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Nonlinear sum operator equations and applications to elastic beam equation and fractional differential equation

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Abstract

In this paper, by studying the solutions of the abstract operator equation $A(x,x)+B(x,x)+e=x$ on ordered Banach spaces, where A, B are two mixed monotone operators and $e\in P$ with $\theta\leq e\leq h$, we prove a class of boundary value problems on elastic beam equation to have a unique solution. Furthermore, we also apply our abstract result to establish the existence and uniqueness theorem of nontrivial solutions for nonlinear fractional boundary value problems. The iterative sequences to approximate unique solutions for the above two classes of problems are also obtained.

Keywords: Mixed monotone operator; Beam equation; Existence and uniqueness; Fractional differential equation; Cone

1 Introduction

Our work is motivated by recent results obtained in [1]. In [1], Cabrera, López, and Sadarangani studied the existence and uniqueness of positive solutions for the following boundary value problem by using a mixed monotone operator method:

$$\begin{cases} u^{(4)}(t)=f(t,u(t),(Hu)(t)), \quad t\in[0,1],\\ u(0)=u'(0)=0,\\ u''(1)=0, \qquad u'''(1)=g(u(1)). \end{cases} \tag{1.1}$$

Problem (1.1) describes an elastic beam of length 1 depending on a nonlinear foundation provided with the function f. The first boundary condition $u(0)=u'(0)=0$ implies that the left end of the beam is fixed. The boundary condition $u''(1)=0$, $u'''(1)=g(u(1))$ implies that the right end of the beam is fastened with a bearing device, given by the function g. In particularly, if $g\equiv 0$, problem (1.1) is called the cantilever beam. It models the deflection of the elastic beam fixed at the left end and free at the right end [2–4].

The assumptions imposed on f and g are the following:

(H1) $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$, $g:[0,+\infty)\to(-\infty,0]$ are continuous functions. Moreover, g is decreasing;

(H2) $f(t,x,y)$ is increasing in x and decreasing in y, for fixed $t\in[0,1]$;

(H3) $g(\lambda x)\leq\lambda g(x)$ for any $\lambda\in(0,1)$ and $x\in[0,+\infty)$;

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(H4) there exists a constant $\gamma \in (0,1)$ such that

$$f\bigl(t, \lambda x, \lambda^{-1} y\bigr) \geq \lambda^{\gamma} f(t,x,y),$$

for every $\lambda \in (0,1)$, $t \in [0,1]$ and $x, y \in [0,+\infty)$;

(H5) $g(1) < 0$ and there exists a constant $\xi > 0$ such that

$$-g(x) \leq \xi \leq \frac{2}{3} \int_0^1 s^2 f(s,0,y)\,ds,$$

for every $x, y \in [0,+\infty)$;

(H6) $H : C[0,1] \to C[0,1]$ and satisfies the following conditions:
   (a) $Hu \geq 0$ for any $u \in P$;
   (b) for $u, v \in P$, $u \leq v \Rightarrow Hu \leq Hv$;
   (c) for $\lambda \in (0,1)$ and $u \in P$, $H(\lambda u) \geq \lambda Hu$,
   where $P = \{x \in C[0,1] : x(t) \geq 0, t \in [0,1]\}$.

In recent years, much attention has been paid to elastic beam equations. Various tools ad methods have been applied to study the existence, uniqueness and multiplicity of solutions for problem (1.1), for example topological degree theory [5–8], the monotone iteration method [9–13], partial order theory [14, 15], and critical point theory [16, 17]. We would like to mention some results of [5, 6, 9, 14], which motivated us to consider problem (1.1). In [9], Alves, Ma and Pelicer studied the following boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t,u,u'), & t \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, & u'''(1) = g(u(1)), \end{cases} \tag{1.2}$$

where the conditions imposed on f and g are local. The authors established the existence of monotone solutions to problem (1.2). Furthermore, Zhai and Anderson [14] considered the uniqueness of positive solutions for (1.2) when $f(t,u,u')$ is replaced by $f(t,u)$, and the iterative sequences of approximating the unique solution were also constructed. In [5], Wang et al. were concerned with the following boundary value problem with a parameter:

$$\begin{cases} u^{(4)}(t) = \lambda f(t,u), & t \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, & u'''(1) = g(u(1)), \end{cases} \tag{1.3}$$

where $\lambda \geq 0$ is a parameter. The authors proved how the parameter λ affects the number of monotone solutions of (1.3). Very recently, Cianciaruso, Infante and Pietramala [6] transformed problem (1.2) into the following Hammerstein integral equation with perturbation:

$$u(t) = \gamma(t) h\bigl(u(1)\bigr) + \int_0^1 k(t,s) f\bigl(s, u(s), u'(s)\bigr)\,ds,$$

where γ and $k(t,s)$ are defined on Sect. 4. We should point out that Hammerstein integral equations can be ascribed to general nonlinear operator equations discussed in this paper [18].

Since Guo and Lakshmikantham [19] introduced mixed monotone operators, many authors have investigated various types of nonlinear mixed monotone operators in Banach spaces and many interesting theorems have been established. In [20], Bhaskar and Lakshmikantham were concerned with some coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces. Moreover, Harjani, López and Sadarangani [21] generalized the main results of [20] using the altering distance functions. We note that Li and Zhao [22] considered a class of $\tau-\varphi$-mixed monotone operators. On the other hand, mixed monotone operators with perturbation have been extensively studied. In [23], Liu et al. considered the existence and uniqueness of positive solutions to the following operator equation on ordered Banach spaces:

$$A(x,x) + B(x,x) = x,$$

where A and B are two mixed monotone operators. The authors also gave an application to nonlinear fractional differential equation with two-point boundary conditions. Very recently, Wardowski [24] introduced the definition of (e,u)-concave-convex operator, and proved a fixed point theorem of such operator by analyzing some of its properties. By comparing with main result obtained in [25], we find that the above new operator is the same as $\varphi-(h,e)$-concave operator defined in Zhai and Wang [25].

In this paper, we firstly consider the existence and uniqueness of solution to the following operator equation on ordered Banach spaces E:

$$A(x,x) + B(x,x) + e = x, \tag{1.4}$$

where A and B are two mixed monotone operators, and $e \in P$ with P a cone in E.

Secondly, based on main results of [1], we will apply the abstract result for (1.1) to improve and generalize conditions (H1)–(H6). More specifically, we will study the existence and uniqueness of solutions for the following boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t,u(t),(Hu)(t)) - b, \quad t \in [0,1], \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = g(u(1)), \end{cases} \tag{1.5}$$

where $b > 0$ is a constant, $f : [0,1] \times (-\infty,+\infty) \times (-\infty,+\infty) \longrightarrow (-\infty,+\infty)$, $g : (-\infty,+\infty) \longrightarrow (-\infty,+\infty)$ are continuous functions and H is an operator.

The rest of the paper consists of the following sections. In Sect. 2, we present some preliminaries and lemmas to be used to prove our main result. In Sect. 3, we establish the existence and uniqueness theorems of solution for (1.1). In Sect. 4, to demonstrate the applicability of our abstract theorem, we give an application to nonlinear fourth-order two-point boundary value problems and give an example to explain our theoretical result. In Sect. 5, we use our abstract result to prove fractional boundary value problem to have a unique solution.

2 Preliminaries and lemmas

In this section, we give some definitions and lemmas to be used in the proof of our main result [19, 26, 27].

Throughout this paper, E is a real Banach space with norm $\|\cdot\|$, P is a cone in E, θ is the zero element in E. A partially ordered relation in E is given by $x \leq y$ iff $y - x \in P$. P is said to be normal if there exists a positive constant N, such that $\theta \leq x \leq y \Longrightarrow \|x\| \leq N\|y\|$, the smallest N is called the normal constant of P. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), we denote by C_h the set

$$C_h = \{x \in E \mid \text{ there exist } \lambda > 0 \text{ and } \mu > 0 \text{ such that } \lambda h \leq x \leq \mu h\}.$$

We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay$ ($Ax \geq Ay$).

Let $e \in P$ with $\theta \leq e \leq h$. Define

$$C_{h,e} = \{x \in E | x + e \in C_h\}.$$

Lemma 2.1 ([25]) *If $x \in C_{h,e}$, then $\lambda x + (\lambda - 1)e \in C_{h,e}$ for $\lambda > 0$.*

Lemma 2.2 ([25]) *If $x, y \in C_{h,e}$, then there exist $0 < \mu < 1$, $\gamma > 1$ such that*

$$\mu y + (\mu - 1)e \leq x \leq \gamma y + (\gamma - 1)e.$$

Further, we can choose a small number $r \in (0, 1)$, such that

$$ry + (r - 1)e \leq x \leq r^{-1}y + \left(r^{-1} - 1\right)e.$$

Definition 2.1 ([27]) Let $A : C_{h,e} \times C_{h,e} \longrightarrow E$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x, and decreasing in y, i.e., $u_i, v_i \in C_{h,e}$ ($i = 1, 2$), $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. Element $x \in C_{h,e}$ is called a fixed point of A if $A(x, x) = x$.

