Zhou *Boundary Value Problems* (2018) 2018:185
https://doi.org/10.1186/s13661-018-1106-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Inhomogeneous boundary value problems for steady compressible magnetohydrodynamic equations

Chunhui Zhou[1*]

[*]Correspondence:
zhouchunhui@seu.edu.cn
[1]School of Mathematics, Southeast University, Nanjing, China

Abstract

This paper will be concerned with the compressible perturbation to steady magnetohydrodynamic equations near a uniform flow. In particular, the velocity of the basic flow under consideration can be any non-zero constant. We prove that there exists a stationary strong solution around a given basic flow with inhomogeneous boundary condition.

MSC: 35Q30; 76N10

Keywords: Magnetohydrodynamic equations; Steady compressible fluid; Inhomogeneous boundary value

1 Introduction

The steady compressible isentropic magnetohydrodynamic equations in 3D can be described as follows:

$$\operatorname{div}(\rho\mathbf{v}) = 0, \tag{1.1}$$

$$-\mu\triangle\mathbf{v} - \tilde{\mu}\nabla\operatorname{div}\mathbf{v} + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) + \nabla P = (\nabla\times\mathbf{B})\times\mathbf{B} + \rho\mathbf{F}, \tag{1.2}$$

$$\nabla\times(\nu\nabla\times\mathbf{B}) - \nabla\times(\mathbf{v}\times\mathbf{B}) = 0, \tag{1.3}$$

$$\operatorname{div}\mathbf{B} = 0. \tag{1.4}$$

Here ρ denotes the density, $\mathbf{v} = (v_1, v_2, v_3)$ the velocity and $P = a\rho^\gamma$ the pressure of the fluid with a a constant; as we will study the strong solutions to system (1.1)–(1.4), without loss of generality, we will assume $a = \gamma = 1$ in the following, $\mathbf{B} = (B_1, B_2, B_3)$ is the magnetic field, $\mu > 0$, $\tilde{\mu} > 0$ are the first and second viscosity constants, $\nu > 0$ is the magnetic diffusion coefficient, $\mathbf{F} = (F_1, F_2, F_3)$ the external force.

In this paper, we will study the simplified model of system (1.1)–(1.4) in 2D and consider the compressible perturbation of a uniform flow in the domain $\Omega = [0,1]\times[0,1]$. Moreover, as the velocity of the basic flow under consideration can be any non-zero constant, there will be an inflow ($\mathbf{v}\cdot\mathbf{n} < 0$) and outflow ($\mathbf{v}\cdot\mathbf{n} > 0$) part on the boundary, where $\mathbf{n}$ is the outer normal vector to the boundary. More precisely, we will denote

$$\Gamma_{\text{in}} = \{x_1 = 0, 0 \le x_2 \le 1\},$$

$$\Gamma_{\text{out}} = \{x_1 = 1, 0 \le x_2 \le 1\},$$
$$\Gamma_0 = \{0 \le x_1 \le 1, x_2 = 0\} \cup \{0 \le x_1 \le 1, x_2 = 1\}.$$

Using the mass equation (1.1) and the divergence-free condition (1.4) of the magnetic field, the complete boundary value problem we consider will be as follows:

$$\operatorname{div}(\rho\mathbf{v}) = 0, \tag{1.5}$$
$$-\mu\triangle v_1 - \tilde{\mu}\partial_1 \operatorname{div}\mathbf{v} + \rho\mathbf{v}\cdot v_1 + \partial_1 P = -B_2(\partial_1 B_2 - \partial_2 B_1) + \rho F_1, \tag{1.6}$$
$$-\mu\triangle v_2 - \tilde{\mu}\partial_2 \operatorname{div}\mathbf{v} + \rho\mathbf{v}\cdot v_2 + \partial_2 P = B_1(\partial_1 B_2 - \partial_2 B_1) + \rho F_2, \tag{1.7}$$
$$-\nu\triangle B_1 - \partial_2(v_1 B_2 - v_2 B_1) = 0, \tag{1.8}$$
$$-\nu\triangle B_2 + \partial_1(v_1 B_2 - v_2 B_1) = 0, \tag{1.9}$$
$$\operatorname{div}\mathbf{B} = 0. \tag{1.10}$$
$$\rho = \rho_0 \quad \text{on } \Gamma_{\text{in}}, \tag{1.11}$$
$$\mathbf{v} = \mathbf{v}_0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \tag{1.12}$$
$$\mathbf{v}\cdot\mathbf{n} = \operatorname{curl}\mathbf{v} = 0 \quad \text{on } \Gamma_0, \tag{1.13}$$
$$\mathbf{B} = \mathbf{B}_0 \quad \text{on } \partial\Omega, \tag{1.14}$$

