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Infinitely many geometrically distinct solutions for periodic Schrödinger–Poisson systems

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Abstract

This paper is dedicated to studying the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $V(x)$, $K(x)$, and $f(x, u)$ are periodic in x . By using the non-Nehari manifold method, we establish the existence of ground state solutions for the above problem under some weak assumptions. Moreover, when f is odd in u , we prove that the above problem admits infinitely many geometrically distinct solutions. Our results improve and complement some related literature.

MSC: 35J10; 35J20

Keywords: Schrödinger–Poisson system; Nehari manifold; Ground state; Geometrically distinct solutions

1 Introduction

In this paper we are concerned with the nonlinear Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $V, K : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $f : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following basic assumptions:

(V0) $V, K \in C(\mathbb{R}^3, (0, \infty))$, $V(x)$, and $K(x)$ are 1-periodic in x_1 , x_2 , and x_3 ;

(F0) $f(x, t)$ is 1-periodic in x_1 , x_2 , and x_3 ;

(F1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $f(x, t) = o(|t|)$ uniformly in x as $t \rightarrow 0$, and there exist constants $C_0 > 0$ and $p \in (2, 6)$ such that

$$|f(x, t)| \leq C_0(1 + |t|^{p-1}), \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}.$$

Schrödinger–Poisson system (also called Schrödinger–Maxwell system) appears in the quantum mechanics model or Hartree–Fock model, which is related to the study of the

interaction of a charged non-relativistic quantum mechanical particle with the electromagnetic field. System (1.1) can be described by coupling a nonlinear Schrödinger and a Poisson equation, from physical point of view, the unknown terms u and ϕ are the fields associated to the particle and the electric potential, respectively, the functions V and K are, respectively, an external potential and nonnegative density charge, the nonlinear term f simulates the interaction effect between particles or external nonlinear perturbations, and the coupled term $\phi(x)u$ concerns the interaction with the field. For more details on the physical aspects, we refer the readers to [5–8, 20, 23].

Note that when $\phi \equiv 0$, (1.1) reduces to the well-known Schrödinger equation, which has been the object of various investigations; see, for example, [26, 33–35, 39, 40] and the references therein.

Under assumption (V0), the set

$$E = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx < +\infty \right\}$$

is a Hilbert space equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx \right)^{1/2}.$$

It is well known that the Poisson equation is solved by using the Lax–Milgram theorem. Indeed, as we shall see in Sect. 2, for every $u \in E$, unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is obtained, such that $-\Delta\phi = K(x)u^2$ and so (1.1) can be reduced to a single equation with a nonlocal term

$$-\Delta u + V(x)u + K(x)\phi_u(x)u = f(x, u). \tag{1.2}$$

Moreover, (1.2) is variational and its solutions are the critical points of the functional Φ defined on E by

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx, \tag{1.3}$$

where $F(x, t) = \int_0^t f(x, s) \, ds$. Define

$$\mathcal{N} := \{ u \in E : \langle \Phi'(u), u \rangle = 0, u \neq 0 \}, \tag{1.4}$$

which is the Nehari manifold of Φ . Let $*$ denote the action of \mathbb{Z}^3 on $H^1(\mathbb{R}^3)$ given by

$$(k * u)(x) = u(x - k), \quad k \in \mathbb{Z}^3.$$

We note that if u_0 is a solution of (1.1), then so are $k * u_0$ for all $k \in \mathbb{Z}^3$. Set

$$\mathcal{O}(u) := \{ k * u_0, k \in \mathbb{Z}^3 \},$$

which is called the orbit of u_0 with respect to the action of \mathbb{Z}^3 . Two solutions u_1 and u_2 are said to be geometrically distinct if $\mathcal{O}(u_1)$ and $\mathcal{O}(u_2)$ are disjoint.

In recent years, there have been rich results for Schrödinger–Poisson systems like (1.1) on the existence of nontrivial solutions, positive solutions, ground states, semi-classical states, and multiple solutions; we refer to [1–3, 6, 9, 14–17, 25, 29] for the case $f(x, u) \sim |u|^{q-2}u$ with $q \in (4, 6)$; [4, 28, 31, 36, 41] for the case $f(x, u) \sim |u|^{q-2}u$ with $q \in (3, 4)$; [13] for the convolution nonlinearity; [18, 38] for the critical growth nonlinearity. [19] deals with the multiplicity of solutions for the fractional Schrödinger–Poisson systems. In this paper, we focus on the existence of ground state solutions and infinitely many geometrically distinct solutions for (1.1) in a periodic setting. Let us recall some previous results that led us to the present research.

When the potential and nonlinearity are periodic, that is (V0) and (F0) are satisfied, Zhao and Zhao [41] proved that (1.1) with $K(x) = 1$ has a ground solution and infinitely many geometrically distinct solutions by using the Nehari manifold approach, where f and $\partial f/\partial u$ are continuous and satisfy suitable conditions. Based on the generalized Nehari manifold approach developed by Szulkin and Weth [32], Sun and Ma [31] obtained similar results as those in [41], where f satisfies (F0), (F1) and the following assumptions:

- (SC) $\lim_{|t| \rightarrow \infty} \frac{F(x,t)}{|t|^4} = +\infty$ uniformly in $x \in \mathbb{R}^3$;
- (MT) $f(x, t)/|t|^3$ is increasing in t on $\mathbb{R} \setminus \{0\}$ for every $x \in \mathbb{R}^3$.

Later, Chen and Tang relaxed (SC) and (MT) to the following weaker conditions:

- (F2) $\lim_{|t| \rightarrow \infty} \frac{F(x,t)}{|t|^3} = \infty$ uniformly in $x \in \mathbb{R}^3$;
- (F3') there exists $\theta \in (0, 1)$ such that

$$\left[\frac{f(x, \tau)}{\tau^3} - \frac{f(x, t\tau)}{(t\tau)^3} \right] \text{sign}(1 - t) + \theta V(x) \frac{|1 - t^2|}{(t\tau)^2} \geq 0, \quad \forall x \in \mathbb{R}^3, t > 0, \tau \neq 0,$$

and established the existence of ground state solutions for (1.1) by means of the non-Nehari manifold method developed by Tang [33–35].

To the best of our knowledge, except for [31, 41], there seems to be no result about the existence of infinitely many geometrically distinct solutions for (1.1). Motivated by the work of [9, 10, 31, 33], in the present paper, we shall establish the existence of ground state solutions and infinitely many geometrically distinct solutions for (1.1) under weaker assumptions than previous works.

