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A study on the thermoelasticity of three-phase-lag dipolar materials with voids

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Abstract

With a wide applicability in solid state mechanics, the theory of continuous dipolar materials with voids is a distinctive part of microstructure theory, a theory that emerged with the necessity of eliminating the discrepancies between the classical theory and its experimental applications. These inconsistencies are caused by the influence of the microstructure of materials on the general deformations of the bodies, such as ceramics, graphite or polymers. In the present work, we study the mixed boundary value problem with initial data for the thermoelasticity of three-phase-lag dipolar materials with voids, in order to obtain the corresponding constitutive laws, using the same method as in the classical elasticity. By going deeper into the study of these materials, we achieve a uniqueness result and a reciprocal relation, using less common methods, like the dissipative inequality regarding the result of uniqueness. Along with these aspects, we demonstrate a variational principle for anisotropic and non-homogeneous materials, derived from a well-known variational principle, but studied in the context of thermoelasticity of three-phase-lag dipolar materials with voids.

Keywords: Thermoelasticity; Dipolar; Voids; Three-phase-lag

1 Introduction

The purpose of removing the incompatibility between the classical theory of thermoelasticity and its specific experiments, in both cases that of the heat conduction equation and the classical heat equation, led to the modification of the classical heat equation (the Fourier equation), which was of parabolic type, involving the propagation of the waves at infinite speed, and which becomes, after several stages of modification, of the hyperbolic type.

A three-phase-lag model of linearized theory of the coupled thermoelasticity is formulated by considering the heat law involving the temperature gradient and thermal displacement gradient, among the constitutive variables. The Fourier law is replaced by an approximation to a modification on the Fourier law with three different translations for the heat flux vector, for the temperature gradient and for the thermal displacement gradient.

Significant stages of the classical heat equation transformation are due to Lord and Shulman in [26] who opened the path to this transformation, followed by Green and Naghdi, with their three distinct theories [16–18], and the theory of Tzou, see [43], the heat equation becoming hyperbolic, involving both the gradient of the temperature and the gradient of the thermal displacement. In other, more recent studies, see [6], the law of Fourier is also replaced by an approximation of the equation

$$q_i(x, t + t_q) = -\left[K_{ij}\theta_{,j}(x, t + t_T) + K^*_{ij}\alpha_{,j}(x, t + t_\alpha)\right],$$

where t_α, t_q and t_T are the notations for the thermal displacement gradient, the heat flux, respectively the temperature gradient, in the three-phase-lag heat conduction theory, K_{ij} and K^*_{ij} representing the tensor of the thermal conductivity, respectively the tensor of the conductivity rate. The results of applying the three-phase-lag theory to particular materials, a theory that comes from the dual-phase-lag theory, see [2, 29, 43], have been studied in many papers, for example in [3, 5, 6, 12, 13, 24, 25, 28, 30, 36, 38], other studies presenting the advantages of this theory, such as [1, 9, 39].

In this work, the authors approach the theorems of uniqueness, reciprocity or variational principles, providing a broad notional support for practical applications. Due to the fact that the classical theory of elasticity proved its limits in some situations, such as the one in which the microstructure of materials is involved in general deformations of the bodies, for example in the case of ceramics, graphite, bones, mineral wool or polymers, the theory of the microstructure media appears precisely in order to eliminate the discrepancies between the classical theory and its experimental applications.

The process of manufacturing these materials, with such various reliabilities in the environment we live in, involves the knowledge of the mechanical behavior, of their specific properties and their mathematical description, and the study by means of specific techniques where materials are the result of a reaction between at least two other materials. In order for the study to be as complete as possible, it should be extended on the materials presented at a small scale, because the information from the micro scale leads us their projection on the macro scale, highlighting the benefits in many of their applications.

The promoter of the microstructure theory was Eringen, who laid the foundation for the research of this theory in [10, 11], followed by many other authors who approached this subject; see for example [31, 41]. The theory of elastic materials with voids has been studied, starting with Goodman and Cowin in [15], continuing with Nunziato and Cowin in [37], Ieşan in [23] and being extended into more recent studies; we mention here [4, 7, 21].

Being distinctive parts of the microstructure theory, the theory of continuous dipolar materials, respectively, the theory of dipolar materials with voids have a wide applicability, and many valuable works highlight the importance of these material structures, from Mindlin in [35] and Green and Rivlin in [19], who opened the way in the dipolar theory, to Gurtin and Fried, see [14], the authors who approached the theory from the perspective of the interdependence between the medium and what surrounds it, and to more recent works that concerned the dipolar theory, the dipolar theory with voids or with double porosity, such as [8, 27, 32, 34]. Another topic related to particular materials with microstructure approached by many researchers was that of vibrations and their study in the context of the three-phase-lag theory and its implications on the specific structures described above, some examples of work in this domain being [30, 33, 40, 42].

As far as the present paper is concerned, it is dedicated to the thermoelastic dipolar structure with voids under the influence of the three-phase-lag theory, studying the mixed boundary value problem with initial data for this type of media, in order to obtain the corresponding constitutive laws, using the same method as in the classical elasticity. By going deeper into the study of these materials, we achieve a uniqueness result and a reciprocal relation, using less common methods, like the dissipative inequality regarding the result of uniqueness. Along with these aspects, we demonstrate a variational principle for an anisotropic and non-homogeneous material, derived from Gurtin's well-known variational principle, see [20], but studied in the context of the thermoelasticity of three-phase-lag dipolar materials with voids.

2 Basic equations

It is considered that the thermoelastic dipolar material with voids occupies at time t_0 the domain $\mathcal{D}$ of the three-dimensional Euclidian space $\mathbb{R}^3$, a domain which is a regular region, its boundary $\partial\mathcal{D}$ being a smooth surface and denoting its closure $\overline{\mathcal{D}}$. Using a fixed rectangular Cartesian system of axes, each point of the domain $\mathcal{D}$ is characterized by three rectangular coordinates, with the specification that we will use the notation x for (x_1, x_2, x_3) and t for the time. During this work, the functions will be seen as functions of (x, t) defined on the cylinder $\overline{\mathcal{D}} \times (0, \infty)$, where $\overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}$. Both arguments will be omitted if confusion cannot occur. In the following, when an index is repetitive in a monomial, the well-known Einstein summation convention will be used and the values taken by the Greek and Latin indices are 1, 2 and 1, 2, 3, respectively. The partial differentiation of a function in relation to time will be represented by a point above the function and an index preceded by a comma signifies the partial differentiation of the function in relation to the corresponding Cartesian coordinate. The behavior of a thermoelastic dipolar material with voids is characterized by the displacement vector field $u = (u_i)_{1\leq i\leq 3}$, the dipolar displacement tensor field $\varphi = (\varphi_{ij})_{1\leq i,j\leq 3}$, the volume fraction field ν corresponding to voids, and the temperature variation θ:

$$u_i = u_i(x,t), \qquad \varphi_{ij} = \varphi_{ij}(x,t), \qquad \nu = \nu(x,t), \qquad \theta = \theta(x,t) \quad (x,t) \in \mathcal{D} \times [0, t_0),$$

with bringing up that the components of both the displacement, the temperature variation and the partial differentiations in relation to the Cartesian coordinates and to the time are small. This study is developed according to the theory of linear thermoelasticity of the dipolar bodies with voids. The Green and Rivlin method is applied, overlapping a rigid rotation motion with constant angular velocity, this rigid motion not influencing in any way the body characteristics. The geometric equations, expressing the strain tensors ε_{ij}, γ_{ij}, χ_{ijk} and the vector υ_k, in relation to the motion variables, and the continuity equation have the following forms, according to [10]:

$$\varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}), \qquad \gamma_{ij} = u_{j,i} - \varphi_{ij}, \qquad \chi_{ijk} = \varphi_{jk,i}, \qquad \upsilon_k = \nu_{,k}, \tag{1}$$

$$\frac{d\rho}{dt} + \rho\dot{u}_{j,j} = 0, \tag{2}$$

where ρ is the density of the body in the reference configuration. Just as in [16] we will define, using the temperature, the thermal displacement α and the thermal gradient β_i through the following expressions:

$$\alpha(x,t) = \int_{t_0}^{t} \theta(x,s)\,ds, \qquad \beta_i(x,t) = \int_{t_0}^{t} \theta_{,i}(x,s)\,ds, \tag{3}$$

where we used the notations $\dot{\alpha}(x,t) = \theta(x,t)$, $\alpha(x,t_0) = 0$, $\beta_i(x,t) = \alpha_{,i}(x,t)$ and

$$\alpha(x,0) = 0, \qquad \beta_i(x,0) = 0 \tag{4}$$

if t_0 is zero.

The fundamental system governing the linear theory of the dipolar materials with voids thermoelasticity consists of the following equations:

– the motion equations:

$$\begin{aligned} &(\tau_{ji} + \sigma_{ji})_{,j} + \rho f_i = \rho \ddot{u}_i, \\ &\mu_{ijk,i} + \sigma_{jk} + \rho g_{jk} = I_{ks}\ddot{\varphi}_{js}, \end{aligned} \quad \text{in } \mathcal{D} \times (0,\infty), \tag{5}$$

– the equilibrated forces balance:

$$\lambda_{i,i} + \xi + \rho \ell = \rho k \ddot{\nu}, \quad \text{in } \mathcal{D} \times (0,\infty), \tag{6}$$

– the energy equation:

$$T\dot{\eta} = Q - q_{i,i}, \tag{7}$$

– the geometric equations (1), where the notation is:

- τ_{ji}, σ_{ji}, μ_{ijk} are the stress tensors components,
- f_i, g_{jk} are the body force components, respectively, the body couple force components,
- I_{ij} are the microinertia coefficients,
- ℓ is the extrinsic equilibrated body force related to the voids,
- λ_i are the components of equilibrated stress vector corresponding to ν,
- k is the equilibrated inertia coefficient,
- ξ is the intrinsic equilibrated body force.

At the same time, the surface traction components t_i, the surface couple components μ_{jk}, the surface heat flux q, and the equilibrated stress vector λ corresponding to the voids, are defined in the following forms:

$$\begin{aligned} &t_i := (\tau_{ji} + \sigma_{ji})n_j, \\ &\mu_{jk} := \mu_{ijk}n_i, \\ &\lambda := \lambda_i n_i, \\ &q := q_i n_i \end{aligned} \quad \text{on } \partial\mathcal{D}, \tag{8}$$

for all regular points which belong to the surface $\partial\mathcal{D}$, noting that q_i are the components of the heat flux vector and n_i are the components of the unit outward normal vector to the surface $\partial\mathcal{D}$.

3 Analysis

In the following, we will deduce some thermodynamics law consequences, in order to obtain the main results presented by the theorems of the next section. For this purpose, using the procedure presented in [28], it is necessary that the first two laws of thermodynamics hold at any point of the domain $\mathcal{D}$ and at each moment t. The first law of thermodynamics has the following form:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathcal{D}}\left(\rho e+\frac{1}{2}\rho\dot{u}_i\dot{u}_i+\frac{1}{2}I_{ks}\dot{\varphi}_{jk}\dot{\varphi}_{js}+\frac{1}{2}k\dot{\nu}\dot{\nu}\right)dV\\&\quad=\int_{\mathcal{D}}(\rho f_i\dot{u}_i+\rho g_{jk}\dot{\varphi}_{jk}+\rho\ell\dot{\nu}+Q)\,dV+\int_{\partial\mathcal{D}}(t_i\dot{u}_i+\mu_{jk}\dot{\varphi}_{jk}+\lambda\dot{\nu}-q)\,dA,\end{aligned}\tag{9}$$

where e represents the internal energy per mass unit and Q represents the intensity of applied heat source per volume unit.