where $\mathbf{v}_0 = (v_0^1, v_0^2)$ and $\mathbf{B}_0 = (B_0^1, B_0^2)$.

Before stating the main result, we will briefly review some related work. In one space dimension, for the initial-boundary value problem with general large initial data for compressible magnetohydrodynamic equations one may refer to [1, 16]. In [10] Kawashima proved the global existence of strong solutions for general electromagnetic fluid equations in two dimension when the initial date are small perturbations of given constant state. In [11], the authors proved that there exist classical solutions global in time and they also obtained the optimal decay rate for three-dimensional compressible MHD equations when the initial data are small perturbations of given constant state. For the low Mach limit or the vanishing viscous limit for MHD equations in different cases refer to [2, 6, 8, 9]. In [5, 7], the authors considered the global weak solutions and large-time behavior of solutions to the three-dimensional compressible MHD equations with general large initial data. In the case of a steady flow, the author in [18] proved the global existence of weak solutions to steady compressible isentropic MHD equations with large force. In [17], the authors proved that there exist strong solutions to steady compressible MHD equations with homogeneous boundary condition and a small force. To the best of the author's knowledge, there is no result dealing with the nonhomogeneous boundary value problem as regards steady compressible MHD equations. However, there are a lot of studies on the inhomogeneous boundary value problems to the steady compressible Navier–Stokes equations [4, 12–15]. In this paper we will study the nonhomogeneous boundary value problem for steady compressible MHD equations in two dimensions.

Notation: Let $K \subset \mathbb{R}^N$ be an open set, $1 \le p < \infty$, we denote by $L^p(K)$ the Lebesgue space equipped with the norm $\|\cdot\|_{L^p(K)} = \|\cdot\|_{0,p,K}$; by $W^{s,p}(K)$, $s \in \mathbb{R}$ (see [3]) the Sobolev space equipped with the norm $\|\cdot\|_{s,p,K}$, by $|\cdot|_{s,p}$ the Sobolev norm on the boundary, when $p = 2$, we also denote by H^s the Sobolev space. Besides, the set Ξ is defined by

$$\Xi = \left\{\mathbf{u} = (u_1, u_2) \in H^1(\Omega), \mathbf{u} = 0 \text{ on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_0\right\}.$$

Let $\mathcal{U}_0 = (1,0)$, $\mathcal{B}_0 = (1,0)$. We define

$$M_0 = |\mathbf{v}_0 - \mathcal{U}_0|_{2-1/p,p;\Gamma_{\text{in}}\cup\Gamma_{\text{out}}} + |\rho_0 - 1|_{1,p;\Gamma_{\text{in}}} + |\mathbf{B}_0 - \mathcal{B}_0|_{2-1/p,p;\partial\Omega} + \|\mathbf{F}\|_{L^p}. \tag{1.15}$$

Moreover, we assume that v_1^0 satisfies the following compatibility conditions:

$$\partial_{x_2} v_0^1(0,0) = \partial_{x_2} v_0^1(0,1) = \partial_{x_2} v_0^1(1,0) = \partial_{x_2} v_0^1(1,1) = 0 \tag{1.16}$$

The main result is the following.

