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Laplace's equation with concave and convex boundary nonlinearities on an exterior region

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Abstract

This paper studies Laplace's equation $-\Delta u = 0$ in an exterior region $U \subsetneq \mathbb{R}^N$, when $N \geq 3$, subject to the nonlinear boundary condition $\frac{\partial u}{\partial \nu} = \lambda|u|^{q-2}u + \mu|u|^{p-2}u$ on ∂U with $1 < q < 2 < p < 2^*$. In the function space $\mathcal{H}(U)$, one observes that, when $\lambda > 0$ and $\mu \in \mathbb{R}$ arbitrary, then there exists a sequence $\{u_k\}$ of solutions with negative energy converging to 0 as $k \rightarrow \infty$; on the other hand, when $\lambda \in \mathbb{R}$ and $\mu > 0$ arbitrary, then there exists a sequence $\{\tilde{u}_k\}$ of solutions with positive and unbounded energy. Also, associated with the p -Laplacian equation $-\Delta_p u = 0$, the exterior p -harmonic Steklov eigenvalue problems are described.

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1 Introduction

This paper discusses the existence of infinitely many harmonic functions in an exterior region $U \subsetneq \mathbb{R}^N$ when $N \geq 3$, subject to a nonlinear boundary condition on ∂U that combines concave and convex terms with $1 < q < 2 < p < 2^*$, described as below

$$\begin{cases} -\Delta u(x) = 0 & \text{in } U, \\ \frac{\partial u}{\partial \nu}(z) = \lambda|u(z)|^{q-2}u(z) + \mu|u(z)|^{p-2}u(z) & \text{on } \partial U, \end{cases} \quad (1.1)$$

in the space $E^1(U)$ of functions where $u \in L^{2^*}(U)$ and $\nabla u \in L^2(U; \mathbb{R}^N)$. Here, $2^* := \frac{2N}{N-2}$ is the critical Sobolev index and $\nabla u := (D_1u, D_2u, \dots, D_Nu)$ is the weak gradient of u .

A region is a nonempty, open, connected subset U of \mathbb{R}^N , and is said to be an exterior region provided that its complement $\mathbb{R}^N \setminus U$ is a nonempty, compact subset. Without loss of generality, we simply assume that $0 \notin U$. The boundary of a set A is denoted by ∂A .

Our general assumption on U is the following condition.

Condition B.1 $U \subsetneq \mathbb{R}^N$ is an exterior region, with $0 \notin U$, whose boundary ∂U is the union of finitely many disjoint, closed, Lipschitz surfaces, each of finite surface area.

One may want to notice here that the prototypical problem

$$-\Delta u(x) = \lambda|u(x)|^{q-2}u(x) + \mu|u(x)|^{p-2}u(x) \quad \text{in } \Omega$$

has originally been investigated by Ambrosetti, Brézis and Cerami [1] in 1994 and then in 1995 by Bartsch and Willem [7], in the function space $H_0^1(\Omega)$ on a bounded region Ω with a smooth boundary $\partial\Omega$. Since then, there have been a large number of papers appearing on some related problems; nevertheless, the description of the existence of solutions to problem (1.1) is missing as a reasonable decomposition result of the associated Hilbert function space is required for application of the dual fountain theorem, as discussed in [7, 28].

The aim of this paper is to solve problem (1.1) using a recent decomposition result by Auchmuty and Han [3, 12]. To state our result, we first define the energy functional

$$\varphi(u) := \frac{1}{2} \int_U |\nabla u|^2 dx - \frac{\lambda}{q} \int_{\partial U} |u|^q d\sigma - \frac{\mu}{p} \int_{\partial U} |u|^p d\sigma. \quad (1.2)$$

Here, dx is the Lebesgue volume element of \mathbb{R}^N while $d\sigma$ is the Hausdorff $(N-1)$ -dimensional surface element of ∂U . The main result of this paper is described as below.

Theorem 1.1 *Assume condition (B1) holds and $1 < q < 2 < p < 2_*$.*

- When $\lambda \in \mathbb{R}$ and $\mu > 0$ arbitrary, then problem (1.1) has a sequence $\{u_k\}$ of solutions in $E^1(U)$ such that $\varphi(u_k) > 0$ and $\varphi(u_k) \rightarrow \infty$ as $k \rightarrow \infty$.*
- When $\lambda > 0$ and $\mu \in \mathbb{R}$ arbitrary, then problem (1.1) has a sequence $\{v_k\}$ of solutions in $E^1(U)$ such that $\varphi(v_k) < 0$ and $\varphi(v_k) \rightarrow 0^-$ as $k \rightarrow \infty$.*

We recall in Sect. 2 some necessary results to carry out the proofs that are detailed in Sect. 3; Sect. 4 is devoted to the description of the p -harmonic Steklov eigenvalue problems on an exterior region U in a Banach space $E^{1,p}(U)$ when $1 < p < N$.

We remark all solutions considered in this paper are weak or distributional solutions.

It is interesting to see that some nice properties of the first exterior p -harmonic Steklov eigenvalue problem are described in Han [12, 13, 15] and he [14] also studied an exterior harmonic boundary value problem with some oscillating boundary condition. However, there is no result for the sequence of p -harmonic Steklov eigenvalue problems on an exterior region U , so we will study this in Sect. 4. See Torné [27] for the bounded region case.

Finally, one notices that we only want to simply present an application of some result in [3, 12]. Theorem 1.1 remains true when the special nonlinearity in (1.1) is replaced by more general ones as mentioned in [7]. On the other hand, it is very interesting to know more results like this using the fountain theorems of Yan and Yang [29], Zou [33], Du and Mao [8], Sun, Liu and Wu [26]. Other important results can be found in Polidoro and Ragusa [22], Han [16, 17], Feng, Li and Sun [9], and Phung and Minh [21], Mao and Zhao [18–20], Guan, Zhao and Lin [10, 11], Zhang [30], Zhang, Liu and Wu [31, 32].

