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Blow-up of solution for quasilinear viscoelastic wave equation with boundary nonlinear damping and source terms

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Abstract

In this paper, we consider the blow-up result of solution for a quasilinear viscoelastic wave equation with strong damping and boundary nonlinear damping. We prove a finite time blow-up result of solution with positive initial energy as well as nonpositive initial energy under suitable conditions on the initial data and positive function g.

MSC: 35L05; 35B44

Keywords: Blow-up; Wave equation; Viscoelasticity

1 Introduction

In this paper we investigate a blow-up result for the following quasilinear viscoelastic wave equation:

$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u_t = 0, \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,t) = 0, \quad \text{on } \Gamma_0 \times [0,\infty), \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)\,ds + f(u_t) = |u|^{p-2}u, \quad \text{on } \Gamma_1 \times [0,\infty), \tag{1.3}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, where Γ_0 and Γ_1 are measurable over $\partial\Omega$, ν is the unit outward normal to $\partial\Omega$, and g is a positive function.

For the case of $\rho = 0$, problem (1.1)–(1.4) arises in the theory of viscoelasticity and describes the spread of strain waves in a viscoelastic configuration [1–3]. Messaoudi [4] studied the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + a|u_t|^{m-2}u_t = b|u|^{p-2}u, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,\infty), \end{cases}$$

where $m \geq 1$, $p > 2$, $a, b > 0$ are constants and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing function. Under suitable conditions on g, he proved that any weak solution with negative initial energy blows up in finite time if $p > m$. He [5] also extended the blow-up result to certain solutions with positive initial energy. Song and Zhong [6] considered the viscoelastic wave equation with strong damping:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u_t = |u|^{p-2}u, & \text{in } \Omega \times [0,T], \\ u(x,t) = 0, & x \in \partial\Omega. \end{cases}$$

Recently, Park et al. [7] showed the blow-up result of solution for the following viscoelastic wave equation with nonlinear damping:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + h(u_t) = |u|^{p-2}u, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

They obtained the blow-up of solution with positive initial energy as well as nonpositive initial energy under a weaker assumption on the damping term. Liu and Yu [8] studied the following viscoelastic wave equation with boundary damping and source terms:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = 0, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \Gamma_0 \times [0,\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)\,ds + |u_t|^{m-2}u_t = |u|^{p-2}u, & \text{on } \Gamma_1 \times [0,\infty). \end{cases}$$

They proved the blow-up result of solutions with nonpositive initial energy as well as positive initial energy for both the linear and nonlinear damping cases. In the absence of the viscoelastic term ($g = 0$), the related problem has been extensively investigated, and results concerning the global existence of solution and nonexistence have been studied (see [9–14]).

For the case of $\rho > 0$, Cavalcanti et al. [15] considered the following problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds - \gamma\Delta u_t = 0, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

They showed a global existence result for $\gamma \geq 0$ and an exponential decay result for $\gamma > 0$. In the case of $\gamma = 0$, Liu [16] proved the nonlinear viscoelastic problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds = b|u|^{p-2}u, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

He discussed the general decay result for the global solution and the finite time blow-up of solution. In the absence of the dispersion term $-\Delta u_{tt}$, Song [17] investigated the nonexistence of global solutions with positive initial energy for the viscoelastic wave equation with nonlinear damping:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + |u_t|^{m-2}u_t = |u|^{p-2}u, & \text{in } \Omega \times [0,T], \\ u(x,t) = 0, & x \in \partial\Omega. \end{cases}$$

Recently, Hao and Wei [18] established the blow-up result of solution with negative initial energy and some positive initial energy for the quasilinear viscoelastic wave equation with strong damping:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u_t = |u|^{p-2}u, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

To our knowledge, there are few blow-up results of solution for quasilinear viscoelastic wave equation with boundary nonlinear damping and source terms. Motivated by the previous work, we study the blow-up of solution with nonpositive initial energy as well as positive initial energy for the quasilinear viscoelastic wave equation with strong damping and boundary weak damping.

This paper is organized as follows. In Sect. 2, we recall the notation, hypotheses, and some necessary preliminaries. In Sect. 3, we prove the blow-up of solution for (1.1)–(1.4).

2 Preliminaries

In this section we give notations, hypotheses, and some lemmas needed in our main result.