Lemma 2.3 *Let P be a normal cone and $T : C_{h,e} \times C_{h,e} \longrightarrow E$ be a mixed monotone operator with $T(h, h) \in C_{h,e}$, and the following condition is satisfied*:

(H) *there exists a mapping $\varphi : (0, 1) \longrightarrow (0, +\infty)$ with $\varphi(\lambda) > \lambda$ such that*

$$T\left(\lambda u + (\lambda - 1)e, \lambda^{-1}v + \left(\lambda^{-1} - 1\right)e\right) \geq \varphi(\lambda)T(u, v) + \left(\varphi(\lambda) - 1\right)e,$$

for all $u, v \in C_{h,e}$ and $\lambda \in (0, 1)$. Then:

(1) *there exist $u_0, v_0 \in C_{h,e}$, and $s \in (0, 1)$ such that*

$$sv_0 \leq u_0 < v_0, u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0;$$

(2) *T has a unique fixed point x^* in $C_{h,e}$;*

(3) *for any initial values $x_0, y_0 \in C_{h,e}$, by making the sequences as follows*:

$$x_n = T(x_{n-1}, y_{n-1}), \qquad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Proof Firstly, by (H), we have

$$T\big(t^{-1}u+\big(t^{-1}-1\big)e,tv+(t-1)e\big)\le\varphi(t)^{-1}T(u,v)+\big(\varphi(t)^{-1}-1\big)e, \tag{2.1}$$

for every $t\in(0,1)$, $u,v\in C_{h,e}$. For every $u,v\in C_{h,e}$, there exist $\sigma_1,\sigma_2\in(0,1)$ such that

$$\sigma_1h+(\sigma_1-1)e\le u\le\sigma_1^{-1}h+\big(\sigma_1^{-1}-1\big)e,$$
$$\sigma_2h+(\sigma_2-1)e\le u\le\sigma_2^{-1}h+\big(\sigma_2^{-1}-1\big)e.$$

Let $\sigma=\min\{\sigma_1,\sigma_2\}$. Then $\sigma\in(0,1)$, from (2.1) and the mixed monotone properties of operator T, we have

$$T(u,v)\ge T\big(\sigma h+(\sigma-1)e,\sigma^{-1}h+\big(\sigma^{-1}-1\big)e\big)\ge\varphi(\sigma)T(h,h)+\big(\varphi(\sigma)-1\big)e,$$
$$T(u,v)\le T\big(\sigma^{-1}h+\big(\sigma^{-1}-1\big)e,\sigma h+(\sigma-1)e\big)\le\varphi(\sigma)^{-1}T(h,h)+\big(\varphi(\sigma)^{-1}-1\big)e.$$

It follows from $T(h,h)\in C_{h,e}$ that $T(u,v)\in C_{h,e}$. Hence we have $T:C_{h,e}\times C_{h,e}\longrightarrow C_{h,e}$. Since $T(h,h)\in C_{h,e}$, we can choose a small enough number $t_0\in(0,1)$ such that

$$t_0h+(t_0-1)e\le T(h,h)\le t_0^{-1}h+\big(t_0^{-1}-1\big)e.$$

Nothing that $\varphi(t)>t$, we can take a positive integer k such that

$$\left(\frac{\varphi(t_0)}{t_0}\right)^k\ge\frac{1}{t_0}.$$

Let

$$x_n=t_0^nh+\big(t_0^n-1\big)e,\qquad y_n=t_0^{-n}h+\big(t_0^{-n}-1\big)e,\quad n=1,2,\ldots.$$

Thus

$$x_n=t_0x_{n-1}+(t_0-1)e,\qquad y_n=t_0^{-1}y_{n-1}+\big(t_0^{-1}-1\big)e,\quad n=1,2,\ldots.$$

Denote $u_0:=x_k$, $v_0:=y_k$, then $u_0,v_0\in C_{h,e}$,

$$\begin{aligned}
T(u_0,v_0)&=T(x_k,y_k)\\
&=T\big(t_0x_{k-1}+(t_0-1)e,t_0^{-1}y_{k-1}+\big(t_0^{-1}-1\big)e\big)\\
&\ge\varphi(t_0)T(x_{k-1},y_{k-1})+\big(\varphi(t_0)-1\big)e\\
&\ge\varphi(t_0)T\big(t_0x_{k-2}+(t_0-1)e,t_0^{-1}y_{k-2}+\big(t_0^{-1}-1\big)e\big)+\big(\varphi(t_0)-1\big)e\\
&\ge\varphi(t_0)\big(\varphi(t_0)T(x_{k-2},y_{k-2})+\big(\varphi(t_0)-1\big)e\big)+\big(\varphi(t_0)-1\big)e\\
&=\varphi(t_0)^2T(x_{k-2},y_{k-2})+\big(\varphi(t_0)^2-1\big)e\ge\cdots\ge\varphi(t_0)^kT(h,h)+\big(\varphi(t_0)^k-1\big)e\\
&\ge\varphi(t_0)^k\big(t_0h+(t_0-1)e\big)+\big(\varphi(t_0)^k-1\big)e
\end{aligned}$$

$$
\begin{aligned}
&\geq t_0^{k-1}\big(t_0 h + (t_0-1)e\big) + \big(t_0^{k-1}-1\big)e \\
&= t_0^k h + \big(t_0^k - 1\big)e = x_k = u_0
\end{aligned}
$$

and

$$
\begin{aligned}
T(v_0,u_0) &= T(y_k,x_k) \\
&= T\big(t_0^{-1}y_{k-1} + \big(t_0^{-1}-1\big)e, t_0 x_{k-1} + (t_0-1)e\big) \\
&\leq \varphi(t_0)^{-1}T(y_{k-1},x_{k-1}) + \big(\varphi(t_0)^{-1}-1\big)e \\
&\leq \varphi(t_0)^{-1}T\big(t_0^{-1}y_{k-2}, t_0 x_{k-2} + (t_0-1)e\big) + \big(\varphi(t_0)^{-1}-1\big)e \\
&\leq \varphi(t_0)^{-1}\big[\varphi(t_0)^{-1}T(y_{k-2},x_{k-2}) + \big(\varphi(t_0)^{-1}-1\big)e\big] + \big(\varphi(t_0)^{-1}-1\big)e \\
&= \varphi(t_0)^{-2}T(y_{k-2},x_{k-2}) + \big(\varphi(t_0)^{-2}-1\big)e \\
&\leq \cdots \leq \varphi(t_0)^{-k}T(h,h) + \big(\varphi(t_0)^{-k}-1\big)e \\
&\leq \varphi(t_0)^{-k}\big(t_0^{-1}h + \big(t_0^{-1}-1\big)e\big) + \big(\varphi(t_0)^{-k}-1\big)e \\
&\leq t_0^{1-k}\big(t_0^{-1}h + \big(t_0^{-1}-1\big)e\big) + \big(t_0^{1-k}-1\big)e \\
&= t_0^{-k}h + \big(t_0^{-k}-1\big)e = y_k = v_0.
\end{aligned}
$$

Therefore we conclude that there exist $u_0, v_0 \in C_{h,e}$ and $s \in (0,1)$ such that $sv_0 \leq u_0 \leq v_0$ and $u_0 \leq T(u_0,v_0) \leq T(v_0,u_0) \leq v_0$.

Construct the sequences

$$u_n = T(u_{n-1},v_{n-1}), \qquad v_n = T(v_{n-1},u_{n-1}), \quad n = 1,2,\ldots.$$

It is clear that $u_1 \leq v_1$. Combining with the mixed monotone properties of T, we have $u_n \leq v_n$, $n = 1,2,\ldots$. We obtain for all $n \in \mathbb{N}$

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{2.2}$$

Furthermore

$$u_n \geq u_0 \geq sv_0 + (s-1)e \geq sv_n + (s-1)e, \quad n = 1,2,\ldots.$$

Let

$$t_n = \sup\big\{t > 0 | u_n \geq tv_n + (t-1)e\big\}.$$

Thus we have $u_n \geq t_n v_n + (t_n - 1)e$, $n = 1,2,\ldots$, and then $u_{n+1} \geq u_n \geq t_n v_n + (t_n-1)e \geq t_n v_{n+1} + (t_n-1)e$, $n = 1,2,\ldots$. Therefore $t_{n+1} \geq t_n$, i.e. $\{t_n\}$ is increasing with $\{t_n\} \subset (0,1]$. Assume that $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. If not, $0 < t^* < 1$.

In the following, we prove that $t^* = 1$. If $0 < t^* < 1$, we need to discuss the following two cases.

Case one: there exists an integer N such that $t_N = t^*$. For all $n > N$, we have $t_n = t^*$. Then

$$
\begin{aligned}
u_{n+1} &= T(u_n, v_n) \geq T\big(t_n v_n + (t_n - 1)e, t_n^{-1} u_n + (t_n^{-1} - 1)e\big) \\
&= T\big(t^* v_n + (t^* - 1)e, (t^*)^{-1} u_n + ((t^*)^{-1} - 1)e\big) \\
&\geq \varphi(t^*) T(v_n, u_n) + (\varphi(t^*) - 1)e, \quad \text{for } n \geq N.
\end{aligned}
$$

We see from the definition of t_{n+1} that $t^* = t_{n+1} \geq \varphi(t^*) > t^*$, which is a contradiction.

