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Unique iterative positive solutions for a singular p-Laplacian fractional differential equation system with infinite-point boundary conditions

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Abstract

By using the method of mixed monotone operator, a unique positive solution is obtained for a singular p-Laplacian boundary value system with infinite-point boundary conditions in this paper. Green's function is derived and some useful properties of the Green's function are obtained. Based upon these new properties and by using mixed monotone operator, the existence results of the positive solutions for the boundary value problem are established. Moreover, the unique positive solution that we obtained in this paper is dependent on λ, μ, and an iterative sequence and convergence rate, which are important for practical application, are given. An example is given to demonstrate the application of our main results.

MSC: 34B08; 34B16; 34B18

Keywords: Fractional differential equation system; Infinite-point; Singular p-Laplacian; Iterative positive solution

1 Introduction

Fractional calculus has been shown to be more accurate and realistic than integer order models, and it also provides an excellent tool to describe the hereditary properties of materials and processes, particularly in viscoelasticity, electrochemistry, porous media, and so on. Fractional differentials arise in a variety of different areas such as physics, chemistry, electrical networks, economics, rheology, biology chemical, image processing, and so on. There has been a significant development in the study of fractional differential equations in recent years, for an extensive collection of such literature, readers can refer to [2, 4, 6–11, 13, 16–18, 21–23, 26, 28, 32–43, 45–49], and there are many types of fractional differential equations, such as values at infinite points are involved in the boundary conditions that we refer the reader to [6, 7, 34, 38, 39]. For some differential equations in which fractional derivatives are involved in the nonlinear terms the reader can refer to [6–10, 23]. In order to meet the needs, the p-Laplacian equation is introduced in some boundary value problems, fractional differential equation system of p-Laplacian, and we refer the reader to [3, 9–12, 15, 17, 22, 24, 28, 30, 31, 35–37, 41] for some relevant work. Besides, there are a lot of methods to study fractional differential equations such as mixed monotone operator (see [8, 10, 18, 21, 26, 32]), degree theory (see [20, 48]), spectral analysis (see [2, 20, 29, 41, 42, 49]), bifurcation method (see [19, 20, 29, 31]), and so on. Motivated by the excellent results above, in this paper, we consider the following infinite-point singular p-Laplacian fractional differential equation boundary value system:

$$\begin{cases} D_{0^+}^{\alpha}(\varphi_{p_1}(D_{0^+}^{\gamma}u))(t) + \lambda^{\frac{1}{q_1-1}} f(t, u(t), D_{0^+}^{\mu_1}u(t), D_{0^+}^{\mu_2}u(t), \ldots, D_{0^+}^{\mu_{n-2}}u(t), v(t)) = 0, \\ \quad 0 < t < 1, \\ D_{0^+}^{\beta}(\varphi_{p_2}(D_{0^+}^{\delta}v))(t) + \overline{\lambda}^{\frac{1}{q_2-1}} g(t, u(t), D_{0^+}^{\overline{\mu}_1}u(t), D_{0^+}^{\overline{\mu}_2}u(t), \ldots, D_{0^+}^{\overline{\mu}_{m-2}}u(t)) = 0, \\ \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \quad j = 0, 1, 2, \ldots, n-2; \\ D_{0^+}^{\gamma}u(0) = 0, \qquad D_{0^+}^{r_1}u(1) = \sum_{j=1}^{\infty} \eta_j D_{0^+}^{r_2}u(\xi_j), \\ v^{(j)}(0) = 0, \quad j = 0, 1, 2, \ldots, m-2; \\ D_{0^+}^{\delta}v(0) = 0, \qquad D_{0^+}^{\overline{r}_1}v(1) = \sum_{j=1}^{\infty} \overline{\eta}_j D_{0^+}^{\overline{r}_2}v(\overline{\xi}_j), \end{cases} \tag{1.1}$$

where $\frac{1}{2} < \alpha, \beta \le 1$, $n-1 < \gamma \le n (n \ge 3), m-1 < \delta \le m (m \ge 3), m \le n$, $r_1, r_2 \in [2, n-2], \overline{r}_1, \overline{r}_2 \in [2, m-2]$, $r_2 \le r_1, \overline{r}_2 \le \overline{r}_1$, p-Laplacian operator φ_{p_i} is defined as $\varphi_{p_i}(s) = |s|^{p_i-2}s, p_i, q_i > 1, \frac{1}{p_i} + \frac{1}{q_i} = 1$ $(i = 1, 2), i-1 < \mu_i \le i$ $(i = 1, 2, \ldots, n-2), k-1 < \overline{\mu}_i \le k (k = 1, 2, \ldots, m-2)$, and $\eta_j, \overline{\eta}_j \ge 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1, 0 < \overline{\xi}_1 < \overline{\xi}_2 < \cdots < \overline{\xi}_{j-1} < \overline{\xi}_j < \cdots < 1$ $(j = 1, 2 \cdots)$ are parameters, $f \in C((0,1) \times (0, +\infty)^n, \mathbb{R}_+^1))(\mathbb{R}_+^1 = [0, +\infty)$ and $f(t, x_1, x_2, \ldots, x_n)$ has singularity at $x_i = 0$ $(i = 1, 2, \ldots, n)$ and $t = 0, 1$, $g \in C((0,1) \times (0, +\infty)^{m-1}, \mathbb{R}_+^1)), D_{0^+}^{\alpha}u, D_{0^+}^{\gamma}u, D_{0^+}^{\mu}u, D_{0^+}^{\beta}u, D_{0^+}^{\delta}u, D_{0^+}^{\overline{\mu}}u, D_{0^+}^{r_i}u, D_{0^+}^{\overline{r}_i}u$ $(i = 1, 2)$ are the standard Riemann–Liouville derivatives.

In this paper, we investigate the existence of positive solutions for a singular infinite-point p-Laplacian boundary value system. Compared with our papers [8, 10], values at infinite points are involved in the boundary conditions of BVP (1.1), but integral boundary conditions are involved in the boundary value conditions in [8, 10], $p_1 \ne p_2$ in the p-Laplacian system of (1.1), but $p_1 = p_2$ in the p-Laplacian system in [8, 10]. Moreover, the method that we used in [8, 10] is reducing order, but the method that we use in this paper is introducing an order relation without reducing order; and at last we still get an iterative solution for infinite-point fractional differential equation system. Compared with [38], fractional derivatives are involved in the nonlinear terms and boundary conditions for BVP (1.1), and we get a more precise result, that is, the positive solution is obtained by iterative sequences which begin with a simple function. Compared with [6, 39], the equation in this paper is a p-Laplacian boundary value system which is a great extension from the general fractional differential equation, and at the same time we get an iterative solution. Compared with [39], the equation in our paper is an equation system and the uniqueness of an iterative positive solution of equation (1.1) that we obtained is dependent on λ, μ.

For convenience of presentation, we here list some conditions to be used throughout the paper.

(S_1) $f(t, x_1, x_2, \ldots, x_n) = \phi(t, x_1, x_2, \ldots, x_n) + \psi(t, x_1, x_2, \ldots, x_n)$, where $\phi : (0,1) \times (0, +\infty)^n \to \mathbb{R}_+^1$ is continuous, $\phi(t, x_1, x_2, \ldots, x_n)$ may be singular at $t = 0, 1$, and is nondecreasing on $x_i > 0$ $(i = 1, 2, \ldots, n)$. $\psi : (0,1) \times (0, +\infty)^n \to \mathbb{R}_+^1$ is continuous, $\psi(t, x_1, x_2, \ldots, x_n)$ may be singular at $t = 0, 1$, $x_i = 0$ $(i = 1, 2, \ldots, n)$ and is nonincreasing on $x_i > 0$ $(i = 1, 2, \ldots, n)$.

(S_2) There exists $0 < \sigma < 1$ such that, for all $x_i > 0$ $(i = 1, 2, \ldots, n)$ and $t, l \in (0, 1)$,

$$\phi(t, lx_1, lx_2, \ldots, lx_n) \ge l^{\sigma^{\frac{1}{q_1-1}}} \phi(t, x_1, x_2, \ldots, x_n),$$
$$\psi\left(t, l^{-1}x_1, l^{-1}x_2, \ldots, l^{-1}x_n\right) \ge l^{\sigma^{\frac{1}{q_1-1}}} \psi(t, x_1, x_2, \ldots, x_n),$$

where q_1 is defined by (1.1).

(S_3) $g \in C((0,1) \times \mathbb{R}_+^{m-1}, \mathbb{R}_+^1)$, $g(t, x_1, x_2, \ldots, x_{m-1})$ is nondecreasing on $x_i > 0$ $(i = 1, 2, \ldots, m-1)$ and $g(t, 1, 1, \ldots, 1) \not\equiv 0$, $t \in (0, 1)$. Moreover, there exists $\varsigma \in (0, 1)$ such that, for all $x_i > 0$ $(i = 1, 2, \ldots, m-1)$ and $t, l \in (0, 1)$,

$$g(t, lx_1, lx_2, \ldots, lx_{m-1}) \ge l^{\varsigma^{\frac{1}{q_2-1}}} g(t, x_1, x_2, \ldots, x_{m-1}),$$

where q_2 is defined by (1.1).

(S_4)

$$0 < \int_0^1 \phi^2(\tau, 1, 1, \ldots, 1)\, d\tau < +\infty,$$
$$0 < \int_0^1 \tau^{-2(\gamma-1)\sigma^{\frac{1}{q_1-1}}} \psi^2(\tau, 1, 1, \ldots, 1)\, d\tau < +\infty,$$
$$0 < \int_0^1 g^2(\tau, 1, \ldots, 1)\, d\tau < +\infty.$$

Remark 1.1 According to (S_2) and (S_3), for all $x_i > 0$ $(i = 1, 2, \ldots, n)$, $\sigma, t \in (0, 1)$, and $l \ge 1$, we have

$$\phi(t, lx_1, lx_2, \ldots, lx_n) \le l^{\sigma^{\frac{1}{q_1-1}}} \phi(t, x_1, x_2, \ldots, x_n),$$
$$\psi\left(t, l^{-1}x_1, l^{-1}x_2, \ldots, l^{-1}x_n\right) \le l^{\sigma^{\frac{1}{q_1-1}}} \psi(t, x_1, x_2, \ldots, x_n),$$
$$g(t, lx_1, \ldots, lx_{m-1}) \le l^{\varsigma^{\frac{1}{q_2-1}}} g(t, x_1, \ldots, x_{m-1}),$$

where q_i $(i = 1, 2)$ is defined by (1.1).