Theorem 1.1 *Let M_0 defined in (1.15) be small enough, $2 < p < \infty$, v_0^1 satisfy the compatibility conditions (1.16), then system (1.5)–(1.14) admits a unique solution $(\mathbf{v}, \rho, \mathbf{B}) \in W^{2,p} \times W^{1,p} \times W^{2,p}$ satisfying*

$$\|\mathbf{v} - \mathcal{U}_0\|_{2,p;\Omega} + \|\rho - 1\|_{1,p;\Omega} + \|\mathbf{B} - \mathcal{B}_0\|_{2,p;\Omega} \le CM_0, \tag{1.17}$$

here C is a constant depending on p.

The proof of Theorem 1.1 basically relies on the energy estimate and L^p estimate of the linearized system to system (1.5)–(1.14). As the compressible condition results in the loss of regularity in the mass equation (the term $\mathbf{v} \cdot \nabla\rho$), it seems hard to obtain the strong solutions to both the linear and the nonlinear system directly by using the fixed point theories. To overcome this problem, we will prove the existence of a weak solution to the linear system by taking the limit of a sequence of viscous solutions to an approximate system, and to the nonlinear system by proving that the sequence of solutions to the linear system is a Cauchy sequence in $H^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$. In the case of the inhomogeneous boundary conditions, the total mass of the fluid is unknown, i.e. the L^1 norm of the density is unknown, thus we cannot obtain the L^2 norm of the density by the classical method of energy from the momentum equations. To overcome this, we will introduce a transformation to "straighten" the stream line. Compared with [4], here we have to take more care of the terms including $\mathbf{B}$ after the transformation in the process of energy estimate. Then, by the L^p estimate of linearized system, we find that the weak solutions are also the strong solutions. Finally, we point out that our method failed in the case of the Dirichlet boundary condition of $\mathbf{v}$ on the rigid wall (Γ_0) as one needs more dedicated estimates of ρ and $\mathbf{v}$ around the corners.

2 The linearized system

In this section we will study a linearized system satisfied by the the perturbation. First, we will homogenize the boundary condition of the linearized system. By the theory of elliptic partial differential equations of second order, we can find $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2)$, $\tilde{\mathbf{B}} = (\tilde{B}_1, \tilde{B}_2)$

satisfying

$$\begin{aligned} &\tilde{\mathbf{v}} = \mathbf{v}_0 - \mathcal{U}_0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \qquad \tilde{\mathbf{v}} \cdot \mathbf{n} = \operatorname{curl} \tilde{\mathbf{v}} = 0 \quad \text{on } \Gamma_0, \\ &\|\tilde{\mathbf{v}}\|_{2,p;\Omega} \le C(p) M_0, \\ &\tilde{\mathbf{B}} = \mathbf{B}_0 - \mathcal{B}_0 \quad \text{on } \partial\Omega, \qquad \|\tilde{\mathbf{B}}\|_{2,p;\Omega} \le C(p) M_0, \end{aligned} \tag{2.1}$$

In fact, we can consider the following boundary problem:

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = \varphi_0 & \text{on } \Gamma_0, \\ u = \varphi_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ u = \varphi_{\text{out}} & \text{on } \Gamma_{\text{out}}, \end{cases} \tag{2.2}$$

and by the theory of elliptic equations ([3]), we have the following.

Lemma 2.1 *Let $1 < p < \infty$, $f \in L^p(\Omega)$, $\varphi_0 \in W^{2,2-1/p}(\Gamma_0)$, $\varphi_{\text{in}} \in W^{2,2-1/p}(\Gamma_{\text{in}})$, $\varphi_{\text{out}} \in W^{2,2-1/p}(\Gamma_{\text{out}})$, be given such that $\varphi_0(0,0) = \varphi_{\text{in}}(0,0)$, $\varphi_0(0,1) = \varphi_{\text{in}}(0,1)$, $\varphi_0(1,0) = \varphi_{\text{out}}(1,0)$, $\varphi_0(1,1) = \varphi_{\text{in}}(1,1)$, then there exists a unique solution $u \in W^{2,p}(\Omega)$ satisfying the boundary problem* (2.2) *and the following estimate holds*:

$$\|u\|_{2,p;\Omega} \le C\bigl(\|\varphi_0\|_{2,2-1/p;\Gamma_0} + \|\varphi_{\text{in}}\|_{2,2-1/p;\Gamma_{\text{in}}} + \|\varphi_{\text{out}}\|_{2,2-1/p;\Gamma_{\text{out}}} + \|f\|_{p;\Omega}\bigr), \tag{2.3}$$

where C is a constant depending on p.