Case two: for all integers n, $t_n < t^*$, then we get

$$
\begin{aligned}
u_{n+1} &= T(u_n, v_n) \geq T\big(t_n v_n + (t_n - 1)e, t_n^{-1} u_n + (t_n^{-1} - 1)e\big) \\
&= T\left(\frac{t_n}{t^*}\big(t^* v_n + (t^* - 1)e\big) + \left(\frac{t_n}{t^*} - 1\right)e, \left(\frac{t_n}{t^*}\right)^{-1}\big((t^*)^{-1} u_n + ((t^*)^{-1} - 1)e\big)\right. \\
&\quad \left. + \left(\left(\frac{t_n}{t^*}\right)^{-1} - 1\right)e\right) \\
&\geq \varphi\left(\frac{t_n}{t^*}\right) T\big(t^* v_n + (t^* - 1)e, (t^*)^{-1} u_n + ((t^*)^{-1} - 1)e\big) + \left(\varphi\left(\frac{t_n}{t^*}\right) - 1\right)e \\
&\geq \varphi\left(\frac{t_n}{t^*}\right)\big(\varphi(t^*) T(v_n, u_n) + (\varphi(t^*) - 1)e\big) + \left(\varphi\left(\frac{t_n}{t^*}\right) - 1\right)e \\
&\geq \varphi\left(\frac{t_n}{t^*}\right)\varphi(t^*) T(v_n, u_n) + \left(\varphi\left(\frac{t_n}{t^*}\right)\varphi(t^*) - 1\right)e.
\end{aligned}
$$

Again, it follows from the definition of t_{n+1} that

$$
t_{n+1} \geq \varphi\left(\frac{t_n}{t^*}\right)\varphi(t^*) > \frac{t_n}{t^*}\varphi(t^*).
$$

Taking $n \to \infty$, we have $t^* \geq \varphi(t^*) > t^*$, which is also a contradiction. Consequently $t^* = 1$. Since P is normal, we have

$$
\begin{aligned}
&\|u_{n+p} - u_n\| \leq M(1 - t_n)\|v_0 + e\|, \\
&\|v_n - v_{n+p}\| \leq M(1 - t_n)\|v_0 + e\|,
\end{aligned}
\tag{2.3}
$$

where M is the normality constant. Let $n \to \infty$ in (2.3), we deduce that

$$
\|u_{n+p} - u_n\| \to 0, \qquad \|v_n - v_{n+p}\| \to 0.
$$

Therefore u_n and v_n are Cauchy sequences. Note that E is complete, there exist $u^*, v^* \in E$ such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$. By (2.3), we get

$$
u_0 \leq u_n \leq u^* \leq v^* \leq v_n \leq v_0.
$$

Thus $u^*, v^* \in C_{h,e}$ and

$$
\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - t_n)(v_0 + e).
$$

The normality of P implies that

$$\left\|v^* - u^*\right\| \le M(1 - t_n)\|v_0 + e\| \to 0 \quad (n \to \infty).$$

Consequently $u^* = v^*$. Put $x^* = u^* = v^*$ and we have

$$u_{n+1} = T(u_n, v_n) \le T\left(x^*, x^*\right) \le T(v_n, u_n) = v_{n+1}.$$

Taking $n \to \infty$, we deduce that $x^* = T(x^*, x^*)$. That is, x^* is a fixed point of T in $C_{h,e}$.

Secondly, we prove that x^* is the unique fixed point of T in $C_{h,e}$. Assume that y^* is any fixed point of T in $C_{h,e}$. It follows from Lemma 2.2 and $x^*, y^* \in C_{h,e}$ that there exists $\tau > 0$ such that

$$\tau y^* + (\tau - 1)e \le x^* \le \tau^{-1} y^* + \left(\tau^{-1} - 1\right)e.$$

Let

$$\widetilde{t} = \sup\left\{\tau > 0 | \tau y^* + (\tau - 1)e \le x^* \le \tau^{-1} y^* + \left(\tau^{-1} - 1\right)e\right\}.$$

Next we show that $\widetilde{t} \ge 1$. If $0 < \widetilde{t} < 1$, then

$$\begin{aligned} x^* &= T\left(x^*, x^*\right) \\ &\ge T\left(\widetilde{t} y^* + (\widetilde{t} - 1)e, \widetilde{t}^{-1} y^* + \left(\widetilde{t}^{-1} - 1\right)e\right) \\ &\ge \varphi(\widetilde{t}) T\left(y^*, y^*\right) + \left(\varphi(\widetilde{t}) - 1\right)e \\ &= \varphi(\widetilde{t}) y^* + \left(\varphi(\widetilde{t}) - 1\right)e. \end{aligned}$$

Combining with the definition of $\widetilde{t}$, we have $\widetilde{t} \ge \varphi(\widetilde{t}) > \widetilde{t}$, which is a contradiction. Hence $\widetilde{t} \ge 1$. Furthermore

$$x^* \ge \widetilde{t} y^* + (\widetilde{t} - 1)e \ge \widetilde{t} y^* \ge y^*.$$

Similarly, we also deduce that $y^* \ge x^*$. Therefore $x^* = y^*$.

Lastly, we construct successively the sequences $x_n = T(x_{n-1}, y_{n-1})$, $y_n = T(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, for any initial points $x_0, y_0 \in C_{h,e}$. Thus, we can take small numbers $\tau_2, \tau_3 \in (0,1)$ such that

$$\begin{aligned} &\tau_2 h + (\tau_2 - 1)e \le x_0 \le \tau_2^{-1} h + \left(\tau_2^{-1} - 1\right)e, \\ &\tau_3 h + (\tau_3 - 1)e \le y_0 \le \tau_3^{-1} h + \left(\tau_3^{-1} - 1\right)e. \end{aligned}$$

Let $\tau^* = \min\{\tau_2, \tau_3\}$. Then $\tau^* \in (0,1)$ and

$$\tau^* h + \left(\tau^* - 1\right)e \le x_0, \qquad y_0 \ge \left(\tau^*\right)^{-1} h + \left(\left(\tau^*\right)^{-1} - 1\right)e.$$

We can choose a sufficiently large positive integer m such that

$$\left(\frac{\varphi(\tau^*)}{\tau^*}\right)^m \ge \frac{1}{\tau^*}.$$

Set

$$\overline{u}_0 = (\tau^*)^m h + ((\tau^*)^m - 1)e, \qquad \overline{v}_0 = (\tau^*)^{-m} h + ((\tau^*)^{-m} - 1)e.$$

Obviously, $\overline{u}_0, \overline{v}_0 \in C_{h,e}$, and $\overline{u}_0 < x_0, y_0 < \overline{v}_0$. Let

$$\overline{u}_n = T(\overline{u}_{n-1}, \overline{v}_{n-1}), \qquad \overline{v}_n = T(\overline{v}_{n-1}, \overline{u}_{n-1}), \quad n = 1, 2, \ldots.$$

Similarly, there exists $\overline{y} \in C_{h,e}$ such that

$$T(\overline{y}, \overline{y}) = \overline{y}, \qquad \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} \overline{v}_n = \overline{y}.$$

Since fixed points of T in $C_{h,e}$ is unique, we have $x^* = \overline{y}$. And by induction, $\overline{u}_n < x_n, y_n < \overline{v}_n$, $n = 1, 2, \ldots$. By the normality of P, we deduce that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$. □

3 Main result

In this section we consider the existence and uniqueness of a solution for the operator equation (1.1).

Theorem 3.1 *Let P be a normal cone in E, and let $A, B : C_{h,e} \times C_{h,e} \longrightarrow E$ be two mixed monotone operators and satisfy the following conditions*:

(i) *for all $t \in (0,1)$, there exists $\psi(t) \in (t,1)$ such that*

$$A(tx + (t-1)e, t^{-1}y + (t^{-1} - 1)e) \geq \psi(t)A(x,y) + (\psi(t) - 1)e, \quad \forall x, y \in C_{h,e};$$

(ii) *for all $t \in (0,1)$ and $x, y \in C_{h,e}$*

$$B(tx + (t-1)e, t^{-1}y + (t^{-1} - 1)e) \geq tB(x,y) + (t-1)e;$$

(iii) *$A(h,h) \in C_{h,e}$ and $B(h,h) \in C_{h,e}$*;

(iv) *there exists a constant $\delta > 0$, such that for all $x, y \in C_{h,e}$*

$$A(x,y) \geq \delta B(x,y) + (\delta - 1)e.$$

Then the operator equation (1.1) has a unique solution x^* in $C_{h,e}$, and for any initial values $x_0, y_0 \in C_{h,e}$, by setting the sequences

$$x_n = A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}) + e,$$
$$y_n = A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}) + e, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ in E as $n \to \infty$.

Proof Firstly, from conditions (i) and (ii), for every $t \in (0,1)$ and $x, y \in C_{h,e}$, we have

$$A(t^{-1}x + (t^{-1} - 1)e, ty + (t-1)e) \leq \psi(t)^{-1}A(x,y) + (\psi(t)^{-1} - 1)e, \tag{3.1}$$
$$B(t^{-1}x + (t^{-1} - 1)e, ty + (t-1)e) \leq t^{-1}B(x,y) + (t^{-1} - 1)e. \tag{3.2}$$