We impose the assumptions for problem (1.1)–(1.4):

(H_1) *Hypotheses on g*

The kernel $g : [0,\infty) \to [0,\infty)$ is a nonincreasing and differentiable function satisfying

$$1 - \int_0^\infty g(s)\,ds := l > 0. \tag{2.1}$$

(H_2) *Hypotheses on f*

Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function with $f(0) = 0$. Assume that there exists a strictly increasing and odd function $\xi : [-1,1] \to \mathbb{R}$ such that

$$\bigl|\xi(s)\bigr| \le \bigl|f(s)\bigr| \le \bigl|\xi^{-1}(s)\bigr| \quad \text{for } |s| \le 1, \tag{2.2}$$

$$a_1|s|^{m-1} \le \bigl|f(s)\bigr| \le a_2|s|^{m-1} \quad \text{for } |s| > 1, \tag{2.3}$$

where a_1, a_2 are positive constants and ξ^{-1} represents the inverse function of ξ.

(H_3) *Hypotheses on p, m, and ρ*

$$2 < m, 2 < p \quad \text{if } n = 1,2, \qquad 2 < m, p \le \frac{2(n-1)}{n-2} \quad \text{if } n \ge 3, \tag{2.4}$$

$$0 < \rho \quad \text{if } n = 1,2, \qquad 0 < \rho \le \frac{2}{n-2} \quad \text{if } n \ge 3. \tag{2.5}$$

As usual, $(\cdot,\cdot)$ and $\|\cdot\|_p$ denote the inner product in the space $L^2(\Omega)$ and the norm of the space $L^p(\Omega)$, respectively. For brevity, we denote $\|\cdot\|_2$ by $\|\cdot\|$. We introduce the notations:

$$(u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)\,d\Gamma, \qquad \|\cdot\|_{q,\Gamma_1} = \|\cdot\|_{L^q(\Gamma_1)}, \quad 1 \le q \le \infty,$$

the Hilbert space

$$V = \bigl\{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\bigr\}$$

($u|_{\Gamma_0}$ is the trace sense), endowed the equivalent norm $\|\nabla u\|$. We recall the trace Sobolev embedding

$$V \hookrightarrow L^q(\Gamma_1) \quad \text{for } 2 \le q < r = \frac{2(n-1)}{(n-2)}$$

and the embedding inequality

$$\|v\|_{q,\Gamma_1} \le B\|\nabla v\| \quad \text{for } v \in V, \tag{2.6}$$

where $B > 0$ is the optimal constant. We define the energy associated with problem (1.1)–(1.4) by

$$\begin{aligned} E(t) = {}& \frac{1}{\rho+2}\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 \\ & + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\big\|u(t)\big\|_{p,\Gamma_1}^p, \end{aligned} \tag{2.7}$$

where $(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2\,ds$. It is easy to find that

$$E'(t) = -\big(f\big(u_t(t)\big), u_t(t)\big)_{\Gamma_1} + \frac{1}{2}\big(g' \circ \nabla u\big)(t) - \frac{g(t)}{2}\big\|\nabla u(t)\big\|^2 - \|\nabla u_t\|^2 \le 0. \tag{2.8}$$

Therefore, E is a nonincreasing function.

Next, we define the functionals:

$$I(t) = \left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 + (g \circ \nabla u)(t) - \big\|u(t)\big\|_{p,\Gamma_1}^p,$$

$$H(t) = \frac{1}{2}\left[\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 + (g \circ \nabla u)(t)\right] - \frac{1}{p}\big\|u(t)\big\|_{p,\Gamma_1}^p.$$

Similar as in [8], for $t \ge 0$, we define

$$h(t) = \inf_{u \in V, u \ne 0} \sup_{\lambda \ge 0} H(\lambda u).$$

Then, we have

$$0 < h_0 \le h(t) \le \sup_{\lambda \ge 0} H(\lambda u) \quad \text{for } t \ge 0,$$

where

$$h_0 = \frac{p-2}{2p}\left(\frac{l}{B^2}\right)^{p/(p-2)},$$

$$\sup_{\lambda \ge 0} H(\lambda u) = \frac{p-2}{2p}\left(\frac{(1 - \int_0^t g(s)\,ds)\|\nabla u(t)\|^2 + (g \circ \nabla u)(t)}{\|u(t)\|_{p,\Gamma_1}^2}\right)^{p/(p-2)}.$$