Before presenting our theorems, in addition to (V0), (F0), (F1), and (F2), we introduce the following assumptions:

- (F3) $\frac{f(x,t)-V(x)t}{|t|^3}$ is nondecreasing in t on both $(-\infty, 0)$ and $(0, \infty)$ for every $x \in \mathbb{R}^3$;
- (F4) there exists a constant $\theta \in [0, 1)$ such that

$$f(x, t)t - 4F(x, t) + \theta V(x)t^2 \geq 0, \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R};$$

- (F5) $f(x, -t) = -f(x, t), \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}$.

Theorem 1.1 *Assume that $V, K,$ and f satisfy (V0), (F0), and (F1)–(F3). Then Problem (1.1) has a solution $u_0 \in E$ such that $\Phi(u_0) = \inf_N \Phi > 0$.*

Theorem 1.2 *Assume that $V, K,$ and f satisfy (V0), (F0), and (F1)–(F5). Then Problem (1.1) admits infinitely many pairs of geometrically distinct solutions.*

Remark 1.3 Note that the assumptions of Theorems 1.1 and 1.2 are weaker than those of [10, 31, 41]. A simple example of a function satisfying our assumptions but not conditions

in [31, 41] is $f(x, u) = b(x)u^3 - |u|^{3/2}u + |u|u$ for all $(x, u) \in \mathbb{R}^3 \times \mathbb{R}$, where $b(x)$ is 1-periodic in x_1, x_2 , and x_3 and $\inf_{\mathbb{R}^3} b > 0$. In this sense, our results improve and complement those of [10, 31, 41].

To prove Theorem 1.1, following the idea of [10], we apply the non-Nehari manifold method. Unlike the Nehari manifold approach, the key point of this method lies in finding a minimizing Cerami sequence for Φ outside \mathcal{N} by using a diagonal method. However, the fact that (F3) is weaker than (F3') used in [10] would require our extra efforts. To prove Theorem 1.2, inspired by [31, 32], we use deformation type arguments and Lusternik–Schnirelman theory. However, since $t \mapsto \frac{f(x,t)}{|t|^3}$ is not increasing, the generalized Nehari approach developed by [32] does not work. To circumvent this obstacle, we borrow the idea of [11, 12] in which Kirchhoff-type problems and Klein–Gordon–Maxwell systems were considered respectively. However, the competing effect of the nonlocal term $\int_{\mathbb{R}^3} \phi_u u^2 \, dx$ and the nonlinear term $\int_{\mathbb{R}^3} F(u) \, dx$ in the expression of Φ makes our problem more complicated.

The paper is organized as follows. In Sect. 2, we introduce some notation and preliminaries. We complete the proofs of Theorems 1.1 and 1.2 in Sects. 3 and 4, respectively.

Throughout this paper, we denote the norm of $L^s(\mathbb{R}^3)$ by $\|u\|_s = (\int_{\mathbb{R}^3} |u|^s \, dx)^{1/s}$ for $s \geq 2$, $B_r(x) = \{y \in \mathbb{R}^3 : |y - x| < r\}$, and positive constants possibly different in different places by C_1, C_2, \dots .

2 Notation and preliminaries

Hereafter, $H^1(\mathbb{R}^3)$ is the usual Sobolev space with the standard scalar product and norm

$$(u, v)_{H^1} = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) \, dx, \quad \|u\|_{H^1}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx,$$

and

$$D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

equipped with the norm defined by

$$\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx.$$

It is easy to show that (1.1) can be reduced to a single equation with a nonlocal term. Namely, for any $Ku^2 \in L^1_{loc}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x - y|} \, dx \, dy < \infty,$$

the distributional solution

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x - y|} \, dy = \frac{1}{|x|} * Ku^2 \tag{2.1}$$

of the Poisson equation

$$-\Delta \phi = K(x)u^2, \quad x \in \mathbb{R}^3$$

belongs to $D^{1,2}(\mathbb{R}^3)$ and is the unique weak solution in $D^{1,2}(\mathbb{R}^3)$ (see, e.g., [29] for more details), and

$$\int_{\mathbb{R}^3} \nabla \phi_u \nabla v \, dx = \int_{\mathbb{R}^3} K(x) u^2 v \, dx, \quad \forall v \in H^1(\mathbb{R}^3), \tag{2.2}$$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(x)K(y)}{|x-y|} u^2(x)u^2(y) \, dx \, dy = \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx. \tag{2.3}$$

Moreover, $\phi_u(x) > 0$ when $u \neq 0$, because K does (see (V0)). By using Hardy–Littlewood–Sobolev inequality (see [21] or [22, p. 98]), we have the following inequality:

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)v(y)|}{|x-y|} \, dx \, dy \leq \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}} \|u\|_{6/5} \|v\|_{6/5}, \quad u, v \in L^{6/5}(\mathbb{R}^3). \tag{2.4}$$

Formally, the solutions of (1.1) are then the critical points of the reduced functional (1.3). Indeed, (V0), (F0), and (2.4) imply that Φ is a well-defined functional of class C^1 and that

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv) \, dx + \int_{\mathbb{R}^3} [K(x)\phi_u(x)u - f(x, u)]v \, dx. \tag{2.5}$$

Hence if $u \in E$ is a critical point of Φ , then the pair (u, ϕ_u) , with ϕ_u as in (2.1), is a solution of (1.1).