Since $A(h,h) \in C_{h,e}$, $B(h,h) \in C_{h,e}$, there exist constants $a_i > 0$ and $b_i > 0$ ($i = 1,2$) such that

$$a_1 h + (a_1 - 1)e \le A(h,h) \le b_1 h + (b_1 - 1)e, \tag{3.3}$$

$$a_2 h + (a_2 - 1)e \le B(h,h) \le b_2 h + (b_2 - 1)e. \tag{3.4}$$

Next we show that $A : C_{h,e} \times C_{h,e} \to C_{h,e}$. For every $x, y \in C_{h,e}$, we can take two small enough numbers $\alpha_1, \alpha_2 \in (0,1)$ such that

$$\begin{aligned} &\alpha_1 h + (\alpha_1 - 1)e \le x \le \alpha_1^{-1} h + \bigl(\alpha_1^{-1} - 1\bigr)e, \\ &\alpha_2 h + (\alpha_2 - 1)e \le y \le \alpha_2^{-1} h + \bigl(\alpha_2^{-1} - 1\bigr)e. \end{aligned} \tag{3.5}$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$, then $\alpha \in (0,1)$, by (3.1), (3.3)–(3.5), we obtain

$$\begin{aligned} A(x,y) &\ge A\bigl(\alpha h + (\alpha - 1)e, \alpha^{-1} h + \bigl(\alpha^{-1} - 1\bigr)e\bigr) \ge \psi(\alpha) A(h,h) + \bigl(\psi(\alpha) - 1\bigr)e \\ &\ge \psi(\alpha)\bigl(a_1 h + (a_1 - 1)e\bigr) + \bigl(\psi(\alpha) - 1\bigr)e \\ &\ge \psi(\alpha) a_1 h + \bigl(\psi(\alpha) a_1 - 1\bigr)e \end{aligned}$$

and

$$\begin{aligned} A(x,y) &\le A\bigl(\alpha^{-1} h + \bigl(\alpha^{-1} - 1\bigr)e, \alpha h + (\alpha - 1)e\bigr) \le \psi(\alpha)^{-1} A(h,h) + \bigl(\psi(\alpha)^{-1} - 1\bigr)e \\ &\le \psi(\alpha)^{-1}\bigl(b_1 h + (b_1 - 1)e\bigr) + \bigl(\psi(\alpha)^{-1} - 1\bigr)e \\ &\le \psi(\alpha)^{-1} b_1 h + \bigl(\psi(\alpha)^{-1} b_1 - 1\bigr)e. \end{aligned}$$

Hence $A(x,y) \in C_{h,e}$, that is $A : C_{h,e} \times C_{h,e} \to C_{h,e}$.

Finally, we prove that $B : C_{h,e} \times C_{h,e} \to C_{h,e}$. For every $x, y \in C_{h,e}$, we can choose two sufficiently small numbers $\beta_1, \beta_2 \in (0,1)$ such that

$$\begin{aligned} &\beta_1 h + (\beta_1 - 1)e \le x \le \beta_1^{-1} h + \bigl(\beta_1^{-1} - 1\bigr)e, \\ &\beta_2 h + (\beta_2 - 1)e \le y \le \beta_2^{-1} h + \bigl(\beta_2^{-1} - 1\bigr)e. \end{aligned} \tag{3.6}$$

Let $\beta = \min\{\beta_1, \beta_2\}$, then $\beta \in (0,1)$, by (3.2), (3.3), (3.4) and (3.6), we deduce that

$$\begin{aligned} B(x,y) &\ge B\bigl(\beta h + (\beta - 1)e, \beta^{-1} h + \bigl(\beta^{-1} - 1\bigr)e\bigr) \ge \beta B(h,h) + (\beta - 1)e \\ &\ge \beta\bigl(a_2 h + (a_2 - 1)e\bigr) + (\beta - 1)e \\ &\ge \beta a_2 h + (\beta a_2 - 1)e, \end{aligned}$$

and

$$\begin{aligned} B(x,y) &\le B\bigl(\beta^{-1} h + \bigl(\beta^{-1} - 1\bigr)e, \beta h + (\beta - 1)e\bigr) \le \beta^{-1} B(h,h) + \bigl(\beta^{-1} - 1\bigr)e \\ &\le \beta^{-1}\bigl(b_2 h + (b_2 - 1)e\bigr) + \bigl(\beta^{-1} - 1\bigr)e \\ &\le \beta^{-1} b_2 h + \bigl(\beta^{-1} b_1 - 1\bigr)e. \end{aligned}$$

Therefore $B : C_{h,e} \times C_{h,e} \to C_{h,e}$.

Now we define the operator $T = A + B + e : C_{h,e} \times C_{h,e} \to E$ by

$$T(x,y) = A(x,y) + B(x,y) + e, \quad \forall x,y \in C_{h,e}. \tag{3.7}$$

Then $T : C_{h,e} \times C_{h,e} \to E$ is a mixed monotone operator. Note that $A(h,h) \in C_{h,e}$, $B(h,h) \in C_{h,e}$, we have $T(h,h) = A(h,h) + B(h,h) + e \in C_{h,e}$.

In the following, we show that, for every $t \in (0,1)$, there exists $\varphi(t) \in (t,1]$, such that, for all $x,y \in C_{h,e}$, $T(tx + (t-1), t^{-1}y + (t^{-1}-1)e) \geq \varphi(t)T(x,y) + (\varphi(t)-1)e$. For every $x,y \in C_{h,e}$, by condition (iv), we have

$$\begin{aligned} &A(x,y) + \delta A(x,y) + \delta e \geq \delta B(x,y) + (\delta - 1)e + \delta A(x,y) + \delta e, \\ &A(x,y) \geq \frac{\delta}{1+\delta} T(x,y) - \frac{e}{1+\delta}. \end{aligned} \tag{3.8}$$

By the conditions (i), (ii), (3.7) and (3.8), for every $x,y \in C_{h,e}$, we obtain

$$\begin{aligned} &T\big(tx + (t-1)e, t^{-1}y + \big(t^{-1}-1\big)e\big) - tT(x,y) \\ &\quad = A\big(tx + (t-1)e, t^{-1}y + \big(t^{-1}-1\big)e\big) + B\big(tx + (t-1)e, t^{-1}y + \big(t^{-1}-1\big)e\big) \\ &\qquad + e - t\big(A(x,y) + B(x,y) + e\big) \\ &\quad \geq \psi(t)A(x,y) + \big(\psi(t)-1\big)e + tB(x,y) + (t-1)e + (t-1)e + e \\ &\qquad - tA(x,y) - tB(x,y) - te \\ &\quad \geq \big(\psi(t)-t\big)A(x,y) + \big(\psi(t)-1\big)e \\ &\quad \geq \big(\psi(t)-t\big)\left(\frac{\delta}{1+\delta}T(x,y) - \frac{e}{1+\delta}\right) + \big(\psi(t)-1\big)e \\ &\quad = \frac{\delta(\psi(t)-t)}{1+\delta}T(x,y) + \left(\psi(t) - 1 - \frac{\psi(t)-t}{1+\delta}\right)e. \end{aligned} \tag{3.9}$$

Thus

$$\begin{aligned} &T\big(tx + (t-1)e, t^{-1}y + \big(t^{-1}-1\big)e\big) \\ &\quad \geq \left(\frac{\delta(\psi(t)-t)}{1+\delta} + t\right)T(x,y) + \left(\psi(t) - 1 - \frac{\psi(t)-t}{1+\delta}\right)e \\ &\quad = \frac{\delta\psi(t)+t}{1+\delta}T(x,y) + \left(\frac{\delta\psi(t)+t}{1+\delta} - 1\right)e, \quad \text{for } x,y \in C_{h,e}. \end{aligned} \tag{3.10}$$

Let $\varphi(t) = \frac{\delta\psi(t)+t}{1+\delta}$, then $\varphi(t) \in (t,\psi(t)) \subset (t,1]$, $t \in (0,1)$, by (3.10), we conclude that

$$T\big(tx + (t-1)e, t^{-1}y + \big(t^{-1}-1\big)e\big) \geq \varphi(t)T(x,y) + \big(\varphi(t)-1\big)e, \quad \forall x,y \in C_{h,e}.$$

According to Lemma 2.3, we obtain the conclusion of Theorem 3.1. □

4 Application to nonlinear fourth-order two-point boundary value problem

Lemma 4.1 ([9]) *Suppose that* $h \in C[0,1]$. *Then the boundary value problem*

$$\begin{cases} u^{(4)} = h(t), & t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{4.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)\,ds,$$

where $G(t,s)$ *is the Green function given by*

$$G(t,s) = \frac{1}{6}\begin{cases} s^2(3t-s), & 0 \le s \le t \le 1, \\ t^2(3s-t), & 0 \le t \le s \le 1. \end{cases}$$

By (4.1), we know that the following boundary value problem:

$$\begin{cases} u^{(4)} = h(t), & t \in (0,1), \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = g(u(1)), \end{cases} \tag{4.2}$$

where $h, g \in C[0,1]$ has the following integral expression:

$$u(t) = \int_0^1 G(t,s)h(s)\,ds - g\big(u(1)\big)\left(\frac{t^2}{2} - \frac{t^3}{6}\right), \quad t \in [0,1],$$

where $G(t,s)$ is introduced in Lemma 4.1.