2 The function space $E^{1,p}(U)$

First, let us fix the notations that will be used in this paper. Given $p, q \in [1, \infty]$, $L^p(U)$ and $L^q(\partial U, d\sigma)$ are the usual spaces of extended, real-valued, Lebesgue measurable functions on U and ∂U , with their standard norms written as $\|\cdot\|_{p,U}$ and $\|\cdot\|_{q,\partial U}$, respectively.

Auchmuty and Han [3, 4, 12] recently introduced a new function space $E^{1,p}(U)$ suitable for the study of harmonic boundary value problems on an exterior region U which satisfies

the boundary regularity condition (B1)—that is, each function $u \in E^{1,p}(U)$ satisfies $u \in L^{p^*}(U)$ and $|\nabla u| \in L^p(U)$ with $N \geq 2$ and $p^* := \frac{Np}{N-p}$ when $1 < p < N$.

The gradient L^p -norm provides a norm to guarantee $E^{1,p}(U)$ a Banach function space—that is, $E^{1,p}(U)$ is a Banach function space with respect to the norm

$$\|u\|_{p,\nabla} := \left(\int_U |\nabla u|^p dx \right)^{\frac{1}{p}} \quad \text{for all } u \in E^{1,p}(U). \tag{2.1}$$

Notice when $p \geq N$, Auchmuty and Han [4] showed, with an interesting example, that $E^{1,p}(U)$ is not complete with respect to the gradient L^p -norm in general.

When $N \geq 3$ and $p = 2$, they instead used the notation $E^1(U)$ to denote the associated Hilbert function space with respect to the gradient L^2 -inner product

$$\langle u, v \rangle_{\nabla} := \int_U \nabla u \cdot \nabla v dx \quad \text{for all } u, v \in E^1(U), \tag{2.2}$$

whose norm is thus written as $\|u\|_{\nabla}$. In addition, one has the direct sum

$$E^1(U) = E_0^1(U) \oplus \mathcal{H}(U), \tag{2.3}$$

where $\mathcal{H}(U)$ denotes the Hilbert subspace of $E^1(U)$ of all functions u satisfying

$$\langle u, v \rangle_{\nabla} = \int_U \nabla u \cdot \nabla v dx = 0 \quad \text{for all } v \in C_c^1(U)$$

and $E_0^1(U)$ denotes the closure of $C_c^1(U)$ with respect to this ∇ -norm. Here, $C_c^1(U)$ is the set of functions that are continuously differentiable and have compact support in U .

Let us recall some results as regards the space $E^{1,p}(U)$ which will be used later.

Lemma 2.1 *Suppose that $N \geq 2$, $1 < p < N$ and condition (B1) holds. Then the embedding of $E^{1,p}(U)$ into $L^{p^*}(U)$ is continuous, where $p^* := \frac{Np}{N-p}$ is the critical Sobolev index; besides, the embedding of $E^{1,p}(U)$ into $L^q(\partial U, d\sigma)$ is continuous when $1 \leq q \leq p_*$ and also compact when $1 \leq q < p_*$, where $p_* := \frac{(N-1)p}{N-p}$ is the trace critical Sobolev index.*

Obviously, Lemma 2.1 shows us some concrete function spaces that are contained in the dual space of $E^{1,p}(U)$. The preceding results can be found, with details, in [3, 4, 12].

Below, we give the fountain theorems. Given a compact group \mathfrak{G} and a normed vector space \mathcal{X} with norm $\|\cdot\|$, we say \mathfrak{G} acts *isometrically* on \mathcal{X} provided $\|gu\| = \|u\|$ for all $g \in \mathfrak{G}$ and $u \in \mathcal{X}$; also, a subset $\tilde{\mathcal{X}} \subseteq \mathcal{X}$ is said to be *invariant* with respect to \mathfrak{G} provided $gu \in \tilde{\mathcal{X}}$ for every $u \in \tilde{\mathcal{X}}$ and $g \in \mathfrak{G}$. On the other hand, given \mathfrak{G} and a finite dimensional space \mathbf{V} , we say the action of \mathfrak{G} on \mathbf{V} is *admissible* when each continuous, equivariant map $\wp : \partial \mathbf{O} \rightarrow \mathbf{V}^k$ has a zero, where \mathbf{O} is an open, bounded, invariant (with respect to \mathfrak{G}) neighborhood of 0 in \mathbf{V}^{k+1} for some $k \geq 1$; here, the map \wp is said to be *equivariant* provided $g \circ \wp = \wp \circ g$ for all $g \in \mathfrak{G}$, with $g(v_1, v_2, \dots, v_k) := (gv_1, gv_2, \dots, gv_k)$ for any $v = (v_1, v_2, \dots, v_k) \in \mathbf{V}^k$.

Next, given a Banach space \mathcal{X} , a functional $\psi : \mathcal{X} \rightarrow \mathbb{R}$ is said to belong to $C^1(\mathcal{X}, \mathbb{R})$, provided its first *Fréchet derivative* exists and is continuous on \mathcal{X} ; when ψ has a continuous first *Gateaux derivative* ψ' on \mathcal{X} , then one observes $\psi \in C^1(\mathcal{X}, \mathbb{R})$. Clearly, the functional φ defined in (1.2) is in $C^1(\mathcal{H}(U), \mathbb{R})$ and we shall assume this from now on.

Also, $\psi : \mathcal{X} \rightarrow \mathbb{R}$ is said to be *invariant* with respect to \mathfrak{G} provided $\psi \circ g = \psi$ for every $g \in \mathfrak{G}$.

Now, let \mathcal{X} be a Banach space with $\mathcal{X} = \overline{\bigoplus_{j \in \mathbb{N}} \mathcal{X}(j)}$ and write, for each $k \in \mathbb{N}$,

$$\mathcal{Y}_k := \overline{\bigoplus_{j=0}^k \mathcal{X}(j)} \quad \text{and} \quad \mathcal{Z}_k := \overline{\bigoplus_{j=k}^{\infty} \mathcal{X}(j)}. \tag{2.4}$$

Then one has the following results: the *fountain theorem* and the *dual fountain theorem*.