2 Preliminaries and lemmas

For some basic definitions and lemmas about the theory of fractional calculus, which are useful for the following research, the reader can refer to the recent literature such as [14, 25, 27], we omit some definitions and properties of fractional calculus here.

Lemma 2.1 *Let $y, \overline{y} \in L^1(0,1) \cap C(0,1)$, then the equation of the BVPs*

$$\begin{cases} -D_{0^+}^{\gamma} u(t) = y(t), \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \quad j = 0, 1, 2, \ldots, n-2; \qquad D_{0^+}^{r_1} u(1) = \sum_{j=1}^{\infty} \eta_j D_{0^+}^{r_2} u(\xi_j), \end{cases} \tag{2.1}$$

$$\begin{cases} -D_{0^+}^{\delta} v(t) = \overline{y}(t), \quad 0 < t < 1, \\ v^{(j)}(0) = 0, \quad j = 0, 1, 2, \ldots, m-2; \qquad D_{0^+}^{\overline{r}_1} v(1) = \sum_{j=1}^{\infty} \overline{\eta}_j D_{0^+}^{\overline{r}_2} v(\overline{\xi}_j) \end{cases} \tag{2.2}$$

has integral representation

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)y(s)\,ds, \\ v(t) &= \int_0^1 H(t,s)\overline{y}(s)\,ds, \end{aligned} \tag{2.3}$$

respectively, where

$$G(t,s) = \frac{1}{\Delta\Gamma(\gamma)} \begin{cases} \Gamma(\gamma)t^{\gamma-1}P(s)(1-s)^{\gamma-r_1-1} - \Delta(t-s)^{\gamma-1}, & 0 \le s \le t \le 1, \\ \Gamma(\gamma)t^{\gamma-1}P(s)(1-s)^{\gamma-r_1-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

$$H(t,s) = \frac{1}{\overline{\Delta}\Gamma(\delta)} \begin{cases} \Gamma(\delta)t^{\delta-1}\overline{P}(s)(1-s)^{\delta-\overline{r}_1-1} - \overline{\Delta}(t-s)^{\delta-1}, & 0 \le s \le t \le 1, \\ \Gamma(\delta)t^{\delta-1}\overline{P}(s)(1-s)^{\delta-\overline{r}_1-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.5}$$

in which

$$P(s) = \frac{1}{\Gamma(\gamma-r_1)} - \frac{1}{\Gamma(\gamma-r_2)} \sum_{s\le\xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\gamma-r_2-1} (1-s)^{r_1-r_2},$$

$$\overline{P}(s) = \frac{1}{\Gamma(\delta-\overline{r}_1)} - \frac{1}{\Gamma(\delta-\overline{r}_2)} \sum_{s\le\overline{\xi}_j} \overline{\eta}_j \left(\frac{\overline{\xi}_j - s}{1-s}\right)^{\delta-\overline{r}_1-1} (1-s)^{\overline{r}_1-\overline{r}_2},$$

$$\Delta = \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)} \sum_{j=1}^{\infty} \eta_j \xi_j^{\gamma-r_2-1} \neq 0,$$

$$\overline{\Delta} = \frac{\Gamma(\delta)}{\Gamma(\delta-\overline{r}_1)} - \frac{\Gamma(\delta)}{\Gamma(\delta-\overline{r}_2)} \sum_{j=1}^{\infty} \overline{\eta}_j \overline{\xi}_j^{\delta-\overline{r}_2-1} \neq 0.$$

Proof We only need to prove (2.4), the proof of (2.5) is similar to the proof of (2.4). By means of the definition of fractional differential integral, we can reduce (2.1) to an equivalent integral equation

$$u(t) = -I_{0^+}^{\gamma} y(t) + C_1 t^{\gamma-1} + C_2 t^{\gamma-2} + \cdots + C_n t^{\gamma-n}$$

for $C_i \in \mathbb{R}$ $(i = 1,2,\ldots,n)$. From $u^{(j)}(0) = 0$ $(j = 0,1,2,\ldots,n-2)$, we have $C_i = 0$ $(i = 2,3,\ldots,n)$. Consequently, we get

$$u(t) = C_1 t^{\gamma-1} - I_{0^+}^{\gamma} y(t).$$

By some properties of the fractional integrals and fractional derivatives, we have

$$\begin{aligned} D_{0^+}^{r_1} u(t) &= C_1 \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)} t^{\gamma-r_1-1} - I_{0^+}^{\gamma-r_1} y(t), \\ D_{0^+}^{r_2} u(t) &= C_1 \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)} t^{\gamma-r_2-1} - I_{0^+}^{\gamma-r_2} y(t). \end{aligned} \tag{2.6}$$

On the other hand, $D_{0^+}^{r_1}u(1) = \sum_{j=1}^{\infty}\eta_j D_{0^+}^{r_2}u(\xi_j)$ combining with (2.6), we get

$$\begin{aligned} C_1 &= \int_0^1 \frac{(1-s)^{\gamma-r_1-1}}{\Gamma(\gamma-p_1)\Delta}y(s)\,ds - \sum_{j=1}^{\infty}\eta_j\int_0^{\xi_j}\frac{(\xi_j-s)^{\gamma-r_2-1}}{\Gamma(\gamma-r_2)\Delta}y(s)\,ds \\ &= \int_0^1 \frac{(1-s)^{\gamma-r_1-1}P(s)}{\Delta}y(s)\,ds, \end{aligned}$$

where

$$\begin{aligned} P(s) &= \frac{1}{\Gamma(\gamma-r_1)} - \frac{1}{\Gamma(\gamma-r_2)}\sum_{s\le\xi_j}\eta_j\left(\frac{\xi_j-s}{1-s}\right)^{\gamma-r_2-1}(1-s)^{r_1-r_2}, \\ \Delta &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)}\sum_{j=1}^{\infty}\eta_j\xi_j^{\gamma-r_2-1}. \end{aligned}$$

Hence,

$$\begin{aligned} u(t) &= C_1 t^{\gamma-1} - I_{0^+}^{\gamma}y(t) \\ &= -\int_0^t \frac{\Delta(t-s)^{\gamma-1}}{\Gamma(\gamma)\Delta}y(s)\,ds + \int_0^1 \frac{(1-s)^{\gamma-r_1-1}t^{\gamma-1}P(s)}{\Delta}y(s)\,ds. \end{aligned}$$

Therefore, we get (2.4), similarly, we get (2.5).

Moreover, by (2.4), for $i = 1, 2, \ldots, n-2$, we have

$$D_{0^+}^{\mu_i}G(t,s) = \frac{1}{\Delta\Gamma(\gamma-\mu_i)}\begin{cases} \Gamma(\gamma)t^{\gamma-\mu_i-1}P(s)(1-s)^{\gamma-r_1-1} - \Delta(t-s)^{\gamma-\mu_i-1}, \\ \quad 0\le s\le t\le 1, \\ \Gamma(\gamma)t^{\gamma-\mu_i-1}P(s)(1-s)^{\gamma-r_1-1}, \\ \quad 0\le t\le s\le 1. \end{cases} \tag{$*$}$$

□

Lemma 2.2 *Let $\Delta, \overline{\Delta} > 0$, then the Green function defined by* (2.3) *satisfies*:

(1) $G, H : [0,1]\times[0,1] \to \mathbb{R}_+^1$ *are continuous and* $G(t,s), H(t,s) > 0$, *for all* $t,s\in(0,1)$;

(2)

$$\frac{1}{\Gamma(\gamma)}t^{\gamma-1}j(s) \le G(t,s) \le a^{\star}t^{\gamma-1}, \quad t,s\in[0,1], \tag{2.7}$$

$$\frac{1}{\Gamma(\delta)}t^{\delta-1}\bar{j}(s) \le H(t,s) \le \overline{a}^{\star}t^{\delta-1}, \quad t,s\in[0,1], \tag{2.8}$$

where

$$\begin{aligned} &j(s) = (1-s)^{\gamma-r_1-1}\big[1-(1-s)^{r_1}\big], \qquad \bar{j}(s) = (1-s)^{\delta-\bar{r}_1-1}\big[1-(1-s)^{\bar{r}_1}\big], \\ &a^{\star} = \frac{1}{\Delta\Gamma(\gamma-r_1)}, \qquad \overline{a}^{\star} = \frac{1}{\overline{\Delta}\Gamma(\delta-\bar{r}_1)}, \end{aligned}$$

in which $\Delta, \overline{\Delta}$ *are defined as in Lemma* 2.1.

Proof Let

$$G_\star(t,s) = \frac{1}{\Gamma(\gamma)} \begin{cases} t^{\gamma-1}(1-s)^{\gamma-r_1-1} - (t-s)^{\gamma-1}, & 0 \le s \le t \le 1, \\ t^{\gamma-1}(1-s)^{\gamma-r_1-1}, & 0 \le t \le s \le 1. \end{cases}$$

From [13], for $r_1 \in [2, n-2]$, we have

$$0 \le t^{\gamma-1}(1-s)^{\gamma-r_1-1}\bigl[1-(1-s)^{r_1}\bigr] \le \Gamma(\alpha)G_\star(t,s) \le t^{\gamma-1}(1-s)^{\gamma-r_1-1}. \tag{2.9}$$

By direct calculation, we get $P'(s) \ge 0$, $s \in [0,1]$, and so $P(s)$ is nondecreasing with respect to s. For $r_2 \le r_1$, $r_1, r_2 \in [2, n-2]$, $s \in [0,1]$, we get

$$\begin{aligned} \Gamma(\alpha)P(s) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)} \sum_{s \le \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\gamma-r_2-1} (1-s)^{r_1-r_2} \\ &\ge \Gamma(\gamma)P(0) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)} \sum \eta_j \xi_j^{\gamma-r_2-1} = \Delta. \end{aligned} \tag{2.10}$$

By (2.4) and (2.10), we have

$$\Delta\Gamma(\gamma)G(t,s) \ge \begin{cases} \Delta t^{\gamma-1}(1-s)^{\gamma-r_1-1} - \Delta(t-s)^{\gamma-1}, & 0 \le s \le t \le 1, \\ \Delta t^{\gamma-1}(1-s)^{\gamma-r_1-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.11}$$