Remark 2.2 If we replace the Dirichlet boundary condition $u = \varphi_0 \in W^{2-1/p,p}$ on Γ_0 in (2.2) by the Neumann boundary condition $\partial_n u = \varphi_0 \in W^{1-1/p,p}$ on Γ_0, and if $p > 2$, also assuming the compatibility conditions $\varphi_0(0,0) = -\partial_{x_2}\varphi_{\text{in}}(0,0)$, $\varphi_0(0,1) = \partial_{x_2}\varphi_{\text{in}}(0,1)$, $\varphi_0(1,0) = -\partial_{x_2}\varphi_{\text{out}}(1,0)$, $\varphi_0(1,1) = \partial_{x_2}\varphi_{\text{out}}(1,1)$, then (2.3) also holds with $\|\varphi_0\|_{2-1/p,p}$ replaced by $\|\varphi_0\|_{1-1/p,p}$. Consequently, we can find functions $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2)$, $\tilde{\mathbf{B}} = (\tilde{B}_1, \tilde{B}_2)$ satisfying (2.1).

Now let $v_1 = 1 + \tilde{v}_1 + \bar{v}_1$, $v_2 = \tilde{v}_2 + \bar{v}_2$, $\rho = 1 + \bar{\rho}$, $B_1 = 1 + \tilde{B}_1 + \bar{B}_1$, $B_2 = \tilde{B}_2 + \bar{B}_2$, $\bar{\mathbf{v}} = (\bar{v}_1, \bar{v}_2)$, $\bar{\mathbf{B}} = (\bar{B}_1, \bar{B}_2)$, then the system satisfied by $(\bar{\mathbf{v}}, \bar{\rho}, \bar{\mathbf{B}})$ can be written as

$$\begin{aligned} &\partial_1 \bar{\rho} + (\bar{\mathbf{v}} + \tilde{\mathbf{v}}) \cdot \nabla \bar{\rho} + \operatorname{div} \bar{\mathbf{v}} = R(\bar{\mathbf{v}}, \bar{\rho}) \quad \text{in } \Omega, \\ &\partial_1 \bar{v}_1 - \mu \Delta \bar{v}_1 - \tilde{\mu} \partial_1 \operatorname{div} \bar{\mathbf{v}} + \partial_1 \bar{\rho} = K_1(\bar{\mathbf{v}}, \bar{\rho}, \bar{\mathbf{B}}) \quad \text{in } \Omega, \\ &\partial_1 \bar{v}_2 - \mu \Delta \bar{v}_2 - \tilde{\mu} \partial_2 \operatorname{div} \bar{\mathbf{v}} + \partial_2 \bar{\rho} - (\partial_1 \bar{B}_2 - \partial_2 \bar{B}_1) = K_2(\bar{\mathbf{v}}, \bar{\rho}, \bar{\mathbf{B}}) \quad \text{in } \Omega, \\ &-\nu \Delta \bar{B}_1 - \partial_2 \bar{B}_2 + \partial_2 \bar{v}_2 = H_1(\bar{\mathbf{v}}, \bar{\rho}, \bar{\mathbf{B}}) \quad \text{in } \Omega, \\ &-\nu \Delta \bar{B}_2 + \partial_1 \bar{B}_2 - \partial_1 \bar{v}_2 = H_2(\bar{\mathbf{v}}, \bar{\rho}, \bar{\mathbf{B}}) \quad \text{in } \Omega \\ &\bar{\mathbf{v}} = 0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\ &\bar{\mathbf{v}} \cdot \mathbf{n} = \operatorname{curl} \bar{\mathbf{v}} = 0 \quad \text{on } \Gamma_0, \\ &\bar{\mathbf{B}} = 0 \quad \text{on } \partial\Omega, \\ &\bar{\rho} = \rho_0 - 1 \quad \text{on } \Gamma_{\text{in}}, \end{aligned} \tag{2.4}$$