Theorem 2.2 ([1]) *Let \mathfrak{G} be a compact group, $\mathcal{X} = \overline{\bigoplus_{j \in \mathbb{N}} \mathcal{X}(j)}$ a Banach space with norm $\|\cdot\|$, and $\psi \in C^1(\mathcal{X}, \mathbb{R})$ an invariant functional; for each $k \in \mathbb{N}$, let $\mathcal{Y}_k, \mathcal{Z}_k$ be defined as in (2.4), and $\rho_k > \varrho_k > 0$ some constants. For every $k \geq k_0$ with a fixed $k_0 \in \mathbb{N}$, we also assume*

- (a1) \mathfrak{G} acts isometrically on \mathcal{X} , the spaces $\mathcal{X}(j)$ are invariant and there is a finite dimensional space \mathbf{V} such that, for all $j \in \mathbb{N}$, $\mathcal{X}(j) \simeq \mathbf{V}$ and the action of \mathfrak{G} on \mathbf{V} is admissible;
- (a2) $\mathfrak{a}_k := \max_{u \in \mathcal{Y}_k, \|u\| = \rho_k} \psi(u) < 0$;
- (a3) $\mathfrak{b}_k := \inf_{u \in \mathcal{Z}_k, \|u\| = \varrho_k} \psi(u) \rightarrow \infty$ as $k \rightarrow \infty$;
- (a4) ψ satisfies the $(PS)_c$ -condition for every $c \in (0, \infty)$.

Then ψ has a sequence of critical values $\{u_k\}$ with $\psi(u_k) > 0$ and $\psi(u_k) \rightarrow \infty$ when $k \rightarrow \infty$.

Theorem 2.3 [1] *Under the hypotheses of Theorem 2.2, suppose again that condition (a1) holds. For every $k \geq k_1$ with a fixed $k_1 \in \mathbb{N}$, we also assume*

- (b1) $\tilde{\mathfrak{a}}_k := \max_{u \in \mathcal{Y}_k, \|u\| = \varrho_k} \psi(u) < 0$;
- (b2) $\tilde{\mathfrak{b}}_k := \inf_{u \in \mathcal{Z}_k, \|u\| = \rho_k} \psi(u) \geq 0$;
- (b3) $\tilde{\mathfrak{c}}_k := \inf_{u \in \mathcal{Z}_k, \|u\| \leq \rho_k} \psi(u) \rightarrow 0^-$ as $k \rightarrow \infty$;
- (b4) ψ satisfies the $(PS)_c^*$ -condition with respect to \mathcal{Y}_k for each $c \in [\tilde{\mathfrak{c}}_{k_1}, 0)$.

Then ψ has a sequence of critical values $\{v_k\}$ with $\psi(v_k) < 0$ and $\psi(v_k) \rightarrow 0^-$ when $k \rightarrow \infty$.

Remark Notice $\tilde{\mathfrak{c}}_k \leq \min_{u \in \mathcal{X}(k), \|u\| = \varrho_k} \psi(u) \leq \max_{u \in \mathcal{X}(k), \|u\| = \varrho_k} \psi(u) \leq \tilde{\mathfrak{a}}_k < 0$ as $\mathcal{Y}_k \cap \mathcal{Z}_k = \mathcal{X}(k)$ —this fact is used in conditions (b3) and (b4) presented above in Theorem 2.3.

A sequence $\{u_k\}$ is said to be a *Palais–Smale sequence* for the functional $\psi \in C^1(\mathcal{X}, \mathbb{R})$ at level c in \mathcal{X} , $(PS)_c$ -sequence for short, if $\psi(u_k) \rightarrow c$ yet $\psi'(u_k) \rightarrow 0$ as $k \rightarrow \infty$; ψ satisfies the $(PS)_c$ -condition provided each $(PS)_c$ -sequence has a strongly convergent subsequence in \mathcal{X} . On the other hand, a sequence $\{\tilde{u}_{k_l}\}$, with \tilde{u}_{k_l} in \mathcal{Y}_{k_l} , is said to be a *generalized Palais–Smale sequence* for ψ at level c , $(PS)_c^*$ -sequence for short, if $\psi(\tilde{u}_{k_l}) \rightarrow c$ yet $\psi|'_{\mathcal{Y}_{k_l}}(\tilde{u}_{k_l}) \rightarrow 0$ as $l \rightarrow \infty$; ψ satisfies the $(PS)_c^*$ -condition with respect to \mathcal{Y}_k provided each $(PS)_c^*$ -sequence has a subsequence that converges strongly to a critical point of ψ in \mathcal{X} .

More details on fountain theorems can be found in [6, 7, 28, 29, 33].

3 Existence results of (1.1)

In this section, we shall present the proofs of Theorem 1.1. Matching with the preceding notations, we can identify $\mathfrak{G} = \mathbb{Z}_2$ —the second order quotient group, $\mathcal{X} = \mathcal{H}(U)$ —the Hilbert subspace of $E^1(U)$ of all finite energy harmonic functions, and $\psi = \varphi \in C^1(\mathcal{X}, \mathbb{R})$. One result in [3] shows $\mathcal{X} = \overline{\bigoplus_{j \in \mathbb{N}} \mathcal{X}(j)}$; here, $\mathcal{X}(j) = \text{span}\{s_j\} \simeq \mathbf{V} = \mathbb{R}$, with $s_j \in \mathcal{H}(U)$ a finite energy *harmonic Steklov eigenfunction* associated with the j th *harmonic Steklov*

eigenvalue $\delta_j > 0$. Noting that the functional φ is even, condition (a1) is trivially satisfied since a classical result of Borsuk–Ulam says that the antipodal action of \mathbb{Z}_2 on \mathbb{R} is admissible.

In the following, we shall deduce conditions (a2)–(a4) and (b1)–(b4) to guarantee the conclusions of the first and the second part of Theorem 1.1, respectively.