So, by (2.9) and (2.11), we have

$$\begin{aligned} \Delta\Gamma(\gamma)G(t,s) &\ge \Delta\Gamma(\gamma)G_\star(t,s) \\ &\ge \Delta t^{\gamma-1}(1-s)^{\gamma-r_1-1}\bigl[1-(1-s)^{r_1}\bigr], \end{aligned} \tag{2.12}$$

hence,

$$G(t,s) \ge \frac{1}{\Gamma(\gamma)} t^{\gamma-1} j(s).$$

On the other hand,

$$P(s) = \frac{1}{\Gamma(\gamma-r_1)} - \frac{1}{\Gamma(\gamma-r_2)} \sum_{s \le \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\gamma-r_2-1} (1-s)^{r_1-r_2} \le \frac{1}{\Gamma(\gamma-r_1)},$$

clearly,

$$\Delta\Gamma(\gamma)G(t,s) \le \Gamma(\gamma)t^{\gamma-1}P(s)(1-s)^{\gamma-r_1-1},$$

hence,

$$G(t,s) \le a^\star t^{\gamma-1}.$$

So the proof of (2.7) is completed. Similarly, (2.8) also holds. □

To study PFDE (1.1), in what follows we consider the associated linear PFDEs:

$$\begin{cases} D_{0^+}^{\alpha}(\varphi_{p_1}(D_{0+}^{\gamma}u))(t)+\rho(t)=0, \quad 0<t<1, \\ u^{(j)}(0)=0, \quad j=0,1,2,\dots,n-2; \\ D_{0+}^{\gamma}u(0)=0, \qquad D_{0^+}^{r_1}u(1)=\sum_{j=1}^{\infty}\eta_j D_{0^+}^{r_2}u(\xi_j), \end{cases} \tag{2.13}$$

and

$$\begin{cases} D_{0^+}^{\beta}(\varphi_{p_2}(D_{0+}^{\delta}v))(t)+\overline{\rho}(t)=0, \quad 0<t<1, \\ v^{(j)}(0)=0, \quad j=0,1,2,\dots,m-2; \qquad D_{0+}^{\delta}v(0)=0, \\ D_{0^+}^{\overline{r}_1}v(1)=\sum_{j=1}^{\infty}\overline{\eta}_j D_{0^+}^{\overline{r}_2}v(\overline{\xi}_j), \end{cases} \tag{2.14}$$

where $\rho(t),\overline{\rho}(t)\in L^1((0,1),\mathbb{R}_+^1)\cap C((0,1),\mathbb{R}_+^1)$.

Lemma 2.3 *PFDEs* (2.13), (2.14) *have the unique positive solution*

$$u(t)=\int_0^1 G(t,s)\left(\int_0^s \overline{a}(s-\tau)^{\alpha-1}\rho(\tau)\,d\tau\right)^{q_1-1}ds, \quad t\in[0,1], \tag{2.15}$$

$$v(t)=\int_0^1 H(t,s)\left(\int_0^s \overline{b}(s-\tau)^{\beta-1}\overline{\rho}(\tau)\,d\tau\right)^{q_2-1}ds, \quad t\in[0,1], \tag{2.16}$$

respectively, in which $\overline{a}=\frac{1}{\Gamma(\alpha)},\overline{b}=\frac{1}{\Gamma(\beta)}$.

Proof Let $h=D_{0^+}^{\gamma}u, k=\varphi_{p_1}(h)$, then the solution of the initial value problem

$$\begin{cases} D_{0^+}^{\alpha}k(t)+\rho(t)=0, \quad 0<t<1, \\ k(0)=0, \end{cases} \tag{2.17}$$

is given by $k(t)=C_1t^{\alpha-1}-I_{0^+}^{\alpha}\rho(t), t\in[0,1]$. By the relations $k(0)=0$, we have $C_1=0$, and hence,

$$k(t)=-I_{0^+}^{\alpha}\rho(t), \quad t\in[0,1]. \tag{2.18}$$

By $D_{0^+}^{\gamma}u=h, h=\varphi_{p_1}^{-1}(k)$, we have from (2.17) that the solution of (2.13) satisfies

$$\begin{cases} D_{0^+}^{\gamma}u(t)=\varphi_{p_1}^{-1}(-I_{0^+}^{\alpha}\rho(t)), \quad 0<t<1, \\ u^{(j)}(0)=0, \quad j=0,1,2,\dots,n-2; \qquad D_{0^+}^{r_1}u(1)=\sum_{j=1}^{\infty}\eta_j D_{0^+}^{r_2}u(\xi_j). \end{cases} \tag{2.19}$$

By (2.3), the solution of Eq. (2.19) can be written as

$$u(t)=-\int_0^1 G(t,s)\varphi_{p_1}^{-1}\big(-I_{0^+}^{\alpha}\rho(s)\big)\,ds, \quad t\in[0,1].$$

Since $\rho(s)\geq 0, s\in[0,1]$, we have $\varphi_{p_1}^{-1}(-I_{0^+}^{\alpha}\rho(s))=-(I_{0^+}^{\alpha}\rho(s))^{q_1-1}, s\in[0,1]$, which implies that the solution of Eq. (2.13) is

$$u(t)=\int_0^1 G(t,s)\left(\int_0^s \overline{a}(s-\tau)^{\alpha-1}\rho(\tau)\,d\tau\right)^{q_1-1}ds, \quad t\in[0,1].$$

Similarly, the solution of Eq. (2.14) is

$$v(t) = \int_0^1 H(t,s)\left(\int_0^s \overline{b}(s-\tau)^{\beta-1}\overline{\rho}(\tau)\,d\tau\right)^{q_2-1} ds, \quad t \in [0,1],$$

where $\overline{a}, \overline{b}$ is defined as Lemma 2.3. □

The vector (u,v) is a solution of system (1.1) if and only if $(u,v) \in E \times E$ (E is defined as (2.23)) is a solution of the following nonlinear integral equation system:

$$\begin{cases} u(t) = \lambda \int_0^1 G(t,s)(\int_0^s \overline{a}(s-\tau)^{\alpha-1} f(\tau, u(\tau), D_{0^+}^{\mu_1} u(\tau), \ldots, \\ \qquad D_{0^+}^{\mu_{n-2}} u(\tau), v(\tau))\,d\tau)^{q_1-1}\,ds, \\ v(\tau) = \overline{\lambda} \int_0^1 H(t,s)(\int_0^s \overline{b}(s-\tau)^{\beta-1} g(\tau, u(\tau), D_{0^+}^{\overline{\mu}_1} u(\tau), \ldots, \\ \qquad D_{0^+}^{\overline{\mu}_{m-1}} u(\tau))\,d\tau)^{q_2-1}\,ds, \\ \quad t \in [0,1]. \end{cases} \tag{2.20}$$

By Lemma 2.1, we know that the unique solution for problem (1.1) has the following integral formulation:

$$\begin{aligned} u(t) = \lambda \int_0^1 G(t,s)(\int_0^s \overline{a}(s-\tau)^{\alpha-1} f\Bigl(\tau, u(\tau), D_{0^+}^{\mu_1} u(\tau), \ldots, D_{0^+}^{\mu_{n-2}} u(\tau), \Bigl[\overline{\lambda} \int_0^1 H(\tau,s) \\ \times \left(\int_0^s \overline{b}(s-w)^{\beta-1} g\bigl(\tau, u(\tau), D_{0^+}^{\overline{\mu}_1} u(\tau), \ldots, D_{0^+}^{\overline{\mu}_{m-1}} u(\tau)\bigr)\,dw\right)^{q_2-1} ds\Bigr]\,d\tau\Bigr)^{q_1-1} ds, \\ t \in [0,1]. \end{aligned} \tag{2.21}$$

Let P be a normal cone of a Banach space E, and $e \in P$, $e > \theta$, where θ is a zero element of E. Define a component of P by $Q_e = \{u \in P|$ there exists a constant $C > 0$ such that $\frac{1}{C}e \preceq u \preceq Ce\}$. $A : Q_e \times Q_e \to P$ is said to be mixed monotone if $A(u,y)$ is nondecreasing in u and nonincreasing in y, i.e., $u_1 \preceq u_2 (u_1, u_2 \in Q_e)$ implies $A(u_1,y) \preceq A(u_2,y)$ for any $y \in Q_e$, and $y_1 \preceq y_2 (y_1, y_2 \in Q_e)$ implies $A(u,y_1) \succeq A(u,y_2)$ for any $u \in Q_e$. The element $u^\star \in Q_e$ is called a fixed point of A if $A(u^\star, u^\star) = u^\star$.

Lemma 2.4 ([1, 5, 21, 44]) *Let E be a Banach space and P be a normal cone of the Banach space E. Suppose that $A : Q_e \times Q_e \to Q_e$ is a mixed monotone operator and there exists a constant σ, $0 < \sigma < 1$, such that*

$$A\left(lx, \frac{1}{l}y\right) \succeq l^\sigma A(x,y), \quad x, y \in Q_e, 0 < l < 1, \tag{2.22}$$

then A has a unique fixed point $x^\star \in Q_e$, and for any $x_0 \in Q_e$, we have

$$\lim_{k\to\infty} x_k = x^\star,$$

where

$$x_k = A(x_{k-1}, x_{k-1}), \quad k = 1, 2, \ldots,$$

and the convergence rate is

$$\|x_k - x^\star\| = o\big(1 - r^{\sigma^k}\big),$$

where r is a constant, $0 < r < 1$, *dependent on* x_0.