Lemma 2.1 (Lemma 4.1 of [8]) *Suppose that* (H_1)–(H_3) *hold. For any fixed number* $\delta < 1$, *assume that* $(u_0, u_1) \in V \times L^2(\Omega)$ *and satisfy*

$$I(0) < 0, \qquad E(0) < \delta h_0. \tag{2.9}$$

Assume further that g satisfies

$$\int_0^\infty g(s)\,ds < \frac{p-2}{p-2+\frac{1}{[(1-\hat{\delta})^2(p-2)+2(1-\hat{\delta})]}}, \tag{2.10}$$

where $\hat{\delta} = \max\{0, \delta\}$. *Then we have* $I(t) < 0$ *for all* $t \in [0, T)$, *and*

$$\begin{aligned} h_0 &< \frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + (g \circ \nabla u)(t)\right] \\ &< \frac{p-2}{2p}\|u(t)\|_{p,\Gamma_1}^p, \quad t \in [0, T). \end{aligned} \tag{2.11}$$

Throughout this paper, we define

$$K(t) = \hat{\delta} h_0 - E(t), \tag{2.12}$$

which, from (2.8), is an increasing function. Using (2.7), (2.9), and (2.11), we see that

$$0 < K(0) \le K(t) \le \hat{\delta} h_0 + \frac{1}{p}\|u(t)\|_{p,\Gamma_1}^p \le p_0\|u(t)\|_{p,\Gamma_1}^p, \quad t \in [0, T), \tag{2.13}$$

where $p_0 = \frac{(p-2)\hat{\delta}}{2p} + \frac{1}{p}$.

Moreover, similar as in [5], we can show the following lemma which is needed later.

Lemma 2.2 *Let the conditions of Lemma* 2.1 *hold. Then the solution u of problem* (1.1)–(1.4) *satisfies*

$$\|u(t)\|_{p,\Gamma_1}^s \le C_0\|u(t)\|_{p,\Gamma_1}^p, \quad t \in [0, T), \text{ for any } 2 \le s \le p, \tag{2.14}$$

where C_0 *is a positive constant.*

Proof If $\|u(t)\|_{p,\Gamma_1} \ge 1$, then $\|u(t)\|_{p,\Gamma_1}^s \le \|u(t)\|_{p,\Gamma_1}^p$.

If $\|u(t)\|_{p,\Gamma_1} \le 1$, then

$$\|u(t)\|_{p,\Gamma_1}^s \le \|u(t)\|_{p,\Gamma_1}^2 \le B^2\|\nabla u(t)\|^2,$$

where we used (2.6). Then there exists a positive constant $C_1 = \max\{1, B^2\}$ such that

$$\|u(t)\|_{p,\Gamma_1}^s \le C_1\left(\|u(t)\|_{p,\Gamma_1}^p + \|\nabla u(t)\|^2\right) \quad \text{for any } 2 \le s \le p. \tag{2.15}$$

By (2.1), (2.7), (2.12), and (2.13), we get

$$\frac{l}{2}\|\nabla u(t)\|^2 \le \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2$$

$$\le \hat{\delta}h_0 - K(t) - \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} - \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{p}\|u(t)\|_{p,\Gamma_1}^{p}$$
$$\le \hat{\delta}h_0 + \frac{1}{p}\|u(t)\|_{p,\Gamma_1}^{p}. \tag{2.16}$$

Using (2.13), (2.15), and (2.16), we have the desired result (2.14). □

We state the well-posedness which can be established by the arguments of [5, 19–21].

Theorem 2.1 *Assume* (H_1)–(H_3) *hold. Then, for every* $(u_0,u_1)\in V\times L^2(\Omega)$, *there exists a unique local solution in the class*

$$u\in C\big([0,T);V\big),\qquad u_t\in C\big([0,T);L^2(\Omega)\big)\cap L^m\big(\Gamma_1\times[0,T)\big)\quad \textit{for some } T>0.$$

3 A blow-up result

To show the blow-up result for solutions with nonpositive initial energy as well as positive initial energy, we use the similar method of [8]. Our main result reads as follows.