At the same time, the local form of the second law of thermodynamics is

$$\dot{\eta}\geq\frac{Q}{T}-\frac{q_{i,i}}{T}+\frac{q_i}{T^2}T_{,i},\tag{10}$$

where η and T are the entropy per volume unit, respectively the absolute temperature, $T=T_0+\theta$, where T_0 is the body absolute temperature in the reference configuration.

In order to obtain another form of the previous inequality, we multiply Eqs. $(5)_1$, $(5)_2$ and (6) by $\dot{u}_i$, $\dot{\varphi}_{jk}$, respectively, $\dot{\nu}$, after that, by integrating on $\mathcal{D}$ the obtained relations and by introducing them into the first principle of thermodynamics, the following relation is obtained:

$$\begin{aligned}&\int_{\mathcal{D}}\left[(\tau_{ji}+\sigma_{ji})_{,j}\dot{u}_i+\rho f_i\dot{u}_i+\mu_{ijk,i}\dot{\varphi}_{jk}+\sigma_{jk}\dot{\varphi}_{jk}+\rho g_{jk}\dot{\varphi}_{jk}+\lambda_{i,i}\dot{\nu}+\xi\dot{\nu}+\rho\ell\dot{\nu}\right]dV\\&\qquad+\int_{\mathcal{D}}\rho\dot{e}\,dV\\&\quad=\int_{\mathcal{D}}(\rho f_i\dot{u}_i+\rho g_{jk}\dot{\varphi}_{jk}+\rho\ell\dot{\nu}+Q)\,dV+\int_{\partial\mathcal{D}}(t_i\dot{u}_i+\mu_{jk}\dot{\varphi}_{jk}+\lambda\dot{\nu}-q)\,dA,\end{aligned}\tag{11}$$

a relation which, after applying the divergence theorem, becomes

$$\begin{aligned}&\int_{\mathcal{D}}\sigma_{jk}\dot{\varphi}_{jk}\,dV+\int_{\mathcal{D}}\xi\dot{\nu}\,dV+\int_{\mathcal{D}}\rho\dot{e}\,dV\\&\quad=\int_{\mathcal{D}}Q\,dV+\int_{\mathcal{D}}(\tau_{ji}+\sigma_{ji})\dot{u}_{i,j}\,dV+\int_{\mathcal{D}}\mu_{ijk}\dot{\varphi}_{jk,i}\,dV+\int_{\mathcal{D}}\lambda_i\dot{\nu}_{,i}\,dV-\int_{\mathcal{D}}q_{i,i}\,dV.\end{aligned}\tag{12}$$

The previous relation can be rewritten in the form:

$$\int_{\mathcal{D}}\left[\rho\dot{e}+\sigma_{jk}\dot{\varphi}_{jk}+\xi\dot{\nu}-Q-(\tau_{ji}+\sigma_{ji})\dot{u}_{i,j}-\mu_{ijk}\dot{\chi}_{ijk}-\lambda_i\dot{\nu}_{,i}+q_{i,i}\right]dV=0,\tag{13}$$

from which, taking into account that $\mathcal{D}$ is an arbitrary domain, we infer the equality

$$\rho\dot{e}=\tau_{ij}\dot{\varepsilon}_{ij}+\sigma_{ij}\dot{\gamma}_{ij}+\mu_{ijk}\dot{\chi}_{ijk}-q_{i,i}+Q-\xi\dot{\nu}+\lambda_i\dot{\nu}_{,i}.\tag{14}$$

The previous equality leads to another form of the inequality (10), namely

$$\rho\dot{e}-\tau_{ij}\dot{\varepsilon}_{ij}-\sigma_{ij}\dot{\gamma}_{ij}-\mu_{ijk}\dot{\chi}_{ijk}+\xi\dot{\nu}-\lambda_i\dot{\nu}_{,i}-T\dot{\eta}+\frac{q_i}{T}T_{,i}\leq 0.\tag{15}$$

The Helmholtz free energy Ψ, given by

$$\Psi = \rho e - T\eta, \tag{16}$$

leads to rewriting Eq. (14) in the form

$$\dot{\Psi} = \tau_{ij}\dot{\varepsilon}_{ij} + \sigma_{ij}\dot{\gamma}_{ij} + \mu_{ijk}\dot{\chi}_{ijk} - q_{i,i} + Q - \xi\dot{\nu} + \lambda_i\dot{\nu}_{,i} - \dot{T}\eta - T\dot{\eta}. \tag{17}$$

Considering the fact that

$$\Psi = \Psi(\varepsilon_{ij}, \gamma_{ij}, \chi_{ijk}, \nu, \nu_{,i}, T, T_{,i}),$$

we have

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial\Psi}{\partial\gamma_{ij}}\dot{\gamma}_{ij} + \frac{\partial\Psi}{\partial\chi_{ijk}}\dot{\chi}_{ijk} + \frac{\partial\Psi}{\partial\nu}\dot{\nu} + \frac{\partial\Psi}{\partial\nu_{,i}}\dot{\nu}_{,i} + \frac{\partial\Psi}{\partial T}\dot{T} + \frac{\partial\Psi}{\partial T_{,i}}\dot{T}_{,i}. \tag{18}$$

By comparing Eqs. (17) and (18), we deduce

$$\begin{aligned} &\tau_{ij} = \frac{\partial\Psi}{\partial\varepsilon_{ij}}, \qquad \sigma_{ij} = \frac{\partial\Psi}{\partial\gamma_{ij}}, \qquad \mu_{ijk} = \frac{\partial\Psi}{\partial\chi_{ijk}}, \qquad \eta = -\frac{\partial\Psi}{\partial T}, \\ &\xi = -\frac{\partial\Psi}{\partial\nu}, \qquad \lambda_i = \frac{\partial\Psi}{\partial\nu_{,i}}, \qquad \frac{\partial\Psi}{\partial T_{,i}} = 0, \end{aligned} \tag{19}$$

and

$$T\dot{\eta} = Q - q_{i,i}.$$

The entropy inequality (15) can be written with the help of the free energy, in the form

$$\dot{\Psi} + \dot{T}\eta - \tau_{ij}\dot{\varepsilon}_{ij} - \sigma_{ij}\dot{\gamma}_{ij} - \mu_{ijk}\dot{\chi}_{ijk} + \xi\dot{\nu} - \lambda_i\dot{\nu}_{,i} + \frac{q_i\theta_{,i}}{T} \leq 0, \tag{20}$$

an inequality that can be written like

$$\begin{aligned} &\frac{\partial\Psi}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial\Psi}{\partial\gamma_{ij}}\dot{\gamma}_{ij} + \frac{\partial\Psi}{\partial\chi_{ijk}}\dot{\chi}_{ijk} + \frac{\partial\Psi}{\partial\nu}\dot{\nu} + \frac{\partial\Psi}{\partial\nu_{,i}}\dot{\nu}_{,i} + \frac{\partial\Psi}{\partial T}\dot{\theta} \\ &\quad + \dot{\theta}\eta - \tau_{ij}\dot{\varepsilon}_{ij} - \sigma_{ij}\dot{\gamma}_{ij} - \mu_{ijk}\dot{\chi}_{ijk} + \xi\dot{\nu} - \lambda_i\dot{\nu}_{,i} + \frac{q_i\theta_{,i}}{T} \leq 0. \end{aligned} \tag{21}$$

Assuming that the materials have a symmetry center and on the one hand, the media is stress free in the reference configuration having null intrinsic equilibrated body forces, body couple, respectively, and on the other hand, there is imposed the quadric form of the internal energy density in relation to its independent constitutive variables required by the linear theory, the form of the free energy is

$$\begin{aligned}\Psi ={}& \frac{1}{2}C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + G_{mnij}\varepsilon_{mn}\gamma_{ij} + \frac{1}{2}B_{ijmn}\gamma_{ij}\gamma_{mn} + F_{mnrij}\varepsilon_{ij}\chi_{mnr} \\ &+ D_{mnijk}\gamma_{mn}\chi_{ijk} + \frac{1}{2}A_{ijkmnr}\chi_{ijk}\chi_{mnr} + d_{ijm}\varepsilon_{ij}\nu_{,m} + e_{ijm}\gamma_{ij}\nu_{,m} \\ &+ f_{ijkm}\chi_{ijk}\nu_{,m} + \frac{1}{2}g_{im}\nu_{,i}\nu_{,m} + a_{ij}\varepsilon_{ij}\nu + b_{ij}\gamma_{ij}\nu + c_{ijk}\chi_{ijk}\nu + d_i\nu_{,i}\nu \\ &+ \frac{1}{2}\omega\nu^2 - \alpha_{ij}\varepsilon_{ij}\theta - \beta_{ij}\gamma_{ij}\theta - \gamma_{ijk}\chi_{ijk}\theta - \frac{1}{2}a\theta^2 - a_i\nu_{,i}\theta - b\nu\theta.\end{aligned} \tag{22}$$

In the previous expression of the free energy, the constitutive coefficients, representing the characteristic functions of the material, are prescribed functions in $C^1(\mathcal{D})$, and fulfill the following symmetry relations:

$$\begin{aligned}&C_{ijmn} = C_{mnij} = C_{jimn}, \qquad B_{ijmn} = B_{mnij}, \qquad G_{ijmn} = G_{ijnm}, \qquad F_{ijkmn} = F_{jikmn}, \\ &A_{ijkmnr} = A_{mnrijk}, \qquad a_{ij} = a_{ji}, \qquad g_{im} = g_{mi}, \qquad \alpha_{ij} = \alpha_{ji}.\end{aligned} \tag{23}$$

Using Eqs. (19), from the Ψ form (22), we obtain the following constitutive equations:

$$\begin{aligned}&\tau_{ij} = C_{ijmn}\varepsilon_{mn} + G_{ijmn}\gamma_{mn} + F_{mnrij}\chi_{mnr} + d_{ijm}\nu_{,m} + a_{ij}\nu - \alpha_{ij}\theta, \\ &\sigma_{ij} = G_{ijmn}\varepsilon_{mn} + B_{ijmn}\gamma_{mn} + D_{ijmnr}\chi_{mnr} + e_{ijm}\nu_{,m} + b_{ij}\nu - \beta_{ij}\theta, \\ &\mu_{ijk} = F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + A_{ijkmnr}\chi_{mnr} + f_{ijkm}\nu_{,m} + c_{ijk}\nu - \gamma_{ijk}\theta, \\ &\eta = \alpha_{ij}\varepsilon_{ij} + \beta_{ij}\gamma_{ij} + \gamma_{ijk}\chi_{ijk} + a\theta + a_i\nu_{,i} + b\nu, \\ &\xi = -a_{ij}\varepsilon_{ij} - b_{ij}\gamma_{ij} - c_{ijk}\chi_{ijk} - d_i\nu_{,i} - \omega\nu + b\theta, \\ &\lambda_i = d_{mni}\varepsilon_{mn} + e_{mni}\gamma_{mn} + f_{mnri}\chi_{mnr} + g_{im}\nu_{,m} + d_i\nu - a_i\theta, \\ &T_0\dot{\eta} = Q - q_{i,i}.\end{aligned} \tag{24}$$