Lemma 2.1 *Under assumptions (V0), (F1), and (F2),*

$$\Phi(u) \geq \Phi(tu) + \frac{1-t^4}{4} \langle \Phi'(u), u \rangle + \frac{(1-t^2)^2}{4} \|\nabla u\|_2^2, \quad \forall u \in E, t \geq 0. \tag{2.6}$$

Proof For any $x \in \mathbb{R}^3, t \geq 0, \tau \neq 0$, (F2) yields

$$\begin{aligned} & \frac{1-t^4}{4} \tau f(x, \tau) + F(x, t\tau) - F(x, \tau) + \frac{V(x)}{4} (1-t^2)^2 \tau^2 \\ &= \int_t^1 \left[\frac{f(x, \tau)}{\tau^3} - \frac{f(x, s\tau)}{(s\tau)^3} + V(x) \frac{(1-s^2)}{(s\tau)^2} \right] s^3 \tau^4 \, ds \geq 0. \end{aligned} \tag{2.7}$$

Note that

$$\Phi(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx \tag{2.8}$$

and

$$\langle \Phi'(u), u \rangle = \|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx - \int_{\mathbb{R}^3} f(x, u)u \, dx. \tag{2.9}$$

Thus, by (2.7), (2.8), and (2.9), one has

$$\begin{aligned} \Phi(u) - \Phi(tu) &= \frac{1-t^2}{2} \|u\|^2 + \frac{1-t^4}{4} \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx + \int_{\mathbb{R}^3} [F(x, tu) - F(x, u)] \, dx \\ &= \frac{1-t^4}{4} \langle \Phi'(u), u \rangle + \frac{(1-t^2)^2}{4} \|\nabla u\|_2^2 \end{aligned}$$

$$\begin{aligned}
 & + \int_{\mathbb{R}^3} \left[\frac{1-t^4}{4} f(x, u)u + F(x, tu) - F(x, u) + \frac{V(x)}{4} (1-t^2)^2 u^2 \right] dx \\
 & \geq \frac{1-t^4}{4} \langle \Phi'(u), u \rangle + \frac{(1-t^2)^2}{4} \|\nabla u\|_2^2, \quad t \geq 0.
 \end{aligned}$$

This shows that (2.6) holds. □

Corollary 2.2 *Under assumptions (V0), (F1), and (F2), for $u \in \mathcal{N}$,*

$$\Phi(u) = \max_{t \geq 0} \Phi(tu). \tag{2.10}$$

Unlike the super-cubic case, to show $\mathcal{N} \neq \emptyset$ in our situation, we have to overcome the competing effect of the nonlocal term. Inspired by Chen and Tang [10], we define a set Λ as follows:

$$\Lambda = \left\{ u \in E : \int_{\mathbb{R}^3} [V(x)u^2 + K(x)\phi_u u^2 - f(x, u)u] dx < 0 \right\}.$$

Lemma 2.3 *Under assumptions (V0) and (F1)–(F3), $\Lambda \neq \emptyset$ and $\mathcal{N} \subset \Lambda$. Then, for any $u \in \Lambda$, there exists unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}$.*

Proof First, we show that $\Lambda \neq \emptyset$. From (2.4) and Sobolev imbedding theorem, there exists $C_1 > 0$ such that $\int_{\mathbb{R}^3} \phi_u u^2 dx \leq C_1 \|u\|^4$ for all $u \in E$. For any fixed $u \in E$ with $u \neq 0$, set $u_t(x) = u(tx)$ for $t > 0$. By (V0), one has

$$\begin{aligned}
 & \int_{\mathbb{R}^3} [V(x)(tu_t)^2 + K(x)\phi_{(tu_t)}(tu_t)^2 - f(x, tu_t)tu_t] dx \\
 & = t^{-1} \int_{\mathbb{R}^3} V(t^{-1}x)u^2 dx + t^{-1} \int_{\mathbb{R}^3} K(t^{-1}x)\phi_u u^2 dx - \int_{\mathbb{R}^3} \frac{f(t^{-1}x, tu)tu}{t^3} dx \\
 & \leq V_\infty t^{-1} \|u\|_2^2 + C_1 K_\infty t^{-1} \|u\|^4 - \int_{\mathbb{R}^3} \frac{f(t^{-1}x, tu)tu}{t^3} dx,
 \end{aligned} \tag{2.11}$$

where $V_\infty = \sup_{x \in \mathbb{R}^3} V(x)$ and $K_\infty = \sup_{x \in \mathbb{R}^3} K(x)$. Note that, for $u(x) \neq 0$, $F(t^{-1}x, tu)/|tu|^3 \rightarrow +\infty$ as $t \rightarrow +\infty$ uniformly in $x \in \mathbb{R}^3$ by (F3), and (2.7) with $t = 0$ yields

$$\frac{1}{4} f(x, \tau)\tau - F(x, \tau) + \frac{V(x)}{4} \tau^2 \geq 0, \quad \forall x \in \mathbb{R}^3, \tau \in \mathbb{R}, \tag{2.12}$$

then we have

$$\frac{f(t^{-1}x, tu)tu}{|tu|^3} \rightarrow +\infty, \quad \text{as } t \rightarrow +\infty \text{ uniformly in } x \in \mathbb{R}^3. \tag{2.13}$$

Thus, it follows from (V0), (2.11), and (2.13) that

$$\int_{\mathbb{R}^3} [V(x)(tu_t)^2 + K(x)\phi_{(tu_t)}(tu_t)^2 - f(x, tu_t)tu_t] dx \rightarrow -\infty, \quad \text{as } t \rightarrow +\infty.$$

Thus, taking $v = Tu_T$ for T large, we have $v \in \Lambda$. Hence, $\Lambda \neq \emptyset$. From (2.5), it is easy to see that $\mathcal{N} \subset \Lambda$.

Next, we prove the last part of the lemma. Let $u \in \Lambda$ be fixed and define a function $g(t) := \langle \Phi'(tu), tu \rangle$ on $[0, \infty)$. By (F2), one has

$$f(x, t\tau)t\tau \geq f(x, \tau)\tau t^4 - V(x)(t^2 - 1)(t\tau)^2, \quad \forall x \in \mathbb{R}^3, t \geq 1, \tau \in \mathbb{R}, \tag{2.14}$$

which yields

$$\begin{aligned} & \int_{\mathbb{R}^3} [V(x)(t\tau)^2 + K(x)\phi_{t\tau}(t\tau)^2 - f(x, t\tau)t\tau] \, dx \\ & \leq t^4 \int_{\mathbb{R}^3} [V(x)\tau^2 + K(x)\phi_\tau\tau^2 - f(x, \tau)\tau] \, dx, \quad \forall t \geq 1, \tau \in \mathbb{R}. \end{aligned} \tag{2.15}$$

From (2.5) and (2.15) it follows that

$$\begin{aligned} g(t) & \leq t^2 \|u\|^2 + t^4 \int_{\mathbb{R}^3} [V(x)u^2 + K(x)\phi_u u^2 - f(x, u)u] \, dx \\ & \quad - t^2 \int_{\mathbb{R}^3} V(x)u^2 \, dx, \quad \forall t \geq 1. \end{aligned} \tag{2.16}$$