To further simplify notations, set $\|\cdot\| := \|\cdot\|_\nabla$ and $\|\cdot\|_s := \|\cdot\|_{s,\partial U}$ in this section. Take a nonzero $u \in \mathcal{Y}_k$, and use tu and (1.2) to derive, for some sufficiently large $t > 0$,

$$\varphi(tu) := \frac{t^2}{2} \|u\|^2 - \frac{\lambda t^q}{q} \|u\|_q^q - \frac{\mu t^p}{p} \|u\|_p^p < 0 \tag{3.1}$$

in view of the fact that the space \mathcal{Y}_k is of finite dimension—so that all norms are equivalent. As such, condition (a2) is satisfied for every $\rho_k \geq t\|u\| > 0$ when $\mu > 0$.

Next, define

$$\alpha_k := \sup_{u \in \mathcal{Z}_k - \{0\}} \frac{\|u\|_p}{\|u\|} > 0. \tag{3.2}$$

Then one observes $\alpha_k \rightarrow 0$ when $k \rightarrow \infty$. Actually, it is readily seen that $0 < \alpha_{k+1} \leq \alpha_k$, so that $\alpha_k \rightarrow \alpha \geq 0$ as $k \rightarrow \infty$. By hypotheses, there exists a $u_k \in \mathcal{Z}_k$ satisfying $\|u_k\| = 1$ and $\|u_k\|_p \geq \alpha_k/2$ for each k ; by definition of \mathcal{Z}_k , one sees $u_k \rightharpoonup 0$, i.e., u_k converges weakly to 0, in $\mathcal{H}(U)$. Lemma 2.1 then yields $u_{k_l} \rightarrow 0$ in $L^p(\partial U, d\sigma)$ as $l \rightarrow \infty$, for a subsequence $\{u_{k_l}\}$ of $\{u_k\}$. That is, $\alpha = 0$. On each subspace \mathcal{Z}_k with a sufficiently large norm, we have

$$\varphi(u) \geq \frac{1}{2} \|u\|^2 - \frac{|\lambda|c_1^q}{q} \|u\|_q^q - \frac{\mu\alpha_k^p}{p} \|u\|_p^p \geq \frac{1}{2} \left(\frac{1}{2} + \frac{1}{p} \right) \|u\|^2 - \frac{\mu\alpha_k^p}{p} \|u\|_p^p, \tag{3.3}$$

where $c_1 > 0$ is such a constant that $\|u\|_q \leq c_1\|u\|$ for all $u \in E^1(U)$. Take $\varrho_k := (\mu\alpha_k^p)^{-\frac{1}{p-2}}$. Then, via the property of α_k , there exists a $k_0 \in \mathbb{N}$ such that one can always choose $\rho_k = 2\varrho_k$ in the foregoing estimate (3.1) for every $k \geq k_0$; in addition, for (3.3), one derives

$$\varphi(u) \geq \frac{1}{2} \left(\frac{1}{2} - \frac{1}{p} \right) \mu^{-\frac{2}{p-2}} \alpha_k^{-\frac{2p}{p-2}} \rightarrow \infty \tag{3.4}$$

when $k \rightarrow \infty$. As a consequence, condition (a3) is ensured.

Finally, take a $(PS)_c$ -sequence $\{u_k\}$ for the functional φ at level $c > 0$ in $\mathcal{H}(U)$; that is, $\varphi(u_k) \rightarrow c$ yet $\varphi'(u_k) \rightarrow 0$ as $k \rightarrow \infty$. Then one has, for k sufficiently large,

$$\begin{aligned} c + 1 + \|u_k\| &\geq \varphi(u_k) - \frac{1}{p} \varphi'(u_k)(u_k) \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) \int_U |\nabla u_k|^2 dx - \lambda \left(\frac{1}{q} - \frac{1}{p} \right) \int_{\partial U} |u_k|^q d\sigma, \end{aligned} \tag{3.5}$$

from which one deduces immediately that

$$c + 1 + \|u_k\| + |\lambda|c_1^q \left(\frac{1}{q} - \frac{1}{p} \right) \|u_k\|_q^q \geq \left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|^2. \tag{3.6}$$

As a result, these u_k are bounded, and thus, without loss of generality, converge weakly to a function $u \in \mathcal{H}(U)$; via result 2.1 again, we may simply suppose that $u_k \rightarrow u$ in $L^p(\partial U, d\sigma)$

and $L^q(\partial U, d\sigma)$ when $k \rightarrow \infty$; besides, using (1.2), a routine calculation leads to

$$\begin{aligned} \|u_k - u\|^2 &= (\varphi'(u_k) - \varphi'(u))(u_k - u) + \lambda \int_{\partial U} [|u_k|^{q-2}u_k - |u|^{q-2}u](u_k - u) d\sigma \\ &\quad + \mu \int_{\partial U} [|u_k|^{p-2}u_k - |u|^{p-2}u](u_k - u) d\sigma \rightarrow 0 \end{aligned} \tag{3.7}$$

when $k \rightarrow \infty$. Thus, condition (a4) is also derived so that part (a) of Theorem 1.1 is proved.

On the other hand, apply a parallel idea as shown in (3.1) to prove that condition (b1) is satisfied for every $0 < \varrho_k \leq t\|u\|$, with a nonzero $u \in \mathcal{Y}_k$ and some sufficiently small $t > 0$, when $\lambda > 0$ —since \mathcal{Y}_k is of finite dimension. Next, define

$$\beta_k := \sup_{u \in \mathcal{Z}_k - \{0\}} \frac{\|u\|_q}{\|u\|} > 0. \tag{3.8}$$

