,$$

where $\eta_n \circ w_n$ denotes the composition of w_n and g_n. Of course, $\psi_{1,n}, \psi_{2,n} \geq 0$ and $\varphi = \psi_{1,n} + \psi_{2,n}$. Since u_1, u_2 are supersolutions of $(P)_\gamma$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \bigl(u_i(x) - u_i(y)\bigr)\bigl[\psi_{i,n}(x) - \psi_{i,n}(y)\bigr]K(x-y)\,dx\,dy \\ &\quad \geq \lambda \int_\Omega u_i\psi_{i,n}\,dx + \int_\Omega |u_i|^{p-2}u_i\psi_{i,n}\,dx + \gamma \int_\Omega g\psi_{i,n}\,dx,\end{aligned}$$

for $i = 1, 2$. It follows that

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \bigl(u_1(x) - u_1(y)\bigr)\bigl\{\bigl[1 - \eta_n\bigl(w_n(x)\bigr)\bigr]\varphi(x) - \bigl[1 - \eta_n\bigl(w_n(y)\bigr)\bigr]\varphi(y)\bigr\}K(x-y)\,dx\,dy \\ &\quad \geq \lambda \int_\Omega u_1(x)\bigl[1 - \eta_n\bigl(w_n(x)\bigr)\bigr]\varphi(x)\,dx + \int_\Omega |u_1|^{p-2}u_1\bigl[1 - \eta_n\bigl(w_n(x)\bigr)\bigr]\varphi(x)\,dx \\ &\qquad + \gamma \int_\Omega g\bigl[1 - \eta_n\bigl(w_n(x)\bigr)\bigr]\varphi(x)\,dx\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \big(u_2(x) - u_2(y)\big)\big\{\eta_n\big(w_n(x)\big)\varphi(x) - \eta_n\big(w_n(y)\big)\varphi(y)\big\}K(x-y)\,dx\,dy \\
&\geq \lambda \int_\Omega u_2(x)\eta_n\big(w_n(x)\big)\varphi(x)\,dx + \int_\Omega |u_2|^{p-2}u_2\eta_n\big(w_n(x)\big)\varphi(x)\,dx \\
&\quad + \gamma \int_\Omega g\eta_n\big(w_n(x)\big)\varphi(x)\,dx.
\end{aligned} \tag{2.3}$$

For a.e. $x \in \Omega_1 := \{x \in \Omega : u_1(x) > u_2(x)\}$, $w(x) > 0$ and hence $\eta_n(w_n(x)) \to 1$ for a.e. $x \in \Omega_1$. Similarly, $\eta_n(w_n(x)) \to 0$ for a.e. $x \in \Omega_2 := \{x \in \Omega : u_1(x) < u_2(x)\}$. Adding (2.2) and (2.3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \big[u_2(x) - u_2(y) - \big(u_1(x) - u_1(y)\big)\big]\big\{\eta_n\big(w_n(x)\big)\varphi(x) - \eta_n\big(w_n(y)\big)\varphi(y)\big\}K(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^{2N}} \big(u_1(x) - u_1(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \\
&\geq \lambda \int_\Omega u_1(x)\varphi(x)\,dx + \lambda \int_\Omega \big[u_2(x) - u_1(x)\big]\eta_n\big(w_n(x)\big)\varphi(x)\,dx + \int_\Omega |u_1|^{p-2}u_1\varphi\,dx \\
&\quad + \int_\Omega \big[|u_2|^{p-2}u_2 - |u_1|^{p-2}u_1\big]\eta_n\big(w_n(x)\big)\varphi(x)\,dx + \gamma \int_\Omega g\varphi(x)\,dx.
\end{aligned} \tag{2.4}$$

Define

$$\begin{aligned}
A_1 &:= \big\{(x,y) \in \mathbb{R}^{2N} : w(x) > 0, w(y) > 0\big\}, \qquad A_2 := \big\{(x,y) \in \mathbb{R}^{2N} : w(x) > 0, w(y) \leq 0\big\}, \\
A_3 &:= \big\{(x,y) \in \mathbb{R}^{2N} : w(x) \leq 0, w(y) > 0\big\}, \qquad A_4 := \big\{(x,y) \in \mathbb{R}^{2N} : w(x) \leq 0, w(y) \leq 0\big\}.
\end{aligned}$$

By the dominated convergence theorem, we find, as $n \to \infty$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \big[u_2(x) - u_2(y) - \big(u_1(x) - u_1(y)\big)\big]\big\{\eta_n\big(w_n(x)\big)\varphi(x) - \eta_n\big(w_n(y)\big)\varphi(y)\big\}K(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^{2N}} \big(u_1(x) - u_1(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \\
&\to \int_{A_1} \big[u_2(x) - u_2(y) - \big(u_1(x) - u_1(y)\big)\big]\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \\
&\quad + \int_{A_2} \big[u_2(x) - u_2(y) - \big(u_1(x) - u_1(y)\big)\big]\varphi(x)K(x-y)\,dx\,dy \\
&\quad - \int_{A_3} \big[u_2(x) - u_2(y) - \big(u_1(x) - u_1(y)\big)\big]\varphi(y)K(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^{2N}} \big(u_1(x) - u_1(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \\
&= \int_{A_1 \cup A_4} \big[\underline{(u_1 \wedge u_2)(x)} - \underline{(u_1 \wedge u_2)(y)}\big]\big[\varphi(x) - \varphi(y)\big]K(x-y)\,dx\,dy \\
&\quad + \int_{A_2} \big[u_2(x) - u_2(y) - \big(u_1(x) - u_1(y)\big)\big]\varphi(x)K(x-y)\,dx\,dy \\
&\quad - \int_{A_3} \big[u_2(x) - u_2(y) - \big(u_1(x) - u_1(y)\big)\big]\varphi(y)K(x-y)\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{A_2\cup A_3}\big(u_1(x)-u_1(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy\\
&\le \int_{A_1\cup A_4}\big[\underline{(u_1\wedge u_2)(x)}-\underline{(u_1\wedge u_2)(y)}\big]\big[\varphi(x)-\varphi(y)\big]K(x-y)\,dx\,dy\\
&\quad+\int_{A_2}\big[\underline{(u_1\wedge u_2)(x)}-\underline{(u_1\wedge u_2)(y)}\big]\big[\varphi(x)-\varphi(y)\big]K(x-y)\,dx\,dy\\
&\quad+\int_{A_3}\big[\underline{(u_1\wedge u_2)(x)}-\underline{(u_1\wedge u_2)(y)}\big]\big[\varphi(x)-\varphi(y)\big]K(x-y)\,dx\,dy\\
&=\int_{\mathbb{R}^{2N}}\big[\underline{(u_1\wedge u_2)(x)}-\underline{(u_1\wedge u_2)(y)}\big]\big[\varphi(x)-\varphi(y)\big]K(x-y)\,dx\,dy.
\end{aligned}
$$