where $\mathbf{K} = (K_1, K_2)$, $\mathbf{H} = (H_1, H_2)$ and

$$\begin{aligned}
&R(\bar{\mathbf{v}}, \bar{\rho}) = -\operatorname{div}\tilde{\mathbf{v}} - \bar{\rho}\operatorname{div}(\bar{\mathbf{v}} + \tilde{\mathbf{v}}),\\
&K_1(\bar{\mathbf{v}}, \bar{\rho}, \bar{\mathbf{B}}) = (\bar{\rho}+1)F_1 + \mu\Delta\tilde{v}_1 + \tilde{\mu}\partial_1\operatorname{div}\tilde{\mathbf{v}}\\
&\qquad - (\bar{\rho}+1)(\bar{\mathbf{v}} + \tilde{\mathbf{v}} + U_0)\cdot\nabla\tilde{v}_1 - (\bar{\rho}+1)(\bar{\mathbf{v}} + \tilde{\mathbf{v}})\cdot\nabla\bar{v}_1\\
&\qquad - \bar{\rho}\partial_1\bar{v}_1 - (\tilde{B}_2 + \bar{B}_2)\big[\partial_1(\tilde{B}_2 + \bar{B}_2) - \partial_2(\tilde{B}_1 + \bar{B}_1)\big],\\
&K_2(\bar{\mathbf{v}}, \bar{\rho}, \bar{\mathbf{B}}) = (\bar{\rho}+1)F_2 + \mu\Delta\tilde{v}_2 + \tilde{\mu}\partial_2\operatorname{div}\tilde{\mathbf{v}}\\
&\qquad - (\bar{\rho}+1)(\bar{\mathbf{v}} + \tilde{\mathbf{v}} + U_0)\cdot\nabla\tilde{v}_2 - (\bar{\rho}+1)(\bar{\mathbf{v}} + \tilde{\mathbf{v}})\cdot\nabla\bar{v}_2\\
&\qquad - \bar{\rho}\partial_1\bar{v}_2 + (\tilde{B}_1 + \bar{B}_1)\big[\partial_1(\tilde{B}_2 + \bar{B}_2) - \partial_2(\tilde{B}_1 + \bar{B}_1)\big],\\
&H_1(\bar{\mathbf{v}}, \bar{\rho}, \bar{\mathbf{B}}) = \mu\triangle\tilde{B}_1 - \partial_2\tilde{B}_2 + \partial_2\tilde{v}_2 - \partial_2\big[(\tilde{B}_2 + \bar{B}_2)(\tilde{v}_2 + \bar{v}_1) - (\tilde{B}_1 + \bar{B}_1)(\tilde{v}_2 + \bar{v}_2)\big],\\
&H_2(\bar{\mathbf{v}}, \bar{\rho}, \bar{\mathbf{B}}) = \mu\triangle\tilde{B}_2 + \partial_1\tilde{B}_2 - \partial_1\tilde{v}_2 + \partial_1\big[(\tilde{B}_2 + \bar{B}_2)(\tilde{v}_2 + \bar{v}_1) - (\tilde{B}_1 + \bar{B}_1)(\tilde{v}_2 + \bar{v}_2)\big].
\end{aligned} \tag{2.5}$$