Also, we can use the inequality (21) in order to deduce the dissipative inequality,

$$\int_{\mathcal{D}} \frac{q_i\theta_{,i}}{T_0}\,dV \le 0, \tag{25}$$

which, with the help of the divergence theorem, is written in the form

$$\int_{\mathcal{D}} \frac{q_{i,i}\theta}{T_0}\,dV - \int_{\partial\mathcal{D}} \frac{q\theta}{T_0}\,dA \ge 0, \tag{26}$$

the previous inequality taking a new form, if there is no surface heat flux, it means that $q = 0$, as follows:

$$\int_{\mathcal{D}} \frac{q_{i,i}\theta}{T_0}\,dV \ge 0.$$

Regarding the three-phase-lag theory, the constitutive equation of the heat flux vector in this theory has the form, see [6],

$$q_i(x, t + t_q) = -\left[K_{ij}\theta_{,j}(x, t + t_T) + K^*_{ij}\alpha_{,j}(x, t + t_\beta)\right], \tag{27}$$

where t_β, t_q and t_T are the thermal displacement gradient, the heat flux, respectively the temperature gradient in the three-phase-lag heat conduction theory, and the tensors K_{ij} and K^*_{ij} are symmetric. If we develop it in a Taylor series and only keep the terms up to t_q^2, see [28], we are led to the following equation:

$$q_i + t_q\dot{q}_i + \frac{1}{2}t_q^2\ddot{q}_i = -\big[K_{ij}\theta_{,j} + K_{ij}t_T\dot{\theta}_{,j} + K^*_{ij}\alpha_{,j} + K^*_{ij}t_\beta\dot{\alpha}_{,j}\big], \quad 0 < t_\beta \le t_T \le t_q. \tag{28}$$

Given two functions v and w defined on $\mathcal{D} \times (0,\infty)$ and continuous in relation to time, we recall that the convolution product of functions v and w is defined by

$$(v * w)(t) = \int_0^t v(t-s)w(s)\,ds,$$

and this will be used to determine identities and to introduce new variables and notations. In order to construct the mixed initial-boundary value problem, we add, on the domain $\overline{\mathcal{D}}$, the following initial conditions:

$$\begin{aligned}
&u_i(x,0) = u_i^0(x), \qquad \dot{u}_i(x,0) = u_i^1(x),\\
&\varphi_{ij}(x,0) = \varphi_{ij}^0(x), \qquad \dot{\varphi}_{ij}(x,0) = \varphi_{ij}^1(x),\\
&\nu(x,0) = \nu^0(x), \qquad \dot{\nu}(x,0) = \nu^1(x),\\
&\theta(x,0) = \theta^0(x), \qquad \dot{\theta}(x,0) = \theta^1(x),\\
&\eta(x,0) = \eta^0(x),
\end{aligned} \tag{29}$$

and on the cylinder $\partial\mathcal{D} \times (0,\infty)$, we add the boundary conditions as follows:

$$\begin{aligned}
&u_i(x,t) = \widetilde{u}_i, \quad (x,t) \in \partial\mathcal{D}_u \times (0,\infty),\\
&\varphi_{ij}(x,t) = \widetilde{\varphi}_{ij}, \quad (x,t) \in \partial\mathcal{D}_\varphi \times (0,\infty),\\
&\nu(x,t) = \widetilde{\nu}, \quad (x,t) \in \partial\mathcal{D}_\nu \times (0,\infty),\\
&\alpha(x,t) = \widetilde{\alpha}, \quad (x,t) \in \partial\mathcal{D}_\alpha \times (0,\infty),\\
&(\tau_{ij} + \sigma_{ij})(x,t)n_j = \widetilde{t}_i, \quad (x,t) \in \partial\mathcal{D}_u^c \times (0,\infty),\\
&\mu_{ijk}(x,t)n_i = \widetilde{\mu}_{jk}, \quad (x,t) \in \partial\mathcal{D}_\varphi^c \times (0,\infty),\\
&\lambda_i(x,t)n_i = \widetilde{\lambda}, \quad (x,t) \in \partial\mathcal{D}_\nu^c \times (0,\infty),\\
&q_i(x,t)n_i = \widetilde{q}, \quad (x,t) \in \partial\mathcal{D}_\alpha^c \times (0,\infty),
\end{aligned} \tag{30}$$

where $\partial\mathcal{D}_u$, $\partial\mathcal{D}_\varphi$, $\partial\mathcal{D}_\nu$ and $\partial\mathcal{D}_\alpha$, together with their complements, $\partial\mathcal{D}_u^c$, $\partial\mathcal{D}_\varphi^c$, $\partial\mathcal{D}_\nu^c$, respectively, $\partial\mathcal{D}_\alpha^c$, are the surface $\partial\mathcal{D}$ subsets, such that

$$\begin{aligned}
&\partial\mathcal{D}_u \cap \partial\mathcal{D}_u^c = \partial\mathcal{D}_\varphi \cap \partial\mathcal{D}_\varphi^c = \partial\mathcal{D}_\nu \cap \partial\mathcal{D}_\nu^c = \partial\mathcal{D}_\alpha \cap \partial\mathcal{D}_\alpha^c = \varnothing,\\
&\partial\overline{\mathcal{D}}_u \cup \partial\mathcal{D}_u^c = \partial\overline{\mathcal{D}}_\varphi \cup \partial\mathcal{D}_\varphi^c = \partial\overline{\mathcal{D}}_\nu \cup \partial\mathcal{D}_\nu^c = \partial\overline{\mathcal{D}}_\alpha \cup \partial\mathcal{D}_\alpha^c = \partial\mathcal{D}.
\end{aligned}$$

We suppose that:

(i) u_i^0, u_i^1, φ_{ij}^0, φ_{ij}^1, ν^0, ν^1, θ^0, θ^1 and η^0 are continuous and prescribed functions on $\mathcal{D}$;

(ii) $\widetilde{u}_i$, $\widetilde{\varphi}_{ij}$, $\widetilde{\nu}$ and $\widetilde{\alpha}$ are continuous and prescribed functions on their domains of definition;

(iii) $\widetilde{t}_i$, $\widetilde{\mu}_{jk}$, $\widetilde{\lambda}$ and $\widetilde{q}$ are piecewise regular functions on their domains of definition and continuous in the relation to the time variable.

In the following, incorporating the initial conditions in the basic equations, we will formulate some theorems and preliminary identities, which we will use in the next section.

Theorem 1 *The functions u_i, φ_{ij}, η and ν verify Eqs.* (5), (6), (7) *and the initial conditions* (29), *if and only if they fulfill the following relations*:

$$\begin{aligned} &g * (\tau_{ji} + \sigma_{ji})_{,j} + \rho \mathcal{F}_i = \rho u_i, \\ &g * (\mu_{ijk,i} + \sigma_{jk}) + \rho \mathcal{G}_{jk} = I_{ks}\varphi_{js}, \\ &g * (\lambda_{i,i} + \xi) + \rho \mathcal{L} = \rho k \nu, \qquad (x,t) \in \mathcal{D} \times (0,\infty), \\ &\eta = R - \frac{1}{T_0} l * q_{i,i}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} &g(t) = (l * l)(t) = t, \\ &l(t) = 1, \end{aligned} \qquad t \in (0,\infty), \tag{32}$$

and

$$\begin{aligned} &\mathcal{F}_i = g * f_i + \left(t u_i^1 + u_i^0\right), \\ &\mathcal{G}_{jk} = g * g_{jk} + I_{ks}\left(t\varphi_{js}^1 + \varphi_{js}^0\right), \\ &\mathcal{L} = g * \ell + k\left(t\nu^1 + \nu^0\right), \\ &R = \frac{1}{T_0} l * Q + \eta^0. \end{aligned} \tag{33}$$

Proof We will only demonstrate Eq. $(31)_3$, the other relations can be demonstrated by the same pattern, following the method presented by Ieșan in [22] and using the equality

$$g * (\rho k \ddot{\nu}) = \rho k (g * \ddot{\nu}) = \rho k\left[-t\dot{\nu}(x,0) - \nu(x,0) + \nu(x,t)\right], \tag{34}$$

equivalent to

$$g * (\rho k \ddot{\nu}) = \rho k \nu - \rho k\left(t\nu^1 + \nu^0\right). \tag{35}$$

From the balance of equilibrated force equation (6) we obtain

$$g * (\lambda_{i,i} + \xi + \rho \ell) = g * (\rho k \ddot{\nu}), \tag{36}$$

and the use of Eqs. (35) and (36) leads to

$$g * (\lambda_{i,i} + \xi) + \rho\left[(g * \ell) + k\left(t\nu^1 + \nu^0\right)\right] = \rho k \nu. \tag{37}$$

If in the previous relation we use the notation $(33)_3$, we get

$$g * (\lambda_{i,i} + \xi) + \rho\mathcal{L} = \rho k\nu,$$

which ends the proof of Eq. $(31)_3$.

Reciprocally, if the function ν satisfies Eq. $(31)_3$, then, taking into account Eq. (34), we have

$$g * (\lambda_{i,i} + \xi) + \rho(g * \ell) + \rho k\bigl(t\nu^1 + \nu^0\bigr) = \rho k\bigl[g * \ddot{\nu} + t\dot{\nu}(x,0) + \nu(x,0)\bigr], \tag{38}$$

from which, considering $t = 0$, we obtain

$$\nu(x,0) = \nu^0. \tag{39}$$

By taking the derivative of Eq. (38) with respect to the time variable t and then, considering $t = 0$, we will get

$$\dot{\nu}(x,0) = \nu^1, \tag{40}$$

a relation which, together with Eq. (39), represents the initial conditions $(29)_5$ and $(29)_6$. With this, Eq. (38) is reduced to

$$g * (\lambda_{i,i} + \xi + \rho\ell - \rho k\ddot{\nu}) = 0, \tag{41}$$

from which, based on the properties of the convolution product, we will obtain the equation of the equilibrated forces (6). □

We will use the notation, with $l(t) = 1$, $t \in (0,\infty)$,

$$\hat{f} = l * f = \int_0^t f(x,s)\,ds,$$

which allows us to consider two new variables,

$$\zeta_i = \beta_i + t_T\dot{\beta}_i, \qquad \varsigma_i = \hat{\beta}_i + t_\alpha\beta_i, \tag{42}$$

with the purpose of obtaining a simplified form of Eq. (28), concretely,

$$(1 + D_t)q_i = -\bigl(K_{ij}\dot{\zeta}_j + K^*_{ij}\dot{\varsigma}_j\bigr), \tag{43}$$

where D_t is the differential operator

$$D_t = t_q\frac{\partial}{\partial t} + \frac{1}{2}t_q^2\frac{\partial^2}{\partial t^2}. \tag{44}$$

The entropy flux vector r is introduced through the notation $r_i = T_0^{-1}\hat{q}_i$ in the form given by

$$r = r_i n_i = T_0^{-1}\hat{q}_i n_i. \tag{45}$$

Using the newly introduced notations, the properties of the convolution product and the preliminary identities obtained in Theorem 1, we can rewrite the energy equation in the form

$$t * (\eta + r_{i,i} - R) = 0, \quad R = T_0^{-1}\hat{Q} + \eta^0, \tag{46}$$

and the identities $(31)_1$, $(31)_2$ and $(31)_3$, in the following form:

$$\begin{aligned} & t * (\tau_{ji} + \sigma_{ij})_{,j} + \rho \mathcal{F}_i = \rho u_i, \\ & t * (\mu_{ijk,i} + \sigma_{jk}) + \rho \mathcal{G}_{jk} = I_{ks}\varphi_{js}, \\ & t * (\lambda_{i,i} + \xi) + \rho \mathcal{L} = \rho k \nu. \end{aligned} \tag{47}$$

Using Eqs. (43) and (45) we are led to the equation

$$r_i + t_q \dot{r}_i + \frac{1}{2} t_q^2 \ddot{r}_i = -\frac{1}{T_0}\left(K_{ij}\zeta_j + K_{ij}^* \varsigma_j\right), \tag{48}$$

which can be reformulated as

$$(1 + D_t) r_i = -\frac{1}{T_0}\left(K_{ij}\beta_j + K_{ij} t_T \dot{\beta}_j + K_{ij}^* \hat{\beta}_j + K_{ij}^* t_\alpha \beta_j\right). \tag{49}$$