Similarly, as $n\to\infty$,

$$
\lambda\int_\Omega u_1(x)\varphi(x)\,dx+\lambda\int_\Omega\big[u_2(x)-u_1(x)\big]\eta_n\big(w_n(x)\big)\varphi(x)\,dx\to\lambda\int_\Omega\underline{(u_1\wedge u_2)(x)}\varphi(x)\,dx
$$

and

$$
\begin{aligned}
&\int_\Omega |u_1|^{p-2}u_1\varphi\,dx+\int_\Omega\big[|u_2|^{p-2}u_2-|u_1|^{p-2}u_1\big]\eta_n\big(w_n(x)\big)\varphi(x)\,dx\\
&\quad\to\int_\Omega |u_1\wedge u_2|^{p-2}(u_1\wedge u_2)\varphi\,dx.
\end{aligned}
$$

Thus, by (2.4), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}}\big[(u_1\wedge u_2)(x)-(u_1\wedge u_2)(y)\big]\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy\\
&\quad\ge\lambda\int_\Omega(u_1\wedge u_2)\varphi\,dx+\int_\Omega|u_1\wedge u_2|^{p-2}(u_1\wedge u_2)\varphi\,dx+\gamma\int_\Omega g\varphi\,dx
\end{aligned}
$$

for any $\varphi\in C_0^\infty(\Omega)$ with $\varphi\ge 0$. Since $C_0^\infty(\Omega)$ is dense in X_0, for any $\varphi\in X_0$ with $\varphi\ge 0$, we can find $\varphi_n\in C_0^\infty$ such that $\varphi_n\to\varphi$ in the X_0 norm. This completes the proof. □

Remark 2.3 Lemma 2.2 is valid for the following second order quasilinear elliptic operator in divergence form:

$$
\sum_{i=1}^N D_i\big(A_i\big(x,u(x),Du(x)\big)\big),
$$

where A_i $(i=1,\dots,N)$ satisfies some conditions; see [5] for more details.

Lemma 2.4 *For any $\lambda\in[0,\lambda_1)$ problem $(P)_\gamma$ admits at least one positive solutions which is a minimum of all solutions if γ is small enough.*

Proof Set

$$
\varepsilon=\frac12\frac{(\lambda_1-\lambda)^{1/(p-2)}}{\max_{x\in\Omega}\phi_1(x)}
$$

and

$$\rho = \frac{\inf_{x\in \operatorname{supp} g}\{[\lambda_1-\lambda]\varepsilon\phi_1-(\varepsilon\phi_1)^{p-1}\}}{\sup_{x\in\Omega} g(x)},$$

where supp g denotes the closure of $\{x\in\Omega \,|\, g(x)\neq 0\}$. It is easy to verify that $\overline{u}=\varepsilon\phi_1$ is a supersolution of $(P)_\gamma$ if $\gamma\le\rho$ and $\underline{u}=0$ is a subsolution of $(P)_\gamma$ for all $\gamma\ge 0$.

Now let $u_0=\underline{u}$, and then given u_n inductively define u_{n+1} to be the unique weak solution of linear boundary value problem

$$\begin{cases} (-\Delta)^s u_{n+1}=\lambda u_n+|u_n|^{p-2}u_n+\gamma g & \text{in } \Omega, \\ u_{n+1}=0 & \text{in } \mathbb{R}^N\setminus\Omega. \end{cases} \tag{2.5}$$

Similarly let $w_0=\overline{u}$, and then given w_n inductively define w_{n+1} to be the unique weak solution of linear boundary value problem

$$\begin{cases} (-\Delta)^s w_{n+1}=\lambda w_n+|w_n|^{p-2}w_n+\gamma g & \text{in } \Omega, \\ w_{n+1}=0 & \text{in } \mathbb{R}^N\setminus\Omega. \end{cases} \tag{2.6}$$

Claim 1. $\underline{u}=u_0\le u_1\le w_1\le w_0=\overline{u}$.

From (2.5) we deduce

$$\int_{\mathbb{R}^{2N}}\big(u_1(x)-u_1(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy=\gamma\int_\Omega g\varphi\,dx,\quad \forall\varphi\in X_0. \tag{2.7}$$

Similarly from (2.6) we have

$$\begin{aligned} &\int_{\mathbb{R}^{2N}}\big(w_1(x)-w_1(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \\ &\quad=\lambda\int_\Omega \overline{u}\varphi\,dx+\int_\Omega \overline{u}^{p-1}\varphi\,dx+\gamma\int_\Omega g\varphi\,dx,\quad \forall\varphi\in X_0. \end{aligned} \tag{2.8}$$

Subtract (2.8) from (2.7) and set $\varphi=(u_1-w_1)^+$. We obtain

$$\int_{\mathbb{R}^{2N}}\big[\psi_1(x)-\psi_1(y)\big]\big[\psi_1^+(x)-\psi_1^+(y)\big]K(x-y)\,dx\,dy\le 0, \tag{2.9}$$

where $\psi_1(x)=u_1(x)-w_1(x)$, for all $x\in\mathbb{R}^N$. It is easy to see that

$$\big[\psi_1(x)-\psi_1(y)\big]\big[\psi_1^+(x)-\psi_1^+(y)\big]\ge\big|\psi^+(x)-\psi^+(y)\big|^2,\quad \forall x,y\in\mathbb{R}^N.$$

So, by (2.9),

$$\psi_1^+(x)-\psi_1^+(y)=0,\quad \forall x,y\in\mathbb{R}^N.$$

Then, $\psi_1^+(x)=0$ for all $x\in\mathbb{R}^N$ since $\psi(x)=0$ for any $x\in\mathbb{R}^N\setminus\Omega$. So $\psi_1\le 0$ and $u_1\le w_1$ a.e. in Ω.

Similarly, by the definition of supersolution and subsolution, (2.5) and (2.6) we can prove $u_0\le u_1$ and $w_1\le w_0$.

Claim 2. $u_n \le u_{n+1} \le w_{n+1} \le w_n$ a.e. in Ω, $\forall n = 0, 1, 2, \ldots$.

Claim 2 obviously holds for $n = 0$. Assume for induction that

$$u_{n-1} \le u_n \le w_n \le w_{n-1} \quad \text{a.e. in } \Omega.$$

From (2.5) and (2.6) we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \bigl(u_{n+1}(x) - u_{n+1}(y)\bigr)\bigl(\varphi(x) - \varphi(y)\bigr)K(x-y)\,dx\,dy \\
&\quad = \lambda \int_\Omega u_n \varphi\,dx + \int_\Omega u_n^{p-1}\varphi\,dx + \gamma \int_\Omega g\varphi\,dx, \qquad (2.10)\\
&\int_{\mathbb{R}^{2N}} \bigl(w_{n+1}(x) - w_{n+1}(y)\bigr)\bigl(\varphi(x) - \varphi(y)\bigr)K(x-y)\,dx\,\underline{dy} \\
&\quad = \lambda \int_\Omega w_n \varphi\,dx + \int_\Omega w_n^{p-1}\varphi\,dx + \gamma \int_\Omega g\varphi\,dx \qquad (2.11)
\end{aligned}$$

for all $\varphi \in X_0$. Subtract (2.11) from (2.10) and set $\varphi = (u_{n+1} - w_{n+1})^+$. We obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \bigl[\psi_{n+1}(x) - \psi_{n+1}(y)\bigr]\bigl[\psi_{n+1}^+(x) - \psi_{n+1}^+(y)\bigr]K(x-y)\,dx\,dy \\
&\quad = \lambda \int_\Omega (u_n - w_n)\psi\,dx + \int_\Omega \bigl(u_n^{p-1} - w_n^{p-1}\bigr)\varphi\,dx \le 0,
\end{aligned}$$

where $\psi_{n+1}(x) = u_{n+1}(x) - w_{n+1}(x)$, for all $x \in \mathbb{R}^N$. Thus $u_{n+1} \le w_{n+1}$ a.e. in Ω. Similarly we can get $u_n \le u_{n+1}$ and $w_{n+1} \le w_n$.