Similarly, one observes that $\beta_k \rightarrow 0$ when $k \rightarrow \infty$ again from Lemma 2.1. On each subspace \mathcal{Z}_k with a sufficiently small norm, we have

$$\varphi(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda\beta_k^q}{q}\|u\|^q - \frac{|\mu|c_2^p}{p}\|u\|^p \geq \frac{1}{4}\|u\|^2 - \frac{\lambda\beta_k^q}{q}\|u\|^q, \tag{3.9}$$

where $c_2 > 0$ is such a constant that $\|u\|_p \leq c_2\|u\|$ for any $u \in E^1(U)$. Take $\rho_k := (\frac{4\lambda\beta_k^q}{q})^{\frac{1}{2-q}}$ to derive $\varphi(u) \geq 0$. Via the property of β_k , there exists a $k_1 \in \mathbb{N}$ such that we may select the above $\varrho_k = \frac{\rho_k}{2}$ to be sufficiently small for all $k \geq k_1$. As such, condition (b2) is ensured; also, condition (b3) follows in view of the fact $\rho_k \rightarrow 0$ as $k \rightarrow \infty$. Finally, take a $(PS)_c^*$ -sequence $\{\tilde{u}_{k_l}\}$, with $\tilde{u}_{k_l} \in \mathcal{Y}_{k_l}$, for φ at level $c \in [\tilde{c}_{k_1}, 0)$; that is, $\varphi(\tilde{u}_{k_l}) \rightarrow c$ whereas $\varphi'|_{\mathcal{Y}_{k_l}}(\tilde{u}_{k_l}) \rightarrow 0$ as $l \rightarrow \infty$. So, one infers $\frac{1}{p}|\varphi'(\tilde{u}_{k_l})(\tilde{u}_{k_l})| \leq \|\tilde{u}_{k_l}\|$ for sufficiently large l . Thus, (3.6) holds again. As a result, these \tilde{u}_{k_l} are bounded, and thus converge weakly without loss of generality to a function $\tilde{u} \in \mathcal{H}(U)$; via Lemma 2.1 again, we may simply assume that $\tilde{u}_{k_l} \rightarrow \tilde{u}$ in $L^p(\partial U, d\sigma)$ and $L^q(\partial U, d\sigma)$ as $l \rightarrow \infty$; by use of (1.2) again, one observes, just like (3.7), $\|\tilde{u}_{k_l} - \tilde{u}\| \rightarrow 0$ when $l \rightarrow \infty$. Thus, condition (b4) is also derived so that part (b) of Theorem 1.1 is proved.

All the above discussions finish the proof of Theorem 1.1 completely.

We do not know whether $\mathfrak{v}_k \rightarrow 0$ as $k \rightarrow \infty$; this is the case if 0 is the only solution of problem (1.1) with energy 0. However, we can derive the following result.

Proposition 3.1 *Assume condition (B1) holds and $1 < q < 2 < p < 2_*$.*

(a) *When $\lambda \in \mathbb{R}$ arbitrary yet $\mu \leq 0$, then (1.1) has no solution with positive energy; also,*

$$\inf\{\|u\| : u \text{ solves (1.1) with } \varphi(u) > 0\} \rightarrow \infty \quad \text{as } \mu \rightarrow 0^+.$$

(b) *When $\mu \in \mathbb{R}$ arbitrary yet $\lambda \leq 0$, then (1.1) has no solution with negative energy; also,*

$$\sup\{\|v\| : v \text{ solves (1.1) with } \varphi(v) < 0\} \rightarrow 0 \quad \text{as } \lambda \rightarrow 0^+.$$

Proof Take $u \in \mathcal{H}(U)$ to be such that $\varphi(u) \geq 0$ and $\varphi'(u) = 0$. Then one has

$$\varphi(u) - \frac{1}{q}\varphi'(u)(u) = \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 - \mu\left(\frac{1}{p} - \frac{1}{q}\right)\|u\|^p \geq 0. \tag{3.10}$$

When $\mu \leq 0$, then $u = 0$ follows immediately. Accordingly, we need to assume $\mu > 0$ in general. In this case, we correspondingly have

$$\mu c_2^p \left(\frac{1}{q} - \frac{1}{p}\right) \|u\|^p \geq \mu \left(\frac{1}{q} - \frac{1}{p}\right) \|u\|_p^p \geq \left(\frac{1}{q} - \frac{1}{2}\right) \|u\|^2,$$

from which one can deduce that

$$\|u\| \geq \left\{ \mu^{-1} \frac{(\frac{1}{q} - \frac{1}{2})}{c_2^p (\frac{1}{q} - \frac{1}{p})} \right\}^{\frac{1}{p-2}} \rightarrow \infty \tag{3.11}$$

when $\mu \rightarrow 0^+$. This finishes the proof for part (a) of Proposition 3.1.

In addition, let $v \in \mathcal{H}(U)$ be such that $\varphi(v) \leq 0$ and $\varphi'(v) = 0$. Similarly, one has

$$\varphi(v) - \frac{1}{p} \varphi'(v)(v) = \left(\frac{1}{2} - \frac{1}{p}\right) \|v\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{p}\right) \|v\|_q^q \leq 0. \tag{3.12}$$

When $\lambda \leq 0$, then $v = 0$ follows immediately. Accordingly, we need to assume $\lambda > 0$ in general. In this case, we correspondingly have

$$\lambda c_1^q \left(\frac{1}{q} - \frac{1}{p}\right) \|v\|^q \geq \lambda \left(\frac{1}{q} - \frac{1}{p}\right) \|v\|_q^q \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|v\|^2,$$

from which one finds readily that

$$\|v\| \leq \left(\lambda \frac{c_1^q (\frac{1}{q} - \frac{1}{p})}{(\frac{1}{2} - \frac{1}{p})} \right)^{\frac{1}{2-q}} \rightarrow 0 \tag{3.13}$$

when $\lambda \rightarrow 0^+$. This finishes the proof for part (b) of Proposition 3.1. □

Finally, we consider the following analogous problem:

$$\begin{cases} -\Delta u(x) + u(x) = 0 & \text{in } U, \\ \frac{\partial u}{\partial \nu}(z) = \lambda |u(z)|^{q-2} u(z) + \mu |u(z)|^{p-2} u(z) & \text{on } \partial U, \end{cases} \tag{3.14}$$

in the standard Hilbert–Sobolev space $H^1(U)$, where all $u \in H^1(U)$ satisfy $u, |\nabla u| \in L^2(U)$, and we define the associated energy functional

$$\phi(u) := \frac{1}{2} \int_U [|\nabla u|^2 + |u|^2] dx - \frac{\lambda}{q} \int_{\partial U} |u|^q d\sigma - \frac{\mu}{p} \int_{\partial U} |u|^p d\sigma. \tag{3.15}$$

Note that in view of some result in [2], we have the following direct sum:

$$H^1(U) = H_0^1(U) \oplus \mathcal{N}(U), \tag{3.16}$$

where $\mathcal{N}(U)$ is the Hilbert subspace of $H^1(U)$ of all functions u satisfying

$$\langle u, v \rangle_{1,2} = \int_U [\nabla u \cdot \nabla v + uv] dx = 0 \quad \text{for all } v \in C_c^1(U)$$

and $H_0^1(U)$ is the closure of $C_c^1(U)$ with respect to the standard H^1 -norm. Applying a similar procedure to the proof of Theorem 1.1, we can obtain the following result.