Lemma 4.2 ([14]) *For* $(t,s) \in [0,1] \times [0,1]$, *we have*

$$\frac{1}{3}s^2t^2 \le G(t,s) \le \frac{1}{2}st^2.$$

In the following, we consider in the Banach space

$$C[0,1] = \{x : [0,1] \longrightarrow \mathbb{R} \text{ is continuous}\}$$

equipped with the norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. It is obvious that $C[0,1]$ can be endowed with a partial order

$$x, y \in C[0,1], \quad x \le y \iff x(t) \le y(t), \quad \forall t \in [0,1].$$

Set $P = \{x \in C[0,1] | x(t) \ge 0, t \in [0,1]\}$. We see that P is a normal cone in $C[0,1]$. Let

$$e(t) = \frac{b}{24}t^4 + \frac{b}{4}t^2 - \frac{b}{6}t^3, \quad t \in [0,1].$$

Theorem 4.1 *Suppose that the following assumptions are satisfied*:

(H1)′ $f:[0,1]\times[-e^*,+\infty)\times[-e^*,+\infty)\to(-\infty,+\infty)$ *and* $g:[-e^*,+\infty)\to(-\infty,0]$ *are continuous functions. Moreover, g is decreasing, where* $e^*=\max\{e(t):t\in[0,1]\}$;

(H2)′ $f(t,x,y)$ *is increasing in x and decreasing in y for fixed* $t\in[0,1]$;

(H3)′ $g(\lambda x+(\lambda-1)c)\le\lambda g(x)$ *for every* $\lambda\in(0,1)$, $x\in[-e^*,+\infty)$ *and* $c\in[0,e^*]$;

(H4)′ *for every* $\lambda\in(0,1)$, *there is* $\psi(\lambda)>\lambda$ *such that*

$$f\bigl(t,\lambda x+(\lambda-1)c,\lambda^{-1}y+\bigl(\lambda^{-1}-1\bigr)c\bigr)\ge\psi(\lambda)f(t,x,y),$$

for every $t\in[0,1]$, $x\in(-\infty,+\infty)$, $y\in(-\infty,+\infty)$, $c\in[0,e^*]$;

(H5)′ $g(q)<0$ *with* $q\ge\frac{7b}{24}$, *and there exists a constant* $\xi>0$ *such that*

$$-g(x)\le\xi\le\frac{2}{3}\int_0^1 s^2f(s,0,y)\,ds,\quad \textit{for every } x,y\in\bigl[-e^*,+\infty\bigr);$$

(H6)′ $H:C[0,1]\to C[0,1]$ *and satisfies the following assumptions*:

(a)′ $Hu\ge0$ *for every* $u\in C_{h,e}$;

(b)′ *for* $u,v\in C_{h,e}$, $u\le v\Rightarrow Hu\le Hv$;

(c)′ *for* $\lambda\in(0,1)$ *and* $u\in C_{h,e}$,

$$H\bigl(\lambda u+(\lambda-1)c\bigr)\ge\lambda(Hu)+(\lambda-1)c,\quad c\in\bigl[0,e^*\bigr].$$

Then the problem (1.5) *has a unique nontrivial solution* u^* *in* $C_{h,e}$, *where* $h(t)=qt^2$, $t\in[0,1]$.

Proof Firstly, for $t\in[0,1]$

$$e(t)=\frac{b}{24}t^4-\frac{b}{6}t^3+\frac{b}{4}t^2=bt^2\left(\frac{t^2}{24}-\frac{t}{6}+\frac{1}{4}\right)\ge0.$$

That is, $e\in P$. Furthermore, for $t\in[0,1]$

$$e(t)=\frac{b}{24}t^4-\frac{b}{6}t^3+\frac{b}{4}t^2\le\frac{b}{24}t^4+\frac{b}{4}t^2\le\frac{b}{24}t^2+\frac{b}{4}t^2=\frac{7b}{24}t^2\le qt^2=h(t).$$

Hence, $0<e(t)\le h(t)$. In addition, $C_{h,e}=\{u\in C[0,1]|u+e\in C_h\}$. From Lemma 4.1, we rewrite the problem (1.5) as a Hammerstein integral equation

$$\begin{aligned}u(t)&=\int_0^1 G(t,s)f\bigl(s,u(s),(Hu)(s)\bigr)\,ds-b\int_0^1 G(t,s)\,ds-g\bigl(u(1)\bigr)\left(\frac{t^2}{2}-\frac{t^3}{6}\right)\\&=\int_0^1 G(t,s)f\bigl(s,u(s),(Hu)(s)\bigr)\,ds-\left(\frac{b}{24}t^4-\frac{b}{6}t^3+\frac{b}{4}t^2\right)-g\bigl(u(1)\bigr)\left(\frac{t^2}{2}-\frac{t^3}{6}\right)\\&=\int_0^1 G(t,s)f\bigl(s,u(s),(Hu)(s)\bigr)\,ds-e(t)-g\bigl(u(1)\bigr)\left(\frac{t^2}{2}-\frac{t^3}{6}\right)\\&=\int_0^1 G(t,s)f\bigl(s,u(s),(Hu)(s)\bigr)\,ds-e(t)-g\bigl(u(1)\bigr)\left(\frac{t^2}{2}-\frac{t^3}{6}\right)-e(t)+e(t).\end{aligned}$$

Define

$$A(u,v)(t) = \int_0^1 G(t,s)f\big(s,u(s),(Hu)(s)\big)\,ds - e(t),$$

$$(Bu)(t) = -g\big(u(1)\big)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) - e(t),$$

for every $t \in [0,1]$ and $u, v \in C_{h,e}$. We know that the solutions of the problem (1.5) are fixed points of the operator $A + B + e$.

Next, we check that assumptions of Theorem 3.1 are satisfied. We prove that A is a mixed monotone operator. In fact, for $u_1, u_2, v \in C_{h,e}$, with $u_1 \le u_2$, we have, for all $t \in [0,1]$,

$$\begin{aligned} A(u_2,v)(t) &= \int_0^1 G(t,s)f\big(s,u_2(s),(Hv)(s)\big)\,ds - e(t) \\ &\ge \int_0^1 G(t,s)f\big(s,u_1(s),(Hv)(s)\big)\,ds - e(t) \\ &= A(u_1,v)(t). \end{aligned}$$

Similarly, for $u, v_1, v_2 \in C_{h,e}$, with $v_1 \ge v_2$, we have, for all $t \in [0,1]$,

$$\begin{aligned} A(u,v_2)(t) &= \int_0^1 G(t,s)f\big(s,u(s),(Hv_2)(s)\big)\,ds - e(t) \\ &\ge \int_0^1 G(t,s)f\big(s,u(s),(Hv_1)(s)\big)\,ds - e(t) \\ &= A(u,v_1)(t). \end{aligned}$$

That is, A is a mixed monotone operator. In order to prove that B is an increasing operator, we take $u, v \in C_{h,e}$, with $u \le v$. Since g is decreasing, $g(v(1)) \le g(u(1))$ and it follows that

$$(Bu)(t) = -g\big(u(1)\big)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) - e(t) \le -g\big(v(1)\big)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) - e(t) = (Bv)(t),$$

for any $t \in [0,1]$. Therefore, B is an increasing operator.

For $\lambda \in (0,1)$ and $u \in C_{h,e}$, together with assumption (H6)′ ((c)′), and noting that

$$H\big(\lambda\big(\lambda^{-1}u + \big(\lambda^{-1} - 1\big)e\big) + (\lambda - 1)e\big) \ge \lambda H\big(\lambda^{-1}u + \big(\lambda^{-1} - 1\big)e\big) + (\lambda - 1)e,$$

we obtain

$$H\big(\lambda^{-1}u + \big(\lambda^{-1} - 1\big)e\big) \le \lambda^{-1}(Hu) + \big(\lambda^{-1} - 1\big)e. \tag{4.3}$$

Now, for $\lambda \in (0,1)$, $u, v \in C_{h,e}$ and $t \in [0,1]$, it follows from (4.3) (H2)′ and (H4)′ that

$$\begin{aligned} &A\big(\lambda u + (\lambda - 1)e, \lambda^{-1}v + \big(\lambda^{-1} - 1\big)e\big)(t) \\ &\quad = \int_0^1 G(t,s)f\big(s, \lambda u(s) + (\lambda - 1)e(s), H\big(\lambda^{-1}v(s) + \big(\lambda^{-1} - 1\big)e(s)\big)\big)\,ds - e(t) \end{aligned}$$

$$\begin{aligned}
&\geq \int_0^1 G(t,s)f\big(s,\lambda u(s)+(\lambda-1)e(s),\lambda^{-1}(Hv)(s)+\big(\lambda^{-1}-1\big)e(s)\big)\,ds-e(t)\\
&\geq \psi(\lambda)\int_0^1 G(t,s)f\big(s,u(s),(Hv)(s)\big)\,ds-e(t)\\
&=\psi(\lambda)\left(\int_0^1 G(t,s)f\big(s,u(s),(Hv)(s)\big)\,ds-e(t)\right)+\big(\psi(\lambda)-1\big)e(t)\\
&=\psi(\lambda)A(u,v)(t)+\big(\psi(\lambda)-1\big)e(t).
\end{aligned}$$

Thus

$$A\big(\lambda u+(\lambda-1)e,\lambda^{-1}v+\big(\lambda^{-1}-1\big)e\big)\geq \psi(\lambda)A(u,v)+\big(\psi(\lambda)-1\big)e,$$

for every $u,v\in C_{h,e}$, $\lambda\in(0,1)$, and $\psi(\lambda)>\lambda$. Taking $\lambda\in(0,1)$, $u\in C_{h,e}$ and $t\in[0,1]$, by assumption (H3)$'$, we have

$$\begin{aligned}
B\big(\lambda u+(\lambda-1)e\big)(t)&=-g\big(\lambda u(1)+(\lambda-1)e\big)\left(\frac{t^2}{2}-\frac{t^3}{6}\right)-e(t)\\
&\geq -\lambda g\big(u(1)\big)\left(\frac{t^2}{2}-\frac{t^3}{6}\right)-e(t)\\
&=\lambda\left(-g\big(u(1)\big)\left(\frac{t^2}{2}-\frac{t^3}{6}\right)-e(t)\right)+(\lambda-1)e(t)\\
&=\lambda(Bu)(t)+(\lambda-1)e(t).
\end{aligned}$$

Hence, we obtain

$$B\big(\lambda u+(\lambda-1)e\big)\geq \lambda(Bu)+(\lambda-1)e,\quad \forall u\in C_{h,e},\lambda\in(0,1).$$