By Claims 1 and 2 we have

$$\underline{u} = u_0 \le u_1 \le \cdots \le u_n \le u_{n+1} \le \cdots \le w_{n+1} \le w_n \cdots \le w_1 \le w_0 = \overline{u}.$$

Set

$$u(x) = \lim_{n\to\infty} u_n(x), \qquad w(x) = \lim_{n\to\infty} w_n(x).$$

Clearly, $u(x) \le w(x)$ a.e. in Ω. Taking $\varphi = u_{n+1}$ in (2.10) we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \bigl(u_{n+1}(x) - u_{n+1}(y)\bigr)^2 K(x-y)\,dx\,dy \\
&\quad = \lambda \int_\Omega u_n u_{n+1}\,dx + \int_\Omega u_n^{p-1} u_{n+1}\,dx + \gamma \int_\Omega g u_{n+1}\,dx \\
&\quad \le \lambda \int_\Omega \overline{u}^2\,dx + \int_\Omega \overline{u}^p\,dx + \gamma \biggl(\int_\Omega g^2\,dx\biggr)^{1/2} \biggl(\int_\Omega \overline{u}^2\,dx\biggr)^{1/2}.
\end{aligned}$$

This shows $\{\|u_n\|_{X_0}\}$ is bounded. So, going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in X_0. The Lebesgue's dominated convergence theorem yields

$$\int_\Omega u_n^{p-1}\varphi\,dx \to \int_\Omega u^{p-1}\varphi\,dx, \quad \forall \varphi \in X_0$$

as $n \to \infty$.

Letting $n \to \infty$ in (2.10) we have

$$\int_{\mathbb{R}^{2N}} \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x - y)\,dx\,dy$$
$$= \lambda \int_\Omega u\varphi\,dx + \int_\Omega u^{p-1}\varphi\,dx + \gamma \int_\Omega g\varphi\,dx, \quad \forall \varphi \in X_0.$$

Similarly we can verify that w is a weak solution of $(P)_\gamma$. However, we cannot rule out the possibility that u and w are the same solution. Note that, since $u \le \varepsilon\phi_1$ and $\phi_1 \in L^\infty(\Omega)$, we get $u \in L^\infty(\Omega)$. It is easy to see that $u(x) > 0$ in Ω.

Next we show that u is a minimal solution. Assume that U is any weak solution of $(P)_\gamma$. By Lemma 2.2, $U \wedge \bar{u} := \min\{U, \bar{u}\}$ is a supersolution of $(P)_\gamma$. Using the same method of monotonic iteration we get a positive solution v of $(P)_\gamma$ such that $v \le U \wedge \bar{u} \le \bar{u}$. Using the same argument as proof of Claim 2 above we obtain

$$u_n \le v \le w_n \quad \text{for all } n.$$

Passing to the limit we have

$$u \le v \le w.$$

Consequently, $u \le v \le U$. This shows that u is a minimal solution. □

Lemma 2.5 *For $\lambda \in [0, \lambda_1)$ there exists a positive constant γ^* such that $(P)_\gamma$ has a positive minimal solution for all $\gamma \in (0, \gamma^*)$, and $(P)_\gamma$ has no positive solutions if $\gamma > \gamma^*$.*

Proof Set

$$\gamma^* = \sup\big\{\overline{\gamma} > 0 | (P)_\gamma \text{ has at least one positive solution for all } \gamma \in (0, \overline{\gamma})\big\}.$$

Lemma 2.1 and Lemma 2.4 imply that γ^* is well defined.

For any fixed $\gamma_0 \in (0, \gamma^*)$, we take $\delta > 0$ such that $\gamma_0 + \delta < \gamma^*$. Let $u_{\gamma_0+\delta}$ be a positive solution of $(P)_{\gamma_0+\delta}$. It is easy to verify that 0 is a subsolution and $u_{\gamma_0+\delta}$ is a supersolution of $(P)_{\gamma_0}$. Using the same method of monotonic iteration as that in proof of Lemma 2.4 we find a minimal solution u_{γ_0} of $(P)_{\gamma_0}$.

By similar arguments we can show there is no positive solution of $(P)_\gamma$ for any $\gamma > \gamma^*$. □

Lemma 2.6 *Assume that $\lambda \in [0, \lambda_1)$, $\gamma \in (0, \gamma^*)$, where γ^* is the one in Lemma 2.5. Let u_γ be the positive minimal solution of $(P)_\gamma$. Then*

$$\tau = \inf\bigg\{\int_{\mathbb{R}^{2N}} \big(\psi(x) - \psi(y)\big)^2 K(x - y)\,dx\,dy - \lambda \int_\Omega \psi^2\,dx \,\bigg|\, (p - 1) \int_\Omega u_\gamma^{p-2}\psi^2\,dx = 1, \quad \psi \in X_0\bigg\} \tag{2.12}$$

can be attained and $\tau > 1$.

Proof Clearly, $0 \leq \tau < +\infty$. Let $\{\psi_n\} \subset X_0$ be a minimizing sequence of (2.12). Then

$$[\lambda_1 - \lambda] \int_\Omega \psi_n^2 \, dx \leq \int_{\mathbb{R}^{2N}} \big(\psi_n(x) - \psi_n(y)\big)^2 K(x-y) \, dx \, dy - \lambda \int_\Omega \psi_n^2 \, dx = \tau + o(1).$$

So $|\psi_n|_2$ is bounded. Since

$$\int_{\mathbb{R}^{2N}} \big(\psi_n(x) - \psi_n(y)\big) K(x-y) \, dx \, dy = \lambda \int_\Omega \psi_n^2 \, dx + \tau + o(1),$$

we see that $\|\psi_n\|_{X_0}$ is bounded. Consequently, we may assume that there is a subsequence, still denoted by ψ_n, such that

$$\begin{aligned} &\psi_n \rightharpoonup \psi_0 \quad \text{in } X_0, \\ &\psi_n \to \psi_0 \quad \text{in } L^2(\mathbb{R}^N), \\ &\psi_n \to \psi_0 \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Hence, as $n \to \infty$,

$$\begin{aligned} \tau &= \liminf_{n\to\infty} \biggl[\int_{\mathbb{R}^{2N}} \big(\psi_n(x) - \psi_n(y)\big)^2 K(x-y) \, dx \, dy - \lambda \int_\Omega \psi_n^2 \, dx\biggr] \\ &\geq \int_{\mathbb{R}^{2N}} \big(\psi_0(x) - \psi_0(y)\big)^2 K(x-y) \, dx \, dy - \lambda \int_\Omega \psi_0 \, dx. \end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$(p-1) \int_\Omega u_\gamma^{p-2} \psi_0^2 \, dx = \lim_{n\to\infty} (p-1) \int_\Omega u_\gamma^{p-2} \psi_n^2 \, dx = 1.$$