Theorem 3.1 *Let* (H_1)–(H_3) *hold and* $p>m$, *and the conditions of Lemma* 2.1 *hold. Moreover, we assume*

$$\xi^{-1}(1) < \left(\frac{\theta\hat{\delta}h_0 p\eta}{(p-1)|\Gamma_1|}\right)^{\frac{p-1}{p}}, \tag{3.1}$$

where $0<\theta<\min\{2c_1,2c_2\}$, $0<\eta<\theta^{\frac{1}{p-1}}$, *and* c_1 *and* c_2 *will be specified later. Then the solution of problem* (1.1)–(1.4) *blows up in finite time.*

Proof To show this theorem, we use the idea given in [4, 5]. We assume that there exists some positive constant M such that, for $t>0$, the solution $u(t)$ of (1.1)–(1.4) satisfies

$$\|u_t(t)\|_{\rho+2}^{\rho+2} + \|\nabla u(t)\|^2 + \|u(t)\|_{p,\Gamma_1}^{p} \le M. \tag{3.2}$$

Let us consider the following function:

$$L(t) = K^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t(t)|^\rho u_t(t)u(t)\,dx + \frac{\varepsilon}{2}\|\nabla u(t)\|^2,\quad \varepsilon>0, \tag{3.3}$$

where

$$0<\sigma<\min\left\{1,\frac{1}{\rho+2},\frac{p-m}{p(m-1)}\right\}. \tag{3.4}$$

From (1.1)–(1.3), we obtain

$$L'(t) = (1-\sigma)K^{-\sigma}(t)K'(t) + \frac{\varepsilon}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} - \varepsilon\|\nabla u(t)\|^2$$
$$+\varepsilon\int_0^t g(t-s)\big(\nabla u(s),\nabla u(t)\big)\,ds - \varepsilon\big(f\big(u_t(t)\big),u(t)\big)_{\Gamma_1} + \varepsilon\|u(t)\|_{p,\Gamma_1}^{p}. \tag{3.5}$$

Using Young's inequality, we get

$$\begin{aligned}
&\int_0^t g(t-s)\big(\nabla u(s), \nabla u(t)\big)\,ds \\
&\quad = \int_0^t g(t-s)\big\|\nabla u(t)\big\|^2\,ds + \int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t), \nabla u(t)\big)\,ds \\
&\quad \geq \left(1-\frac{1}{4\tau}\right)\int_0^t g(s)\,ds\big\|\nabla u(t)\big\|^2 - \tau(g\circ\nabla u)(t)
\end{aligned} \tag{3.6}$$

for some number $\tau > 0$. By (3.5) and (3.6), we have

$$\begin{aligned}
L'(t) \geq{}& (1-\sigma)K^{-\sigma}(t)K'(t) + \frac{\varepsilon}{\rho+1}\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} - \varepsilon\big\|\nabla u(t)\big\|^2 \\
&+ \varepsilon\left(1-\frac{1}{4\tau}\right)\int_0^t g(s)\,ds\big\|\nabla u(t)\big\|^2 \\
&- \varepsilon\tau(g\circ\nabla u)(t) - \varepsilon\big(f\big(u_t(t)\big), u(t)\big)_{\Gamma_1} + \varepsilon\big\|u(t)\big\|_{p,\Gamma_1}^p.
\end{aligned} \tag{3.7}$$

Applying (2.7) and (2.12) to the last term $\|u(t)\|_{p,\Gamma_1}^p$ in the right-hand side of (3.7), we find that

$$\begin{aligned}
L'(t) \geq{}& (1-\sigma)K^{-\sigma}(t)K'(t) + \frac{\varepsilon}{\rho+1}\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} - \varepsilon\big\|\nabla u(t)\big\|^2 \\
&+ \varepsilon\left(1-\frac{1}{4\tau}\right)\int_0^t g(s)\,ds\big\|\nabla u(t)\big\|^2 - \varepsilon\tau(g\circ\nabla u)(t) - \varepsilon\big(f\big(u_t(t)\big), u(t)\big)_{\Gamma_1} \\
&+ \varepsilon\left(pK(t) + \frac{p}{\rho+2}\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} + \frac{p}{2}\left(1-\int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2\right. \\
&\left.+ \frac{p}{2}(g\circ\nabla u)(t) - p\hat{\delta}h_0\right) \\
={}& (1-\sigma)K^{-\sigma}(t)K'(t) + \varepsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} \\
&+ \varepsilon\left(\frac{p}{2}-\tau\right)(g\circ\nabla u)(t) + \varepsilon pK(t) - \varepsilon\big(f\big(u_t(t)\big), u(t)\big)_{\Gamma_1} \\
&+ \varepsilon\left\{\left(\frac{p}{2}-1\right) - \left(\frac{p}{2}-1+\frac{1}{4\tau}\right)\int_0^t g(s)\,ds\right\}\big\|\nabla u(t)\big\|^2 - \varepsilon p\hat{\delta}h_0.
\end{aligned}$$