The corresponding linearized system to system (2.4) can be written as

$$\partial_1\rho + \hat{\mathbf{v}}\cdot\nabla\rho + \operatorname{div}\mathbf{v} = f \quad \text{in } \Omega, \tag{2.6}$$

$$\partial_1 v_1 - \mu\Delta v_1 - \tilde{\mu}\partial_1\operatorname{div}\mathbf{v} + \partial_1\rho = g_1 \quad \text{in } \Omega, \tag{2.7}$$

$$\partial_1 v_2 - \mu\Delta v_2 - \tilde{\mu}\partial_2\operatorname{div}\mathbf{v} + \partial_2\rho - (\partial_1 B_2 - \partial_2 B_1) = g_2 \quad \text{in } \Omega, \tag{2.8}$$

$$-\nu\Delta B_1 - \partial_2 B_2 + \partial_2 v_2 = h_1 \quad \text{in } \Omega, \tag{2.9}$$

$$-\nu\Delta B_2 + \partial_1 B_2 - \partial_1 v_2 = h_2 \quad \text{in } \Omega, \tag{2.10}$$

$$\rho = \varrho_0 \quad \text{on } \Gamma_{\text{in}}, \tag{2.11}$$

$$\mathbf{v} = 0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \tag{2.12}$$

$$\mathbf{v}\cdot\mathbf{n} = \operatorname{curl}\mathbf{v} = 0 \quad \text{on } \Gamma_0, \tag{2.13}$$

$$\mathbf{B} = 0 \quad \text{on } \partial\Omega, \tag{2.14}$$

where $f \in W^{1,p}(\Omega)$, $\mathbf{g} = (g_1, g_2)$, $\mathbf{h} = (h_1, h_2) \in L^p(\Omega)$, $\hat{\mathbf{v}} \in W^{2,p}(\Omega)$, $\varrho_0 \in W^{1,p}(\Gamma_{\text{in}})$ are given functions satisfying $\hat{\mathbf{v}}\cdot\mathbf{n} = \operatorname{curl}\hat{\mathbf{v}} = 0$ on Γ_0 and $\|\hat{\mathbf{v}}\|_{2,p;\Omega}$ sufficiently small.

As the term $\hat{\mathbf{v}}\cdot\nabla\rho$ will result in a loss of regularity in $W^{1,p}(\Omega)$ in the right side of (2.6), it seems hard to obtain the strong solutions to the system (2.6)–(2.14) directly by the contraction mapping principle. Instead, we will first prove the weak solvability of the system (2.6)–(2.14).

Definition 2.3 If $(\mathbf{v}, \rho, \mathbf{B}) \in H^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ satisfies

$$(1) \quad -\int_{\Gamma_{\text{in}}} \varrho_0(s)\omega(0,s)\,ds + \int_\Omega \big[\omega\operatorname{div}\mathbf{v} - \rho\partial_1\omega - \rho\operatorname{div}(\omega\hat{\mathbf{v}})\big]\,dx = \int_\Omega f\omega\,dx, \tag{2.15}$$

for $\forall\omega \in C^1(\bar{\Omega})$ satisfying $\omega|_{\Gamma_{\text{out}}} = 0$;

$$\begin{aligned}
(2) \quad &\int_\Omega \big[\partial_1\mathbf{v}\cdot\boldsymbol{\varpi} - (\partial_1 B_2 - \partial_2 B_1)\varpi_2 - \rho\operatorname{div}\boldsymbol{\varpi}\big]\,dx\\
&\quad + \int_\Omega [\mu\nabla\mathbf{v} : \nabla\boldsymbol{\varpi} + \tilde{\mu}\operatorname{div}\mathbf{v}\operatorname{div}\boldsymbol{\varpi}]\,dx = \int_\Omega \mathbf{g}\varpi\,dx,
\end{aligned} \tag{2.16}$$

for $\forall\varpi = (\varpi_1, \varpi_2) \in \Xi$;

$$\begin{aligned} (3) \quad & \nu \int_\Omega \nabla B_1 \cdot \nabla\phi \, dx + \int_\Omega [-\partial_2 B_2 + \partial_2 v_2]\phi \, dx = \int_\Omega h_1 \phi \, dx, \\ & \nu \int_\Omega \nabla B_2 \cdot \nabla\phi \, dx + \int_\Omega [\partial_1 B_2 - \partial_1 v_2]\phi \, dx = \int_\Omega h_2 \phi \, dx, \end{aligned} \tag{2.17}$$

for $\forall\phi \in H_0^1(\Omega)$,

then we call $(\mathbf{v}, \rho, \mathbf{B})$ a weak solution to the system (2.6)–(2.14)