Lemma 2.5 ([1, 5, 21, 44]) *Let E be a Banach space and P be a normal cone of the Banach space E. Suppose that* $A : Q_e \times Q_e \to Q_e$ *is a mixed monotone operator and there exists a constant* $\sigma \in (0,1)$ *such that* (2.22) *holds. If* $x^\star_\lambda$ *is a unique solution of the equation*

$$\lambda A(x,x) = x, \quad \lambda > 0,$$

in Q_e, *then*

(1) *For any* $\lambda_0 \in (0,+\infty)$, $\|x^\star_\lambda - x^\star_{\lambda_0}\| \to 0, \lambda \to \lambda_0$;
(2) *If* $0 < \sigma < \frac{1}{2}$, *then* $0 < \lambda_1 < \lambda_2$ *implies* $x^\star_{\lambda_1} \preceq x^\star_{\lambda_2}, x^\star_{\lambda_1} \neq x^\star_{\lambda_2}$, *and*

$$\lim_{\lambda\to 0}\|u^\star_\lambda\| = 0, \qquad \lim_{\lambda\to+\infty}\|u^\star_\lambda\| = +\infty.$$

Let

$$E = \big\{u | u \in C[0,1], D^{\mu_i}_{0^+}u, D^{\overline{\mu}_j}_{0^+}u \in C[0,1], i = 1,2,\ldots,n-2; j = 1,2,\ldots,m-2\big\} \tag{2.23}$$

be a Banach space with the norm

$$\begin{aligned}\|u\| = \max\Big\{&\max_{t\in[0,1]}\big|u(t)\big|, \max_{t\in[0,1]}\big|D^{\mu_i}_{0^+}u(t)\big|, \max_{t\in[0,1]}\big|D^{\overline{\mu}_j}_{0^+}u(t)\big|,\\ &i = 1,2,\ldots,n-2; j = 1,2,\ldots,m-2\Big\}.\end{aligned}$$

Moreover, we define a cone of E by

$$\begin{aligned}P = \big\{&u \in E : u(t) \geq 0, D^{\mu_i}_{0^+}u(t) \geq 0, D^{\overline{\mu}_j}_{0^+}u(t) \geq 0,\\ &t \in [0,1], i = 1,2,\ldots,n-2; j = 1,2,\ldots,m-2\big\}.\end{aligned}$$

Clearly, P is a normal cone, and E is endowed with an order relation $u \preceq v$ if and only if $u(t) \leq v(t), D^{\mu_i}_{0^+}u(t) \leq D^{\mu_i}_{0^+}v(t), D^{\overline{\mu}_j}_{0^+}u(t) \leq D^{\overline{\mu}_j}_{0^+}v(t)$ $(i = 1,2,\ldots,n-2; j = 1,2,\ldots,m-2), t \in [0,1]$. Let $e(t) = t^{\gamma-1}$ for $t \in [0,1]$, also define a component of P by

$$Q_e = \left\{u \in P : \text{ there exists } M \geq 1, \frac{1}{M}e(t) \leq u(t) \leq Me(t), t \in [0,1]\right\}.$$

3 Main results

Theorem 3.1 *Suppose that* (S_1)–(S_4) *hold. Then PFDE* (1.1) *has a unique positive solution* $(u^\star_\lambda, v^\star_\mu)$ *which satisfies*

$$\frac{1}{M}t^{\gamma-1} \leq u^\star_\lambda(t) \leq Mt^{\gamma-1}, \quad t \in [0,1],$$

$$\frac{\overline{\lambda}}{\Gamma(\delta)}t^{\delta-1}\overline{b}^{q_2-1}\left(\frac{\overline{\rho}}{M}\right)^{\varsigma}\int_0^1 \bar{j}(s)\left[\int_0^s (s-w)^{\beta-1}w^{(\gamma-1)\varsigma\frac{1}{q_2-1}}g(w,1,\dots,1)\,dw\right]^{q_2-1}ds \le v_\mu^\star(t)$$
$$\le \overline{\lambda}\overline{a}^\star t^{\delta-1}\overline{b}^{q_2-1}(M\rho+1)^{\varsigma}\int_0^1\left[\int_0^s (s-w)^{\beta-1}g(w,1,\dots,1)\,dw\right]^{q_2-1}ds,$$
$$t\in[0,1], \tag{3.1}$$

and $v_{\overline{\lambda}}^\star$ is dependent on $u_\lambda^\star$,

$$v_{\overline{\lambda}}^\star(t) = \overline{\lambda}\int_0^1 H(t,s)\left(\int_0^s \overline{b}(s-\tau)^{\beta-1}g\big(\tau,u_\lambda^\star(\tau),D_{0^+}^{\overline{\mu}_1}u_\lambda^\star(\tau),\dots,D_{0^+}^{\overline{\mu}_{m-2}}u_\lambda^\star(\tau)\big)\,d\tau\right)^{q_2-1}ds,$$
$$t\in[0,1]. \tag{$\star$}$$

Moreover, $u_\lambda^\star$ satisfies

(1) *For $\lambda_0\in(0,\infty)$, $\|u_\lambda^\star - u_{\lambda_0}^\star\|\to 0, \lambda\to\lambda_0$;*
(2) *If $0<\sigma<\frac{1}{2}$, then $0<\lambda_1<\lambda_2$ implies $u_{\lambda_1}^\star\preceq u_{\lambda_2}^\star, u_{\lambda_1}^\star\ne u_{\lambda_2}^\star$, and*

$$\lim_{\lambda\to 0}\|u_\lambda^\star\| = 0,\qquad \lim_{\lambda\to+\infty}\|u_\lambda^\star\| = +\infty.$$

Moreover, for any $u_0(t)\in Q_e$, constructing a successive sequence:

$$u_{k+1}(t) = \lambda\int_0^1 G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}(\phi\big(\tau,u_k(\tau),D_{0^+}^{\mu_1}u_k(\tau),\dots,\right.$$
$$D_{0^+}^{\mu_{n-2}}u_k(\tau),Au_k(\tau)\big)$$
$$\left.+\psi\big(\tau,u_k(\tau),D_{0^+}^{\mu_1}u_k(\tau),\dots,D_{0^+}^{\mu_{n-2}}u_k(\tau),Au_k(\tau)\big)\,d\tau\right]^{q_1-1}ds,$$
$$k=1,2,\dots,t\in[0,1],$$

and we have $\|u_k-u_\lambda^\star\|\to 0$ as $k\to\infty$, the convergence rate is

$$\|u_k-u_\lambda^\star\| = o\big(1-r^{\sigma^k}\big),$$

where r is a constant, $0<r<1$, dependent on u_0 (A is defined by (3.2)).

Proof We now consider the existence of a positive solution to problem (1.1). From the discussion in Sect. 2, we only need to consider the existence of a positive solution to PFDE (2.21). In order to realize this purpose, define the operator $A: Q_e\to P$ by

$$Au(\tau) = \overline{\lambda}\int_0^1 H(\tau,s)\left[\int_0^s \overline{b}(s-w)^{\beta-1}g\big(w,u(w),D_{0^+}^{\overline{\mu}_1}u(w),\dots,D_{0^+}^{\overline{\mu}_{m-2}}u(w)\big)\,dw\right]^{q_2-1}ds,$$
$$\tau\in[0,1], \tag{3.2}$$

define the operator $T_\lambda: Q_e\times Q_e\to P$ by

$$T_\lambda(u,z)(t) = \lambda\int_0^1 G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\dots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big)\right.$$

$$
\begin{aligned}
&+\psi\big(\tau,z(\tau),D_{0^+}^{\mu_1}z(\tau),\dots,D_{0^+}^{\mu_{n-2}}z(\tau),Au(\tau)\big)\,d\tau\Bigg]^{q_1-1}ds,\\
&t\in[0,1].
\end{aligned}
\tag{3.3}
$$

Now we prove that $T_\lambda : Q_e \times Q_e \to P$ is well defined. For any $u \in Q_e$, that is, there exists $M \geq 1$ such that $\frac{1}{M}e(w) \leq u(w) \leq Me(w)$, by (3.2), (S_3), and Remark 1.1, for all $\tau \in [0,1]$, we have

$$
\begin{aligned}
&Au(\tau)\\
&=\overline{\lambda}\int_0^1 H(\tau,s)\biggl[\int_0^s \overline{b}(s-w)^{\beta-1}g\big(w,u(w),D_{0^+}^{\overline{\mu}_1}u(w),\dots,D_{0^+}^{\overline{\mu}_{m-2}}u(w)\big)\,dw\biggr]^{q_2-1}ds\\
&\leq\overline{\lambda}\overline{a}^\star\tau^{\delta-1}\int_0^1\biggl[\int_0^s\overline{b}(s-w)^{\beta-1}g\biggl(w,Mw^{\gamma-1},\frac{M\Gamma(\gamma)}{\Gamma(\gamma-\overline{\mu}_1)}w^{\gamma-1-\overline{\mu}_1},\dots,\\
&\quad\frac{M\Gamma(\gamma)}{\Gamma(\gamma-\overline{\mu}_{m-2})}w^{\gamma-1-\overline{\mu}_{m-2}}\biggr)\,dw\biggr]^{q_2-1}ds\\
&\leq\overline{\lambda}\overline{a}^\star\tau^{\delta-1}\int_0^1\biggl[\int_0^s\overline{b}(s-w)^{\beta-1}g\biggl(w,Mw^{\gamma-1}+1,\frac{M\Gamma(\gamma)}{\Gamma(\gamma-\overline{\mu})}w^{\gamma-1-\overline{\mu}_1}+1,\dots,\\
&\quad\frac{M\Gamma(\gamma)}{\Gamma(\gamma-\overline{\mu})}w^{\gamma-1-\overline{\mu}_{m-2}}+1\biggr)\,dw\biggr]^{q_2-1}ds\\
&\leq\overline{\lambda}\overline{a}^\star\tau^{\delta-1}\int_0^1\biggl[\int_0^s\overline{b}(s-w)^{\beta-1}g\biggl(w,M+1,\frac{M\Gamma(\gamma)}{\Gamma(\gamma-\overline{\mu}_1)}+1,\dots,\\
&\quad\frac{M\Gamma(\gamma)}{\Gamma(\gamma-\overline{\mu}_{m-2})}+1\biggr)\,dw\biggr]^{q_2-1}ds\\
&\leq\overline{\lambda}\overline{a}^\star\tau^{\delta-1}\int_0^1\biggl[\int_0^s\overline{b}(s-w)^{\beta-1}g(w,M\rho+1,\dots,M\rho+1,)\,dw\biggr]^{q_2-1}ds\\
&\leq\overline{\lambda}\overline{a}^\star\tau^{\delta-1}\int_0^1\biggl[\int_0^s(M\rho+1)^{\varsigma\frac{1}{q_2-1}}\overline{b}(s-w)^{\beta-1}g(w,1,\dots,1)\,dw\biggr]^{q_2-1}ds\\
&\leq\overline{\lambda}\overline{a}^\star\tau^{\delta-1}\overline{b}^{q_2-1}(M\rho+1)^{\varsigma}\int_0^1\biggl[\int_0^s(s-w)^{\beta-1}g(w,1,\dots,1)\,dw\biggr]^{q_2-1}ds,
\end{aligned}
\tag{3.4}
$$