Considering Eqs. $(24)_4$ and $(24)_7$, Eq. (49) can be rewritten as follows:

$$\begin{aligned} & (1 + D_t)\left(\alpha_{ij}\dot{\varepsilon}_{ij} + \beta_{ij}\dot{\gamma}_{ij} + \gamma_{ijk}\dot{\chi}_{ijk} + a\dot{\theta} + a_i \dot{\nu}_{,i} + b\dot{\nu} - \frac{1}{T_0} Q\right) \\ & \quad = \frac{1}{T_0}\left(K_{ij}\dot{\zeta}_j + K_{ij}^* \dot{\varsigma}_j\right)_{,i}, \end{aligned} \tag{50}$$

the previous equation being known as *the equation of the heat transport*. By multiplying the previous equation by θ and then integrating on $\mathcal{D}$ the newly acquired relation, we have

$$\begin{aligned} & \int_{\mathcal{D}} \theta \left[D_t(\alpha_{ij}\dot{\varepsilon}_{ij} + \beta_{ij}\dot{\gamma}_{ij} + \gamma_{ijk}\dot{\chi}_{ijk} + a\dot{\theta} + a_i\dot{\nu}_{,i} + b\dot{\nu}) - (1 + D_t)\frac{Q}{T_0} \right. \\ & \qquad \left. - \frac{1}{T_0}\left(K_{ij}\dot{\zeta}_j + K_{ij}^*\dot{\varsigma}_j\right)_{,i} \right] dV \\ & \quad = - \int_{\mathcal{D}} \theta (\alpha_{ij}\dot{\varepsilon}_{ij} + \beta_{ij}\dot{\gamma}_{ij} + \gamma_{ijk}\dot{\chi}_{ijk} + a\dot{\theta} + a_i\dot{\nu}_{,i} + b\dot{\nu})\, dV, \end{aligned} \tag{51}$$

which, with the help of Eq. (24), has the simplified form

$$\int_{\mathcal{D}} \theta \left(\dot{\eta} - \frac{Q}{T_0}\right) dV = \int_{\mathcal{D}} \theta \left[\frac{1}{T_0}\left(K_{ij}\dot{\zeta}_j + K_{ij}^*\dot{\varsigma}_j\right)_{,i} + D_t\left(\frac{Q}{T_0} - \dot{\eta}\right)\right] dV. \tag{52}$$

Concluding, the geometric equations (1), the constitutive equations (24), (28) and (46), the equations of motion (47), the initial and the boundary conditions (29) and (30), respectively, represent the general form for the mixed initial-boundary values problem $\mathcal{P}$, in the context of three-phase-lag thermoelasticty theory of dipolar materials with voids.

4 Main results

To begin with, it is necessary that the elasticity tensors appearing in Eqs. (23) and (24) fulfill the condition according to which there exists a strictly positive constant c_0 so that

$$\begin{aligned} &C_{ijmn}\mathrm{u}_{ij}\mathrm{u}_{mn} + 2G_{ijmn}\mathrm{u}_{ij}\mathrm{v}_{mn} + 2F_{ijmnr}\mathrm{u}_{ij}\mathrm{w}_{mnr} + 2d_{ijm}\mathrm{u}_{ij}\upsilon_m + 2a_{ij}\mathrm{u}_{ij}\nu \\ &\quad + B_{ijmn}\mathrm{v}_{ij}\mathrm{v}_{mn} + 2D_{ijmnr}\mathrm{v}_{ij}\mathrm{w}_{mnr} + 2e_{ijm}\mathrm{v}_{ij}\upsilon_m + 2b_{ij}\mathrm{v}_{ij}\nu + A_{ijkmnr}\mathrm{w}_{ijk}\mathrm{w}_{mnr} \\ &\quad + 2f_{ijkm}\mathrm{w}_{ijk}\upsilon_m + 2c_{ijk}\mathrm{w}_{ijk}\nu + 2d_i\upsilon_i\nu + g_{im}\upsilon_i\upsilon_m + \omega\nu^2 \\ &\geq c_0\left(\mathrm{u}_{ij}\mathrm{u}_{ij} + \mathrm{v}_{ij}\mathrm{v}_{ij} + \mathrm{w}_{ijk}\mathrm{w}_{ijk} + \upsilon_i\upsilon_i + \nu^2\right), \end{aligned} \tag{53}$$

for any five tensors u_{ij}, v_{ij}, w_{ijk}, υ_i and ν.

4.1 Uniqueness

The uniqueness of the mixed initial-boundary value problem $\mathcal{P}$ is related to the result provided by the following theorem.

Theorem 2 *If we assume that*:
- (i) $\rho(x) > 0$, $\beta(x) > 0$, $T_0(x) > 0$, $\forall x \in \mathcal{D}$;
- (ii) *the tensor* I_{ij} *is positively defined*;
- (iii) *the condition* (53) *is fulfilled*, *then the problem* $\mathcal{P}$ *has at most one solution*.

Proof We assume, on the contrary, that our problem $\mathcal{P}$ admits two solutions $(u'_i, \varphi'_{ij}, \nu', \theta')$ and $(u''_i, \varphi''_{ij}, \nu'', \theta'')$, which leads us to the conclusion that the difference between two solutions is in turn a solution to the mixed problem $\mathcal{P}$, due to the linearity. We will have the following notations:

$$u_i^D = u''_i - u'_i, \qquad \varphi_{ij}^D = \varphi''_{ij} - \varphi'_{ij}, \qquad \nu^D = \nu'' - \nu', \qquad \theta^D = \theta'' - \theta'.$$

It is noticed that if all the elements of the external data, the initial data and the boundary data are null then the requirements of the mixed problem $\mathcal{P}$ are verified by $s^D = (u_i^D, \varphi_{ij}^D, \nu^D, \theta^D)$. The integral

$$\int_{\mathcal{D}} (\tau_{ij}\dot{\varepsilon}_{ij} + \sigma_{ij}\dot{\gamma}_{ij} + \mu_{ijk}\dot{\chi}_{ijk} + \lambda_i\dot{\nu}_{,i} - \xi\dot{\nu})\,dV$$

leads to an estimate of the difference $(u_i^D, \varphi_{ij}^D, \nu^D, \theta^D)$. We emphasize that in the following we will renounce to writing the upper index ^{D} for all the functions since confusion cannot occur.

For the purpose of estimating the difference of solutions, we will obtain for the previous integral an equality by using the equations of motion (5), by applying the divergence theorem and taking into account the fact that the boundary conditions are null, having

$$\begin{aligned} &\int_{\mathcal{D}} (\tau_{ij}\dot{\varepsilon}_{ij} + \sigma_{ij}\dot{\gamma}_{ij} + \mu_{ijk}\dot{\chi}_{ijk} + \lambda_i\dot{\nu}_{,i} - \xi\dot{\nu})\,dV \\ &\quad = -\int_{\mathcal{D}} (\rho\ddot{u}_i\dot{u}_i + I_{ks}\ddot{\varphi}_{js}\dot{\varphi}_{jk} + \rho k\ddot{\nu}\dot{\nu})\,dV. \end{aligned} \tag{54}$$

By the instrumentality of the constitutive equations (24), the previous equality (54) can be reformulated as

$$\begin{aligned}
&\int_{\mathcal{D}}\big[(C_{ijmn}\varepsilon_{mn}\dot{\varepsilon}_{ij}+G_{ijmn}\gamma_{mn}\dot{\varepsilon}_{ij}+F_{mnrij}\chi_{mnr}\dot{\varepsilon}_{ij}+d_{ijm}\nu_{,m}\dot{\varepsilon}_{ij}+a_{ij}\nu\dot{\varepsilon}_{ij})\\
&\quad+(G_{ijmn}\varepsilon_{mn}\dot{\gamma}_{ij}+B_{ijmn}\gamma_{mn}\dot{\gamma}_{ij}+D_{ijmnr}\chi_{mnr}\dot{\gamma}_{ij}+e_{ijm}\nu_{,m}\dot{\gamma}_{ij}+b_{ij}\nu\dot{\gamma}_{ij})\\
&\quad+(F_{ijkmn}\varepsilon_{mn}\dot{\chi}_{ijk}+D_{mnijk}\gamma_{mn}\dot{\chi}_{ijk}+A_{ijkmnr}\chi_{mnr}\dot{\chi}_{ijk}+f_{ijkm}\nu_{,m}\dot{\chi}_{ijk}+c_{ijk}\nu\dot{\chi}_{ijk})\\
&\quad+(d_{mni}\varepsilon_{mn}\dot{\nu}_{,i}+e_{mni}\gamma_{mn}\dot{\nu}_{,i}+f_{mnri}\chi_{mnr}\dot{\nu}_{,i}+g_{im}\nu_{,m}\dot{\nu}_{,i}+d_{i}\nu\dot{\nu}_{,i})\\
&\quad+(a_{ij}\varepsilon_{ij}\dot{\nu}+b_{ij}\gamma_{ij}\dot{\nu}+c_{ijk}\chi_{ijk}\dot{\nu}+d_{i}\nu_{,i}\dot{\nu}+\omega\nu\dot{\nu})\\
&\quad-\theta(\alpha_{ij}\dot{\varepsilon}_{ij}+\beta_{ij}\dot{\gamma}_{ij}+\gamma_{ijk}\dot{\chi}_{ijk}+a_{i}\dot{\nu}_{,i}+b\dot{\nu})\big]\,dV\\
&=-\int_{\mathcal{D}}(\rho\ddot{u}_{i}\dot{u}_{i}+I_{jk}\ddot{\varphi}_{js}\dot{\varphi}_{jk}+\rho k\ddot{\nu}\dot{\nu})\,dV, \qquad (55)
\end{aligned}$$

which, by using the properties of symmetry (23) and the constitutive equation $(24)_4$ becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigg[\int_{\mathcal{D}}\big(C_{ijmn}\varepsilon_{ij}\varepsilon_{mn}+2G_{ijmn}\varepsilon_{ij}\gamma_{mn}+2F_{ijmnr}\varepsilon_{ij}\chi_{mnr}+2d_{ijm}\varepsilon_{ij}\nu_{,m}\\
&\quad+2a_{ij}\varepsilon_{ij}\nu+B_{ijmn}\gamma_{ij}\gamma_{mn}+2D_{ijmnr}\gamma_{ij}\chi_{mnr}+2e_{ijm}\gamma_{ij}\nu_{,m}+2b_{ij}\gamma_{ij}\nu\\
&\quad+A_{ijkmnr}\chi_{ijk}\chi_{mnr}+2f_{ijkm}\chi_{ijk}\nu_{,m}+2c_{ijk}\chi_{ijk}\nu+2d_{i}\nu_{,i}\nu+g_{im}\nu_{,i}\nu_{,m}\\
&\quad+\omega\nu^{2}+\rho\dot{u}_{i}\dot{u}_{i}+I_{jk}\dot{\varphi}_{js}\dot{\varphi}_{jk}+\rho k\dot{\nu}\dot{\nu}+a\theta^{2}\big)\,dV\bigg]=\int_{\mathcal{D}}\dot{\eta}\theta\,dV. \qquad (56)
\end{aligned}$$