2.1 Solutions to the linearized system

In this section, we will first prove the weak solvability of the system (2.6)–(2.14), then by the theory of elliptic systems and the method of proving the L^p estimate of steady compressible Navier–Stokes system in [4], we can find that the weak solutions $(\mathbf{v}, \rho, \mathbf{B}) \in W^{2,p} \times W^{1,p} \times W^{2,p}$. The main result reads as follows.

Theorem 2.4 *Assume that* $\varrho_0 \in W^{1,p}(\Gamma_{\text{in}})$, $f \in W^{1,p}(\Omega)$, $\mathbf{h}$, $\mathbf{g} \in L^p(\Omega)$, $\hat{\mathbf{v}} \in W^{2,p}(\Omega)$ *satisfying* $\hat{\mathbf{v}} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$ *and* $\|\hat{\mathbf{v}}\|_{2,p;\Omega}$ *small enough, then system* (2.6)–(2.14) *admits a strong solution* $(\mathbf{v}, \rho, \mathbf{B})$ *with the following estimate*:

$$\|\rho\|_{1,p;\Omega} + \|\mathbf{v}\|_{2,p;\Omega} + \|\mathbf{B}\|_{2,p;\Omega} \le C\big(|\varrho_0|_{1,p;\Gamma_{\text{in}}} + \|f\|_{1,p;\Omega} + \|\mathbf{g}\|_{L^p(\Omega)} + \|\mathbf{h}\|_{L^p(\Omega)}\big), \tag{2.18}$$

here C *is a constant depending on* μ, $\tilde{\mu}$, ν *and* p.

In the case of the inhomogeneous boundary conditions on Γ_{in} and Γ_{out}, the total mass of the fluid is unknown, i.e. the L^1 norm of the density is unknown, thus we cannot obtain the L^2 norm of the density by the classical method of energy from the momentum equations. Instead, we will get the L^2 norm of ρ from the mass equation by using the following transformation $\Pi : \Omega \to \Omega$ to "straighten" the stream line:

$$\begin{cases} x_1 = \bar{x}_1, \\ x_2 = \bar{x}_2 + \int_0^{\bar{x}_1} \frac{\hat{v}_2}{1+\hat{v}_1}(s, x_2(s, \bar{x}_2))\, ds, \end{cases} \tag{2.19}$$

and obviously one has

$$\left\| \frac{\partial(\bar{x}_1, \bar{x}_2)}{\partial(x_1, x_2)} - I \right\|_{1,p;\Omega} \le \|\hat{\mathbf{v}}\|_{2,p;\Omega}. \tag{2.20}$$

Then direct computation shows that in the new coordinate system, system (2.6)–(2.14) is transformed to the following system:

$$\partial_1\rho + \operatorname{div}\mathbf{v} = \frac{1}{1+\hat{v}_1} f + \tilde{f}(\mathbf{v}) \quad \text{in } \Omega, \tag{2.21}$$

$$\partial_1 v_1 - \mu\Delta v_1 - \tilde{\mu}\partial_1 \operatorname{div}\mathbf{v} - \tilde{g}_1(\mathbf{v}) + \partial_1\rho = g + \hat{g}_1(\rho) \quad \text{in } \Omega, \tag{2.22}$$

$$\partial_1 v_2 - \mu\Delta v_2 - \tilde{\mu}\partial_2 \operatorname{div}\mathbf{v} - (\partial_1 B_2 - \partial_2 B_1) - \tilde{g}_2(\mathbf{v}) + \partial_2\rho = g + \hat{g}_2(\rho, \mathbf{B}) \quad \text{in } \Omega, \tag{2.23}$$