where $\rho=\max\{1,\frac{\Gamma(\gamma)}{\Gamma(\gamma-\overline{\mu}_{m-2})}\}$, and

$$
\begin{aligned}
&Au(\tau)\\
&=\overline{\lambda}\int_0^1 H(\tau,s)\biggl[\int_0^s \overline{b}(s-w)^{\beta-1}g\big(w,u(w),D_{0^+}^{\overline{\mu}_1}u(w),\dots,D_{0^+}^{\overline{\mu}_{m-2}}u(w)\big)\,dw\biggr]^{q_2-1}ds\\
&\geq\overline{\lambda}\frac{1}{\Gamma(\delta)}\tau^{\delta-1}\int_0^1\overline{j}(s)\biggl[\int_0^s\overline{b}(s-w)^{\beta-1}g\biggl(w,\frac{1}{M}w^{\gamma-1},\frac{\Gamma(\gamma)}{M\Gamma(\gamma-\overline{\mu}_1)}w^{\gamma-1-\overline{\mu}_1},\dots,\\
&\quad\frac{\Gamma(\gamma)}{M\Gamma(\gamma-\overline{\mu}_{n-2})}w^{\gamma-1-\overline{\mu}_{n-2}}\biggr)\,dw\biggr]^{q_2-1}ds\\
&\geq\overline{\lambda}\frac{1}{\Gamma(\delta)}\tau^{\delta-1}\int_0^1\overline{j}(s)\biggl[\int_0^s\overline{b}(s-w)^{\beta-1}g\biggl(w,\frac{\overline{\rho}}{M}w^{\gamma-1},\dots,\frac{\overline{\rho}}{M}w^{\gamma-1}\biggr)\,dw\biggr]^{q_2-1}ds
\end{aligned}
$$

$$\begin{aligned}
&\geq \overline{\lambda}\frac{1}{\Gamma(\delta)}\tau^{\delta-1}\int_0^1 \overline{j}(s)\left[\int_0^s \overline{b}(s-w)^{\beta-1}w^{(\gamma-1)\varsigma^{\frac{1}{q_2-1}}}\left(\frac{\overline{\rho}}{M}\right)^{\varsigma^{\frac{1}{q_2-1}}} g(w,1,\ldots,1)\,dw\right]^{q_2-1} ds \\
&\geq \overline{\lambda}\frac{1}{\Gamma(\delta)}\tau^{\delta-1}\overline{b}^{q_2-1}\left(\frac{\overline{\rho}}{M}\right)^{\varsigma} \\
&\quad\times \int_0^1 \overline{j}(s)\left[\int_0^s (s-w)^{\beta-1}w^{(\gamma-1)\varsigma^{\frac{1}{q_2-1}}} g(w,1,\ldots,1)\,dw\right]^{q_2-1} ds,
\end{aligned}\tag{3.5}$$

where $\overline{\rho}=\min\{1,\frac{\Gamma(\gamma)}{\Gamma(\gamma-\overline{\mu}_1)}\}$. By (S_4), we get that $A:P_e\to P$ is well defined. From (3.4), (S_1), and Remark 1.1, we have

$$\begin{aligned}
&\phi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\ldots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big) \\
&\quad\leq \phi\bigg(\tau,Me(\tau),D_{0^+}^{\mu_1}Me(\tau),\ldots,D_{0^+}^{\mu_{n-2}}Me(\tau),\mu\overline{a}^{\star}\tau^{\delta-1}\overline{b}^{q_2-1}(M\rho+1)^{\varsigma} \\
&\qquad\times \int_0^1\left[\int_0^s (s-w)^{\beta-1}g(w,1,1)\,dw\right]^{q_2-1} ds\bigg) \\
&\quad\leq \phi\big(\tau,\ldots,Mb+1,Mb+1,M^{\varsigma}b+1\big) \\
&\quad\leq (Mb+1)^{\sigma^{\frac{1}{q_2-1}}}\phi(\tau,1,\ldots,1) \\
&\quad\leq 2^{\sigma^{\frac{1}{q_1-1}}}b^{\sigma^{\frac{1}{q_1-1}}}M^{\sigma^{\frac{1}{q_1-1}}}\phi(\tau,1,\ldots,1),\quad \tau\in(0,1),
\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}
M>\max&\bigg\{\bigg\{\lambda a^{\star}\overline{a}^{q_1-1}\frac{1}{(2\alpha-1)^{\frac{q_1-1}{2}}}\int_0^1\bigg[2^{\sigma^{\frac{1}{q_1-1}}}b^{\sigma^{\frac{1}{q_1-1}}}\left(\int_0^s \phi^2(\tau,1,\ldots,1)\right)^{\frac{1}{2}}+c^{-\sigma^{\frac{1}{q_1-1}}} \\
&\times\left(\int_0^s \tau^{-2(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\psi^2(\tau,1,1,\ldots,1)\,d\tau\right)^{\frac{1}{2}}\bigg]^{q_1-1} ds\bigg\}^{\frac{1}{1-\sigma}},1,2c,b^{-1}, \\
&\bigg\{\lambda\frac{1}{\Gamma(\gamma)}\int_0^1 j(s)\bigg[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(c^{-\sigma^{\frac{1}{q_1-1}}}\tau^{(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\phi(\tau,1,\ldots,1)+2^{-\sigma^{\frac{1}{q_1-1}}}b^{-\sigma^{\frac{1}{q_1-1}}} \\
&\psi(\tau,1,1,\ldots,1)\big)\,d\tau\bigg]^{q_1-1} ds\bigg\}^{-\frac{1}{1-\sigma}}\bigg\},
\end{aligned}$$

in which

$$\begin{aligned}
b=\max&\left\{\frac{\Gamma(\gamma)}{\Gamma(\gamma-\mu_{n-2})},1,\overline{\lambda}\overline{a}^{\star}\tau^{\delta-1}\overline{b}^{q_2-1}(\rho+1)^{\varsigma}\right. \\
&\left.\times\int_0^1\left[\int_0^s (s-w)^{\beta-1}g(w,1,\ldots,1)\,dw\right]^{q_2-1} ds\right\}.
\end{aligned}$$

By (3.4), (S_1), and (S_2), we also have

$$\begin{aligned}
&\psi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\ldots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big) \\
&\quad\leq \psi\bigg(\tau,\frac{1}{M}\tau^{\gamma-1},\frac{\Gamma(\gamma)}{M\Gamma(\gamma-\mu_1)}\tau^{\gamma-1-\mu_1},\ldots,
\end{aligned}$$

$$
\begin{aligned}
&\frac{\Gamma(\gamma)}{M\Gamma(\gamma-\mu_{n-2})}\tau^{\gamma-1-\mu_{n-2}}, \frac{\overline{\lambda}}{\Gamma(\delta)}\tau^{\delta-1}\overline{b}^{q_2-1}\left(\frac{\overline{\rho}}{M}\right)^{\varsigma} \\
&\quad\times\int_0^1 \bar{j}(s)\left(\int_0^s (s-w)^{\beta-1}w^{(\gamma-1)\varsigma^{\frac{1}{q_2-1}}}g(w,1,1)\,dw\right)^{q_2-1}ds\Big) \\
&\le \psi\left(\tau,\frac{c}{M}\tau^{\gamma-1},\dots,\frac{c}{M}\tau^{\gamma-1}\right) \\
&\le \left(\frac{c}{M}\tau^{\gamma-1}\right)^{-\sigma^{\frac{1}{q_1-1}}}\psi(\tau,1,\dots,1) \\
&= c^{-\sigma^{\frac{1}{q_1-1}}}M^{\sigma^{\frac{1}{q_1-1}}}\tau^{-(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\psi(\tau,1,\dots,1),\quad \tau\in(0,1),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
c = \min\bigg\{&\frac{\Gamma(\gamma)}{\Gamma(\gamma-\mu_1)},1,\frac{\overline{\lambda}}{\Gamma(\delta)}\tau^{\delta-1}\overline{b}^{q_2-1}\overline{\rho}^{\varsigma} \\
&\times\int_0^1\bar{j}(s)\left[\int_0^s (s-w)^{\beta-1}w^{(\gamma-1)\varsigma^{\frac{1}{q_2-1}}}g(w,1,1)\,dw\right]^{q_2-1}ds\bigg\}.
\end{aligned}
$$

Noting $\frac{c}{D}\tau^{\gamma-1}<1$, and by (3.4), (S_1), and (S_2), we have

$$
\begin{aligned}
&\phi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\dots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big) \\
&\ge \phi\left(\tau,\frac{1}{M}\tau^{\gamma-1},\frac{\Gamma(\gamma)}{M\Gamma(\gamma-\mu_1)}\tau^{\gamma-1-\mu_1},\dots,\right. \\
&\quad\frac{\Gamma(\gamma)}{M\Gamma(\gamma-\mu_{n-2})}\tau^{\gamma-1-\mu_{n-2}},\frac{\mu}{\Gamma(\delta)}\tau^{\delta-1}\overline{b}^{q_2-1}\left(\frac{\overline{\rho}}{M}\right)^{\varsigma} \\
&\quad\left.\times\int_0^1\bar{j}(s)\left[\int_0^s (s-w)^{\beta-1}w^{(\gamma-1)\varsigma^{\frac{1}{q_2-1}}}g(w,1,1)\,dw\right]^{q_2-1}ds\right) \\
&\ge \phi\left(\tau,\frac{c}{M}\tau^{\gamma-1},\dots,\frac{c}{M}\tau^{\gamma-1}\right) \\
&\ge \left(\frac{c}{M}\tau^{\gamma-1}\right)^{\sigma^{\frac{1}{q_1-1}}}\phi(\tau,1,\dots,1) \\
&= c^{\sigma^{\frac{1}{q_1-1}}}M^{-\sigma^{\frac{1}{q_1-1}}}\tau^{(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\phi(\tau,1,\dots,1),\quad\tau\in(0,1).
\end{aligned}
\tag{3.8}
$$