In order to obtain a new inequality, we will consider both inequality (26), in the case of $q = 0$, and the constitutive equation $(24)_7$, which means that

$$\int_{\mathcal{D}}\frac{1}{T_0}q_{i,i}\theta\,dV=-\int_{\mathcal{D}}\dot{\eta}\theta\,dV\geq 0, \qquad (57)$$

and from this inequality (57), along with Eq. (56), we will deduce the new inequality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigg[\int_{\mathcal{D}}\big(C_{ijmn}\varepsilon_{ij}\varepsilon_{mn}+2G_{ijmn}\varepsilon_{ij}\gamma_{mn}+2F_{ijmnr}\varepsilon_{ij}\chi_{mnr}+2d_{ijm}\varepsilon_{ij}\nu_{,m}\\
&\quad+2a_{ij}\varepsilon_{ij}\nu+B_{ijmn}\gamma_{ij}\gamma_{mn}+2D_{ijmnr}\gamma_{ij}\chi_{mnr}+2e_{ijm}\gamma_{ij}\nu_{,m}+2b_{ij}\gamma_{ij}\nu\\
&\quad+A_{ijkmnr}\chi_{ijk}\chi_{mnr}+2f_{ijkm}\chi_{ijk}\nu_{,m}+2c_{ijk}\chi_{ijk}\nu+2d_{i}\nu_{,i}\nu+g_{im}\nu_{,i}\nu_{,m}\\
&\quad+\omega\nu^{2}+\rho\dot{u}_{i}\dot{u}_{i}+I_{jk}\dot{\varphi}_{js}\dot{\varphi}_{jk}+\rho k\dot{\nu}\dot{\nu}+a\theta^{2}\big)\,dV\bigg]\leq 0, \qquad (58)
\end{aligned}$$

for all $(x,t)\in\mathcal{D}\times[0,\infty)$. From Eq. (58), taking into account the condition (53), we get

$$\dot{u}_i = 0, \qquad \dot{\varphi}_{ij} = 0, \qquad \dot{\nu} = 0, \quad \text{and} \quad \theta = 0.$$

So, considering that the initial conditions are null for the difference of the solutions, we see that for s^{D} we have $u_i = 0$, $\varphi_{ij} = 0$, $\nu = 0$ and $\theta = 0$, therefore, all its elements are null, which means that the solution is unique, thus the proof of this theorem is finished. □

4.2 Reciprocity

In the following, we consider two external data systems $S^{(\delta)}$ that act successively on the thermoelastic dipolar body with voids, defined as follows:

$$S^{(\delta)} = \{\mathcal{F}_i^{(\delta)}, \mathcal{G}_{jk}^{(\delta)}, \mathcal{L}^{(\delta)}, R^{(\delta)}, \widetilde{u}_i^{(\delta)}, \widetilde{\varphi}_{ij}^{(\delta)}, \widetilde{\nu}^{(\delta)}, \widetilde{\alpha}^{(\delta)}, \widetilde{t}_i^{(\delta)}, \widetilde{\mu}_{ij}^{(\delta)}, \widetilde{\lambda}^{(\delta)}, \widetilde{\theta}^{(\delta)}, \\ \widetilde{q}^{(\delta)}, u_i^{0(\delta)}, u_i^{1(\delta)}, \varphi_{ij}^{0(\delta)}, \varphi_{ij}^{1(\delta)}, \nu^{0(\delta)}, \nu^{1(\delta)}, \theta^{0(\delta)}, \theta^{1(\delta)}, \eta^{0(\delta)}\}, \tag{59}$$

where $\delta = 1, 2$. According to each system $S^{(\delta)}$, the solutions of the mixed problem will receive the notation

$$s^{(\delta)} = \{u_i^{(\delta)}, \varphi_{ij}^{(\delta)}, \nu^{(\delta)}, \theta^{(\delta)}, \alpha^{(\delta)}\}, \quad \delta = 1, 2.$$

The connection between the loading systems and their corresponding solutions is reflected by the next theorem.

Theorem 3 *Between the systems of charges $S^{(\delta)}$ and their corresponding solutions $s^{(\delta)}$, the next Betti-type relation of reciprocity holds*:

$$\begin{aligned}
&\int_{\mathcal{D}} \biggl(\rho \mathcal{F}_i^{(1)} * u_i^{(2)} + \rho \mathcal{G}_{ij}^{(1)} * \varphi_{ij}^{(2)} + \rho \mathcal{L}^{(1)} * \nu^{(2)} - t * R^{(1)} * \theta^{(2)} \\
&\quad - \frac{1}{T_0} t * q_i^{(1)} * \beta_i^{(2)}\biggr) dV + \int_{\partial\mathcal{D}} \biggl(t * t_i^{(1)} * u_i^{(2)} + t * \mu_{ij}^{(1)} * \varphi_{ij}^{(2)} \\
&\quad + t * \lambda^{(1)} * \nu^{(2)} + \frac{1}{T_0} t * q^{(1)} * \alpha^{(2)}\biggr) dA \\
&= \int_{\mathcal{D}} \biggl(\rho \mathcal{F}_i^{(2)} * u_i^{(1)} + \rho \mathcal{G}_{ij}^{(2)} * \varphi_{ij}^{(1)} + \rho \mathcal{L}^{(2)} * \nu^{(1)} \\
&\quad - t * R^{(2)} * \theta^{(1)} - \frac{1}{T_0} t * q_i^{(2)} * \beta_i^{(1)}\biggr) dV + \int_{\partial\mathcal{D}} \biggl(t * t_i^{(2)} * u_i^{(1)} + t * \mu_{ij}^{(2)} * \varphi_{ij}^{(1)} \\
&\quad + t * \lambda^{(2)} * \nu^{(1)} + \frac{1}{T_0} t * q^{(2)} * \alpha^{(1)}\biggr) dA,
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
&\mathcal{F}_i^{(\delta)} = g * f_i^{(\delta)} + (t u_i^{1(\delta)} + u_i^{0(\delta)}), && t_i^{(\delta)} = (\tau_{ji}^{(\delta)} + \sigma_{ji}^{(\delta)}) n_j, \\
&\mathcal{G}_{jk}^{(\delta)} = g * g_{jk}^{(\delta)} + I_{ks}(t \varphi_{js}^{1(\delta)} + \varphi_{js}^{0(\delta)}), && \mu_{jk}^{(\delta)} = \mu_{ijk}^{(\delta)} n_i, \\
&\mathcal{L}^{(\delta)} = g * \ell^{(\delta)} + k(t \nu^{1(\delta)} + \nu^{0(\delta)}), \quad \textit{and} && \lambda^{(\delta)} = \lambda_i^{(\delta)} n_i, \\
&R^{(\delta)} = \frac{1}{T_0} l * Q^{(\delta)} + \eta^{0(\delta)}, && q^{(\delta)} = q_i^{(\delta)} n_i,
\end{aligned}$$

obtained by using Eq. (33), *respectively, Eq.* (8).

Proof We introduce the notation

$$\mathcal{I}_{\delta\beta} = \tau_{ij}^{(\delta)} * \varepsilon_{ij}^{(\beta)} + \sigma_{ij}^{(\delta)} * \gamma_{ij}^{(\beta)} + \mu_{ijk}^{(\delta)} * \chi_{ijk}^{(\beta)} \\ - \eta^{(\delta)} * \theta^{(\beta)} - \xi^{(\delta)} * \nu^{(\beta)} + \lambda_i^{(\delta)} * \nu_{,i}^{(\beta)}. \tag{61}$$

In order to demonstrate the symmetry property of the previous relation, namely

$$\mathcal{I}_{\delta\beta} = \mathcal{I}_{\beta\delta}, \tag{62}$$

we will use the constitutive equations (24), and the symmetry relations (23), obtaining the relations

$$\begin{aligned}
&\tau_{ij}^{(1)} * \varepsilon_{ij}^{(2)} - G_{ijmn}\gamma_{mn}^{(1)} * \varepsilon_{ij}^{(2)} - F_{mnrij}\chi_{mnr}^{(1)} * \varepsilon_{ij}^{(2)} - d_{ijm}\nu_{,m}^{(1)} * \varepsilon_{ij}^{(2)} \\
&\quad - a_{ij}\nu^{(1)} * \varepsilon_{ij}^{(2)} + \alpha_{ij}\theta^{(1)} * \varepsilon_{ij}^{(2)} \\
&\quad = \tau_{ij}^{(2)} * \varepsilon_{ij}^{(1)} - G_{ijmn}\gamma_{mn}^{(2)} * \varepsilon_{ij}^{(1)} - F_{mnrij}\chi_{mnr}^{(2)} * \varepsilon_{ij}^{(1)} \\
&\quad - d_{ijm}\nu_{,m}^{(2)} * \varepsilon_{ij}^{(1)} - a_{ij}\nu^{(2)} * \varepsilon_{ij}^{(1)} + \alpha_{ij}\theta^{(2)} * \varepsilon_{ij}^{(1)},
\end{aligned} \tag{63}$$

$$\begin{aligned}
&\sigma_{ij}^{(1)} * \gamma_{ij}^{(2)} - G_{ijmn}\varepsilon_{mn}^{(1)} * \gamma_{ij}^{(2)} - D_{ijmnr}\chi_{mnr}^{(1)} * \gamma_{ij}^{(2)} \\
&\quad - e_{ijm}\nu_{,m}^{(1)} * \gamma_{ij}^{(2)} - b_{ij}\nu^{(1)} * \gamma_{ij}^{(2)} + \beta_{ij}\theta^{(1)} * \gamma_{ij}^{(2)} \\
&\quad = \sigma_{ij}^{(2)} * \gamma_{ij}^{(1)} - G_{ijmn}\varepsilon_{mn}^{(2)} * \gamma_{ij}^{(1)} - D_{ijmnr}\chi_{mnr}^{(2)} * \gamma_{ij}^{(1)} \\
&\quad - e_{ijm}\nu_{,m}^{(2)} * \gamma_{ij}^{(1)} - b_{ij}\nu^{(2)} * \gamma_{ij}^{(1)} + \beta_{ij}\theta^{(2)} * \gamma_{ij}^{(1)},
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\mu_{ijk}^{(1)} * \chi_{ijk}^{(2)} - F_{ijkmn}\varepsilon_{mn}^{(1)} * \chi_{ijk}^{(2)} - D_{mnijk}\gamma_{mn}^{(1)} * \chi_{ijk}^{(2)} - f_{ijkm}\nu_{,m}^{(1)} * \chi_{ijk}^{(2)} \\
&\quad - c_{ijk}\nu^{(1)} * \chi_{ijk}^{(2)} + \gamma_{ijk}\theta^{(1)} * \chi_{ijk}^{(2)} \\
&\quad = \mu_{ijk}^{(2)} * \chi_{ijk}^{(1)} - F_{ijkmn}\varepsilon_{mn}^{(2)} * \chi_{ijk}^{(1)} - D_{mnijk}\gamma_{mn}^{(2)} * \chi_{ijk}^{(1)} \\
&\quad - f_{ijkm}\nu_{,m}^{(2)} * \chi_{ijk}^{(1)} - c_{ijk}\nu^{(2)} * \chi_{ijk}^{(1)} + \gamma_{ijk}\theta^{(2)} * \chi_{ijk}^{(1)},
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\eta^{(1)} * \theta^{(2)} - \alpha_{ij}\varepsilon_{ij}^{(1)} * \theta^{(2)} - \beta_{ij}\gamma_{ij}^{(1)} * \theta^{(2)} - \gamma_{ijk}\chi_{ijk}^{(1)} * \theta^{(2)} \\
&\quad - a_i\nu_{,i}^{(1)} * \theta^{(2)} - b\nu^{(1)} * \theta^{(2)} \\
&\quad = \eta^{(2)} * \theta^{(1)} - \alpha_{ij}\varepsilon_{ij}^{(2)} * \theta^{(1)} - \beta_{ij}\gamma_{ij}^{(2)} * \theta^{(1)} \\
&\quad - \gamma_{ijk}\chi_{ijk}^{(2)} * \theta^{(1)} - a_i\nu_{,i}^{(2)} * \theta^{(1)} - b\nu^{(2)} * \theta^{(1)},
\end{aligned} \tag{66}$$