Hence ψ_0 reaches τ. Since

$$\begin{aligned} \tau &\leq \int_{\mathbb{R}^{2N}} \big(|\psi_0(x)| - |\psi_0(y)|\big)^2 K(x-y) \, dx \, dy - \lambda \int_\Omega |\psi_0|^2 \, dx \\ &\leq \int_{\mathbb{R}^{2N}} \big(\psi_0(x) - \psi_0(y)\big)^2 K(x-y) \, dx \, dy - \lambda \int_\Omega \psi_0^2 \, dx = \tau, \end{aligned}$$

$|\psi_0|$ also achieves τ. So we can assume $\psi_0 \geq 0$ in Ω. It follows from the Lagrange multiplier rule that

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \big(\psi_0(x) - \psi_0(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y) \, dx \, dy \\ &\quad = \lambda \int_\Omega \psi_0 \varphi \, dx + \tau(p-1) \int_\Omega u_\gamma^{p-2} \psi_0 \varphi \, dx, \quad \forall \varphi \in X_0. \end{aligned}$$

We take $\delta > 0$ such that $\gamma + \delta < \gamma^*$. Set $\overline{u} = u_{\gamma+\delta}$, where $u_{\gamma+\delta}$ is a positive solution of $(P)_{\gamma+\delta}$. Then $\overline{u}$ is a supersolution of $(P)_\gamma$. Taking $\varphi = \overline{u} - u_\gamma$ in the equation above we get

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} (\psi_0(x) - \psi_0(y)\big[(\overline{u} - u_\gamma)(x) - (\overline{u} - u_\gamma)(y)\big] \, dx \\ &\quad = \lambda \int_\Omega \psi_0(\overline{u} - u_\gamma) \, dx + \tau(p-1) \int_\Omega u_\gamma^{p-2} \psi_0(\overline{u} - u_\gamma) \, dx. \end{aligned} \tag{2.13}$$

On the other hand, by the definition of $\overline{u}$ and u_γ, we have

$$\begin{aligned}&\lambda\int_\Omega(\overline{u}-u_\gamma)\psi_0\,dx+\int_\Omega\bigl[\overline{u}^{p-1}-u_\gamma^{p-1}\bigr]\psi_0\,dx\\&\quad\le\int_{\mathbb{R}^{2N}}(\psi_0(x)-\psi_0(y)\bigl[(\overline{u}-u_\gamma)(x)-(\overline{u}-u_\gamma)(y)\bigr]\,dx.\end{aligned}\tag{2.14}$$

By (2.13) and (2.14) we have

$$\begin{aligned}\tau(p-1)\int_\Omega u_\gamma^{p-2}(\overline{u}-u_\gamma)\psi_0\,dx&\ge\int_\Omega\bigl[\overline{u}^{p-1}-u_\gamma^{p-1}\bigr]\psi_0\,dx\\&>(p-1)\int_\Omega u_\gamma^{p-2}(\overline{u}-u_\gamma)\psi_0\,dx.\end{aligned}$$

Hence $\tau>1$. □

Lemma 2.7 *There results*

$$\sup_{u_\gamma\in\mathcal{S}}\|u_\gamma\|_{X_0}<\infty,$$

where

$$\mathcal{S}=\bigl\{u_\gamma|\gamma\in\bigl(0,\gamma^*\bigr),u_\gamma\text{ is the minimal solution of }(P)_\gamma\bigr\}.$$

Proof For any $u_\gamma\in\mathcal{S}$, from Lemma 2.6 we get

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}\bigl(u_\gamma(x)-u_\gamma(y)\bigr)^2K(x-y)\,dx\,dy-\lambda\int_\Omega u_\gamma^2\,dx-(p-1)\int_\Omega u_\gamma^p\,dx\\&\quad\ge(\tau-1)(p-1)\int_\Omega u_\gamma^p\,dx\ge0.\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^{2N}}\bigl(u_\gamma(x)-u_\gamma(y)\bigr)^2K(x-y)\,dx\,dy\ge\lambda\int_\Omega u_\gamma^2\,dx+(p-1)\int_\Omega u_\gamma^p\,dx.\tag{2.15}$$

Clearly,

$$\int_{\mathbb{R}^{2N}}\bigl(u_\gamma(x)-u_\gamma(y)\bigr)^2K(x-y)\,dx\,dy=\lambda\int_\Omega u_\gamma^2\,dx+\int_\Omega u_\gamma^p\,dx+\gamma\int_\Omega gu_\gamma\,dx.\tag{2.16}$$

By (2.15) and (2.16), we have

$$\begin{aligned}&(p-2)\biggl[\int_{\mathbb{R}^{2N}}\bigl(u_\gamma(x)-u_\gamma(y)\bigr)^2K(x-y)\,dx\,dy-\lambda\int_\Omega u_\gamma^2\,dx\biggr]\\&\quad\le(p-1)\gamma\int_\Omega gu_\gamma\,dx.\end{aligned}\tag{2.17}$$

Since

$$\int_{\mathbb{R}^{2N}}\bigl(u_\gamma(x)-u_\gamma(y)\bigr)^2K(x-y)\,dx\,dy-\lambda\int_\Omega u_\gamma^2\,dx\ge[\lambda_1-\lambda]\int_\Omega u_\gamma^2\,dx,$$

we deduce

$$\begin{aligned}\int_\Omega u_\gamma^2\,dx &\le \frac{(p-1)\gamma}{(p-2)[\lambda_1-\lambda]}\int_\Omega g u_\gamma\,dx\\ &\le \frac{(p-1)\gamma}{(p-2)[\lambda_1-\lambda]}\left[\frac{1}{2\delta}\int_\Omega g^2\,dx+\frac{\delta}{2}\int_\Omega u_\gamma^2\,dx\right],\end{aligned}$$

for $\delta > 0$ small enough such that

$$\delta < \frac{2(p-2)[\lambda_1-\lambda]}{(p-1)\gamma}.$$

So there exists a positive constant C_2 such that

$$\int_\Omega u_\gamma^2\,dx \le C_2, \tag{2.18}$$

where C_2 depends only on λ_1, λ, p, γ, and g.

By (2.17) and (2.18) we have

$$\begin{aligned}\underline{\int_{\mathbb{R}^{2N}}\big(u_\gamma(x)-u_\gamma(y)\big)^2K(x-y)\,dx\,dy} &\le \lambda\int_\Omega u_\gamma^2+\frac{(p-1)\gamma}{p-2}\int_\Omega g u_\gamma\,dx\\ &\le \left[\lambda+\frac{(p-1)\gamma}{2(p-2)}\right]\int_\Omega u_\gamma^2\,dx+\frac{(p-1)\gamma}{2(p-2)}\int_\Omega g^2\,dx\\ &\le \left[\lambda+\frac{(p-1)\gamma^*}{2(p-2)}\right]\int_\Omega u_\gamma^2\,dx+\frac{(p-1)\gamma^*}{2(p-2)}\int_\Omega g^2\,dx.\end{aligned}$$

So there exists a positive constant C independent of γ such that

$$\|u_\gamma\|_{X_0} \le C. \tag{2.19}$$

□

Now we prove Theorem 1.1.