Using (F0), (2.5), and (2.16), it is easy to verify that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for t large due to $u \in \Lambda$. Therefore, there exists $t_0 = t(u) > 0$ so that $g(t_0) = 0$ and $t(u)u \in \mathcal{N}$. We claim that $t(u)$ is unique for any $u \in \Lambda$. In fact, for any given $u \in \Lambda$, let $t_1, t_2 > 0$ such that $g(t_1) = g(t_2) = 0$. Jointly with (2.6), we have

$$\begin{aligned} \Phi(t_1u) & \geq \Phi(t_2u) + \frac{t_1^4 - t_2^4}{4t_1^4} \langle \Phi'(t_1u), t_1u \rangle + \frac{(t_1^2 - t_2^2)^2}{4t_1^2} \|\nabla u\|_2^2 \\ & = \Phi(t_2u) + \frac{(t_1^2 - t_2^2)^2}{4t_1^2} \|\nabla u\|_2^2 \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \Phi(t_2u) & \geq \Phi(t_1u) + \frac{t_2^4 - t_1^4}{4t_2^4} \langle \Phi'(t_2u), t_2u \rangle + \frac{(t_2^2 - t_1^2)^2}{4t_2^2} \|\nabla u\|_2^2 \\ & = \Phi(t_1u) + \frac{(t_2^2 - t_1^2)^2}{4t_2^2} \|\nabla u\|_2^2. \end{aligned} \tag{2.18}$$

(2.17) and (2.18) imply $t_1 = t_2$. Hence, $t(u) > 0$ is unique for any $u \in \Lambda$. □

Lemma 2.4 *Under assumptions (V0) and (F1)–(F3), then*

$$\inf_{u \in \mathcal{N}} \Phi(u) := c = \inf_{u \in \Lambda, u \neq 0} \max_{t \geq 0} \Phi(tu) > 0.$$

Proof Both Corollary 2.2 and Lemma 2.3 imply that $c = \inf_{u \in \Lambda, u \neq 0} \max_{t \geq 0} \Phi(tu)$. Using Lemma 2.1, it is easy to see that $c > 0$. □

Lemma 2.5 *Under assumptions (V0) and (F1)–(F3), there exist a constant $c_* \in (0, c]$ and a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \rightarrow c_*, \quad \|\Phi'(u_n)\| (1 + \|u_n\|) \rightarrow 0. \tag{2.19}$$

Proof By (F1) and (1.3), we know that there exist $\delta_0 > 0$ and $\rho_0 > 0$ such that

$$\Phi(u) \geq \rho_0, \quad \|u\| = \delta_0. \tag{2.20}$$

In view of Lemmas 2.3 and 2.4, we may choose $v_k \in \mathcal{N} \subset \Lambda$ such that

$$c - \frac{1}{k} < \Phi(v_k) < c + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.21}$$

Using Lemma 2.1 and (2.20), it is easy to check that $\Phi(tv_k) \geq \rho_0$ for small $t > 0$ and $\Phi(tv_k) < 0$ for large $t > 0$ due to $v_k \in \Lambda$. Since $\Phi(0) = 0$, then the mountain pass lemma implies that there exists a sequence $\{u_{k,n}\}_{n \in \mathbb{N}} \subset E$ satisfying

$$\Phi(u_{k,n}) \rightarrow c_k, \quad \|\Phi'(u_{k,n})\|(1 + \|u_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}, \tag{2.22}$$

where $c_k \in [\rho_0, \sup_{t \geq 0} \Phi(tv_k)]$. By virtue of Corollary 2.2, one has $\Phi(v_k) = \sup_{t \geq 0} \Phi(tv_k)$. Hence, by (2.21) and (2.22), one has

$$\Phi(u_{k,n}) \rightarrow c_k \in \left[\rho_0, c + \frac{1}{k} \right), \quad \|\Phi'(u_{k,n})\|(1 + \|u_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}. \tag{2.23}$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\Phi(u_{k,n_k}) \in \left[\rho_0, c + \frac{1}{k} \right), \quad \|\Phi'(u_{k,n_k})\|(1 + \|u_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.24}$$

Let $u_k = u_{k,n_k}$, $k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\Phi(u_n) \rightarrow c_* \in [\rho_0, c], \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \rightarrow 0. \quad \square$$

Lemma 2.6 *Under assumptions (V0) and (F1)–(F3), any sequence $\{u_n\} \subset E$ satisfying (2.19) is bounded in E .*

Proof By (2.6) with $t = 0$, one has

$$c_* + o(1) = \Phi(u_n) - \frac{1}{4} \langle \Phi'(u_n), u_n \rangle \geq \frac{1}{4} \|\nabla u_n\|_2^2. \tag{2.25}$$

By (V0), it is easy to see that there exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^3$. Since $\langle \Phi'(u_n), u_n \rangle = o(1)$, it follows from (V0), (F1), and the Sobolev embedding inequality that

$$\begin{aligned} V_0 \|u_n\|_2^2 &\leq \|u_n\|^2 + \int_{\mathbb{R}^3} K(x) \phi_{u_n} u_n^2 \, dx \\ &= \int_{\mathbb{R}^3} f(u_n) u_n \, dx + o(1) \\ &\leq \frac{1}{2} V_0 \|u_n\|_2^2 + C_2 \|u_n\|_6^6 + o(1) \\ &\leq \frac{1}{2} V_0 \|u_n\|_2^2 + C_2 S^{-3} \|\nabla u_n\|_2^6 + o(1), \end{aligned} \tag{2.26}$$

which together with (2.25) implies that $\{u_n\}$ is bounded in E . □

3 Ground state solutions

In this section, we give the proof of Theorem 1.1.