$$\begin{aligned}
&\xi^{(1)} * \nu^{(2)} + a_{ij}\varepsilon_{ij}^{(1)} * \nu^{(2)} + b_{ij}\gamma_{ij}^{(1)} * \nu^{(2)} + c_{ijk}\chi_{ijk}^{(1)} * \nu^{(2)} \\
&\quad + d_i\nu_{,i}^{(1)} * \nu^{(2)} - b\theta^{(1)} * \nu^{(2)} \\
&\quad = \xi^{(2)} * \nu^{(1)} + a_{ij}\varepsilon_{ij}^{(2)} * \nu^{(1)} + b_{ij}\gamma_{ij}^{(2)} * \nu^{(1)} \\
&\quad + c_{ijk}\chi_{ijk}^{(2)} * \nu^{(1)} + d_i\nu_{,i}^{(2)} * \nu^{(1)} - b\theta^{(2)} * \nu^{(1)},
\end{aligned} \tag{67}$$

$$\begin{aligned}
&\lambda_i^{(1)} * \nu_{,i}^{(2)} - d_{mni}\varepsilon_{mn}^{(1)} * \nu_{,i}^{(2)} - e_{mni}\gamma_{mn}^{(1)} * \nu_{,i}^{(2)} - f_{mnri}\chi_{mnr}^{(1)} * \nu_{,i}^{(2)} \\
&\quad - d_i\nu^{(1)} * \nu_{,i}^{(2)} + a_i\theta^{(1)} * \nu_{,i}^{(2)} \\
&\quad = \lambda_i^{(2)} * \nu_{,i}^{(1)} - d_{mni}\varepsilon_{mn}^{(2)} * \nu_{,i}^{(1)} - e_{mni}\gamma_{mn}^{(2)} * \nu_{,i}^{(1)} \\
&\quad - f_{mnri}\chi_{mnr}^{(2)} * \nu_{,i}^{(1)} - d_i\nu^{(2)} * \nu_{,i}^{(1)} + a_i\theta^{(2)} * \nu_{,i}^{(1)}.
\end{aligned} \tag{68}$$

By summing, member by member, Eqs. (63), (64), (65) and (68), and by decreasing Eqs. (66) and (67), the property of symmetry (62) for the expression $\mathcal{I}_{\delta\beta}$ will be obtained. With the support of both the geometric equations (1) and Eq. $(31)_4$, the expression $\mathcal{I}_{\delta\beta}$, represented by Eq. (61), can be rewritten as well thus:

$$\begin{aligned}
\mathcal{I}_{\delta\beta} &= \tau_{ij}^{(\delta)} * u_{j,i}^{(\beta)} + \sigma_{ij}^{(\delta)} * \left(u_{j,i}^{(\beta)} - \varphi_{ij}^{(\beta)}\right) + \mu_{ijk}^{(\delta)} * \varphi_{jk,i}^{(\beta)} \\
&\quad - \left(R^{(\delta)} - \frac{1}{T_0} l * q_{i,i}^{(\delta)}\right) * \theta^{(\beta)} - \xi^{(\delta)} * \nu^{(\beta)} + \lambda_i^{(\delta)} * \nu_{,i}^{(\beta)} \\
&= \Big\{\left[\left(\tau_{ij}^{(\delta)} + \sigma_{ij}^{(\delta)}\right) * u_j^{(\beta)}\right] + \left(\mu_{ijk}^{(\delta)} * \varphi_{jk}^{(\beta)}\right) + \left(\lambda_i^{(\delta)} * \nu^{(\beta)}\right) \\
&\quad + \left[\frac{1}{T_0}\left(l * q_i^{(\delta)}\right) * \theta^{(\beta)}\right]\Big\}_{,i} - \left[\left(\tau_{ij}^{(\delta)} + \sigma_{ij}^{(\delta)}\right)_{,i} * u_i^{(\beta)}\right] - \left(\mu_{ijk,i}^{(\delta)}\right. \\
&\quad \left. + \sigma_{jk}^{(\delta)}\right) * \varphi_{jk}^{(\beta)} - \left(\lambda_{i,i}^{(\delta)} + \xi^{(\delta)}\right) * \nu^{(\beta)} - R^{(\delta)} * \theta^{(\beta)} - \left[\frac{1}{T_0}\left(l * q_i^{(\delta)}\right) * \theta_{,i}^{(\beta)}\right],
\end{aligned} \tag{69}$$

a relation that alongside with Eqs. (8), (46), (47), the theorem of divergence and the properties of the convolution product, will lead to the relation

$$\begin{aligned}
&\int_{\mathcal{D}} (t * \mathcal{I}_{\delta\beta})\, dV \\
&\quad = \int_{\partial\mathcal{D}} \Big[t * t_i^{(\delta)} * u_i^{(\beta)} + t * \mu_{ijk}^{(\delta)} * \varphi_{jk}^{(\beta)} + t * \lambda^{(\delta)} * \nu^{(\beta)} \\
&\qquad + t * \left(\frac{1}{T_0}\hat{q}^{(\delta)}\right) * \theta^{(\beta)}\Big]\, dA + \int_{\mathcal{D}} \Big[\rho \mathcal{F}_i^{(\delta)} * u_i^{(\beta)} + \rho \mathcal{G}_{jk}^{(\delta)} * \varphi_{jk}^{(\beta)} \\
&\qquad + \rho \mathcal{L}^{(\delta)} * \nu^{(\beta)} - t * R^{(\delta)} * \theta^{(\beta)} - t * \left(\frac{1}{T_0}\hat{q}_i^{(\delta)}\right) * \theta_{,i}^{(\beta)} - \rho u_i^{(\delta)} * u_i^{(\beta)} \\
&\qquad - I_{ks}\varphi_{js}^{(\delta)} * \varphi_{jk}^{(\beta)} - \rho k \nu^{(\delta)} * \nu^{(\beta)}\Big]\, dV.
\end{aligned} \tag{70}$$

Taking into account that $I_{ks} = I_{sk}$ and $r_i = \frac{1}{T_0}\hat{q}_i$, we see that r is also of the form $r = \frac{1}{T_0}\hat{q}$, from Eq. (70), we deduce the following identity:

$$\begin{aligned}
&\int_{\mathcal{D}} \left[\rho \mathcal{F}_i^{(1)} * u_i^{(2)} + \rho \mathcal{G}_{ij}^{(1)} * \varphi_{ij}^{(2)} + \rho \mathcal{L}^{(1)} * \nu^{(2)} - t * R^{(1)} * \theta^{(2)} - t * r_i^{(1)} * \theta_{,i}^{(2)}\right] dV \\
&\qquad + \int_{\partial\mathcal{D}} \left[t * t_i^{(1)} * u_i^{(2)} + t * \mu_{ij}^{(1)} * \varphi_{ij}^{(2)} + t * \lambda^{(1)} * \nu^{(2)} + t * r^{(1)} * \theta^{(2)}\right] dA \\
&\quad = \int_{\mathcal{D}} \Big[\rho \mathcal{F}_i^{(2)} * u_i^{(1)} + \rho \mathcal{G}_{ij}^{(2)} * \varphi_{ij}^{(1)} + \rho \mathcal{L}^{(2)} * \nu^{(1)} \\
&\qquad - t * R^{(2)} * \theta^{(1)} - t * r_i^{(2)} * \theta_{,i}^{(1)}\Big]\, dV + \int_{\partial\mathcal{D}} \Big[t * t_i^{(2)} * u_i^{(1)} + t * \mu_{ij}^{(2)} * \varphi_{ij}^{(1)} \\
&\qquad + t * \lambda^{(2)} * \nu^{(1)} + t * r^{(2)} * \theta^{(1)}\Big]\, dA.
\end{aligned} \tag{71}$$

Adding $\alpha = \hat{\theta}$ and $\dot{\beta} = \theta_{,i}$ to the previous considerations, from the preceding identity we obtain Eq. (60), which means that Theorem 3 is fully demonstrated. □

4.3 Variational principle

For the goal of establishing the convolutional variational principle, see [20], regarding the three-phase-lag linear thermoelasticity theory of the dipolar materials with voids, we will consider r_i in the form:

$$r_i = r_i^{(a)} + r_i^{(b)} + r_i^{(c)} + r_i^{(d)}, \tag{72}$$

we will suppose that the tensors K_{ij}, respectively, K_{ij}^* are invertible, and we introduce the symmetrical tensors s_{ij} and s_{ij}^* by

$$s_{ij} = [K_{ij}]^{-1}, \qquad s_{ij}^* = \left[K_{ij}^*\right]^{-1}. \tag{73}$$

With the help of Eqs. (28) and (49) we deduce

$$\begin{aligned}
&(1 + D_t)s_{ij}r_j^{(a)} + \frac{1}{T_0}\beta_i = 0, \\
&(1 + D_t)s_{ij}r_j^{(b)} + \frac{t_T}{T_0}\theta_{,i} = 0, \\
&(1 + D_t)s_{ij}^*q_j^{(c)} + \beta_i = 0, \\
&(1 + D_t)s_{ij}^*r_j^{(d)} + \frac{t_\alpha}{T_0}\beta_i = 0,
\end{aligned} \tag{74}$$

where $\beta_i = \hat{\theta}_{,i}$ and $q_i^{(c)} = T_0\frac{\partial r_i^{(c)}}{\partial t}$. The option to choose this form for r_i, represented by Eq. (72), will be warranted later on.

We consider the admissible process, as the well-known ordered array

$$p = (u_i, \varphi_{ij}, \nu, \alpha, \theta, \varepsilon_{ij}, \gamma_{ij}, \chi_{ijk}, \lambda_i, \tau_{ij}, \sigma_{ij}, \beta_i, \eta, r_i, r_{i,i}, q_i), \tag{75}$$

whose components are functions assumed be sufficiently regular on their domain of definition. If we note by $\mathcal{A}$ the linear space, endowed with addition and scalar multiplication, of all admissible processes, we can define the functional $\mathcal{F}_t(p)$ as follows:

$$\begin{aligned}
\mathcal{F}_t(p) = {} & \frac{1}{2}\int_{\mathcal{D}} t * (C_{ijmn}\varepsilon_{mn} * \varepsilon_{ij} + 2G_{ijmn}\gamma_{mn} * \varepsilon_{ij} + 2F_{mnrij}\chi_{mnr} * \varepsilon_{ij} \\
& + 2d_{ijm}\nu_{,m} * \varepsilon_{ij} + 2a_{ij}\nu * \varepsilon_{ij} + B_{ijmn}\gamma_{mn} * \gamma_{ij} + 2D_{ijmnr}\chi_{mnr} * \gamma_{ij} \\
& + 2e_{ijm}\nu_{,m} * \gamma_{ij} + 2b_{ij}\nu * \gamma_{ij} + A_{ijkmnr}\chi_{mnr} * \chi_{ijk} + 2f_{ijkm}\nu_{,m} * \chi_{ijk} \\
& + 2c_{ijk}\nu * \chi_{ijk} + 2d_i\nu * \nu_{,i} + g_{im}\nu_{,i} * \nu_{,m} + \omega\nu * \nu + a\theta * \theta)\, dV \\
& + \int_{\mathcal{D}} \left[\rho u_i * u_i + I_{ks}\varphi_{js} * \varphi_{jk} + \rho k\nu * \nu - t * (\eta - R) * \theta\right] dV \\
& - \int_{\mathcal{D}} \left[\left(t * (\tau_{ji} + \sigma_{ji})_{,j} + \rho\mathcal{F}_i\right) * u_i + t * \tau_{ij} * \varepsilon_{ij} + t * \sigma_{ij} * \gamma_{ij}\right] dV \\
& - \int_{\mathcal{D}} \left[\left(t * (\mu_{ijk,i} + \sigma_{jk}) + \rho\mathcal{G}_{jk}\right) * \varphi_{jk} + t * \mu_{ijk} * \chi_{ijk}\right] dV - \int_{\mathcal{D}} \left[\left(t * (\lambda_{i,i} + \xi)\right.\right. \\
& \left.\left. + \rho\mathcal{L}\right) * \nu + t * \lambda_i * \nu_{,i} - t * \xi * \nu\right] dV + \frac{1}{2}\int_{\mathcal{D}} \left[l * T_0(1 + D_t)s_{ij}r_i^{(a)} * r_j^{(a)}\right. \\
& + l * \frac{T_0}{t_T}(1 + D_t)s_{ij}r_i^{(b)} * r_j^{(b)} + l * (1 + D_t)s_{ij}^*q_i^{(c)} * r_j^{(c)}
\end{aligned}$$