Proof of Theorem 1.1 Assume that $\gamma_j \nearrow \gamma^*$ and $u_{\gamma_j} \in \mathcal{S}$. By Lemma 2.7 there is a subsequence, still denoted by $\{u_{\gamma_j}\}$, such that

$$\begin{aligned}&u_{\gamma_j} \rightharpoonup u^* \quad \text{in } X_0,\\ &u_{\gamma_j} \to u^* \quad \text{in } L^2(\mathbb{R}^N),\\ &u_{\gamma_j} \to u^* \quad \text{a.e. in } \mathbb{R}^N.\end{aligned}$$

It is easy to verify that u^* is a solution of $(P)_{\gamma^*}$. Note that 0 is a subsolution of $(P)_\gamma$ for any $\gamma \ge 0$. So we can use the method of monotone iteration to find a minimal solution. □

3 Existence of the second positive solution

We introduce the following problem:

$$\begin{cases} (-\Delta)^s v = v^{p-1} + a(x)v + h(x,v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

where $a(x) = \lambda + (p-1)u_\gamma^{p-2}(x)$, and

$$h(x,v) = \big(v + u_\gamma(x)\big)^{p-1} - u_\gamma^{p-1}(x) - v^{p-1} - (p-1)u_\gamma^{p-2}v.$$

In order to obtain a second solution of $(P)_\gamma$ it suffices to prove (3.1) has a nontrivial solution. Thus $u_\gamma + v$ is a second solution of $(P)_\gamma$.

For problem (3.1), we define the energy functional $J : X_0 \to \mathbb{R}$ as follows:

$$\begin{aligned} J(v) = {} & \frac{1}{2}\int_{\mathbb{R}^{2N}} \big(v(x) - v(y)\big)^2 K(x-y)\,dx\,dy - \frac{1}{2}\int_\Omega a(x)\big(v^+\big)^2\,dx \\ & - \frac{1}{p}\int_\Omega \big(v^+\big)^p\,dx - \int_\Omega H\big(x,v^+\big)\,dx, \end{aligned}$$

where $H(x,v) = \int_0^v h(x,t)\,dt$, $v^+ = \max\{v,0\}$ denotes the positive part of v. By the maximum principle [2, 16], we know that the nontrivial critical points of energy functional J are the positive solutions of (3.1).

It is easy to see that h satisfies

(i) $\sup\{|h(x,t)| : \text{a.e. } x \in \Omega, t \le M\} < +\infty$ for any $M > 0$;

(ii) $\lim_{t\to 0^+} \frac{h(x,t)}{t} = 0$ uniformly in $x \in \Omega$;

(iii) $\lim_{t\to +\infty} \frac{h(x,t)}{t^{p-1}} = 0$ uniformly in $x \in \Omega$.

The following theorem is a modification of Theorem 3 in [15].

Theorem 3.1 *Let $\lambda \in [0,\lambda_1)$, $\gamma \in (0,\gamma^*)$, if there exists some $v_0 \in X_0 \setminus \{0\}$ with $v_0 \ge 0$ a.e. in $\mathbb{R}^N$, such that*

$$\sup_{t\ge 0} J(tv_0) < \frac{s}{N} S_s^{\frac{N}{2s}}, \tag{3.2}$$

then problem (3.1) admits a solution.

Since the proof of Theorem 3.1 is nearly same as that of Theorem 3 in [15] (cf. Theorem 2.1 in [1]), we omit it.

In the following, we shall verify the crucial condition (3.2) holds for $\lambda \in [0,\lambda_1)$, $\gamma \in (0,\gamma^*)$. To this end, we need some preliminary results.

Consider the following minimization problem:

$$S_s := \inf_{v \in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x-y)\,dx\,dy}{(\int_{\mathbb{R}^N} |v|^p\,dx)^{2/p}}.$$

It is well known from [15] that the infimum in the formula above is attained at $\tilde{u}$, where

$$\tilde{u}(x) = \frac{\kappa}{(\mu^2 + |x - x_0|^2)^{\frac{N-2s}{2}}}, \quad x \in \mathbb{R}^N, \tag{3.3}$$

with $\kappa > 0$, $\mu > 0$ and $x_0 \in \mathbb{R}^N$ fixed constants. Equivalently, the function $\bar{u}$ defined as

$$\bar{u} = \frac{\tilde{u}}{\|\tilde{u}\|_{L^p(\mathbb{R}^N)}}$$

is such that

$$S_s = \int_{\mathbb{R}^{2N}} \left|\bar{u}(x) - \bar{u}(y)\right|^2 K(x-y)\,dx\,dy.$$

The function

$$u^*(x) = \bar{u}\left(\frac{x}{S_s^{1/(2s)}}\right), \quad x \in \mathbb{R}^N,$$

is a solution of

$$(-\Delta)^s u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{3.4}$$

Now, we consider the family of the function U_ε defined as

$$U_\varepsilon(x) = \varepsilon^{-(N-2s)/2} u^*(x/\varepsilon), \quad x \in \mathbb{R}^N,$$

for any $\varepsilon > 0$. The function U_ε is a solution of problem (3.4) and satisfies

$$\int_{\mathbb{R}^{2N}} \left|U_\varepsilon(x) - U_\varepsilon(y)\right|^2 K(x-y)\,dx\,dy = \int_{\mathbb{R}^N} \left|U_\varepsilon(x)\right|^p dx = S_s^{N/(2s)}. \tag{3.5}$$

Without loss of generality we may suppose $0 \in \Omega$. Let us fix $\rho > 0$ such that $B_{4\rho} \subset \Omega$ and let $\eta \in C^\infty$ be such that $0 \leq \eta \leq 1$ in $\mathbb{R}^N$, $\eta(x) = 1$ if $|x| < \rho$; $\eta(x) = 0$ if $|x| \geq 2\rho$. For every $\varepsilon > 0$ we denote by u_ε the following function:

$$u_\varepsilon(x) = \eta(x) U_\varepsilon(x), \quad x \in \mathbb{R}^N. \tag{3.6}$$

In what follows we suppose that up to a translation $x_0 = 0$ in (3.3). From [15] we have the following estimates:

$$\int_{\mathbb{R}^{2N}} \left|u_\varepsilon(x) - u_\varepsilon(y)\right|^2 K(x-y)\,dx\,dy \leq S_s^{N/(2s)} + O\left(\varepsilon^{N-2s}\right), \tag{3.7}$$

$$\int_{\mathbb{R}^N} |u_\varepsilon|^p\,dx = S_s^{N/(2s)} + O\left(\varepsilon^N\right), \tag{3.8}$$

$$\int_{\mathbb{R}^N} u_\varepsilon^2\,dx \geq \begin{cases} C_s\varepsilon^{2s} + O(\varepsilon^{N-2s}), & N > 4s, \\ C_s\varepsilon^{2s}|\ln\varepsilon| + O(\varepsilon^{2s}), & N = 4s, \\ C_s\varepsilon^{N-2s} + O(\varepsilon^{2s}), & N < 4s, \end{cases} \tag{3.9}$$

where C_s is a positive constant depending on s.