Theorem 3.2 *Assume condition (B1) holds and $1 < q < 2 < p < 2_*$.*

- (a) *When $\lambda \in \mathbb{R}$ and $\mu > 0$ arbitrary, then problem (3.14) has a sequence $\{u_k\}$ of solutions in $\mathcal{N}(U)$ such that $\phi(u_k) > 0$ and $\phi(u_k) \rightarrow \infty$ as $k \rightarrow \infty$.*
- (b) *When $\lambda > 0$ and $\mu \in \mathbb{R}$ arbitrary, then problem (3.14) has a sequence $\{v_k\}$ of solutions in $\mathcal{N}(U)$ such that $\phi(v_k) < 0$ and $\phi(v_k) \rightarrow 0^-$ as $k \rightarrow \infty$.*

4 p -Laplacian Steklov eigenvalue problems

As mentioned earlier, the beauty of the paper [3] is the discovery of the generalizations to high dimensions of the classical 3d Laplace’s spherical harmonics exterior to the unit ball: the exterior harmonic Steklov eigenvalue problems whose full spectra are derived there. This section is devoted to the description of the exterior p -harmonic Steklove eigenvalue problems in the function space $E^{1,p}(U)$ when $N \geq 3$ and $1 < p < N$. Similar results on bounded regions may be found in the interesting paper of Torné [27].

Recall the p -Laplacian is defined as $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. The exterior p -harmonic Steklove eigenvalue problems are to seek weak solutions of the problem below

$$\begin{aligned}
 -\Delta_p u(x) &= 0 \quad \text{in } U, \\
 \text{subject to } |\nabla u(z)|^{p-2} \frac{\partial u}{\partial \nu} u(z) &= \delta |u(z)|^{p-2} u(z) \quad \text{on } \partial U,
 \end{aligned}
 \tag{4.1}$$

in $E^{1,p}(U)$. This problem has been well developed on bounded regions over a century since Stekloff [23, 24], yet only been investigated recently in [3–5, 12, 13, 15].

Take $\|\cdot\| := \|\cdot\|_{p,\nabla}$ in this section. Define two functionals on $E^{1,p}(U)$ as

$$\varphi(u) := \frac{1}{p} \int_{\partial U} |u|^p d\sigma \quad \text{and} \quad \psi(u) := \frac{1}{p} \int_U |\nabla u|^p dx.
 \tag{4.2}$$

Accordingly, given $u \in E^{1,p}(U)$, write two linear functionals on $E^{1,p}(U)$ to be

$$\begin{cases} \mathcal{P}_u(v) := p\varphi(u)\psi'(u)(v) \\ \mathcal{B}_u(v) := \varphi'(u)(v) - \mathcal{P}_u(v) \end{cases} \quad \text{for each } v \in E^{1,p}(U).
 \tag{4.3}$$

Since $E^{1,p}(U)$ is a reflexive, uniformly convex Banach space when $p > 1$ (see [4]), there exists a unique element, say, u_B in $E^{1,p}(U)$, from Riesz’s theorem, such that

$$\mathcal{B}_u(u_B) = \|\mathcal{B}_u\|_*^2 = \|u_B\|^2;
 \tag{4.4}$$

therefore, one finds a homeomorphism $\mathfrak{H} : E^{1,p}(U) \rightarrow E^{1,p}(U)$ such as $\mathfrak{H}(u) := u_B$. Noticing \mathfrak{H} is odd, and bounded, uniformly continuous on the set $\mathbf{S}_1 := \{u \in E^{1,p}(U) : \|u\| = 1\}$, there are constants $t_0, \nu_1, \nu_0 > 0$ such that, for all $t \in [-t_0, t_0]$ and $u \in \mathbf{S}_1$, one has

$$\nu_1 \geq \|u + tu_B\| \geq \nu_0.
 \tag{4.5}$$

Now, define a flow $H : \mathbf{S}_1 \times [-t_0, t_0] \rightarrow \mathbf{S}_1$ by

$$H(u, t) := \frac{u + tu_B}{\|u + tu_B\|}. \tag{4.6}$$

Then H is odd in u , with $H(u, 0) = u$, uniformly continuous and verifies the property below.

Lemma 4.1 *There exists a map $\ell(u, t) : \mathbf{S}_1 \times (-t_0, t_0) \rightarrow \mathbb{R}$ such that $\ell(u, t) \rightarrow 0$ as $t \rightarrow 0$, uniformly on \mathbf{S}_1 , and, for each $u \in \mathbf{S}_1$ and $t \in (-t_0, t_0)$, we have*

$$\varphi(H(u, t)) = \varphi(u) + \int_0^t [\|u_B\|^2 + \ell(u, s)] ds. \tag{4.7}$$