$$
\begin{aligned}
&+ l * \frac{T_0}{t_\alpha}(1+D_t) s^*_{ij} r^{(d)}_i * r^{(d)}_j \Big] \, dV \\
&+ \int_{\mathcal{D}} \Big[(l * r_i * \beta_i) + (l * r_i * \alpha_{,i}) - \Big(l * \frac{1}{T_0} q_i * \beta_i \Big) + (l * r_{i,i} * \alpha) \\
&- \Big(l * \frac{1}{T_0} q_{i,i} * \theta \Big) \Big] \, dV + \int_{\partial \mathcal{D}_u} (t * t_i * \widetilde{u}_i) \, dA + \int_{\partial \mathcal{D}^c_u} \big[t * (t_i - \widetilde{t}_i) * u_i \big] \, dA \\
&+ \int_{\partial \mathcal{D}_\varphi} (t * \mu_{ij} * \widetilde{\varphi}_{ij}) \, dA + \int_{\partial \mathcal{D}^c_\varphi} \big[t * (\mu_{ij} - \widetilde{\mu}_{ij}) * \varphi_{ij} \big] \, dA \\
&+ \int_{\partial \mathcal{D}_\nu} (t * \lambda * \widetilde{\nu}) \, dA + \int_{\partial \mathcal{D}^c_\nu} \big[t * (\lambda - \widetilde{\lambda}) * \nu \big] \, dA \\
&- \int_{\partial \mathcal{D}_\alpha} (l * r * \widetilde{\alpha}) \, dA - \int_{\partial \mathcal{D}^c_\alpha} \big[l * (r - \widetilde{r}) * \alpha \big] \, dA, \qquad (76)
\end{aligned}
$$

which can be rewritten in the form

$$
\begin{aligned}
\mathcal{F}_t(p) = {} & \frac{1}{2} \int_{\mathcal{D}} t * (C_{ijmn} \varepsilon_{mn} * \varepsilon_{ij} + 2G_{ijmn} \gamma_{mn} * \varepsilon_{ij} + 2F_{mnrij} \chi_{mnr} * \varepsilon_{ij} \\
&+ 2d_{ijm} \nu_{,m} * \varepsilon_{ij} + 2a_{ij} \nu * \varepsilon_{ij} + B_{ijmn} \gamma_{mn} * \gamma_{ij} + 2D_{ijmnr} \chi_{mnr} * \gamma_{ij} \\
&+ 2e_{ijm} \nu_{,m} * \gamma_{ij} + 2b_{ij} \nu * \gamma_{ij} + A_{ijkmnr} \chi_{mnr} * \chi_{ijk} + 2f_{ijkm} \nu_{,m} * \chi_{ijk} \\
&+ 2c_{ijk} \nu * \chi_{ijk} + 2d_i \nu * \nu_{,i} + g_{im} \nu_{,i} * \nu_{,m} + \omega \nu * \nu) \, dV + \int_{\mathcal{D}} \big[\rho u_i * u_i \\
&+ I_{ks} \varphi_{js} * \varphi_{jk} + \rho k \nu * \nu - t * (\eta - R) * \theta \big] \, dV + \int_{\mathcal{D}} \frac{1}{2a} \big[t * (\eta - \alpha_{ij} \varepsilon_{ij} \\
&- \beta_{ij} \gamma_{ij} - \gamma_{ijk} \chi_{ijk} - a_i \nu_{,i} - b\nu) * (\eta - \alpha_{ij} \varepsilon_{ij} - \beta_{ij} \gamma_{ij} - \gamma_{ijk} \chi_{ijk} - a_i \nu_{,i} - b\nu) \big] \, dV \\
&- \int_{\mathcal{D}} \big[\big(t * (\tau_{ji} + \sigma_{ji})_{,j} + \rho \mathcal{F}_i \big) * u_i + t * \tau_{ij} * \varepsilon_{ij} + t * \sigma_{ij} * \gamma_{ij} \big] \, dV \\
&- \int_{\mathcal{D}} \big[\big(t * (\mu_{ijk,i} + \sigma_{jk}) + \rho \mathcal{G}_{jk} \big) * \varphi_{jk} + t * \mu_{ijk} * \chi_{ijk} \big] \, dV - \int_{\mathcal{D}} \big[\big(t * (\lambda_{i,i} + \xi) \\
&+ \rho \mathcal{L} \big) * \nu + t * \lambda_i * \nu_{,i} - t * \xi * \nu \big] \, dV + \frac{1}{2} \int_{\mathcal{D}} (1+D_t) \Big(\hat{T}_0 s_{ij} r^{(a)}_i * r^{(a)}_j \\
&+ \frac{\hat{T}_0}{t_T} s_{ij} r^{(b)}_i * r^{(b)}_j + s^*_{ij} \hat{q}^{(c)}_i * r^{(c)}_j + \frac{\hat{T}_0}{t_\alpha} s^*_{ij} r^{(d)}_i * r^{(d)}_j \Big) \, dV \\
&+ \int_{\mathcal{D}} \Big(\hat{r}_i * \beta_i + \hat{r}_i * \alpha_{,i} - \frac{1}{T_0} \hat{q}_i * \beta_i + \hat{r}_{i,i} * \alpha - \frac{1}{T_0} \hat{q}_{i,i} * \theta \Big) \, dV \\
&+ \int_{\partial \mathcal{D}_u} (t * t_i * \widetilde{u}_i) \, dA + \int_{\partial \mathcal{D}^c_u} \big[t * (t_i - \widetilde{t}_i) * u_i \big] \, dA + \int_{\partial \mathcal{D}_\varphi} (t * \mu_{ij} * \widetilde{\varphi}_{ij}) \, dA \\
&+ \int_{\partial \mathcal{D}^c_\varphi} \big[t * (\mu_{ij} - \widetilde{\mu}_{ij}) * \varphi_{ij} \big] \, dA + \int_{\partial \mathcal{D}_\nu} (t * \lambda * \widetilde{\nu}) \, dA \\
&+ \int_{\partial \mathcal{D}^c_\nu} \big[t * (\lambda - \widetilde{\lambda}) * \nu) \big] \, dA - \int_{\partial \mathcal{D}_\alpha} (\hat{r} * \widetilde{\alpha}) \, dA - \int_{\partial \mathcal{D}^c_\alpha} \big[(r - \widetilde{r}) * \hat{\alpha} \big] \, dA. \qquad (77)
\end{aligned}
$$

At this stage of our study, we can approach the variational principle of the dipolar materials with voids with linear thermoelasticity under the influence of the three-phase-lag theory, its statement and proof being materialized in the next theorem.

Theorem 4 *If the symmetry relations occur on $\mathcal{D}$ and the symmetric tensors K_{ij}, K^*_{ij} are invertible and $t_\alpha > 0$, $t_T > 0$, then*

$$\delta\mathcal{F}_t(p) = 0, \quad t \geq 0, \tag{78}$$

which holds if p satisfies the mixed initial-boundary values problem $\mathcal{P}$, and reciprocal.

Proof We will develop the demonstration not as usual with the direct implication, but with the inverse implication, which means that if for p, represented by Eq. (75), it is supposed that it satisfies the mixed initial-boundary values problem $\mathcal{P}$, we will demonstrate that $\delta\mathcal{F}_t(p) = 0$. For this purpose, we consider a supplementary admissible process denoted

$$\overset{\geqslant}{p} = (\overset{\geqslant}{u}_i, \overset{\geqslant}{\varphi}_{ij}, \overset{\geqslant}{\nu}, \overset{\geqslant}{\alpha}, \overset{\geqslant}{\theta}, \overset{\geqslant}{\varepsilon}_{ij}, \overset{\geqslant}{\gamma}_{ij}, \overset{\geqslant}{\chi}_{ijk}, \overset{\geqslant}{\lambda}_i, \overset{\geqslant}{\tau}_{ij}, \overset{\geqslant}{\sigma}_{ij}, \overset{\geqslant}{\beta}_i, \overset{\geqslant}{\eta}, \overset{\geqslant}{r}_i, \overset{\geqslant}{r}_{i,i}, \overset{\geqslant}{q}_i),$$

which naturally fulfills the property:

$$p, \overset{\geqslant}{p} \in \mathcal{A} \quad \Rightarrow \quad p + k\overset{\geqslant}{p} \in \mathcal{A}, \quad \forall k \in \mathbb{R},$$

and, through both considered admissible processes p and $\overset{\geqslant}{p}$, we calculate the variation of our functional $\mathcal{F}_t(p)$. Thus