Proof of Theorem 1.1 Lemma 2.5 implies the existence of a sequence $\{u_n\} \subset E$ satisfying (2.19), then

$$\Phi(u_n) \rightarrow c_* > 0, \quad \langle \Phi'(u_n), u_n \rangle \rightarrow 0. \tag{3.1}$$

By Lemma 2.6, $\{u_n\}$ is bounded in E . If

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |u_n|^2 \, dx = 0,$$

then by Lion’s concentration compactness principle [24] or [37, Lemma 1.21], $u_n \rightarrow 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. Moreover, there exists $C_3 > 0$ such that $\|u_n\|_2 \leq C_3$. By (F0), for $\varepsilon = c_*/2C_3^2$, there exists $C_\varepsilon > 0$ such that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \left| \frac{1}{2} f(x, u_n) u_n - F(x, u_n) \right| \, dx \leq \frac{3}{2} \varepsilon C_3^2 + C_\varepsilon \lim_{n \rightarrow \infty} \|u_n\|_p^p = \frac{3c_*}{4}. \tag{3.2}$$

By (V0), (2.3), and (2.4), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} K(x) \phi_{u_n}(x) u_n^2 \, dx &= \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(x)K(y)}{|x-y|} u_n^2(x) u_n^2(y) \, dx \, dy \\ &\leq K_\infty^2 \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} \, dx \, dy \\ &\leq C_1 K_\infty^2 \limsup_{n \rightarrow \infty} \|u_n\|_{12/5}^4 = 0, \end{aligned} \tag{3.3}$$

where, and in the sequel, $C_1 = 8\sqrt[3]{2}/3\sqrt[3]{\pi}$. From (1.3), (2.5), (3.1), (3.2), and (3.3), one has

$$\begin{aligned} c_* &= \Phi(u_n) - \frac{1}{2} \langle \Phi'(u_n), u_n \rangle + o(1) \\ &= -\frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_{u_n}(x) u_n^2 \, dx + \int_{\mathbb{R}^3} \left[\frac{1}{2} f(x, u_n) u_n - F(x, u_n) \right] \, dx + o(1) \\ &\leq \frac{3c_*}{4} + o(1). \end{aligned}$$

This contradiction shows $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^3$ such that

$$\int_{B_2(k_n)} |u_n|^2 \, dx > \frac{\delta}{2}. \tag{3.4}$$

Let $v_n(x) = u_n(x + k_n)$. Then

$$\int_{B_2(0)} |v_n|^2 \, dx > \frac{\delta}{2}. \tag{3.5}$$

Since $V(x), K(x)$, and $f(x, u)$ are periodic on x , we have

$$\Phi(v_n) \rightarrow c_* \in (0, c], \quad \|\Phi'(v_n)\|(1 + \|v_n\|) \rightarrow 0. \tag{3.6}$$

Passing to a subsequence, we have $v_n \rightarrow \bar{v}$ in E , $v_n \rightarrow \bar{v}$ in $L^s_{loc}(\mathbb{R}^3)$, $2 \leq s < 6$, and $v_n(x) \rightarrow \bar{v}(x)$ a.e. on \mathbb{R}^3 . Thus, (3.5) implies that $\bar{v} \neq 0$. For every $\phi \in C^\infty_0(\mathbb{R}^3)$, we have

$$\langle \Phi'(\bar{v}), \phi \rangle = \lim_{n \rightarrow \infty} \langle \Phi'(v_n), \phi \rangle = 0.$$

Hence $\Phi'(\bar{v}) = 0$. This shows that $\bar{v} \in \mathcal{N}$ is a nontrivial solution of Problem (1.1) and $\Phi(\bar{v}) \geq c$. It follows from (F2), (3.6), and Fatou’s lemma that

$$\begin{aligned} c &\geq c_* = \lim_{n \rightarrow \infty} \left[\Phi(v_n) - \frac{1}{4} \langle \Phi'(v_n), v_n \rangle \right] \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{4} \|\nabla v_n\|_2^2 + \int_{\mathbb{R}^3} \left[\frac{1}{4} f(x, v_n) v_n - F(x, v_n) + \frac{V(x)}{4} v_n^2 \right] dx \right\} \\ &\geq \frac{1}{4} \liminf_{n \rightarrow \infty} \|\nabla v_n\|_2^2 + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \left[\frac{1}{4} f(x, v_n) v_n - F(x, v_n) + \frac{V(x)}{4} v_n^2 \right] dx \\ &\geq \frac{1}{4} \|\bar{v}\|^2 + \int_{\mathbb{R}^3} \left[\frac{1}{4} f(x, \bar{v}) \bar{v} - F(x, \bar{v}) \right] dx \\ &= \Phi(\bar{v}) - \frac{1}{4} \langle \Phi'(\bar{v}), \bar{v} \rangle = \Phi(\bar{v}). \end{aligned}$$

This shows that $\Phi(\bar{v}) \leq c$, and so $\Phi(\bar{v}) = c = \inf_{\mathcal{N}} \Phi > 0$. □

4 Infinitely many geometrically distinct solutions

To prove Theorem 1.2, we need some notations. For $d_2 \geq d_1 > -\infty$ and $c \in \mathbb{R}$, we put

$$\begin{aligned} \Phi^{d_2} &:= \{u \in E : \Phi(u) \leq d_2\}, & \Phi_{d_1} &:= \{u \in E : \Phi(u) \geq d_1\}, & \Phi_{d_1}^{d_2} &:= \Phi_{d_1} \cap \Phi^{d_2}; \\ \mathcal{K} &:= \{u \in E \setminus \{0\} : \Phi'(u) = 0\}, & \mathcal{K}_c &:= \{u \in \mathcal{K} : \Phi(u) = c\}. \end{aligned}$$

In view of Theorem 1.1, under (V0), (F0), and (F1)–(F3), (1.1) has a nontrivial solution $\bar{u} \in H^1(\mathbb{R}^N)$ satisfying $\Phi(\bar{u}) = c_0 := \inf_{\mathcal{K}} \Phi > 0$. Therefore, $\mathcal{K} \supseteq \mathcal{K}_{c_0} \neq \emptyset$. Following the strategy of [32], we choose a subset \mathcal{F} of \mathcal{K} such that $\mathcal{F} = -\mathcal{F}$ and each orbit $\mathcal{O}(w) \subset \mathcal{K}$ has a unique representative in \mathcal{F} . It suffices to show that the set \mathcal{F} is infinite. So from now on we assume by contradiction that

$$\mathcal{F} \text{ is a finite set.} \tag{4.1}$$

Lemma 4.1 $\kappa := \inf\{\|u - v\| : u, v \in \mathcal{K}, u \neq v\} > 0$.

Proof Choose $\{u_n\}, \{v_n\} \subset \mathcal{K}$ such that $\|u_n - v_n\| \rightarrow \kappa$. Then there exist $w^1, w^2 \in \mathcal{F}$ and $k_n, l_n \in \mathbb{Z}^3$ such that $u_n = w^1(\cdot - k_n)$ and $v_n = w^2(\cdot - l_n)$. Put $m_n = k_n - l_n$. There are two possible cases.