In the following, we show that $A(h,h)\in C_{h,e}$ and $Bh\in C_{h,e}$. So we need to prove $A(h,h)+e\in C_h$, $Bh+e\in C_h$. We consider the function $h(t)=qt^2$ for every $t\in[0,1]$. Since $0\leq h(t)\leq q$ for every $t\in[0,1]$, by assumption (H6)$'$, we have $0\leq Hh\leq Hq$. For every $t\in[0,1]$, it follows from Lemma 4.2 and (H2)$'$ that

$$\begin{aligned}
A(h,h)(t)+e(t)&=\int_0^1 G(t,s)f\big(s,h(s),(Hh)(s)\big)\,ds\\
&\geq t^2\int_0^1\frac{s^2}{3}f\big(s,0,(Hq)(s)\big)\,ds\\
&=\frac{1}{q}\int_0^1\frac{s^2}{3}f\big(s,0,(Hq)(s)\big)\,ds\cdot h(t),
\end{aligned}$$

and

$$\begin{aligned}
A(h,h)(t)+e(t)&\leq t^2\int_0^1\frac{s}{2}f(s,q,0)\,ds\\
&=\frac{1}{q}\int_0^1\frac{s}{2}f(s,q,0)\,ds\cdot h(t).
\end{aligned}$$

Set

$$l_1 = \frac{1}{q}\int_0^1 \frac{s^2}{3} f\bigl(s,0,(Hq)(s)\bigr)\,ds, \qquad l_2 = \frac{1}{q}\int_0^1 \frac{s}{2} f(s,q,0)\,ds.$$

Since $\frac{1}{q}\int_0^1 \frac{s}{2} f(s,q,0)\,ds \geq \frac{1}{q}\int_0^1 \frac{s^2}{3} f(s,0,(Hq)(s))\,ds \geq \frac{\xi}{2q} > 0$, then $l_2 > l_1 > 0$. Thus $A(h,h) + e \in C_h$.

Now we prove that $Bh + e \in C_h$. In fact, for $t \in [0,1]$, using (H5)$'$, we deduce that

$$(Bh)(t) + e(t) = -g\bigl(h(1)\bigr)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) = -g(q)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) \leq g(q)\frac{t^2}{2} = \left(-\frac{g(q)}{2q}\right) \cdot h(t),$$

and

$$(Bh)(t) + e(t) = -g\bigl(h(1)\bigr)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) \geq g(q)t^2\left(\frac{1}{2} - \frac{1}{6}\right) = \left(-\frac{g(q)}{3q}\right) \cdot h(t).$$

Let

$$\beta_1 = -\frac{g(q)}{2q}, \qquad \beta_2 = -\frac{g(q)}{3q}.$$

Since $g(q) < 0$, $0 < \beta_2 < \beta_1$, combining with (H5)$'$, we have $Bh + e \in C_h$.

For every $u, v \in C_{h,e}$ and $t \in [0,1]$, by Lemma 4.2 and (H5)$'$, we get

$$\begin{aligned}
A(u,v)(t) &= \int_0^1 G(t,s) f\bigl(s,u(s),(Hv)(s)\bigr)\,ds - e(t) \\
&\geq \frac{1}{3}t^2 \int_0^1 s^2 f\bigl(s,0,(Hv)(s)\bigr)\,ds - e(t) \\
&\geq \frac{\xi}{2}t^2 - e(t) \geq -g\bigl(u(1)\bigr)\frac{t^2}{2} - e(t) \\
&\geq -g\bigl(u(1)\bigr)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) - e(t) \\
&= (Bu)(t),
\end{aligned}$$

which implies that, for every $u, v \in C_{h,e}$, $A(u,v) \geq \delta_0 Bu + (\delta_0 - 1)e$, with $\delta_0 = 1$. Therefore all the conditions of Theorem 3.1 are fulfilled. Consequently, the conclusion of Theorem 4.1 holds. □

Now, we give an example to illustrate our main result.

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t,u(t),(Hu)(t)) - 1, & t \in [0,1], \\ u(0) = u'(0) = u''(1) = 0, \\ u'''(1) = g(u(1)), \end{cases} \tag{4.4}$$

where

$$(Hu)(t) = \int_0^t u(s)\,ds,$$

$$f\big(t,u(t),(Hu)(t)\big) = \left[\left(\frac{t^4}{6} - \frac{2t^3}{3} + t^2\right)u(t) + \left(\frac{t^4}{24} - \frac{t^3}{6} + \frac{t^2}{4}\right)\right]^{\frac{1}{5}} + \left[\left(\frac{t^4}{6} - \frac{2t^3}{3} + t^2\right)(Hu)(t) + \left(\frac{t^4}{24} - \frac{t^3}{6} + \frac{t^2}{4}\right) + 1\right]^{-\frac{2}{5}},$$

and $g : [-\frac{1}{4}, +\infty) \to (-\infty, 0]$ is the function defined as

$$g(t) = \begin{cases} -t - \frac{1}{4}, & -\frac{1}{4} \le t \le \frac{7}{24}, \\ -\frac{13}{24}, & t > -\frac{7}{24}. \end{cases}$$

Choose

$$e(t) = \frac{t^4}{24} - \frac{t^3}{6} + \frac{t^2}{4}, \qquad h(t) = qt^2 \quad \text{with } q \ge \frac{7}{24}, \text{for every } t \in [0,1].$$

Then

$$e(t) = t^2\left(\frac{t^2}{24} - \frac{t}{6} + \frac{1}{4}\right) \ge \frac{t^2}{8} \ge 0,$$

and

$$e(t) \le \frac{7t^2}{24} \le qt^2 = h(t), \qquad e^* = \frac{1}{4} \quad \text{for } t \in [0,1].$$

It is easy to check that H satisfies assumption (H6)′ of Theorem 4.1, $f : [0,1] \times [-\frac{1}{4}, +\infty) \times [-\frac{1}{4}, +\infty) \to (-\infty, +\infty)$ is continuous and increasing in x and decreasing in y for fixed $t \in [0,1]$, and $g : [-\frac{1}{4}, +\infty) \to (-\infty, \frac{1}{4})$ is continuous and decreasing.

In the following, we verify the assumptions (H3)′ and (H4)′ of Theorem 4.1. For $\lambda \in (0,1)$, $x, y \in (-\infty, +\infty)$ and $c \in [0, e^*]$, we have

$$\begin{aligned} g\big(\lambda x + (\lambda - 1)c\big) &= -\big(\lambda x + (\lambda - 1)c\big) - e^* = -\lambda x + (1-\lambda)c - e^* \\ &\le -\lambda x + (1-\lambda)e^* - e^* = \lambda\big(-x - e^*\big) = \lambda g(x), \end{aligned}$$

and

$$\begin{aligned} &f\big(t, \lambda x + (\lambda - 1)c, \lambda^{-1}y + \big(\lambda^{-1} - 1\big)c\big) \\ &= \left(\frac{e(t)}{e^*}\big(\lambda x + (\lambda - 1)c\big) + e(t)\right)^{\frac{1}{5}} + \left(\frac{e(t)}{e^*}\big(\lambda^{-1}y + \big(\lambda^{-1} - 1\big)c\big) + e(t) + 1\right)^{-\frac{2}{5}} \\ &= \lambda^{\frac{1}{5}}\left(\frac{e(t)}{e^*}\left(x + \left(1 - \frac{1}{\lambda}\right)c\right) + \frac{1}{\lambda}e(t)\right)^{\frac{1}{5}} + \lambda^{\frac{2}{5}}\left(\frac{e(t)}{e^*}\big(y + (1-\lambda)c\big) + \lambda e(t) + \lambda\right)^{-\frac{2}{5}} \end{aligned}$$

$$\begin{aligned}
&\geq \lambda^{\frac{1}{5}}\left(\frac{e(t)}{e^*}x + \left(1-\frac{1}{\lambda}\right)\frac{e(t)}{e^*}e^* + \frac{1}{\lambda}e(t)\right)^{\frac{1}{5}} \\
&\quad + \lambda^{\frac{2}{5}}\left(\frac{e(t)}{e^*}y + (1-\lambda)\frac{e(t)}{e^*}e^* + \lambda e(t) + \lambda\right)^{-\frac{2}{5}} \\
&= \lambda^{\frac{1}{5}}\left(\frac{e(t)}{e^*}x + e(t)\right)^{\frac{1}{5}} + \lambda^{\frac{2}{5}}\left(\frac{e(t)}{e^*}y + e(t) + \lambda\right)^{-\frac{2}{5}} \\
&> \lambda^{\frac{2}{5}}\left[\left(\frac{e(t)}{e^*}x + e(t)\right)^{\frac{1}{5}} + \left(\frac{e(t)}{e^*}y + e(t) + 1\right)^{-\frac{2}{5}}\right] \\
&= \lambda^{\frac{2}{5}} f(t,x,y).
\end{aligned}$$

Let $\varphi(\lambda) = \lambda^{\frac{2}{5}}$. Then

$$f\bigl(t, \lambda x + (\lambda-1)c, \lambda^{-1}y + \bigl(\lambda^{-1}-1\bigr)c\bigr) \geq \varphi(\lambda)f(t,x,y).$$

Therefore, assumption (H3)′, (H4)′ of Theorem 4.1 are satisfied.

Moreover, taking $x, y \in (-\infty, +\infty)$, we get

$$\begin{aligned}
&\frac{2}{3}\int_0^1 s^2 f(s,0,y)\,ds \\
&\quad = \frac{2}{3}\int_0^1 s^2\left[\left(\frac{s^4}{24} - \frac{s^3}{6} + \frac{s^2}{4}\right)^{\frac{1}{5}} + \left(\left(\frac{s^4}{6} + \frac{2s^3}{3} + t^2\right)y + \left(\frac{s^4}{24} - \frac{s^3}{6} + \frac{s^2}{4}\right) + 1\right)^{-\frac{2}{5}}\right]ds \\
&\quad \geq \frac{2}{3}\int_0^1 s^2\left(\frac{s^4}{24} - \frac{s^3}{6} + \frac{s^2}{4}\right)^{\frac{1}{5}} ds \geq \frac{2}{3}\int_0^1 s^2\,ds = 1 > \frac{13}{24} \geq -g(x).
\end{aligned}$$

Note that $g(\frac{7}{24}) = -\frac{13}{24} < 0$, thus assumption (H5)′ of Theorem 4.1 holds.