Lemma 3.2 *Assume that* $\lambda \in [0, \lambda_1)$, $\gamma \in (0, \gamma^*)$, *where* γ^* *is the one in Lemma 2.5. Let* u_γ *be the positive minimal solution of* $(P)_\gamma$. *Then*

$$\hat{\tau} = \inf\left\{ \int_{\mathbb{R}^{2N}} \big(\psi(x) - \psi(y)\big)^2 K(x-y)\,dx\,dy - \int_\Omega a(x)\psi^2\,dx \,\Big|\, \int_\Omega \psi^2\,dx = 1, \psi \in X_0(\Omega) \right\} \tag{3.10}$$

can be attained and $\hat{\tau} > 0$.

Proof By Lemma 2.6, we have

$$\int_{\mathbb{R}^{2N}} \big(\psi(x) - \psi(y)\big)^2 K(x-y)\,dx\,dy - \lambda \int_\Omega \psi^2\,dx \geq \tau(p-1) \int_\Omega u_\gamma^{p-2}\psi^2\,dx, \quad \text{for any } \psi \in X_0,$$

where $\tau > 1$. So,

$$\int_{\mathbb{R}^{2N}} \big(\psi(x) - \psi(y)\big)^2 K(x-y)\,dx\,dy - \int_\Omega a(x)\psi^2\,dx \geq (\tau - 1)(p-1) \int_\Omega u_\gamma^{p-2}\psi^2\,dx, \quad \text{for any } \psi \in X_0.$$

Thus, $0 \leq \hat{\tau} < +\infty$. Let $\{\psi_n\} \subset X_0$ be a minimizing sequence of (3.10). Then

$$\int_{\mathbb{R}^{2N}} \big(\psi(x) - \psi(y)\big)^2 K(x-y)\,dx\,dy = \int_\Omega a(x)\psi_n^2\,dx + \hat{\tau} + o(1),$$

and $\int_\Omega \psi_n^2\,dx = 1$. Since $a \in L^\infty(\Omega)$, we have $\|\psi_n\|_{X_0}$ is bounded. Consequently, we may assume that there is a subsequence, still denoted by ψ_n, such that

$$\begin{aligned} &\psi_n \rightharpoonup \psi_0 \quad \text{in } X_0, \\ &\psi_n \to \psi_0 \quad \text{in } L^2(\mathbb{R}^N), \\ &\psi_n \to \psi_0 \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Hence,

$$\begin{aligned} \hat{\tau} &= \lim_{n\to\infty} \left(\int_{\mathbb{R}^{2N}} \big(\psi_n(x) - \psi_n(y)\big)^2 K(x-y)\,dx\,dy - \int_\Omega a(x)\psi_n^2\,dx \right) \\ &\geq \lim_{n\to\infty} (\tau - 1)(p-1) \int_\Omega u_\gamma^{p-2}\psi_n^2\,dx \\ &= (\tau - 1)(p-1) \int_\Omega u_\gamma^{p-2}\psi_0^2\,dx > 0. \end{aligned}$$

□

Lemma 3.3 *Let* u_ε *be given by* (3.6). *Then there exists a constant* $t_\varepsilon > 0$ *such that*

$$\sup_{t\geq 0} J(tu_\varepsilon) = J(t_\varepsilon u_\varepsilon) \tag{3.11}$$

and

$$J(t_\varepsilon u_\varepsilon) \leq \frac{s}{N} S_s^{N/(2s)} - \frac{1}{2}\lambda t_\varepsilon^2 \int_\Omega u_\varepsilon^2 \, dx - \int_\Omega Q(x, t_\varepsilon u_\varepsilon) \, dx + O(\varepsilon^{N-2s}), \tag{3.12}$$

where $Q(x,v) = \int_0^v q(x,t)\,dt$ *and* $q(x,t) = (t + u_\gamma(x))^{p-1} - u_\gamma^{p-1} - t^{p-1}$ *for* $t \geq 0$.

Proof Let

$$\begin{aligned}\psi(t) &= J(tu_\varepsilon)\\ &= \frac{1}{2}t^2 \int_{\mathbb{R}^{2N}} \big(u_\varepsilon(x) - u_\varepsilon(y)\big)^2 K(x-y)\,dx\,dy - \frac{1}{2}t^2 \int_\Omega a(x)u_\varepsilon^2\,dx\\ &\quad - \frac{1}{p}t^p \int_\Omega u_\varepsilon^p\,dx - \int_\Omega H(x, tu_\varepsilon)\,dx,\end{aligned}$$

for $t \geq 0$. Let

$$\sigma(t) = \int_\Omega H(x, tu_\varepsilon).$$

Since for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\big|H(x,t)\big| \leq \delta t^2 + C_\delta |t|^p,$$

for all $t \geq 0$ and for a.e. $x \in \Omega$, we have

$$\big|\sigma(t)\big| \leq \delta t^2 \int_\Omega u_\varepsilon^2\,dx + C_\delta t^p \int_\Omega u_\varepsilon^p\,dx. \tag{3.13}$$

By Lemma 3.2, there exists $\hat{\tau} > 0$ such that

$$\int_{\mathbb{R}^{2N}} \big(u_\varepsilon(x) - u_\varepsilon(y)\big)^2 K(x-y)\,dx\,dy - \int_\Omega a(x)u_\varepsilon^2\,dx \geq \hat{\tau} \int_\Omega u_\varepsilon^2\,dx. \tag{3.14}$$

By (3.13) and (3.14), there exists a constant $\alpha > 0$ depending on ε such that

$$\psi(t) = \alpha t^2 + o(t^2) \tag{3.15}$$

for $\varepsilon < \frac{1}{2}\hat{\tau}$ as $t \to 0^+$.

Next we study ψ for t large. Note that

$$\psi(t) \leq \frac{1}{2}t^2 \int_{\mathbb{R}^{2N}} \big(u_\varepsilon(x) - u_\varepsilon(y)\big)^2 K(x-y)\,dx\,dy - \frac{1}{p}t^p \int_\Omega u_\varepsilon^p\,dx$$

and thus $\psi(t) \to -\infty$ as $t \to +\infty$. Therefore, we see that there exists $t_\varepsilon > 0$ such that

$$\sup_{t\geq 0} J(tu_\varepsilon) = J(t_\varepsilon u_\varepsilon).$$

We show $\lim_{\varepsilon\to 0} t_\varepsilon = 1$. Note that for every $\tilde{\delta} > 0$ there exists $C_{\tilde{\delta}}$ such that

$$\big|h(x,t)\big| \leq \tilde{\delta}|t|^{p-1} + C_{\tilde{\delta}}|t|, \qquad \big|H(x,t)\big| \leq \frac{1}{p}\tilde{\delta}|t|^p + \frac{1}{2}C_{\tilde{\delta}}|t|^2 \tag{3.16}$$

for all $t \geq 0$ and for a.e. $x \in \Omega$.