Proof Since

$$\varphi(H(u, t)) = \varphi(u) + \int_0^t \varphi'(H(u, s)) \left(\frac{\partial H(u, s)}{\partial s} \right) ds, \tag{4.8}$$

we can define, in view of (4.4),

$$\ell(u, t) := \varphi'(H(u, t)) \left(\frac{\partial H(u, t)}{\partial t} \right) - \mathcal{B}_u(u_B), \tag{4.9}$$

and we have $\ell(u, 0) = 0$ uniformly on \mathbf{S}_1 via (4.5). Actually, a routine computation leads to

$$\frac{\partial H(u, t)}{\partial t} = \frac{u_B}{\|u + tu_B\|} - \frac{u + tu_B}{\|u + tu_B\|^{p+1}} \int_U |\nabla(u + tu_B)|^{p-2} \nabla(u + tu_B) \cdot \nabla u_B dx,$$

from which one deduces easily that, noticing that $\|u\| = 1$ on \mathbf{S}_1 ,

$$\frac{\partial H(u, 0)}{\partial t} = u_B - u \int_U |\nabla u|^{p-2} \nabla u \cdot \nabla u_B dx; \tag{4.10}$$

this further implies that, remembering the fact that $H(u, 0) = u$ now,

$$\varphi'(H(u, 0)) \left(\frac{\partial H(u, 0)}{\partial t} \right) = \varphi'(u)(u_B) - \varphi'(u)(u) \psi'(u)(u_B) = \varphi'(u)(u_B) - \mathcal{P}_u(u_B),$$

which together with (4.3) and (4.9) gives the desired result as ℓ is bounded by (4.5). \square

Using this result, we can derive a version of *deformation lemma*.

Proposition 4.2 *Given a constant $\kappa > 0$, suppose there are constants $\varsigma > 0$ and $\tau \in (0, \kappa)$, such that $\|u_B\| \geq \varsigma$ on $\mathcal{V}_\tau := \{u \in \mathbf{S}_1 : |\varphi(u) - \kappa| \leq \tau\}$. Then, for every compact, symmetric subset \mathbf{G} of \mathbf{S}_1 , one finds a constant $\epsilon \in (0, \tau)$ and an associated odd map $H_\epsilon : \mathbf{S}_1 \rightarrow \mathbf{S}_1$ that is continuous on $\mathcal{V}_\epsilon \cap \mathbf{G}$ and $H_\epsilon(\mathcal{V}_\epsilon \cap \mathbf{G}) \subseteq \varphi_{\kappa+\epsilon}$, where $\varphi_{\kappa+\epsilon} := \{u \in \mathbf{S}_1 : \varphi(u) \geq \kappa + \epsilon\}$.*

Proof As $\ell(u, 0) = 0$ uniformly on \mathbf{S}_1 , we can choose $t_1 \in (0, t_0)$ such that $|\ell(u, t)| \leq \varsigma^2/2$ for each $u \in \mathbf{G} \subsetneq \mathbf{S}_1$ and $t \in [-t_1, t_1]$ since \mathbf{G} is compact. Write $\epsilon := \min\{\tau, \varsigma^2 t_1/4\}$. Then (4.4) and (4.7) implies that, for every $u \in \mathcal{V}_\epsilon \cap \mathbf{G} \subseteq \mathcal{V}_\tau$, $\varphi(H(u, t_1)) \geq \kappa - \epsilon + \frac{\varsigma^2 t_1}{2} \geq \kappa + \epsilon$;

that is, $H(u, t_1) \in \varphi_{\kappa+\epsilon}$ for all $u \in V_\epsilon \cap \mathbf{G}$. As such, we define an odd map via (4.6) that is continuous on $V_\epsilon \cap \mathbf{G}$ by (because φ is even and $\mathbf{G} = -\mathbf{G}$ so that $V_\epsilon \cap \mathbf{G}$ again is symmetric)

$$H_\epsilon(u) := \begin{cases} H(u, t_1) & \text{when } u \in V_\epsilon \cap \mathbf{G}, \\ u & \text{when } u \in \mathbf{S}_1 \setminus V_\epsilon \cap \mathbf{G}, \end{cases} \tag{4.11}$$

and have $H_\epsilon(V_\epsilon \cap \mathbf{G}) \subseteq \varphi_{\kappa+\epsilon}$, as claimed. This in turn completes the proof. □

Next, define

$$\kappa_n := \sup_{\mathbf{G} \in \mathcal{G}_n} \min_{u \in \mathbf{G}} \varphi(u) \geq 0. \tag{4.12}$$

Here, $\mathcal{G}_n := \{\mathbf{G} \subsetneq \mathbf{S}_1 : \mathbf{G} \text{ compact, } \mathbf{G} = -\mathbf{G} \text{ and } \gamma(\mathbf{G}) \leq n\}$, with γ being the *Krasnoselskii genus* (see, for example, section ii.5 of [25] for more detailed descriptions).

Then we can prove the following main results.

Theorem 4.3 *For every $n \geq 1$, $\kappa_n > 0$ and there exists a function $s_n \in E^{1,p}(U)$ such that $\varphi(s_n) = \kappa_n$; in addition, s_n is a weak solution of (4.1) with $\delta = \delta_n := \frac{1}{p\kappa_n} > 0$.*

Proof As $\gamma(\mathbf{S}_1) = \infty$, κ_n is well defined in the sense that $\mathcal{G}_n \neq \emptyset$ for each $n \in \mathbb{N}$. Select a set $\mathbf{G}_n \in \mathcal{G}_n$ with $u \neq 0$ *a.e.* on ∂U for all $u \in \mathbf{G}_n$ to derive $\kappa_n \geq \min_{u \in \mathbf{G}_n} \varphi(u) > 0$.

Next, given $n \geq 1$, there exists a sequence $\{u_{n,k}\}$ in \mathbf{S}_1 such that $\varphi(u_{n,k}) \rightarrow \kappa_n$. Using a subsequence if necessary, it implies $u_{n,k} \rightharpoonup s_n \in E^{1,p}(U)$ yet $u_{n,k} \rightarrow s_n \in L^p(\partial U, d\sigma)$ in view of Lemma 2.1, when $k \rightarrow \infty$. Thus, one deduces that $\varphi(u_{n,k}) \rightarrow \varphi(s_n) = \kappa_n$.