By (3.4), (S_1), and Remark 1.1, we also get

$$
\begin{aligned}
&\psi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\dots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big) \\
&\ge \psi\bigg(\tau,Me(\tau),D_{0^+}^{\mu_1}Me(\tau),\dots,D_{0^+}^{\mu_{n-2}}Me(\tau), \\
&\quad\overline{\lambda}\overline{a}^{\star}\tau^{\delta-1}\overline{b}^{q_2-1}(M\rho+1)^{\varsigma}\int_0^1\left[\int_0^s (s-w)^{\beta-1}g(w,1,\dots,1)\,dw\right]^{q_2-1}ds\bigg) \\
&\ge \psi\big(\tau,Mb\tau^{\gamma-1}+1,Mb\tau^{\gamma-\mu_1-1}+1,\dots,Mb\tau^{\gamma-\mu_{n-2}-1}+1,Mb\tau^{\delta-1}+1\big) \\
&\ge \psi(\tau,Mb+1,Mb+1,\dots,Mb+1)
\end{aligned}
$$

$$\begin{aligned}
&\geq (Mb+1)^{-\sigma^{\frac{1}{q_1-1}}}\psi(\tau,1,\ldots,1)\\
&\geq 2^{-\sigma^{\frac{1}{q_1-1}}}b^{-\sigma^{\frac{1}{q_1-1}}}M^{-\sigma^{\frac{1}{q_1-1}}}\psi(\tau,1,\ldots,1),\quad \tau\in(0,1).
\end{aligned}\tag{3.9}$$

For $x,z\in Q_e$, by (3.6), (3.7), we have

$$\begin{aligned}
&T_\lambda(u,z)(t)\\
&\quad= \lambda\int_0^1 G(t,s)\biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\bigl(\phi\bigl(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\ldots,D_{0^+}^{\mu_{n-2}}u(\tau),v(\tau)\bigr)\\
&\qquad+\psi\bigl(\tau,z(\tau),D_{0^+}^{\mu_1}z(\tau),\ldots,D_{0^+}^{\mu_{n-2}}z(\tau),v(\tau)\bigr)\bigr)\,d\tau\biggr]^{q_1-1}ds\\
&\quad\leq \lambda\int_0^1 G(t,s)\biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\bigl(2^{\sigma^{\frac{1}{q_1-1}}}b^{\sigma^{\frac{1}{q_1-1}}}M^{\sigma^{\frac{1}{q_1-1}}}\times\phi(\tau,1,\ldots,1)\\
&\qquad+c^{-\sigma^{\frac{1}{q_1-1}}}M^{\sigma^{\frac{1}{q_1-1}}}\tau^{-(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\psi(\tau,1,\ldots,1)\bigr)\,d\tau\biggr]^{q_1-1}ds\\
&\quad\leq \lambda a^\star t^{\gamma-1}M^\sigma\overline{a}^{q_1-1}\int_0^1\biggl[\int_0^s(s-\tau)^{\alpha-1}\bigl(2^{\sigma^{\frac{1}{q_1-1}}}b^{\sigma^{\frac{1}{q_1-1}}}\times\phi(\tau,1,\ldots,1)\\
&\qquad+c^{-\sigma^{\frac{1}{q_1-1}}}\tau^{-(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\psi(\tau,1,\ldots,1)\bigr)\,d\tau\biggr]^{q_1-1}ds\\
&\quad\leq \lambda a^\star t^{\gamma-1}M^\sigma\overline{a}^{q-1}\int_0^1\biggl[2^{\sigma^{\frac{1}{q_1-1}}}b^{\sigma^{\frac{1}{q_1-1}}}\int_0^s(s-\tau)^{\alpha-1}\phi(\tau,1,\ldots,1)\,d\tau\\
&\qquad+c^{-\sigma}\int_0^s(s-\tau)^{\alpha-1}\tau^{-(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\psi(\tau,1,\ldots,1)\,d\tau\biggr]^{q_1-1}ds\\
&\quad\leq \lambda a^\star t^{\gamma-1}M^\sigma\overline{a}^{q_1-1}\int_0^1\biggl[2^{\sigma^{\frac{1}{q_1-1}}}b^{\sigma^{\frac{1}{q_1-1}}}\frac{1}{(2\alpha-1)^{\frac12}}s^{\frac{2\alpha-1}{2}}\biggl(\int_0^s\phi^2(\tau,1,\ldots,1)\,d\tau\biggr)^{\frac12}\\
&\qquad+c^{-\sigma}\|(s-\tau)^{\alpha-1}\|_2\|^{-(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\psi(\tau,1,\ldots,1)\|_2\biggr]^{q_1-1}ds\\
&\quad\leq \lambda a^\star t^{\gamma-1}M^\sigma\overline{a}^{q_1-1}\int_0^1\biggl[2^{\sigma^{\frac{1}{q_1-1}}}b^{\sigma^{\frac{1}{q_1-1}}}\frac{1}{(2\alpha-1)^{\frac12}}s^{\frac{2\alpha-1}{2}}\biggl(\int_0^s\phi^2(\tau,1,\ldots,1)\,d\tau\biggr)^{\frac12}\\
&\qquad+c^{-\sigma}\frac{1}{(2\alpha-1)^{\frac12}}s^{\frac{2\alpha-1}{2}}\biggl(\int_0^s\tau^{-(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\psi^2(\tau,1,\ldots,1)\,d\tau\biggr)^{\frac12}\biggr]^{q_1-1}ds\\
&\quad\leq \lambda a^\star t^{\gamma-1}M^\sigma\overline{a}^{q_1-1}\frac{1}{(2\alpha-1)^{\frac{q_1-1}{2}}}\int_0^1\biggl[2^{\sigma^{\frac{1}{q-1}}}b^{\sigma^{\frac{1}{q_1-1}}}\biggl(\int_0^s\phi^2(\tau,1,\ldots,1)\,d\tau\biggr)^{\frac12}\\
&\qquad+c^{-\sigma}\biggl(\int_0^s\tau^{-(\gamma-1)\sigma^{\frac{1}{q_1-1}}}\psi^2(\tau,1,\ldots,1)\,d\tau\biggr)^{\frac12}\biggr]^{q_1-1}ds\\
&\quad<+\infty,\quad t\in[0,1],
\end{aligned}\tag{3.10}$$

where $\|\cdot\|_2$ is the norm in space $L^2[0,1]$. By (S_4), (3.10), we have that $T_\lambda: Q_e\times Q_e\to P$ is well defined. Next, we will prove $T_\lambda: Q_e\times Q_e\to Q_e$.

Formula (3.10) implies that

$$T_\lambda(u,z)(t) \le Mt^{\gamma-1} = Me(t), \quad t \in [0,1]. \tag{3.11}$$

At the same time, by (3.8) and (3.9), we have

$$\begin{aligned}
&T_\lambda(u,z)(t)\\
&\quad= \lambda \int_0^1 G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\left(\phi\left(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\dots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\right)\right.\right.\\
&\qquad \left.\left.+ \psi\left(\tau,z(\tau),D_{0^+}^{\mu_1}z(\tau),\dots,D_{0^+}^{\mu_{n-2}}z(\tau),Az(\tau)\right)\right)d\tau\right]^{q_1-1} ds\\
&\quad\ge \lambda \int_0^1 G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\left(c^{\sigma\frac{1}{q_1-1}} M^{-\sigma\frac{1}{q_1-1}} \tau^{(\alpha-1)\sigma\frac{1}{q_1-1}}\right.\right.\\
&\qquad \left.\left.\times \phi(\tau,1,\dots,1) + 2^{-\sigma\frac{1}{q_1-1}} b^{-\sigma\frac{1}{q_1-1}} M^{\sigma\frac{1}{q_1-1}} \psi(\tau,1,\dots,1)\right)d\tau\right]^{q_1-1} ds\\
&\quad\ge \frac{\lambda}{\Gamma(\gamma)} t^{\gamma-1} M^{-\sigma}\overline{a}^{q_1-1}\int_0^1 j(s)\left[\int_0^s (s-\tau)^{\alpha-1}\left(c^{\sigma\frac{1}{q_1-1}} \tau^{(\alpha-1)\sigma\frac{1}{q_1-1}}\right.\right.\\
&\qquad \left.\left.\times \phi(\tau,1,\dots,1) + 2^{-\sigma\frac{1}{q_1-1}} b^{-\sigma\frac{1}{q_1-1}} \psi(\tau,1,\dots,1)\right)d\tau\right]^{q_1-1} ds, \quad t \in [0,1].
\end{aligned} \tag{3.12}$$

Formula (3.12) implies that

$$T_\lambda(u,z)(t) \ge \frac{1}{M} t^{\gamma-1} = \frac{1}{M} e(t), \quad t \in [0,1]. \tag{3.13}$$