5 Application to fractional differential equation boundary value problem

In this section, we consider the following fractional boundary value problem:

$$\begin{cases}
-D_{0^+}^{\alpha}u(t) = f(t,u(t),u(t)) + g(t,u(t),u(t)) - b, & 0 < t < 1, n-1 < \alpha \leq n, \\
u^{(i)}(0) = 0, & 0 \leq i \leq n-2, \\
[D_{0^+}^{\beta}u(t)]_{t=1} = 0, & 1 \leq \beta \leq n-2,
\end{cases} \tag{5.1}$$

where $n \geq 3$ ($n \in \mathbb{N}$), $b > 0$ is a constant, $f, g : [0,1] \times (-\infty,+\infty) \times (-\infty,+\infty) \to (-\infty,+\infty)$ are continuous functions, and $D_{0^+}^{\alpha}$ is the Riemann–Liouville fractional derivative of order α, i.e.,

$$D_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_0^t \frac{u(s)}{(t-s)^{\alpha-k+1}}\,ds,$$

where $k = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α. Because of non-local behavior, fractional order boundary value problems are extensively applied to blood flow problems, control theory, the fluid-dynamic traffic model and polymer rheology. It implies that differential operators of arbitrary order can describe memory and hereditary properties of certain important processes [28–31]. There are many tools to deal with the uniqueness and multiplicity of solutions for fractional differential equations such as mixed monotone operators [23, 32–34], Avery–Peterson fixed point theorem [35, 36], Guo–Krasnosel'skii fixed point theorem on a cone [37, 38], the fixed point index theory [39–41], monotone iteration method [42], the critical point theory [43, 44], Schauder's fixed point theory [45] and stability.

Problem (5.1) has caused great attention since it generalizes the well-known elastic beam equation [46]. In [47], Goodrich first obtained some properties of the Green's function corresponding to (5.1). Then, applying these properties and Krasnosel'skii fixed point theorem in cones, the author gave some sufficient conditions under which problem (5.1) when $g \equiv 0$ and $b = 0$ has a positive solution. Note that the nonlinearity discussed in [47] grows sublinearly. Furthermore, through the nonlinearity is again considered to grow sublinearly, Xu, Wei and Dong [39] utilized fixed point index theory to establish existence and uniqueness theorems of problem (5.1) based on a priori estimate. On the other hand, if we replace $g(t, u(t), u(t))$ with $g(t, u(t))$ in problem (5.1), Jleli and Samet [33] studied the existence and uniqueness of positive solutions for problem (5.1) with $b = 0$. The authors proved that problem (5.1) has a unique solution by a mixed monotone fixed point theorem obtained in [14]. Zhang and Tian [34] and Yang, Shen and Xie [35] be both concerned with the case that the derivative of the unknown function is involved in the nonlinear term. Recently, Liu et al. [23] obtained a unique solution of problem with $b = 0$ by the fixed point theorems of a sum operator on a cone. We point out that the assumptions imposed on the nonlinear terms f and g in [23, 33–36, 39] are nonnegative. The interesting point of this paper is to remove above restrictions.

Lemma 5.1 ([23, 47]) *Let $g(t) \in C[0,1]$, then the fractional boundary value problem*

$$\begin{cases} -D_{0^+}^{\alpha} u(t) = g(t), & 0 < t < 1, n-1 < \alpha \le n, \\ u^{(i)}(0) = 0, & 0 \le i \le n-2, \\ [D_{0^+}^{\beta} u(t)]_{t=1} = 0, & 1 \le \beta \le n-2, \end{cases}$$

has a unique positive solution

$$u(t) = \int_0^1 G(t,s) g(s)\, ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1. \end{cases}$$

Lemma 5.2 ([23, 47]) *The Green function $G(t,s)$ in Lemma 5.1 has the following properties*:

(1) *$G(t,s)$ is continuous on* $[0,1] \times [0,1]$;

(2) *for all $(t,s) \in [0,1] \times [0,1]$, we can have $G(t,s) \ge 0$*;

(3) *for all $t, s \in [0,1]$, we have*

$$\big(1 - (1-s)^{\beta}\big)(1-s)^{\alpha-\beta-1} t^{\alpha-1} \le \Gamma(\alpha) G(t,s) \le (1-s)^{\alpha-\beta-1} t^{\alpha-1}. \tag{5.2}$$

Define

$$E = C[0,1], \qquad \|u\| = \sup\{u(t)|t \in [0,1]\},$$

and

$$P = \{u \in E | u(t) \geq 0, t \in [0,1]\}.$$

It is clear that E is a Banach space and P is a normal cone of E. Let

$$e(t) = \frac{b}{(\alpha-\beta)\Gamma(\alpha)}\left(t^{\alpha-1} - \frac{\alpha-\beta}{\alpha}t^{\alpha}\right), \quad t \in [0,1].$$

Theorem 5.1 *Suppose that the following assumptions hold*:

(C1) $f,g : [0,1] \times [-e^*, +\infty) \times [-e^*, +\infty) \to (-\infty, +\infty)$ *are continuous and for all* $t \in [0,1]$, $g(t,0,H) \geq 0$ *with* $g(t,0,H) \not\equiv 0$ *where* $H \geq \frac{b}{(\alpha-\beta)\Gamma(\alpha)}$ *and* $e^* = \max\{e(t) : t \in [0,1]\}$;

(C2) *for fixed* $t \in [0,1]$ *and* $y \in [-e^*, +\infty)$, $f(t,x,y)$, $g(t,x,y)$ *are increasing in* $x \in [-e^*, +\infty)$; *for fixed* $t \in [0,1]$ *and* $x \in [-e^*, +\infty)$, $f(t,x,y)$, $g(t,x,y)$ *are decreasing in* $y \in [-e^*, +\infty)$;

(C3) *for all* $\lambda \in (0,1)$, *there exists* $\psi(\lambda) \in (\lambda, 1)$, *such that, for all* $t \in [0,1]$,

$$\begin{aligned} &f\big(t, \lambda x + (\lambda-1)\rho, \lambda^{-1}y + (\lambda^{-1}-1)\rho\big) \geq \psi(\lambda) f(t,x,y), \\ &\quad x,y \in (-\infty,+\infty), \rho \in [0,e^*], \end{aligned} \tag{5.3}$$

$$\begin{aligned} &g\big(t, \lambda x + (\lambda-1)\rho, \lambda^{-1}y + (\lambda^{-1}-1)\rho\big) \geq \lambda g(t,x,y), \\ &\quad x,y \in (-\infty,+\infty), \rho \in [0,e^*]; \end{aligned} \tag{5.4}$$

(C4) *there exists a constant* $\delta > 0$, *such that*

$$f(t,x,y) \geq \delta g(t,x,y), \quad \textit{for all } t \in [0,1], x,y \in [-e^*, +\infty).$$

Then the problem (5.1) *has a unique nontrivial solution* u^* *in* $C_{h,e}$, *where* $h(t) = Ht^{\alpha-1}$, $t \in [0,1]$. *Moreover, we can construct the following two sequences*:

$$\begin{aligned} \omega_n(t) = {}& \int_0^1 G(t,s)\big(f\big(s,\omega_{n-1}(s),\tau_{n-1}(s)\big) + g\big(s,\omega_{n-1}(s),\tau_{n-1}(s)\big)\big)\,ds \\ & - \frac{b}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1} + \frac{b}{\alpha\Gamma(\alpha)}t^{\alpha}, \quad n = 1,2,\ldots, \\ \tau_n(t) = {}& \int_0^1 G(t,s)\big(f\big(s,\tau_{n-1}(s),\omega_{n-1}(s)\big) + g\big(s,\tau_{n-1}(s),\omega_{n-1}(s)\big)\big)\,ds \\ & - \frac{b}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1} + \frac{b}{\alpha\Gamma(\alpha)}t^{\alpha}, \quad n = 1,2,\ldots, \end{aligned}$$

for any given $\omega_0, \tau_0 \in C_{h,e}$, *and we have* $\{\omega_n(t)\}$ *and* $\{\tau_n(t)\}$ *both converge to* $u^*(t)$ *uniformly for all* $t \in [0,1]$.