Case (1). $\{|m_n|\}$ is bounded. Passing to a subsequence, $m_n = m \in \mathbb{Z}^3$, one has

$$\kappa + o(1) = \|u_n - v_n\| = \|w^1(\cdot - k_n) - w^2(\cdot - l_n)\| = \|w^1 - w^2(\cdot - m)\| > 0.$$

Case (2). $\{|m_n|\}$ is unbounded. Passing to a subsequence, $|m_n| \rightarrow \infty$, one has

$$\begin{aligned} \kappa + o(1) &= \|u_n - v_n\| = \|w^1(\cdot - k_n) - w^2(\cdot - l_n)\| \\ &= \|w^1 - w^2(\cdot - m_n)\| = (\|w^1\|^2 + \|w^2\|^2)^{1/2} + o(1) > 0. \end{aligned}$$

Both Cases (1) and (2) show that $\kappa > 0$. □

Lemma 4.2 (Discreteness of (PS)-sequences) *Let $c \geq c_0$. If $\{u_n^1\}, \{u_n^2\} \subset \Phi^c$ are two (PS)-sequences for Φ , then either $\|u_n^1 - u_n^2\| \rightarrow 0$ as $n \rightarrow \infty$ or $\liminf_{n \rightarrow \infty} \|u_n^1 - u_n^2\| \geq \min\{\kappa, \sqrt{2}c_0\} > 0$.*

Proof First, we prove the boundedness of Palais–Smale sequences for Φ . Let $\{u_n\}$ be such that

$$\Phi(u_n) \rightarrow c > 0, \quad \Phi'(u_n) \rightarrow 0. \tag{4.2}$$

Then it follows from (F4) and (4.2) that, for large $n \in \mathbb{N}$,

$$\begin{aligned} c + 1 + \|u_n\| &\geq \Phi(u_n) - \frac{1}{4} \langle \Phi'(u_n), u_n \rangle \\ &= \frac{1}{4} \|u_n\|^2 + \int_{\mathbb{R}^3} \left[\frac{1}{4} f(x, u_n) u_n - F(x, u_n) \right] dx \\ &\geq \frac{1 - \theta}{4} \|u_n\|^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in E . Thus, $\{u_n^1\}, \{u_n^2\} \subset \Phi^c$ are two bounded (PS)-sequences for Φ . Next, we fix p as in (F1), and we distinguish two cases.

Case (1). $\|u_n^1 - u_n^2\|_p \rightarrow 0$. In this case, we can prove that $\lim_{n \rightarrow \infty} \|u_n^1 - u_n^2\| = 0$ in a similar fashion as [32, Lemma 2.14].

Case (2). $\|u_n^1 - u_n^2\|_p \not\rightarrow 0$. Then again by [37, Lemma 1.21], there exist $\varepsilon_0 > 0$ and $k_n \in \mathbb{Z}^3$ such that, after passing to a subsequence,

$$\int_{B_{1+\sqrt{N}}(k_n)} |u_n^1 - u_n^2|^p dx = \max_{k \in \mathbb{Z}^3} \int_{B_{1+\sqrt{N}}(k)} |u_n^1 - u_n^2|^p dx \geq \varepsilon_0.$$

Using that Φ is equivariant with respect to translations of the form $u \mapsto u(\cdot - k)$ with $k \in \mathbb{Z}^3$, we may assume that $\{k_n\}$ is bounded in \mathbb{Z}^3 . We may pass to a subsequence such that

$$u_n^1 \rightharpoonup u^1, \quad u_n^2 \rightharpoonup u^2, \quad u^1 \neq u^2, \quad \Phi'(u^1) = \Phi'(u^2) = 0.$$

We first consider the case where $u^1 \neq 0$ and $u^2 \neq 0$, so that $u^1, u^2 \in \mathcal{K}$. By Lemma 4.1, one has $\liminf_{n \rightarrow \infty} \|u_n^1 - u_n^2\| \geq \|u^1 - u^2\| \geq \kappa$. It remains to consider the case where either $u^2 = 0$ or $u^1 = 0$. In this case, it is easy to see that $\liminf_{n \rightarrow \infty} \|u_n^1 - u_n^2\| \geq \|u^1 - u^2\| \geq \sqrt{2}c_0$. □

Since Φ is even, it is known (see, e.g., [30, Lemma II.3.9]) that Φ admits an odd pseudo-gradient vector field, i.e., there exists an odd Lipschitz continuous map $W : E \setminus (\mathcal{K} \cup \{0\}) \rightarrow$

E such that

$$\left. \begin{aligned} \|W(u)\| &\leq 2\|\Phi'(u)\|, \\ \langle \Phi'(u), W(u) \rangle &\geq \|\Phi'(u)\|^2. \end{aligned} \right\} \tag{4.3}$$

Now we consider the Cauchy problem:

$$\begin{cases} \frac{d\eta}{dt} = -W(\eta), \\ \eta(0, u) = u. \end{cases} \tag{4.4}$$

The basic existence-uniqueness theorem for ordinary differential equations implies that, for each $u \in E$, (4.4) has a unique solution $\eta(t, u)$ defined for t in a maximal interval $(T^-(u), T^+(u))$, and $\eta(t, u)$ is odd with respect to $u \in E$.

Lemma 4.3 *Let $u \in E \setminus (\mathcal{K} \cup \{0\})$. If $\inf_{t \in [0, T^+(u))} \Phi(\eta(t, u)) > -\infty$, then $\lim_{t \rightarrow T^+(u)} \eta(t, u)$ exists and is a critical point of Φ .*

Proof From (4.3) and (4.4), we have

$$\begin{aligned} \frac{d}{dt} \Phi(\eta(t, u)) &= -\langle \Phi'(\eta(t, u)), W(\eta(t, u)) \rangle \\ &\leq -\|\Phi'(\eta(t, u))\|^2 < 0, \quad \forall t \in [0, T^+(u)). \end{aligned} \tag{4.5}$$

This shows that $\Phi(\eta(t, u))$ is strictly decreasing on $t \in [0, T^+(u))$, and so $\tau := \lim_{t \rightarrow T^+(u)} \Phi(\eta(t, u))$ exists.