Moreover, as linear functionals on $E^{1,p}(U)$, $\mathcal{B}_{u_{n,k}} \rightarrow 0$ when $k \rightarrow \infty$. First, by definition of κ_n , one can find a set $\tilde{\mathbf{G}}_n \in \mathcal{G}_n$, with $\kappa_n - \epsilon \leq \varphi(u) \leq \kappa_n + \epsilon$, for each $u \in \tilde{\mathbf{G}}_n$ and some suitably small $\epsilon \in (0, \frac{\kappa_n}{4})$; now, if we suppose on the contrary $\|\mathcal{B}_u\|_* > \varsigma > 0$ uniformly on $\{u \in \mathbf{S}_1 : \frac{\kappa_n}{2} \leq \varphi(u) \leq \frac{3\kappa_n}{2}\}$, Proposition 4.2 provides us with a continuous, odd map H_ϵ on $\tilde{\mathbf{G}}_n$ such that $H_\epsilon(\tilde{\mathbf{G}}_n) \in \mathcal{G}_n$ and $H_\epsilon(\tilde{\mathbf{G}}_n) \subseteq \varphi_{\kappa_n+\epsilon}$ —a contradiction thus is arrived at.

Since $\mathcal{B}_{u_{n,k}} \rightarrow 0$ when $k \rightarrow \infty$, one infers that, in view of (4.2) and (4.3),

$$\lim_{k \rightarrow \infty} \int_U |\nabla u_{n,k}|^{p-2} \nabla u_{n,k} \cdot \nabla v \, dx = \mathcal{F}(v) \quad \text{for all } v \in E^{1,p}(U),$$

where $\mathcal{F}(v) := \frac{1}{p\kappa_n} \int_{\partial U} |s_n|^{p-2} s_n v \, d\sigma$ is a linear functional on $E^{1,p}(U)$. Using Lemma 2.1 again, plus $u_{n,k} \rightharpoonup s_n \in E^{1,p}(U)$ yet $u_{n,k} \rightarrow s_n \in L^p(\partial U, d\sigma)$ as $k \rightarrow \infty$, we obtain

$$\int_U |\nabla s_n|^{p-2} \nabla s_n \cdot \nabla v \, dx = \delta_n \int_{\partial U} |s_n|^{p-2} s_n v \, d\sigma \quad \text{for all } v \in E^{1,p}(U). \tag{4.13}$$

Here, $\delta_n := \frac{1}{p\kappa_n} > 0$. As such, $s_n \in E^{1,p}(U)$ is a weak solution of problem (4.1). □

Theorem 4.4 *Define κ_n as in (4.12) and δ_n by $\frac{1}{p\kappa_n}$ for each $n \in \mathbb{N}$. Then one has*

$$\lim_{n \rightarrow \infty} \delta_n = \infty. \tag{4.14}$$

Proof The conclusion (4.14) follows if we can show that $\lim_{n \rightarrow \infty} \kappa_n = 0$.

For each $n \geq 1$, choose \mathcal{E}_n to be a linear subspace of $E^{1,p}(U)$ of dimension n such that $u \neq 0$ σ a.e. on ∂U for every $u \in \mathcal{E}_n$, and denote its complement in $E^{1,p}(U)$ by \mathcal{E}_n^c . Without loss of generality, assume further that $\mathcal{E}_1 \subsetneq \mathcal{E}_2 \subsetneq \dots \subsetneq \mathcal{E}_n \subsetneq \dots \subsetneq E^{1,p}(U)$. Note that (4.13) guarantees our choice as clearly $s_n|_{\partial U} \neq 0$ σ a.e. on ∂U for all $n \in \mathbb{N}$. Also, we have

$$\overline{\bigcup_{n \in \mathbb{N}} \mathcal{E}_n} \dot{\cup} E_0^{1,p}(U) = E^{1,p}(U), \tag{4.15}$$

where $E_0^{1,p}(U)$ denotes the subspace of $E^{1,p}(U)$ that is the closure of $C_c^1(U)$ with respect to the gradient L^p -norm (2.1) and the notation $\dot{\cup}$ means disjoint union.

Now, define $\tilde{\kappa}_n := \sup_{\mathbf{G} \in \mathcal{G}_n} \min_{u \in \mathbf{G} \cap \mathcal{E}_n^c} \varphi(u)$ to give $\tilde{\kappa}_n \geq \kappa_n > 0$. Then one proves $\lim_{n \rightarrow \infty} \tilde{\kappa}_n = 0$. Actually, if not, there is a constant $\varepsilon > 0$ such that $\tilde{\kappa}_n \geq \varepsilon$ for all $n \geq 1$. Thus, a set $\check{\mathbf{G}}_n \in \mathcal{G}_n$ exists such that $\tilde{\kappa}_n \geq \min_{u \in \check{\mathbf{G}}_n \cap \mathcal{E}_n^c} \varphi(u) \geq \frac{\varepsilon}{2} > 0$ for each $n \in \mathbb{N}$, so that we find a sequence $\{u_n\}$, with $u_n \in \check{\mathbf{G}}_n \cap \mathcal{E}_n^c$, satisfying $\varphi(u_n) \geq \frac{\varepsilon}{2}$ uniformly. Keep in mind $\check{\mathbf{G}}_n \subsetneq \mathbf{S}_1$; from (4.15) and resorting to a subsequence if necessary, one has $u_n \rightarrow 0 \in L^p(\partial U, d\sigma)$, and thus $\varphi(u_n) \rightarrow 0$, as $n \rightarrow \infty$. A contradiction follows and thereby one finishes the proof. \square

Finally, it is worth to mention here that the problem

$$\begin{aligned} -\Delta_p u(x) + |u(x)|^{p-2} u(x) &= 0 \quad \text{in } U, \\ \text{subject to } |\nabla u(z)|^{p-2} \frac{\partial u}{\partial \nu}(z) &= \delta |u(z)|^{p-2} u(z) \quad \text{on } \partial U, \end{aligned} \tag{4.16}$$

can be studied analogically in the space $W^{1,p}(U)$ by use of some results in [2, 5].

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Abbreviations

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

JXM conceived of the study, carried out the main studies and drafted the manuscript. ZQZ participated in the design of the study and performed the theory analysis. AXQ participated in the study and coordination and helped to draft the manuscript. All authors read and approved the final manuscript.

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