Hence, $T_\lambda : Q_e \times Q_e \to Q_e$. Next, we shall prove that $T_\lambda : Q_e \times Q_e \to Q_e$ is a mixed monotone operator. In fact, for any $u_1, u_2, z \in Q_e$ and $u_1 \preceq u_2$, that is, $u_1(t) \le u_2(t), D_{0^+}^{\mu_i} u_1(t) \le D_{0^+}^{\mu_i} u_2(t)$ $(i = 1,2,\dots,n-2), D_{0^+}^{\overline{\mu}_j} u_1(t) \le D_{0^+}^{\overline{\mu}_j} u_2(t)$ $(j = 1,2,\dots,m-2), t \in [0,1]$, by the monotonicity of A and ϕ, and from (S_1), for all $t \in [0,1]$, we have

$$\begin{aligned}
&\lambda \int_0^1 G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\left(\phi\left(\tau,u_1(\tau),D_{0^+}^{\mu_1}u_1(\tau),\dots,D_{0^+}^{\mu_{n-2}}u_1(\tau),Au_1(\tau)\right)\right.\right.\\
&\qquad \left.\left.+ \psi\left(\tau,z(\tau),D_{0^+}^{\mu_1}z(\tau),\dots,D_{0^+}^{\mu_{n-2}}z(\tau),Az(\tau)\right)d\tau\right]^{q_1-1} ds\\
&\quad\le \lambda \int_0^1 G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\left(\phi\left(\tau,u_2(\tau),D_{0^+}^{\mu_1}u_2(\tau),\dots,D_{0^+}^{\mu_{n-2}}u_2(\tau),Au_2(\tau)\right)\right.\right.\\
&\qquad \left.\left.+ \psi\left(\tau,z(\tau),D_{0^+}^{\mu_1}z(\tau),\dots,D_{0^+}^{\mu_{n-2}}z(\tau),Az(\tau)\right)d\tau\right]^{q_1-1} ds,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&\lambda \int_0^1 D_{0^+}^{\mu_i} G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\left(\phi\left(\tau,u_1(\tau),D_{0^+}^{\mu_1}u_1(\tau),\dots,D_{0^+}^{\mu_{n-2}}u_1(\tau),Au_1(\tau)\right)\right.\right.\\
&\qquad \left.\left.+ \psi\left(\tau,z(\tau),D_{0^+}^{\mu_1}z(\tau),\dots,D_{0^+}^{\mu_{n-2}}z(\tau),Az(\tau)\right)d\tau\right]^{q_1-1} ds\\
&\quad\le \lambda \int_0^1 D_{0^+}^{\mu_i} G(t,s)\left[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\left(\phi\left(\tau,u_2(\tau),D_{0^+}^{\mu_1}u_2(\tau),\dots,D_{0^+}^{\mu_{n-2}}u_2(\tau),Au_2(\tau)\right)\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+\psi\big(\tau,z(\tau),D_{0^+}^{\mu_1}z(\tau),\dots,D_{0^+}^{\mu_{n-2}}z(\tau),Az(\tau)\big)\,d\tau\Big]^{q_1-1}\,ds,\\
&i=1,2,\dots,m-2.
\end{aligned}
\tag{3.15}
$$

Hence, by (3.14),(3.15), we have

$$
T_\lambda(u_1,z)\preceq T_\lambda(u_2,z),\quad z\in Q_e, \tag{3.16}
$$

that is, $T_\lambda(u,z)$ is nondecreasing on u for any $z\in Q_e$. Similarly, for all $u,z_1,z_2\in Q_e$, $z_1\succeq z_2$, that is, $z_1(t)\ge z_2(t), D_{0^+}^{\mu_i}z_1(t)\ge D_{0^+}^{\mu_i}z_2(t)$ $(i=1,2,\dots,n-2), D_{0^+}^{\overline{\mu}_j}z_1(t)\ge D_{0^+}^{\overline{\mu}_j}z_2(t)$ $(j=1,2,\dots,m-2), t\in[0,1]$, from (S_1), for all $t\in[0,1]$, we have

$$
\begin{aligned}
&\lambda\int_0^1 G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\dots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big)\\
&\quad+\psi\big(\tau,z_1(\tau),D_{0^+}^{\mu_1}z_1(\tau),\dots,D_{0^+}^{\mu_{n-2}}z_1(\tau),Az_1(\tau)\big)\big)\,d\tau\Big]^{q_1-1}\,ds\\
&\le\lambda\int_0^1 G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\dots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big)\\
&\quad+\psi\big(\tau,z_2(\tau),D_{0^+}^{\mu_1}z_2(\tau),\dots,D_{0^+}^{\mu_{n-2}}z_2(\tau),Az_2(\tau)\big)\big)\,d\tau\Big]^{q_1-1}\,ds,
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
&\lambda\int_0^1 D_{0^+}^{\mu_i}G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\dots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big)\\
&\quad+\psi\big(\tau,z_1(\tau),D_{0^+}^{\mu_1}z_1(\tau),\dots,D_{0^+}^{\mu_{n-2}}z_1(\tau),Az_1(\tau)\big)\big)\,d\tau\Big]^{q_1-1}\,ds\\
&\le\lambda\int_0^1 D_{0^+}^{\mu_1}G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\dots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big)\\
&\quad+\psi\big(\tau,z_2(\tau),D_{0^+}^{\mu_1}z_2(\tau),\dots,D_{0^+}^{\mu_{n-2}}z_2(\tau),Az_2(\tau)\big)\big)\,d\tau\Big]^{q_1-1}\,ds,\\
&i=1,2,\dots,m-2.
\end{aligned}
\tag{3.18}
$$

Hence, by (3.17), (3.18), we have

$$
T_\lambda(u,z_1)\preceq T_\lambda(u,z_2),\quad u\in Q_e, \tag{3.19}
$$

i.e., $T_\lambda(u,z)$ is nonincreasing on z for any $u\in Q_e$. Hence, by (3.16) and (3.19), we have that $T_\lambda: Q_e\times Q_e\to Q_e$ is a mixed monotone operator.

Finally, we show that the operator T_λ satisfies (2.22). For any $u,z\in Q_e$ and $l\in(0,1)$, by (S_2) and Remark 1.1, for all $t\in[0,1]$, we have

$$
\begin{aligned}
&\lambda\int_0^1 G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\Big(\phi\big(\tau,lu(\tau),D_{0^+}^{\mu_1}lu(\tau),\dots,D_{0^+}^{\mu_{n-2}}lu(\tau),Alu(\tau)\big)\\
&\quad+\psi\Big(\tau,\frac{1}{l}z(\tau),D_{0^+}^{\mu_1}\frac{1}{l}z(\tau),\dots,D_{0^+}^{\mu_{n-2}}\frac{1}{l}z(\tau),A\frac{1}{l}z(\tau)\Big)\Big)\,d\tau\Big]^{q_1-1}\,ds\\
&\ge\lambda\int_0^1 G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}l^{\sigma\frac{1}{q_1-1}}\big(\phi\big(\tau,u(\tau),D_{0^+}^{\mu_1}u(\tau),\dots,D_{0^+}^{\mu_{n-2}}u(\tau),Au(\tau)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \psi\big(\tau, z(\tau), D_{0^+}^{\mu_1} z(\tau), \ldots, D_{0^+}^{\mu_{n-2}} z(\tau), Az(\tau)\big)\big)\, d\tau\Big]^{q_1-1} ds \\
&\geq l^{\sigma} \int_0^1 G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau, u(\tau), D_{0^+}^{\mu_1} u(\tau), \ldots, D_{0^+}^{\mu_{n-2}} u(\tau), Au(\tau)\big) \\
&\quad + \psi\big(\tau, z(\tau), D_{0^+}^{\mu_1} z(\tau), \ldots, D_{0^+}^{\mu_{n-2}} z(\tau), Az(\tau)\big)\big)\, d\tau\Big]^{q_1-1} ds,
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&\lambda \int_0^1 D_{0^+}^{\mu_i} G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\Big(\phi\big(\tau, lu(\tau), D_{0^+}^{\mu_1} lu(\tau), \ldots, D_{0^+}^{\mu_{n-2}} lu(\tau), Alu(\tau)\big) \\
&\quad + \psi\Big(\tau, \frac{1}{l} z(\tau), D_{0^+}^{\mu_1} \frac{1}{l} z(\tau), \ldots, D_{0^+}^{\mu_{n-2}} \frac{1}{l} z(\tau), A\frac{1}{l} z(\tau)\Big)\Big)\, d\tau\Big]^{q_1-1} ds \\
&\geq \lambda \int_0^1 D_{0^+}^{\mu_i} G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1} l^{\sigma\frac{1}{q_1-1}}\big(\phi\big(\tau, u(\tau), D_{0^+}^{\mu_1} u(\tau), \ldots, D_{0^+}^{\mu_{n-2}} u(\tau), Au(\tau)\big) \\
&\quad + \psi\big(\tau, z(\tau), D_{0^+}^{\mu_1} z(\tau), \ldots, D_{0^+}^{\mu_{n-2}} z(\tau), Az(\tau)\big)\big)\, d\tau\Big]^{q_1-1} ds \\
&\geq l^{\sigma} \int_0^1 D_{0^+}^{\mu_i} G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\big(\phi\big(\tau, u(\tau), D_{0^+}^{\mu_1} u(\tau), \ldots, D_{0^+}^{\mu_{n-2}} u(\tau), Au(\tau)\big) \\
&\quad + \psi\big(\tau, z(\tau), D_{0^+}^{\mu_1} z(\tau), \ldots, D_{0^+}^{\mu_{n-2}} z(\tau), Az(\tau)\big)\big)\, d\tau\Big]^{q_1-1} ds.
\end{aligned}
\tag{3.21}
$$

Formulas (3.20),(3.21) imply that

$$
T_\lambda\Big(lu, \frac{1}{l} z\Big) \succeq l^{\sigma} T_\lambda(u,z), \quad u, z \in Q_e. \tag{3.22}
$$

Hence, Lemma 2.4 assumes that there exists a unique positive solution $u_\lambda^\star \in Q_e$ such that $T_\lambda(u_\lambda^\star, u_\lambda^\star) = u_\lambda^\star$. It is easy to check that $u_\lambda^\star$ is a unique positive solution of (2.10) for any given $\lambda > 0$. Moreover, by Lemma 2.5, we have

(1) For any $\lambda_0 \in (0, +\infty)$, $\|u_\lambda^\star - u_{\lambda_0}^\star\| \to 0, \lambda \to \lambda_0$;

(2) If $0 < \sigma < \frac{1}{2}$, then $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^\star \preceq u_{\lambda_2}^\star, u_{\lambda_1}^\star \neq u_{\lambda_2}^\star$, and

$$
\lim_{\lambda \to 0} \|u_\lambda^\star\| = 0, \qquad \lim_{\lambda \to +\infty} \|u_\lambda^\star\| = +\infty.
$$

Moreover, for any $u_0(t) \in Q_e$, by Lemma 2.4, for any $t \in [0,1]$, constructing a successive sequence

$$
\begin{aligned}
u_{k+1}(t) &= \lambda \int_0^1 G(t,s)\Big[\int_0^s \overline{a}(s-\tau)^{\alpha-1}(\phi\big(\tau, u_k(\tau), D_{0^+}^{\mu_1} u_k(\tau), \ldots, D_{0^+}^{\mu_{n-2}} u_k(\tau), Au_k(\tau)\big) \\
&\quad + \psi\big(\tau, u_k(\tau), D_{0^+}^{\mu_1} u_k(\tau), \ldots, D_{0^+}^{\mu_{n-2}} u_k(\tau), Au_k(\tau)\big)\, d\tau\Big]^{q_1-1} ds, \\
&\quad k = 1, 2, \ldots, t \in [0,1],
\end{aligned}
$$

and we have $\|u_k - u_\lambda^\star\| \to 0$ as $k \to \infty$, the convergence rate is

$$
\|u_k - u_\lambda^\star\| = o\big(1 - r^{\sigma^m}\big),
$$

where r is a constant, $0 < r < 1$, dependent on u_0. By (2.20), we easily get ($\star$).