Case (1). $T^+(u) < +\infty$. For $0 \leq t_1 < t_2 < T^+(u)$, from (4.3) and (4.4), we have

$$\begin{aligned} \|\eta(t_2, u) - \eta(t_1, u)\| &\leq \int_{t_1}^{t_2} \|W(\eta(t, u))\| dt \\ &\leq 2 \int_{t_1}^{t_2} \sqrt{\langle \Phi'(\eta(t, u)), W(\eta(t, u)) \rangle} dt \\ &\leq 2\sqrt{t_2 - t_1} \left[\int_{t_1}^{t_2} \langle \Phi'(\eta(t, u)), W(\eta(t, u)) \rangle dt \right]^{1/2} \\ &= 2\sqrt{t_2 - t_1} [\Phi(\eta(t_1, u)) - \Phi(\eta(t_2, u))]^{1/2} \\ &\leq 2\sqrt{t_2 - t_1} [\Phi(u) - \tau]^{1/2}. \end{aligned}$$

Since $T^+(u) < +\infty$, this implies that $\lim_{t \rightarrow T^+(u)} \eta(t, u)$ exists and then it must be a critical point of Φ (otherwise the trajectory $t \mapsto \eta(t, u)$ could be continued beyond $T^+(u)$).

Case (2). $T^+(u) = +\infty$. To prove that $\lim_{t \rightarrow T^+(u)} \eta(t, u)$ exists, it suffices to show that

$$\text{for every } \varepsilon > 0, \text{ there exists } t_\varepsilon > 0 \text{ such that } \|\eta(t_\varepsilon, u) - \eta(t, u)\| < \varepsilon, \quad \forall t \geq t_\varepsilon. \tag{4.6}$$

We suppose by contradiction that (4.6) is false. Then there exist $0 < \varepsilon_0 < \frac{1}{2} \min\{\kappa, \sqrt{2}c_0\}$ and a sequence $\{t_n\} \subset [0, +\infty)$ such that

$$0 \leq t_1 < t_2 < \dots < t_n \rightarrow +\infty \quad \text{and} \quad \|\eta(t_{n+1}, u) - \eta(t_n, u)\| = \varepsilon_0, \quad \forall n \in \mathbb{N}. \tag{4.7}$$

Choose the smallest $t_n^1 \in (t_n, t_{n+1})$ and $s_n^1 \in [t_n, t_n^1]$ such that

$$\|\eta(t_n^1, u) - \eta(t_n, u)\| = \frac{\varepsilon_0}{3} \quad \text{and} \quad \|\Phi'(\eta(s_n^1, u))\| = \min_{t \in [t_n, t_n^1]} \|\Phi'(\eta(t, u))\|. \tag{4.8}$$

Then it follows from (4.3), (4.4), and (4.8) that

$$\begin{aligned} \frac{\varepsilon_0}{3} &= \|\eta(t_n^1, u) - \eta(t_n, u)\| \\ &\leq \int_{t_n}^{t_n^1} \|W(\eta(t, u))\| dt \\ &\leq 2 \int_{t_n}^{t_n^1} \|\Phi'(\eta(t, u))\| dt \\ &\leq \frac{2}{\|\Phi'(\eta(s_n^1, u))\|} \int_{t_n}^{t_n^1} \|\Phi'(\eta(t, u))\|^2 dt \\ &\leq \frac{2}{\|\Phi'(\eta(s_n^1, u))\|} \int_{t_n}^{t_n^1} \langle \Phi'(\eta(t, u)), W(\eta(t, u)) \rangle dt \\ &= \frac{2}{\|\Phi'(\eta(s_n^1, u))\|} [\Phi(\eta(t_n, u)) - \Phi(\eta(t_n^1, u))]. \end{aligned}$$

Since $\Phi(\eta(t_n, u)) - \Phi(\eta(t_n^1, u)) \rightarrow 0$ as $n \rightarrow \infty$, the above implies that $\Phi'(\eta(s_n^1, u)) \rightarrow 0$ as $n \rightarrow \infty$. Similarly we find the largest $t_n^2 \in (t_n, t_{n+1})$ and $s_n^2 \in [t_n^2, t_{n+1}]$ such that $\|\eta(t_{n+1}, u) - \eta(t_n^2, u)\| = \frac{\varepsilon_0}{3}$ and $\Phi'(\eta(s_n^2, u)) \rightarrow 0$ as $n \rightarrow \infty$. Let $u_n^1 := \eta(s_n^1, u)$ and $u_n^2 := \eta(s_n^2, u)$. Then $\{u_n^1\}$ and $\{u_n^2\}$ are two Palais–Smale sequences of Φ such that $\frac{\varepsilon_0}{3} \leq \|u_n^1 - u_n^2\| \leq 2\varepsilon_0 < \min\{\kappa, \sqrt{2}c_0\}$. This, however, contradicts Lemma 4.2, hence (4.6) is true. So $\lim_{t \rightarrow T^+(u)} \eta(t, u)$ exists, and it must be a critical point of Φ . \square

In the following, for a subset $A \subset E$ and $\delta > 0$, we put $U_\delta(A) := \{v \in E : \text{dist}(v, A) < \delta\}$.

Lemma 4.4 *Let $c \geq c_0$. Then, for every $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that*

- (a) $\Phi_{c-\varepsilon}^{c+\varepsilon} \cap \mathcal{K} = \mathcal{K}_c$;
- (b) $\lim_{t \rightarrow T^+(u)} \Phi(\eta(t, u)) < c - \varepsilon$ for $u \in \Phi^{c+\varepsilon} \setminus U_\delta(\mathcal{K}_c)$.

Proof In view of (4.1) and the \mathbb{Z}^3 -translation invariance for Φ , $\Phi(\mathcal{K}) := \{\Phi(w) : w \in \mathcal{K}\}$ is a finite set. Therefore, there exists $\hat{\varepsilon} > 0$ such that (a) is satisfied for $\varepsilon \in (0, \hat{\varepsilon})$.