From (2.11), we see that

$$\begin{aligned}
L'(t) \geq{}& (1-\sigma)K^{-\sigma}(t)K'(t) + \varepsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} \\
&+ \varepsilon\left[(1-\tau) + (1-\hat{\delta})\left(\frac{p}{2}-1\right)\right](g\circ\nabla u)(t) + \varepsilon pK(t) - \varepsilon\big(f\big(u_t(t)\big), u(t)\big)_{\Gamma_1} \\
&+ \varepsilon\left[(1-\hat{\delta})\left(\frac{p}{2}-1\right) - \left\{(1-\hat{\delta})\left(\frac{p}{2}-1\right)+\frac{1}{4\tau}\right\}\int_0^t g(s)\,ds\right]\big\|\nabla u(t)\big\|^2
\end{aligned} \tag{3.8}$$

for some τ with $0 < \tau < 1 + (1-\hat{\delta})(\frac{p}{2}-1)$. Using (2.10), (3.8) can be rewritten by

$$\begin{aligned}
L'(t) \geq{}& (1-\sigma)K^{-\sigma}(t)K'(t) + \varepsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\big\|u_t(t)\big\|_{\rho+2}^{\rho+2} \\
&+ \varepsilon c_1(g\circ\nabla u)(t) + \varepsilon c_2\big\|\nabla u(t)\big\|^2 + \varepsilon pK(t) - \varepsilon\big(f\big(u_t(t)\big), u(t)\big)_{\Gamma_1},
\end{aligned} \tag{3.9}$$

where

$$c_1 = (1-\tau) + (1-\hat{\delta})\left(\frac{p}{2}-1\right) > 0,$$

and

$$c_2 = (1-\hat{\delta})\left(\frac{p}{2}-1\right) - \left\{(1-\hat{\delta})\left(\frac{p}{2}-1\right) + \frac{1}{4\tau}\right\}\int_0^t g(s)\,ds > 0.$$

Let us put $\Gamma_{11} = \{x \in \Gamma_1 : |u_t(x,t)| \le 1\}$ and $\Gamma_{12} = \{x \in \Gamma_1 : |u_t(x,t)| > 1\}$. Then we obtain the inequalities which are given in [21]:

$$\int_{\Gamma_{11}} f\bigl(u_t(x,t)\bigr)u(x,t)\,dx \le \frac{\eta^{p-1}}{p}\bigl\|u(t)\bigr\|_{p,\Gamma_1}^p + \frac{(p-1)|\Gamma_1|}{p\eta}\bigl(\xi^{-1}(1)\bigr)^{\frac{p}{p-1}}, \quad \eta > 0,$$

and

$$\int_{\Gamma_{12}} f\bigl(u_t(x,t)\bigr)u(x,t)\,dx \le \frac{\mu^m}{m}\bigl\|u(t)\bigr\|_{p,\Gamma_1}^m + \frac{(m-1)}{m\mu^{\frac{m}{m-1}}}K'(t), \quad \mu > 0.$$

Inserting these into (3.9), we obtain

$$\begin{aligned} L'(t) \ge{}& \left\{(1-\sigma)K^{-\sigma}(t) - \frac{\varepsilon(m-1)}{m\mu^{\frac{m}{m-1}}}\right\}K'(t) \\ &+ \varepsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2}\right)\bigl\|u_t(t)\bigr\|_{\rho+2}^{\rho+2} + \varepsilon c_1(g\circ\nabla u)(t) \\ &+ \varepsilon c_2\bigl\|\nabla u(t)\bigr\|^2 + \varepsilon pK(t) - \frac{\varepsilon\eta^{p-1}}{p}\bigl\|u(t)\bigr\|_{p,\Gamma_1}^p \\ &- \frac{\varepsilon(p-1)|\Gamma_1|}{p\eta}\bigl(\xi^{-1}(1)\bigr)^{\frac{p}{p-1}} - \frac{\varepsilon\mu^m}{m}\bigl\|u(t)\bigr\|_{p,\Gamma_1}^m. \end{aligned} \tag{3.10}$$