$$\begin{aligned}
&\delta\mathcal{F}_t(p) \\
&\quad = \int_{\mathcal{D}} \biggl\{ t * \biggl[C_{ijmn}\varepsilon_{mn} + G_{ijmn}\gamma_{mn} + F_{mnrij}\chi_{mnr} + d_{ijm}\nu_{,m} + a_{ij}\nu \\
&\qquad - \frac{\alpha_{ij}}{a}(\eta - \alpha_{ij}\varepsilon_{ij} - \beta_{ij}\gamma_{ij} - \gamma_{ijk}\chi_{ijk} - a_i\nu_{,i} - b\nu) - \tau_{ij} \biggr] * \overset{\geqslant}{\varepsilon}_{ij} + t * \biggl[G_{ijmn}\varepsilon_{mn} \\
&\qquad + B_{ijmn}\gamma_{mn} + D_{ijmnr}\chi_{mnr} + e_{ijm}\nu_{,m} + b_{ij}\nu - \frac{\beta_{ij}}{a}(\eta - \alpha_{ij}\varepsilon_{ij} - \beta_{ij}\gamma_{ij} \\
&\qquad - \gamma_{ijk}\chi_{ijk} - a_i\nu_{,i} - b\nu) - \sigma_{ij} \biggr] * \overset{\geqslant}{\gamma}_{ij} + t * \biggl[F_{mnrij}\varepsilon_{ij} + D_{ijmnr}\gamma_{ij} \\
&\qquad + A_{ijkmnr}\chi_{mnr} + f_{ijkm}\nu_{,m} + c_{ijk}\nu - \frac{\gamma_{ijk}}{a}(\eta - \alpha_{ij}\varepsilon_{ij} - \beta_{ij}\gamma_{ij} - \gamma_{ijk}\chi_{ijk} \\
&\qquad - a_i\nu_{,i} - b\nu) - \mu_{ijk} \biggr] * \overset{\geqslant}{\chi}_{ijk} + t * \biggl[d_{ijm}\varepsilon_{ij} + e_{ijm}\gamma_{ij} + f_{ijkm}\chi_{ijk} + g_{im}\nu_{,i} \\
&\qquad + d_i\nu - \frac{a_i}{a}(\eta - \alpha_{ij}\varepsilon_{ij} - \beta_{ij}\gamma_{ij} - \gamma_{ijk}\chi_{ijk} - a_i\nu_{,i} - b\nu) - \lambda_i \biggr] * \overset{\geqslant}{\nu}_{,i} \\
&\qquad + t * \biggl[a_{ij}\varepsilon_{ij} + b_{ij}\gamma_{ij} + c_{ijk}\chi_{ijk} + d_i\nu_{,i} + \omega\nu - \frac{b}{a}(\eta - \alpha_{ij}\varepsilon_{ij} - \beta_{ij}\gamma_{ij} - \gamma_{ijk}\chi_{ijk} \\
&\qquad - a_i\nu_{,i} - b\nu) + \xi \biggr] * \overset{\geqslant}{\nu} \biggr\} \, dV + \int_{\mathcal{D}} \biggl\{ t * \biggl[-\theta + \frac{1}{a}(\eta - \alpha_{ij}\varepsilon_{ij} - \beta_{ij}\gamma_{ij} - \gamma_{ijk}\chi_{ijk} - a_i\nu_{,i} \\
&\qquad - b\nu) \biggr] * \overset{\geqslant}{\eta} \biggr\} \, dV + \int_{\mathcal{D}} \bigl[t * (R - \eta - r_{i,i}) * \overset{\geqslant}{\theta} \bigr] \, dV + \int_{\mathcal{D}} \bigl[\bigl(\rho u_i - t * (\tau_{ji} + \sigma_{ji})_{,j} \\
&\qquad - \rho\mathcal{F}_i \bigr) * \overset{\geqslant}{u}_i \bigr] \, dV + \int_{\mathcal{D}} \bigl[(I_{ks}\varphi_{js} - t * \mu_{ijk,i} - t\sigma_{jk} - \rho\mathcal{G}_{jk}) * \overset{\geqslant}{\varphi}_{jk} \bigr] \, dV \\
&\qquad + \int_{\mathcal{D}} \bigl[(\rho k\nu - t * \lambda_{i,i} - t * \xi - \rho\mathcal{L}) * \overset{\geqslant}{\nu} \bigr] \, dV + \int_{\mathcal{D}} \bigl[\hat{T}_0(1 + D_t)s_{ij}r_j^{(a)} + \hat{\beta}_i \bigr] * \overset{\geqslant}{r}{}_i^{(a)} \, dV
\end{aligned}$$

$$+\int_{\mathcal{D}}\left[\frac{\hat{T}_0}{t_T}(1+D_t)s_{ij}r_j^{(b)}+\beta_i\right]*\overset{\geqslant}{r}{}_i^{(b)}\,dV+\int_{\mathcal{D}}\left[T_0(1+D_t)s_{ij}^*r_j^{(c)}+\hat{\beta}_i\right]*\overset{\geqslant}{r}{}_i^{(c)}\,dV$$
$$+\int_{\mathcal{D}}\left[\frac{\hat{T}_0}{t_\alpha}(1+D_t)s_{ij}^*r_j^{(d)}+\hat{\beta}_i\right]*\overset{\geqslant}{r}{}_i^{(d)}\,dV+\int_{\mathcal{D}}\left[\left(r_i-\frac{q_i}{T_0}\right)*\overset{\geqslant}{\alpha}_{,i}+\left(r_{i,i}-\frac{q_{i,i}}{T_0}\right)\right.$$
$$\left.*\overset{\geqslant}{\alpha}\right]dV+\int_{\mathcal{D}}\left[(\alpha_{,i}-\beta_i)*\overset{\geqslant}{r}_i+(\alpha-\hat{\theta})*\overset{\geqslant}{r}_{i,i}\right]dV+\int_{\mathcal{D}_u}\left[t*(\widetilde{u}_i-u_i)*\overset{\geqslant}{t}_i\right]dA$$
$$+\int_{\partial\mathcal{D}_u^c}\left[t*(t_i-\widetilde{t}_i)*\overset{\geqslant}{u}_i\right]dA+\int_{\partial\mathcal{D}_\varphi}\left[t*(\widetilde{\varphi}_{ij}-\varphi_{ij})*\overset{\geqslant}{\mu}_{ij}\right]dA+\int_{\partial\mathcal{D}_\varphi^c}\left[t*(\mu_{ij}-\widetilde{\mu}_{ij})\right.$$
$$\left.*\overset{\geqslant}{\varphi}_{ij}\right]dA+\int_{\partial\mathcal{D}_\nu}\left[t*(\widetilde{\nu}-\nu)*\overset{\geqslant}{\lambda}\right]dA+\int_{\partial\mathcal{D}_\nu^c}\left[t*(\lambda-\widetilde{\lambda})*\overset{\geqslant}{\nu}\right]dA$$
$$+\int_{\partial\mathcal{D}_\alpha}\left[t*(\widetilde{\alpha}-\alpha)*\overset{\geqslant}{r}\right]dA+\int_{\partial\mathcal{D}_\alpha^c}\left[t*(r-\widetilde{r})*\overset{\geqslant}{\alpha}\right]dA. \tag{79}$$

By the instrumentality of the writing of r as a four element sum, represented by Eq. (72), we reach the aim of calculating the variation for $\hat{r}_i*\beta_i$.

In the case that p satisfies the mixed initial-boundary value problem $\mathcal{P}$, it is obvious that the equations of motion $(47)_1$, $(47)_2$, the equilibrated forces balance equation (47), the energy equation (46), the initial conditions (29) and the boundary conditions (30) are verified, as is Eq. (74). Considering all these equations and conditions described above, from Eq. (79) we deduce that

$$\delta\mathcal{F}_t(p)=0.$$

In order to demonstrate the direct implication, we suppose that

$$\delta\mathcal{F}_t(p)=0,\quad t\geq 0, \tag{80}$$

and we will establish that p satisfies the mixed initial-boundary values problem $\mathcal{P}$.

For this purpose, let us consider, following the procedure presented by Gurtin in [20], an arbitrary displacement $\overset{\geqslant}{u}_i$ in such a way that both this and its partial differentiations related to the Cartesian coordinates vanish on the cylinder $\partial\mathcal{D}\times[0,\infty)$, and choosing the process $\overset{\geqslant}{p}$ which is admissible and has the following components:

$$\overset{\geqslant}{p}=(\overset{\geqslant}{u}_i,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),$$

Eq. (80) becomes as follows:

$$\int_{\mathcal{D}}\left[t*(\tau_{ji}+\sigma_{ji})_{,j}+\rho\mathcal{F}_i-\rho u_i\right]*\overset{\geqslant}{u}_i\,dV=0,$$

which, under the influence of the variational calculus fundamental lemma, leads to the first equation of motion $(47)_1$.

We are continuing with the same reasoning in choosing the particular form of the admissible process $\overset{\geqslant}{p}$, but we impose the condition that $\overset{\geqslant}{u}_i$ vanishes on $\partial\mathcal{D}_u\times[0,\infty)$, and, by reusing the fundamental lemma of the variational calculus, from Eqs. (79) and (80), we obtain

$$t * (t_i - \widetilde{t}_i) = 0 \quad \text{on } \partial\mathcal{D}_u \times [0, \infty),$$

which entails actually the sixth boundary condition of Eq. (30).

By applying successively the reasoning presented above, by the most appropriate decision on the form of the admissible process p, in the context of using the variational calculus fundamental lemma, we will always obtain either an equation or a condition of the mixed initial-boundary value problem $\mathcal{P}$ in relation to each new form customized for the admissible process p.

At this stage of the demonstration it is the optimal moment to warrant writing the entropy flux vector r as a sum of four terms, represented by Eq. (72).

Therefore, opting for the admissible process p having the components:

$$\overset{\geqslant}{p} = \big(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \overset{\geqslant}{r}{}_i^{(a)}, 0, 0\big),$$

and using Eqs. (79) and (80), we obtain, with this previous $\overset{\geqslant}{p}$, the equation

$$\hat{T}_0(1 + D_t)s_{ij}r_j^{(a)} + \hat{\beta}_i = 0,$$

from which we have

$$(1 + D_t)r_j^{(a)} + \frac{1}{T_0}K_{ij}\beta_i = 0. \tag{81}$$

Reconsidering the option for the form of the admissible process $\overset{\geqslant}{p}$ as follows:

$$\overset{\geqslant}{p} = \big(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \overset{\geqslant}{r}{}_i^{(b)}, 0, 0\big),$$

for this distinctive form of $\overset{\geqslant}{p}$, by the instrumentality of Eqs. (79) and (80), we obtain the next equation:

$$\frac{\hat{T}_0}{t_T}(1 + D_t)s_{ij}r_j^{(b)} + \beta_i = 0,$$

from which we infer the following equation:

$$(1 + D_t)r_j^{(b)} + \frac{t_T}{T_0}K_{ij}\theta_{,i} = 0. \tag{82}$$

Furthermore, making the choice for the form of the admissible process $\overset{\geqslant}{p}$ in the following way:

$$\overset{\geqslant}{p} = \big(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \overset{\geqslant}{r}{}_i^{(c)}, 0, 0\big),$$

and taking into account this form of $\overset{\geqslant}{p}$ and Eqs. (79) and (80), we obtain the next equation:

$$T_0(1 + D_t)s_{ij}^{*}r_j^{(c)} + \hat{\beta}_i = 0,$$

a relation that can be reformulated as

$$(1 + D_t)r_j^{(c)} + \frac{1}{T_0}K^*_{ij}\hat{\beta}_i = 0. \tag{83}$$

Considering the last version for the particular form of the admissible process $\overset{\geqslant}{p}$, namely

$$\overset{\geqslant}{p} = \left(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \overset{\geqslant}{r}{}_i^{(d)}, 0, 0\right),$$

through Eqs. (79) and (80), we deduce

$$\frac{\hat{T}_0}{t_\alpha}(1 + D_t)s^*_{ij}r_j^{(d)} + \hat{\beta}_i = 0,$$

from which we have

$$(1 + D_t)r_j^{(d)} + \frac{t_\alpha}{T_0}K^*_{ij}\beta_i = 0. \tag{84}$$

Making the sum of Eqs. (81)–(84), Eq. (49) is deduced, which represents actually the explanation of the choice of writing the entropy flux vector r in the form of the four term sum (72), this motivation completing the demonstration of Theorem 4. □

5 Conclusions

This study deepens and expands the results obtained previously and by modeling the linear theory of the dipolar materials with voids into the three-phase-lag theory we develop this analysis around the support offered by the notion of the volume fraction, which is related to the introduction of an additional kinematic freedom degree. The approach of this theoretical model consists in obtaining results regarding the uniqueness of the solution, different from the classical ones, such as the use of the dissipative inequality instead of the Laplace transform technique, applied for instance in [1, 2, 5, 12, 13, 23]. Along with the other results obtained throughout this article, they provide a basis for developing the study of these structures in multiple directions, one of which is to consider a specific aspect: that of the isotropic and homogeneous materials.

The mathematical model presented throughout this work consists in the study of the mixed initial-boundary value problem for the linear theory of the thermoelastic dipolar materials with voids under the influence of the three-phase-lag theory, deducing in this context the corresponding constitutive equations, based on the methods from the classical theory of elasticity. At the same time, we develop the study of these materials by obtaining, through less-used methods, a uniqueness result and a reciprocal relation. For the same purpose, we generalize a well-known variational principle for anisotropic and non-homogeneous materials, by going deeper into the study of the three-phase-lag theory's influence on the dipolar thermoelastic materials with voids. Modeling the present theoretical mathematical pattern, we establish a starting point and a theoretical basis for numerical applications of the theory of these materials, leading to a better understanding of the dipolar materials with voids, which have such a wide applicability.

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