Proof For $t \in [0,1]$

$$e(t) = \frac{b}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1} - \frac{b}{\alpha\Gamma(\alpha)}t^{\alpha} = \frac{b}{(\alpha-\beta)\Gamma(\alpha)}\left(t^{\alpha-1} - \frac{\alpha-\beta}{\alpha}t^{\alpha}\right) \geq 0.$$

That is $e \in P$. Further, for $t \in [0,1]$

$$e(t) = \frac{b}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1} - \frac{b}{\alpha\Gamma(\alpha)}t^{\alpha} \leq \frac{b}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1} \leq Ht^{\alpha-1} = h(t).$$

Hence, $0 < e(t) \leq h(t)$. In addition, $C_{h,e} = \{u \in C[0,1] | u + e \in C_e\}$. From Lemma 5.1, the problem (5.1) has the integral formulation

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)\bigl(f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s),u(s)\bigr) - b\bigr)\,ds \\
&= \int_0^1 G(t,s)\bigl(f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s),u(s)\bigr)\bigr)\,ds - b\int_0^1 G(t,s)\,ds \\
&= \int_0^1 G(t,s)f\bigl(s,u(s),u(s)\bigr)\,ds + \int_0^1 G(t,s)g\bigl(s,u(s),u(s)\bigr)\,ds \\
&\quad - \frac{b}{(\alpha-\beta)\Gamma(\alpha)}\left(t^{\alpha-1} - \frac{\alpha-\beta}{\alpha}t^{\alpha}\right) \\
&= \int_0^1 G(t,s)f\bigl(s,u(s),u(s)\bigr)\,ds + \int_0^1 G(t,s)g\bigl(s,u(s),u(s)\bigr)\,ds - e(t) \\
&= \int_0^1 G(t,s)f\bigl(s,u(s),u(s)\bigr)\,ds - e(t) + \int_0^1 G(t,s)g\bigl(s,u(s),u(s)\bigr)\,ds - e(t) + e(t).
\end{aligned}$$

For every $t \in [0,1]$ and $u, v \in C_{h,e}$, we consider the operators defined by

$$\begin{aligned}
A(u,v)(t) &= \int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds - e(t), \\
B(u,v)(t) &= \int_0^1 G(t,s)g\bigl(s,u(s),v(s)\bigr)\,ds - e(t).
\end{aligned}$$

It is clear that u is the solution of the problem (5.1) if and only if $u = A(u,u) + B(u,u) + e$.

(1) Firstly, we show that $A, B : C_{h,e} \times C_{h,e} \to E$ are mixed monotone operators. In fact, for all $u_i, v_i \in C_{h,e}$ $(i = 1,2)$ with $u_1 \geq u_2$, $v_1 \leq v_2$, by (C2), we get

$$\begin{aligned}
A(u_1,v_1)(t) &= \int_0^1 G(t,s)f\bigl(s,u_1(s),v_1(s)\bigr)\,ds - e(t) \\
&\geq \int_0^1 G(t,s)f\bigl(s,u_2(s),v_2(s)\bigr)\,ds - e(t) = A(u_2,v_2)(t).
\end{aligned}$$

That is $A(u_1,v_1) \geq A(u_2,v_2)$. In a similar way we get $B(u_1,v_1) \geq B(u_2,v_2)$.

(2) From (C3), for every $\lambda \in [0,1]$ and $t \in [0,1]$, there exists $\psi(\lambda) \in (\lambda,1)$ such that, for every $u, v \in C_{h,e}$, we have

$$\begin{aligned}
&A\bigl(\lambda u + (\lambda-1)e, \lambda^{-1}v + \bigl(\lambda^{-1}-1\bigr)e\bigr)(t) \\
&\quad = \int_0^1 G(t,s)f\bigl(s, \lambda u(s) + (\lambda-1)e, \lambda^{-1}v(s) + \bigl(\lambda^{-1}-1\bigr)e\bigr)\,ds - e(t)
\end{aligned}$$

$$
\begin{aligned}
&\geq \psi(\lambda)\int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds - e(t)\\
&= \psi(\lambda)\int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds - e(t) + \psi(\lambda)e(t) - \psi(\lambda)e(t)\\
&= \psi(\lambda)\biggl(\int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds - e(t)\biggr) + \bigl(\psi(\lambda)-1\bigr)e(t)\\
&= \psi(\lambda)A(u,v)(t) + \bigl(\psi(\lambda)-1\bigr)e(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&B\bigl(\lambda u + (\lambda-1)e, \lambda^{-1}v + \bigl(\lambda^{-1}-1\bigr)e\bigr)(t)\\
&\quad= \int_0^1 G(t,s)g\bigl(s,\lambda u(s) + (\lambda-1)e, \lambda^{-1}v(s) + \bigl(\lambda^{-1}-1\bigr)e\bigr)\,ds - e(t)\\
&\quad\geq \lambda\int_0^1 G(t,s)g\bigl(s,u(s),v(s)\bigr)\,ds - e(t)\\
&\quad= \lambda\biggl(\int_0^1 G(t,s)g\bigl(s,u(s),v(s)\bigr)\,ds - e(t)\biggr) + (\lambda-1)e(t)\\
&\quad= \lambda B(u,v)(t) + (\lambda-1)e(t).
\end{aligned}
$$

(3) Next we show that $A(h,h)\in C_{h,e}$, $B(h,h)\in C_{h,e}$. It is sufficient to prove $A(h,h)+e\in C_h$, $B(h,h)+e\in C_h$. It follows from Lemma 5.2 and (C1), (C3) that

$$
\begin{aligned}
A(h,h)(t)+e(t) &= \int_0^1 G(t,s)f\bigl(s,h(s),h(s)\bigr)\,ds\\
&= \int_0^1 G(t,s)f\bigl(s,Hs^{\alpha-1},Hs^{\alpha-1}\bigr)\,ds\\
&\leq \int_0^1 \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)}f(s,H,0)\,ds\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}f(s,H,0)\,ds\cdot t^{\alpha-1}\\
&= \frac{1}{H\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}f(s,H,0)\,ds\cdot h(t)
\end{aligned}
$$

and

$$
\begin{aligned}
A(h,h)(t)+e(t) &\geq \int_0^1 \frac{(1-(1-s)^{\beta})(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)}f(s,0,H)\,ds\\
&\geq \frac{1}{\Gamma(\alpha)}\int_0^1 \bigl(1-(1-s)^{\beta}\bigr)(1-s)^{\alpha-\beta-1}f(s,0,H)\,ds\cdot t^{\alpha-1}\\
&= \frac{1}{H\Gamma(\alpha)}\int_0^1 \bigl(1-(1-s)^{\beta}\bigr)(1-s)^{\alpha-\beta-1}f(s,0,H)\,ds\cdot h(t).
\end{aligned}
$$

Since $\alpha>\beta$, $\Gamma(\alpha)>0$ and from (C2), (C4), we derive that

$$
f(s,H,0)\geq f(s,0,H)\geq \delta g(s,0,H),\quad \text{for } s\in[0,1].
$$

Note that $g(s,0,H) \not\equiv 0$ and $g(s,0,H) \geq 0$ for every $s \in [0,1]$, we have

$$\int_0^1 f(s,H,0)\,ds \geq \int_0^1 f(s,0,H)\,ds \geq \int_0^1 \delta g(s,0,H)\,ds > 0.$$

Let

$$\begin{aligned} l_1 &= \frac{1}{H\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} f(s,H,0)\,ds > 0, \\ l_2 &= \frac{1}{H\Gamma(\alpha)} \int_0^1 \bigl(1-(1-s)^{\beta}\bigr)(1-s)^{\alpha-\beta-1} f(s,0,H)\,ds > 0. \end{aligned}$$

Thus $l_2 h(t) \leq A(h,h)(t) + e(t) \leq l_1 h(t)$, $t \in [0,1]$ and we have $A(h,h) \in C_{h,e}$. In a similar way we get

$$\begin{aligned} B(h,h)(t) + e(t) &= \int_0^1 G(t,s) g\bigl(s,h(s),h(s)\bigr)\,ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s,H,0)\,ds \cdot t^{\alpha-1} \\ &= \frac{1}{H\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s,H,0)\,ds \cdot h(t) \end{aligned}$$

and

$$\begin{aligned} B(h,h)(t) + e(t) &= \int_0^1 G(t,s) g\bigl(s,h(s),h(s)\bigr)\,ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_0^1 \bigl(1-(1-s)^{\beta}\bigr)(1-s)^{\alpha-\beta-1} g(s,0,H)\,ds \cdot t^{\alpha-1} \\ &= \frac{1}{H\Gamma(\alpha)} \int_0^1 \bigl(1-(1-s)^{\beta}\bigr)(1-s)^{\alpha-\beta-1} g(s,0,H)\,ds \cdot h(t). \end{aligned}$$

Let

$$\begin{aligned} l_3 &= \frac{1}{H\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s,H,0)\,ds > 0, \\ l_4 &= \frac{1}{H\Gamma(\alpha)} \int_0^1 \bigl(1-(1-s)^{\beta}\bigr)(1-s)^{\alpha-\beta-1} g(s,0,H)\,ds > 0. \end{aligned}$$

Thus $l_4 h(t) \leq B(h,h)(t) + e(t) \leq l_3 h(t)$, $t \in [0,1]$ and we have $B(h,h) \in C_{h,e}$.

(4) For every $u, v \in C_{h,e}$, $t \in [0,1]$, from (C4) we know that

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s) f\bigl(s,u(s),v(s)\bigr)\,ds - e(t) \\ &\geq \delta \int_0^1 G(t,s) g\bigl(s,u(s),v(s)\bigr)\,ds - e(t) - \delta e(t) + \delta e(t) \\ &\geq \delta \biggl(\int_0^1 G(t,s) g\bigl(s,u(s),v(s)\bigr)\,ds - e(t) \biggr) + (\delta - 1) e(t) \\ &= \delta B(u,v)(t) + (\delta - 1) e(t), \end{aligned}$$

so we get $A(u, v) \geq \delta B(u, v) + (\delta - 1)e$. Therefore all the conditions of Theorem 3.1 are satisfied. Consequently the conclusion of Theorem 5.1 holds. □

6 Conclusions

In this paper, we firstly consider the existence and uniqueness of solution to the operator equation (1.4) on ordered Banach spaces E. Secondly, based on main results of [1], we apply our abstract result for (1.1) to improve and generalize conditions (H1)–(H6). Finally, we use Theorem 3.1 to prove fractional boundary value problem (5.1) to have a unique solution.

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