Adding and subtracting $\varepsilon\theta K(t)$ in the right-hand side of (3.10) and applying (2.7) and (2.12), we get

$$\begin{aligned} L'(t) \ge{}& \left\{(1-\sigma)K^{-\sigma}(t) - \frac{\varepsilon(m-1)}{m\mu^{\frac{m}{m-1}}}\right\}K'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2} - \frac{\theta}{\rho+2}\right)\bigl\|u_t(t)\bigr\|_{\rho+2}^{\rho+2} \\ &+ \varepsilon\left(c_1 - \frac{\theta}{2}\right)(g\circ\nabla u)(t) + \varepsilon\left\{c_2 - \frac{\theta}{2}\left(1 - \int_0^t g(s)\,ds\right)\right\}\bigl\|\nabla u(t)\bigr\|^2 + \varepsilon(p-\theta)K(t) \\ &+ \varepsilon\left(\frac{\theta}{p} - \frac{\eta^{p-1}}{p}\right)\bigl\|u(t)\bigr\|_{p,\Gamma_1}^p - \frac{\varepsilon\mu^m}{m}\bigl\|u(t)\bigr\|_{p,\Gamma_1}^m + \varepsilon\theta\hat{\delta}h_0 \\ &- \frac{\varepsilon(p-1)|\Gamma_1|}{p\eta}\bigl(\xi^{-1}(1)\bigr)^{\frac{p}{p-1}}. \end{aligned} \tag{3.11}$$

Now, we choose $\mu = (kK^{-\sigma}(t))^{-\frac{m-1}{m}}$, $k > 0$ will be specified later. Using (2.13), (2.14), and (3.4), the seventh term in the right-hand side of (3.11) is estimated as

$$\begin{aligned} -\frac{\varepsilon\mu^m}{m}\bigl\|u(t)\bigr\|_{p,\Gamma_1}^m &= -\frac{\varepsilon k^{1-m}}{m}K^{\sigma(m-1)}(t)\bigl\|u(t)\bigr\|_{p,\Gamma_1}^m \\ &\ge -\frac{\varepsilon k^{1-m}}{m}p_0^{\sigma(m-1)}\bigl\|u(t)\bigr\|_{p,\Gamma_1}^{\sigma p(m-1)+m} \ge -\varepsilon C_2k^{1-m}\bigl\|u(t)\bigr\|_{p,\Gamma_1}^p, \end{aligned} \tag{3.12}$$

where $C_2 = \frac{C_0 p_0^{\sigma(m-1)}}{m}$. From (3.11) and (3.12), we obtain

$$\begin{aligned} L'(t) \geq{} & \left\{(1-\sigma) - \frac{\varepsilon k(m-1)}{m}\right\} K^{-\sigma}(t)K'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2} - \frac{\theta}{\rho+2}\right)\|u_t(t)\|_{\rho+2}^{\rho+2} \\ & + \varepsilon\left(c_1 - \frac{\theta}{2}\right)(g\circ\nabla u)(t) + \varepsilon\left(c_2 - \frac{\theta}{2}\right)\|\nabla u(t)\|^2 \\ & + \varepsilon\left(\frac{\theta}{p} - \frac{\eta^{p-1}}{p} - C_2 k^{1-m}\right)\|u(t)\|_{p,\Gamma_1}^p \\ & + \varepsilon(p-\theta)K(t) + \varepsilon\theta\hat{\delta}h_0 - \frac{\varepsilon(p-1)|\Gamma_1|}{p\eta}\big(\xi^{-1}(1)\big)^{\frac{p}{p-1}}. \end{aligned} \tag{3.13}$$

We take θ such that

$$0 < \theta < \min\{2c_1, 2c_2\}, \tag{3.14}$$

then we can choose $\eta > 0$ sufficiently small so that $\theta - \eta^{p-1} > 0$. And then, we select $k > 0$ large enough such that

$$\frac{\theta}{p} - \frac{\eta^{p-1}}{p} - C_2 k^{1-m} > 0,$$

and then take $\varepsilon > 0$ with

$$(1-\sigma) - \frac{\varepsilon k(m-1)}{m} > 0, \qquad K^{1-\sigma}(0) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_1|^\rho u_1 u_0\,dx + \frac{\varepsilon}{2}\|\nabla u_0\|^2 > 0.$$