Without loss of generality, we may assume $U_\delta(\mathcal{K}_c) \subset \Phi^{c+1}$ and $\delta < \min\{\kappa, \sqrt{2}c_0\}$. Next we find $\varepsilon \in (0, \hat{\varepsilon})$ such that (b) holds. Let $u \in \Phi^{c+\hat{\varepsilon}} \setminus U_\delta(\mathcal{K}_c)$. Since $\Phi(\eta(t, u))$ is strictly decreasing on $t \in [0, T^+(u))$, if $\Phi(\eta(t_0, u)) \leq c - \hat{\varepsilon}$ for some $t_0 \in [0, T^+(u))$, then $\lim_{t \rightarrow T^+(u)} \Phi(\eta(t, u)) < c - \hat{\varepsilon}$. Thus, we only consider the case where $\Phi(\eta(t, u)) > c - \hat{\varepsilon}$ for all $t \in [0, T^+(u))$. In this case, it follows from Lemma 4.3 that $\lim_{t \rightarrow T^+(u)} \eta(t, u)$ exists and is a critical point of Φ . Set

$$\alpha := \inf\{\|\Phi'(w)\| : w \in U_\delta(\mathcal{K}_c) \setminus U_{\delta/2}(\mathcal{K}_c)\}. \tag{4.9}$$

We claim that $\alpha > 0$. Indeed, suppose by contradiction that there exists a sequence $\{u_n^1\} \subset U_\delta(\mathcal{K}_c) \setminus U_{\delta/2}(\mathcal{K}_c)$ such that $\Phi'(u_n^1) \rightarrow 0$. Passing to a subsequence, using the finiteness

condition (4.1) and the \mathbb{Z}^3 -invariance of Φ , we may assume $\{u_n^1\} \subset U_\delta(w_0) \setminus U_{\delta/2}(w_0)$ for some $w_0 \in \mathcal{K}_c$. Let $u_n^2 = w_0$. Then

$$\frac{\delta}{2} \leq \limsup_{n \rightarrow \infty} \|u_n^1 - u_n^2\| \leq \delta < \min\{\kappa, \sqrt{2}c_0\},$$

which contradicts Lemma 4.2. Hence $\alpha > 0$.

Let $\beta := \sup\{\|\Phi'(w)\| : w \in U_\delta(\mathcal{K}_c) \setminus U_{\delta/2}(\mathcal{K}_c)\}$ and $0 < \varepsilon < \frac{\alpha^2\delta}{4\beta}$. By Lemma 4.3 and (a), the only way (b) can fail is that $\eta(t, u) \rightarrow \tilde{w} \in \mathcal{K}_c$ as $t \rightarrow T^+(u)$ for some $u \in \Phi^{c+\varepsilon} \setminus U_\delta(\mathcal{K}_c)$. In this case we let

$$\begin{aligned} t_1 &:= \sup\{t \in [0, T^+(u)] : \eta(t, u) \notin U_\delta(\tilde{w})\} \quad \text{and} \\ t_2 &\in \{t \in [0, T^+(u)] : \eta(t, u) \in U_{\frac{\delta}{2}}(\tilde{w})\}. \end{aligned} \tag{4.10}$$

Then from (4.3), (4.4), and (4.10), we have

$$\frac{\delta}{2} \leq \|\eta(t_2, u) - \eta(t_1, u)\| \leq \int_{t_1}^{t_2} \|W(\eta(t, u))\| dt \leq 2 \int_{t_1}^{t_2} \|\Phi'(\eta(t, u))\| dt \leq 2\beta(t_2 - t_1)$$

and

$$\Phi(\eta(t_2, u)) - \Phi(\eta(t_1, u)) \leq - \int_{t_1}^{t_2} \|\Phi'(\eta(t, u))\|^2 dt \leq -\alpha^2(t_2 - t_1) \leq -\frac{\alpha^2\delta}{4\beta}.$$

Hence $\lim_{t_2 \rightarrow T^+(u)} \Phi(\eta(t_2, u)) \leq c + \varepsilon - \frac{\alpha^2\delta}{4\beta} < c$, contrary to our assumption. □

Proof of Theorem 1.2 For $j \in \mathbb{N}$, we consider the family Σ_j of all closed and symmetric subsets $A \subset E \setminus \{0\}$ (i.e., $A = -A = \bar{A}$) with $\gamma(A) \geq j$, where γ denotes the usual Krasnoselskii genus (see, e.g., [27, 30]). Moreover, we consider the nondecreasing sequence of Lusternik–Schnirelman values for Φ defined by $c_k := \{c \in \mathbb{R} : \gamma(\Phi^c) \geq k\}$ for $k \in \mathbb{N}$. We claim:

$$\mathcal{K}_{c_k} \neq \emptyset \quad \text{and} \quad c_k < c_{k+1}, \quad k \in \mathbb{N}. \tag{4.11}$$

To prove this, let $k \in \mathbb{N}$ and $c = c_k$. In view of Lemma 4.1, $\gamma(\mathcal{K}_c) = 0$ if $\mathcal{K}_c = \emptyset$ or $\gamma(\mathcal{K}_c) = 1$ if $\mathcal{K}_c \neq \emptyset$. By the continuity property of the genus, there exists $\delta > 0$ such that $\gamma(U_\delta(\mathcal{K}_c)) = \gamma(\mathcal{K}_c)$. Choose $\varepsilon = \varepsilon(\delta) > 0$ such that the properties of Lemma 4.4 hold. Then, for every $u \in \Phi^{c+\varepsilon} \setminus U_\delta(\mathcal{K}_c)$, $t_u \in [0, T^+(u))$, where $t_u := \inf\{t \in [0, T^+(u)) : \Phi(\eta(t, u)) \leq c - \varepsilon\}$. Since $\eta(t, u)$ is odd with respect to u and Φ is even, it implies that $t_u = t_{-u}$. Define a map

$$h : \Phi^{c+\varepsilon} \setminus U_\delta(\mathcal{K}_c) \rightarrow \Phi^{c-\varepsilon}, \quad h(u) = \eta(t_u, u).$$

Then h is odd and continuous. Hence $\gamma(\Phi^{c+\varepsilon} \setminus U_\delta(\mathcal{K}_c)) \leq \gamma(\Phi^{c-\varepsilon}) \leq k - 1$ and therefore

$$\gamma(\Phi^{c+\varepsilon}) \leq \gamma(\overline{U_\delta(\mathcal{K}_c)}) + k - 1 = \gamma(\mathcal{K}_c) + k - 1.$$

The definition of $c = c_k$ and of c_{k+1} implies that $\gamma(\mathcal{K}_c) \geq 1$ if $c_{k+1} > c_k$ and $\gamma(\mathcal{K}_c) > 1$ if $c_{k+1} = c_k$. Since $\gamma(\mathcal{F}) = \gamma(\mathcal{K}_c) \leq 1$, (4.11) follows.

It follows now from (4.11) that there is an infinite sequence $\{u_k\}$ of pairs of geometrically distinct critical points of Φ with $\Phi(u_k) = c_k$, contrary to (4.1). The proof is finished. \square

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