By (3.4), (3.5), and $u_\lambda^\star \in Q_e$, we get (3.1). Therefore, the proof of Theorem 3.1 is completed. □

4 Example

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{3}{4}}(\varphi_3(D_{0^+}^{\frac{5}{2}}u))(t) + \lambda^2 f(t,u(t),D_{0^+}^{\frac{1}{2}}u(t),v(t)) = 0, \quad 0 < t < 1, \\ D_{0^+}^{\frac{3}{4}}(\varphi_2(D_{0^+}^{\frac{3}{2}}v))(t) + \mu g(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad D_{0+}^{\gamma}u(0) = 0, \qquad D_{0^+}^{r_1}u(1) = \sum_{j=1}^{\infty}\eta_j D_{0^+}^{r_2}u(\xi_j), \\ v(0) = v'(0) = 0, \qquad D_{0+}^{\delta}v(0) = 0, \qquad D_{0^+}^{\overline{r}_1}v(1) = \sum_{j=1}^{\infty}\overline{\eta}_j D_{0^+}^{\overline{r}_2}u(\overline{\xi}_j), \end{cases} \tag{4.1}$$

where $\gamma = \frac{5}{2}, \delta = \frac{3}{2}, \alpha = \beta = \frac{3}{4}$, $r_1 = r_2 = \frac{1}{2}, \overline{r}_1 = \overline{r}_2 = \frac{1}{2}$, $\eta_j = \overline{\eta}_j = \frac{1}{2j^5}$, $\xi_j = \overline{\xi}_j = \frac{1}{j^2}$, $p_1 = 3, q_1 = \frac{3}{2}, p_2 = 2, q_2 = 2$, and

$$\begin{aligned} &\phi(t,x_1,x_2,x_3) = \left(t^{-\frac{1}{4}} + \cos t\right)x_1^{\frac{1}{9}} + 2tx_2^{\frac{1}{8}} + 2x_3^{\frac{1}{16}}, \\ &\psi(t,x_1,x_2,x_3) = t^{-\frac{1}{16}}x_1^{-\frac{1}{8}} + x_2^{-\frac{1}{16}} + (2-t)x_3^{-\frac{1}{15}}, \\ &g(t,u) = \left(3t + t^2\right)u^{\frac{3}{5}} + (t\sin t + t)u^{\frac{2}{3}}. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{\Gamma(\gamma)}{\Gamma(\gamma - r_2)}\sum_{j=1}^{\infty}\eta_j\xi_j^{\gamma - r_2 - 1} = \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})}\sum_{j=1}^{\infty}\eta_j(\xi_j)^{-\frac{1}{2}} = 0.5412 < \frac{\Gamma(\gamma)}{\Gamma(\gamma - r_1)} = \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})}, \\ &\frac{\Gamma(\delta)}{\Gamma(\delta - \overline{r}_2)}\sum_{j=1}^{\infty}\overline{\eta}_j\overline{\xi}_j^{\alpha - \overline{r}_2 - 1} = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})}\sum_{j=1}^{\infty}\overline{\eta}_j(\overline{\xi}_j)^{-\frac{1}{2}} = 0.5412 < \frac{\Gamma(\delta)}{\Gamma(\delta - \overline{r}_1)} = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})}. \end{aligned}$$

Moreover, for any $(t,x_1,x_2,x_3) \in (0,1) \times (0,\infty)^3$ and $0 < l < 1$, we have

$$\begin{aligned} &\phi(t,lx_1,lx_2,lx_3) \\ &\quad = \left(t^{-\frac{1}{4}} + \cos t\right)(lx_1)^{\frac{1}{9}} + 2t(lx_2)^{\frac{1}{8}} + 2(lx_3)^{\frac{1}{16}} \\ &\quad \geq l^{\frac{1}{8}}\left(\left(t^{-\frac{1}{4}} + \cos t\right)x_1^{\frac{1}{9}} + 2tx_2^{\frac{1}{8}} + 2x_3^{\frac{1}{16}}\right) \\ &\quad = l^{\frac{1}{8}}\phi(t,x_1,x_2,x_3) = l^{\sigma^{\frac{1}{q_1-1}}}\phi(t,x_1,x_2,x_3), \\ &\psi\left(t,l^{-1}x_1,l^{-1}x_2,l^{-1}x_3\right) \\ &\quad = t^{-\frac{1}{16}}\left(l^{-1}x_1\right)^{-\frac{1}{8}} + \left(l^{-1}x_2\right)^{-\frac{1}{16}} + (2-t)\left(l^{-1}x_3\right)^{-\frac{1}{15}} \\ &\quad \geq l^{\frac{1}{8}}\left(t^{-\frac{1}{16}}x_1^{-\frac{1}{8}} + x_2^{-\frac{1}{16}} + (2-t)x_3^{-\frac{1}{15}}\right) \\ &\quad = l^{\frac{1}{8}}\psi(t,x_1,x_2,x_3) = l^{\sigma^{\frac{1}{q_1-1}}}\psi(t,x_1,x_2,x_3), \\ &g(t,lu) \\ &\quad = \left(3t + t^2\right)(lu)^{\frac{3}{5}} + (t\sin t + t)(lu)^{\frac{2}{3}} \end{aligned}$$

$$
\begin{aligned}
&\geq l^{\frac{2}{3}}\left((3t+t^2)u^{\frac{3}{5}}+(t\mathrm{sin}t+t)u^{\frac{2}{3}}\right)\\
&= l^{\frac{2}{3}}g(t,u)=l^{\varsigma^{\frac{1}{q_2-1}}}g(t,u).
\end{aligned}
$$

Noting $\sigma = \frac{1}{2\sqrt{2}} < 1, \varsigma = \frac{2}{3}, \psi(\tau,1,1,1) = \tau^{-\frac{1}{16}} + 3 - \tau$, $\phi(\tau,1,1,1) = \tau^{-\frac{1}{4}} + \cos\tau + 2\tau + 2$, $g(\tau,1) = 3\tau + \tau^2 + \tau\sin\tau + \tau$, we have

$$
\begin{aligned}
0 &< \int_0^1 \phi^2(\tau,1,1,1)\,d\tau\\
&= \int_0^1 \left(\tau^{-\frac{1}{4}} + \cos\tau + 2\tau + \tau\right)^2 d\tau\\
&\leq 19 + 8 + \frac{16}{7} < +\infty,\\
0 &< \int_0^1 \tau^{-2(\gamma-1)\sigma^{\frac{1}{q-1}}}\psi^2(\tau,1,1,1)\,d\tau\\
&= \int_0^1 \tau^{-\frac{3}{8}}\left(\tau^{-\frac{1}{16}} + 3 - \tau\right)^2 d\tau\\
&\leq \int_0^1 \tau^{-\frac{3}{8}}\left(\tau^{-\frac{1}{16}} + 3\right)^2 d\tau\\
&= 2 + \frac{96}{9} + 9 < +\infty,\\
0 &< \int_0^1 g^2(\tau,1,1,1)\,d\tau\\
&\leq \int_0^1 \left(3\tau + \tau^2 + \tau\sin\tau + \tau\right)^2 d\tau\\
&\leq 17 \times \frac{1}{3} + 6 \times \frac{1}{4} + \frac{1}{5} + 12 + 16 < +\infty.
\end{aligned}
$$

Thus, assumptions (S_1)–(S_4) of Theorem 3.1 hold. Then Theorem 3.1 implies that problem (4.1) has a unique solution. Furthermore, when $\lambda \to \lambda_0$, $\lambda_0 \in (0,+\infty)$, we have

$$
\left\|u_\lambda^\star - u_{\lambda_0}^\star\right\| \to 0,
$$

and $0 < \lambda_1 < \lambda_2$ implies

$$
u_{\lambda_1}^\star(t) \leq u_{\lambda_2}^\star(t), \qquad u_{\lambda_1}^\star(t) \not\equiv u_{\lambda_2}^\star(t), \quad t \in [0,1].
$$

Since $\sigma = \frac{1}{2\sqrt{2}} \in (0,\frac{1}{2})$, then

$$
\lim_{\lambda\to 0}\left\|u_\lambda^\star\right\| = 0, \qquad \lim_{\lambda\to+\infty}\left\|u_\lambda^\star\right\| = +\infty.
$$

In addition, for any initial $u_0 \in Q_e$, we construct a successive sequence:

$$
\begin{aligned}
u_{k+1}(t) = \int_0^1 \lambda G(t,s)\biggl[\int_0^s \overline{a}(s-\tau)^{\alpha-1}\Bigl(\phi\bigl(t,u_k(t),D_{0^+}^{\frac{1}{2}}u_k(t),Au_k'(t)\bigr)\\
+ \psi\bigl(t,u_k(t),D_{0^+}^{\frac{1}{2}}u_k(t),Au_k'(t)\bigr)\Bigr)\,d\tau\biggr]^{q_1-1} ds, \quad t \in [0,1], k = 1,2,\ldots,
\end{aligned}
$$

and we have $\|u_k - u_\lambda^\star\| \to 0$ as $k \to \infty$, the convergence rate is

$$\left\|u_k - u_\lambda^\star\right\| = o\left(1 - r^{\sigma^k}\right),$$

where r is a constant, $0 < r < 1$, dependent on u_0.

5 Conclusions

In this paper, we study the existence of positive solutions for a singular p-Laplacian fractional order differential equation boundary value system. By using the method of mixed monotone operator, some existence results are obtained for the case where the nonlinearity is allowed to be singular in regard to not only time variable but also space variable and the fractional orders are involved in the nonlinearity of the boundary value problem (1.1). Moreover, our equation system contains many types of equation system because there are many parameters in our p-Laplacian equation system, and also the uniqueness of positive solution of equation (1.1) is dependent on λ, μ. An iterative sequence and convergence rate, which are important for practical application, are given.

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Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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