Clearly,

$$
\begin{aligned}
0 &= \frac{d\psi}{dt}\bigg|_{t=t_\varepsilon} \\
&= t_\varepsilon \int_{\mathbb{R}^{2N}} \big(u_\varepsilon(x) - u_\varepsilon(y)\big)^2 K(x-y)\,dx\,dy - t_\varepsilon \int_\Omega a(x)u_\varepsilon^2\,dx \\
&\quad - t_\varepsilon^{p-1} \int_\Omega u_\varepsilon^p\,dx - \int_\Omega h(x, tu_\varepsilon)u_\varepsilon\,dx.
\end{aligned} \tag{3.17}
$$

Then, by (3.7) and (3.8), we have

$$
\begin{aligned}
t_\varepsilon^{p-2} &= \frac{\int_{\mathbb{R}^{2N}} (u_\varepsilon(x) - u_\varepsilon(y))^2 K(x-y)\,dx\,dy - \lambda \int_\Omega u_\varepsilon^2\,dx - \frac{1}{t_\varepsilon}\int_\Omega q(x, t_\varepsilon u_\varepsilon)u_\varepsilon\,dx}{\int_\Omega u_\varepsilon^p\,dx} \\
&\leq \frac{\int_{\mathbb{R}^{2N}} (u_\varepsilon(x) - u_\varepsilon(y))^2 K(x-y)\,dx\,dy}{\int_\Omega u_\varepsilon^p\,dx} \\
&= \frac{S_s^{N/(2s)} + O(\varepsilon^{N-2s})}{S_s^{N/(2s)} + O(\varepsilon^N)} = 1 + O\big(\varepsilon^{N-2s}\big) \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{3.18}
$$

On the other hand, by (3.17) and (3.16), we have

$$
\begin{aligned}
t_\varepsilon^{p-2} &= \frac{\int_{\mathbb{R}^{2N}} (u_\varepsilon(x) - u_\varepsilon(y))^2 K(x-y)\,dx\,dy - \int_\Omega a(x)u_\varepsilon^2\,dx - \frac{1}{t_\varepsilon}\int_\Omega h(x, t_\varepsilon u_\varepsilon)u_\varepsilon\,dx}{\int_\Omega u_\varepsilon^p\,dx} \\
&\geq \frac{\int_{\mathbb{R}^{2N}} (u_\varepsilon(x) - u_\varepsilon(y))^2 K(x-y)\,dx\,dy - \int_\Omega a(x)u_\varepsilon^2\,dx - \int_\Omega (\tilde{\delta} t_\varepsilon^{p-2} u_\varepsilon^p + C_{\tilde{\delta}} u_\varepsilon^2)\,dx}{\int_\Omega u_\varepsilon^p\,dx} \\
&= \frac{\int_{\mathbb{R}^{2N}} (u_\varepsilon(x) - u_\varepsilon(y))^2 K(x-y)\,dx\,dy - \int_\Omega a(x)u_\varepsilon^2\,dx\,dx}{\int_\Omega u_\varepsilon^p\,dx} - \tilde{\delta} t_\varepsilon^{p-2} - C_{\tilde{\delta}} \frac{\int_\Omega u_\varepsilon^2\,dx}{\int_\Omega u_\varepsilon^p\,dx},
\end{aligned}
$$

consequently, by (3.7)–(3.9), we get

$$
\begin{aligned}
t_\varepsilon^{p-2} &\geq \frac{1}{1+\tilde{\delta}} \left(\frac{\int_{\mathbb{R}^{2N}} (u_\varepsilon(x) - u_\varepsilon(y))^2 K(x-y)\,dx\,dy - \int_\Omega a(x)u_\varepsilon^2\,dx\,dx}{\int_\Omega u_\varepsilon^p\,dx} - C_{\tilde{\delta}} \frac{\int_\Omega u_\varepsilon^2\,dx}{\int_\Omega u_\varepsilon^p\,dx} \right) \\
&\to \frac{1}{1+\tilde{\delta}},
\end{aligned} \tag{3.19}
$$

as $\varepsilon \to 0$. Combining (3.18) and (3.19), we have $\lim_{\varepsilon\to 0} t_\varepsilon = 1$.

Let

$$
d_\varepsilon = \frac{\int_{\mathbb{R}^{2N}} (u_\varepsilon(x) - u_\varepsilon(y))^2 K(x-y)\,dx\,dy}{\int_\Omega u_\varepsilon^p\,dx}.
$$

Since the function $t \mapsto \frac{1}{2}t^2 \int_{\mathbb{R}^{2N}} (u_\varepsilon(x) - u_\varepsilon(y))^2 K(x-y)\,dx\,dy - \frac{1}{p}t^p \int_\Omega u_\varepsilon^p\,dx$ is increasing on the interval $[0, d_\varepsilon]$, we have, by (3.18),

$$
\begin{aligned}
J(t_\varepsilon u_\varepsilon) &\leq \frac{1}{2} d_\varepsilon^2 \int_{\mathbb{R}^{2N}} \big(u_\varepsilon(x) - u_\varepsilon(y)\big)^2 K(x-y)\,dx\,dy - \frac{1}{p} d_\varepsilon^p \int_\Omega u_\varepsilon^p\,dx \\
&\quad - \frac{1}{2} t_\varepsilon^2 \int_\Omega a(x)u_\varepsilon^2\,dx - \int_\Omega H(x, t_\varepsilon u_\varepsilon)\,dx \\
&= \frac{s}{N} S_s^{N/(2s)} - \frac{1}{2}\lambda t_\varepsilon^2 \int_\Omega u_\varepsilon^2\,dx - \int_\Omega Q(x, t_\varepsilon u_\varepsilon)\,dx + O\big(\varepsilon^{N-2s}\big).
\end{aligned}
$$

□

Lemma 3.4 *The condition* (3.2) *holds.*

Proof We consider three cases.

Case 1. $N > 4s$.

By Lemma 3.5 in [4], there exist $\delta > 0$ and $T > 0$ such that

$$q(x,t) \geq t^{\delta} \quad \text{for } x \in B_{4\rho}, t \geq T.$$

Taking $\bar{q}(t) = T^{\delta}\chi_{[T,+\infty)}(t)$, then

$$q(x,t) \geq \bar{q}(t) \geq 0 \quad \text{for } x \in B_{4\rho}, t \geq 0, \tag{3.20}$$

where $\chi_{[T,+\infty)}$ denotes the characteristic function of $[T,+\infty)$. Thus

$$\bar{Q}(t) := \int_0^t \bar{q}(s)\,ds \geq T^{1+\delta} \quad \text{for } t \geq 2T.$$

Direct computation yields

$$\int_{|x|<\rho} \bar{Q}(t_\varepsilon u_\varepsilon)\,dx = \omega_{N-1}\varepsilon^N \int_0^{\rho/(S^{1/(2s)}\varepsilon)} \bar{Q}\left(t_\varepsilon A\kappa\left(\frac{\varepsilon^{-1}}{\mu^2+t^2}\right)^{\frac{N-2s}{2}}\right) t^{N-1}\,dt, \tag{3.21}$$

where ω_{N-1} is the area of S^{N-1}, $A = (\int_{\mathbb{R}^N} \tilde{u}^p\,dx)^{1/p}$, $\kappa, \mu > 0$ are constants. By (3.20) and (3.21), we have

$$\begin{aligned}
\frac{1}{\varepsilon^{N-2s}} \int_{|x|<\rho} Q(x, t_\varepsilon u_\varepsilon)\,dx &\geq \frac{1}{\varepsilon^{N-2s}} \int_{|x|<\rho} \bar{Q}(t_\varepsilon u_\varepsilon)\,dx \\
&\geq \omega_{N-1}\varepsilon^{2s} \int_0^{C\varepsilon^{-1/2}} T^{1+\delta} t^{N-1}\,dt \\
&= \frac{T^{\delta+1}C^N}{N}\varepsilon^{2s-\frac{N}{2}},
\end{aligned}$$

where $C > 0$ is a some constant such that $t_\varepsilon A\kappa(\frac{\varepsilon^{-1}}{\mu^2+t^2})^{\frac{N-2s}{2}} \geq 2T$ for all $t \leq C\varepsilon^{-1/2}$ and ε is small enough. Thus,

$$\frac{1}{\varepsilon^{N-2s}} \int_{|x|<\rho} Q(x, t_\varepsilon u_\varepsilon)\,dx \to +\infty$$

as $\varepsilon \to 0$ since $N > 4s$.