Condition (3.1) gives that

$$\theta\hat{\delta}h_0 - \frac{(p-1)|\Gamma_1|}{p\eta}\big(\xi^{-1}(1)\big)^{\frac{p}{p-1}} > 0. \tag{3.15}$$

Using (3.13)–(3.15) and $2c_1 < p$, we have

$$L'(t) \geq C\big(\|u_t(t)\|_{\rho+2}^{\rho+2} + \|\nabla u(t)\|^2 + \|u(t)\|_{p,\Gamma_1}^p + K(t)\big), \tag{3.16}$$

here and in the sequel C denotes a generic positive constant. By arguments similar to those in [18], we get

$$\big(L(t)\big)^{\frac{1}{1-\sigma}} \leq C\big(K(t) + \|u_t(t)\|_{\rho+2}^{\rho+2} + \|\nabla u(t)\|^2 + \|u(t)\|_{p,\Gamma_1}^p\big). \tag{3.17}$$

Indeed, using Young's inequality and

$$\left|\int_\Omega |u_t(t)|^\rho u_t(t)u(t)\,dx\right| \leq \|u_t(t)\|_{\rho+2}^{\rho+1}\|u(t)\|_{\rho+2},$$

we obtain

$$\left(\left|\int_\Omega |u_t(t)|^\rho u_t(t)u(t)\,dx\right|\right)^{\frac{1}{1-\sigma}} \leq C\big(\|u_t(t)\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}\gamma} + \|u(t)\|_{\rho+2}^{\frac{1}{1-\sigma}\beta}\big), \tag{3.18}$$

where $\frac{1}{\gamma}+\frac{1}{\beta}=1$. By using (3.4) and taking $\gamma=\frac{(1-\sigma)(\rho+2)}{\rho+1}>1$, we get $\frac{\beta}{1-\sigma}=\frac{\rho+2}{(1-\sigma)(\rho+2)-(\rho+1)}$. Since K is an increasing function, from (2.13) and (3.2), we have

$$\|u(t)\|_{\rho+2}^{\frac{\beta}{1-\sigma}}\le B_1^{\frac{\beta}{1-\sigma}}\|\nabla u(t)\|^{\frac{\beta}{1-\sigma}}\le\frac{(B_1^2M)^{\frac{\beta}{2(1-\sigma)}}}{K(0)}K(t)\le\frac{p_0(B_1^2M)^{\frac{\beta}{2(1-\sigma)}}}{K(0)}\|u(t)\|_{p,\Gamma_1}^p,\tag{3.19}$$

where B_1 is the embedding constant. Similarly, from (2.13) and (3.2), we obtain

$$\|\nabla u(t)\|^{\frac{2}{1-\sigma}}\le M^{\frac{1}{1-\sigma}}\le\frac{p_0M^{\frac{1}{1-\sigma}}}{K(0)}\|u(t)\|_{p,\Gamma_1}^p.\tag{3.20}$$

From (3.3), (3.18)–(3.20), we find that (3.17) holds. Combining (3.16) and (3.17), we conclude that

$$L'(t)\ge C\big(L(t)\big)^{\frac{1}{1-\sigma}}\quad\text{for } t\ge 0.$$

Consequently, $L(t)$ blows up in time $T^*\le\frac{1-\sigma}{c\sigma(L(0))^{\frac{\sigma}{1-\sigma}}}$. Furthermore, we have

$$\lim_{t\to T^{*-}}\big(\|u_t(t)\|_{\rho+2}^{\rho+2}+\|\nabla u(t)\|^2+\|u(t)\|_{p,\Gamma_1}^p\big)=\infty.$$

This leads to a contradiction with (3.2). Thus the solution of (1.1)–(1.4) blows up in finite time. □

4 Conclusions

In this paper, we study the blow-up of solutions for the quasilinear viscoelastic wave equation with strong damping and boundary nonlinear damping. In recent years, there has been published much work concerning the wave equation with nonlinear boundary damping. But as far as we know, there was no blow-up result of solutions to the viscoelastic wave equation with nonlinear boundary damping and source terms. Therefore, we prove a finite time blow-up result of solution with positive initial energy as well as nonpositive initial energy. Moreover, we generalize the earlier result under a weaker assumption on the damping term.

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Abbreviations
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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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