By (3.12), we have

$$\begin{aligned}
J(t_\varepsilon u_\varepsilon) &\leq \frac{s}{N}S_s^{N/(2s)} - \frac{1}{2}\lambda t_\varepsilon^2 \int_\Omega u_\varepsilon^2\,dx - \int_\Omega Q(x, t_\varepsilon u_\varepsilon)\,dx + O\big(\varepsilon^{N-2s}\big) \\
&= \frac{s}{N}S_s^{N/(2s)} - \frac{1}{2}\lambda t_\varepsilon^2 \int_\Omega u_\varepsilon^2\,dx - \varepsilon^{N-2s}\left(\frac{1}{\varepsilon^{N-2s}} \int_{|x|<\rho} Q(x, t_\varepsilon u_\varepsilon)\,dx + O(1)\right) \\
&< \frac{s}{N}S_s^{N/(2s)},
\end{aligned}$$

as $\varepsilon \to 0$.

Case 2. $N = 4s$.

Clearly, $p - 1 = \frac{N+2s}{N-2s} = 3$. Note that there exists $C_1 > 0$ such that

$$q(x,t) \geq C_1 t \quad \text{for } x \in B_{4\rho}, t \geq 0.$$

Thus,

$$\begin{aligned}
&\frac{1}{\varepsilon^{N-2s}} \int_{|x|<\rho} Q(x, t_\varepsilon u_\varepsilon)\,dx \\
&\geq \frac{1}{\varepsilon^{N-2s}} \omega_{N-1} \varepsilon^N \int_0^{\rho/(S^{1/(2s)}\varepsilon)} \frac{1}{2} C_1 \left(t_\varepsilon A\kappa \left(\frac{\varepsilon^{-1}}{\mu^2 + t^2} \right)^{\frac{N-2s}{2}} \right)^2 t^{N-1}\,dt \\
&= \frac{1}{2} C_1 \omega_{N-1} (t_\varepsilon A\kappa)^2 \int_0^{\rho/(S^{1/(2s)}\varepsilon)} \frac{1}{(\mu^2 + t^2)^{2s}} t^{N-1}\,dt \\
&\geq \frac{1}{2} C_1 \omega_{N-1} (t_\varepsilon A\kappa)^2 4^{-s} \int_{\rho/(S^{1/(2s)}\varepsilon^{1/2})}^{\rho/(S^{1/(2s)}\varepsilon)} \frac{1}{t^{4s}} t^{4s-1}\,dt \\
&= \frac{1}{4^{1+s}} C_1 \omega_{N-1} (t_\varepsilon A\kappa)^2 |\ln \varepsilon| \to +\infty,
\end{aligned}$$

as $\varepsilon \to 0$.

By (3.12), we have

$$\begin{aligned}
J(t_\varepsilon u_\varepsilon) &\leq \frac{s}{N} S_s^{N/(2s)} - \frac{1}{2} \lambda t_\varepsilon^2 \int_\Omega u_\varepsilon^2\,dx - \int_\Omega Q(x, t_\varepsilon u_\varepsilon)\,dx + O(\varepsilon^{2s}) \\
&= \frac{s}{N} S_s^{N/(2s)} - \frac{1}{2} \lambda t_\varepsilon^2 \int_\Omega u_\varepsilon^2\,dx - \varepsilon^{N-2s} \left(\frac{1}{4^{1+s}} C_1 \omega_{N-1} (t_\varepsilon A\kappa)^2 |\ln \varepsilon| + O(1) \right) \\
&< \frac{s}{N} S_s^{N/(2s)},
\end{aligned}$$

as $\varepsilon \to 0$.

Case 3. $N < 4s$.

Clearly, $p - 1 = \frac{N+2s}{N-2s} > 3$. By Lemma 3.4 in [4], there exists $C_2 > 0$ such that

$$q(x,t) \geq C_2 t^{p-2} \quad \text{for } x \in B_{4\rho}, t \geq 0.$$

Thus,

$$\begin{aligned}
&\frac{1}{\varepsilon^{N-2s}} \int_{|x|<\rho} Q(x, t_\varepsilon u_\varepsilon)\,dx \\
&\geq \frac{1}{\varepsilon^{N-2s}} \omega_{N-1} \varepsilon^N \int_0^{\rho/(S^{1/(2s)}\varepsilon)} \frac{1}{p-1} C_2 \left(t_\varepsilon A\kappa \left(\frac{\varepsilon^{-1}}{\mu^2 + t^2} \right)^{\frac{N-2s}{2}} \right)^{p-1} t^{N-1}\,dt \\
&= \frac{1}{p-1} C_2 \omega_{N-1} (t_\varepsilon A\kappa)^{p-1} \varepsilon^{-\frac{N-2s}{2}} \int_0^{\rho/(S^{1/(2s)}\varepsilon)} \frac{1}{(\mu^2 + t^2)^{\frac{N+2s}{2}}} t^{N-1}\,dt \\
&\geq \frac{1}{p-1} C_2 \omega_{N-1} (t_\varepsilon A\kappa)^{p-1} \varepsilon^{-\frac{N-2s}{2}} \int_0^1 \frac{1}{(\mu^2 + t^2)^{\frac{N+2s}{2}}} t^{N-1}\,dt \to +\infty,
\end{aligned}$$

as $\varepsilon \to 0$.

By (3.12), we have

$$J(t_\varepsilon u_\varepsilon) \le \frac{s}{N} S_s^{N/(2s)} - \frac{1}{2}\lambda t_\varepsilon^2 \int_\Omega u_\varepsilon^2\,dx - \int_\Omega Q(x, t_\varepsilon u_\varepsilon)\,dx + O(\varepsilon^{N-2s})$$
$$= \frac{s}{N} S_s^{N/(2s)} - \frac{1}{2}\lambda t_\varepsilon^2 \int_\Omega u_\varepsilon^2\,dx - \varepsilon^{N-2s}\left(\frac{1}{\varepsilon^{N-2s}} \int_{|x|<\rho} Q(x, t_\varepsilon u_\varepsilon)\,dx + O(1)\right)$$
$$< \frac{s}{N} S_s^{N/(2s)},$$

as $\varepsilon \to 0$. □

Proof of Theorem 1.2 By Lemma 3.3 and Theorem 3.1, we see that problem (3.1) has a solution v for $\lambda \in [0, \lambda_1)$ and $\gamma \in (0, \gamma^*)$. We can obtain the second solution of $(P)_\gamma$ by taking $u = u_\gamma + v$. Combining with Lemma 2.5 we complete our proof. □

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Authors' contributions
YZ contributed the central idea, and wrote the initial draft of the paper. The other authors contributed to refining the ideas, carrying out additional analyses and finalizing this paper. All authors read and approved the final manuscript.

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