$$-\nu\Delta B_1 - \partial_2 B_2 + \tilde{h}_1(\mathbf{B}) + \partial_2 v_2 = h_1 + \hat{h}_1(\mathbf{v}) \quad \text{in } \Omega, \tag{2.24}$$

$$-\nu\Delta B_2+\partial_1 B_2+\tilde{h}_1(\mathbf{B})-\partial_1 v_2=h_2+\hat{h}_2(\mathbf{v})\quad \text{in } \Omega \tag{2.25}$$

$$\rho=\varrho_0\quad \text{on } \Gamma_{\text{in}}, \tag{2.26}$$

$$\mathbf{v}=0\quad \text{on } \Gamma_{\text{in}}\cup\Gamma_{\text{out}}, \tag{2.27}$$

$$\mathbf{v}\cdot\mathbf{n}=\operatorname{curl}\mathbf{v}=0\quad \text{on } \Gamma_0, \tag{2.28}$$

$$\mathbf{B}=0\quad \text{on } \partial\Omega, \tag{2.29}$$

where $\hat{\mathbf{g}}=(\hat{g}_1,\hat{g}_2)$, $\tilde{\mathbf{g}}=(\tilde{g}_1,\tilde{g}_2)$ and

$$\begin{aligned}
\tilde{f}(\mathbf{v})&=-\frac{1}{1+\hat{v}_1}\frac{\partial\bar{x}_2}{\partial x_1}\partial_{\bar{x}_2}v_1-\left(\frac{1}{1+\hat{v}_1}\frac{\partial\bar{x}_2}{\partial x_2}-1\right)\partial_{\bar{x}_2}v_2+\frac{\hat{v}_1}{1+\hat{v}_1}\partial_{\bar{x}_1}v_1,\\
\hat{g}_1(\rho)&=-\partial_{\bar{x}_2}\rho\frac{\partial\bar{x}_2}{\partial x_1},\\
\hat{g}_2(\rho,\mathbf{B})&=-\partial_{\bar{x}_2}\rho\left(\frac{\partial\bar{x}_2}{\partial x_2}-1\right)+\frac{\partial\bar{x}_2}{\partial x_1}\partial_{\bar{x}_2}B_2-\left(\frac{\partial\bar{x}_2}{\partial x_2}-1\right)\partial_{\bar{x}_2}B_2,\\
\tilde{g}_1(\mathbf{v})&=-\partial_{\bar{x}_2}v_1\frac{\partial\bar{x}_2}{\partial x_1}-2(\mu+\tilde{\mu})\partial_{\bar{x}_1\bar{x}_2}v_1\frac{\partial\bar{x}_2}{\partial x_1}-\partial_{\bar{x}_2}v_1\left[\mu\Delta_x\bar{x}_2+\tilde{\mu}\frac{\partial^2\bar{x}_2}{\partial x_1^2}\right]\\
&\quad-\tilde{\mu}\left[\partial_{\bar{x}_1\bar{x}_2}v_2\left(\frac{\partial\bar{x}_2}{\partial x_2}-1\right)+\partial_{\bar{x}_2}^2v_2\frac{\partial\bar{x}_2}{\partial x_1}\frac{\partial\bar{x}_2}{\partial x_2}+\partial_{\bar{x}_2}v_2\frac{\partial^2\bar{x}_2}{\partial x_1\partial x_2}\right]\\
&\quad-\left((\mu+\tilde{\mu})\left(\frac{\partial\bar{x}_2}{\partial x_1}\right)^2+\mu\left[\left(\frac{\partial\bar{x}_2}{\partial x_2}\right)^2-1\right]\right)\partial_{\bar{x}_2}^2v_1,\\
\tilde{g}_2(\mathbf{v})&=g_2-\partial_{\bar{x}_2}v_2\frac{\partial\bar{x}_2}{\partial x_1}-2\mu\partial_{\bar{x}_1\bar{x}_2}v_2\frac{\partial\bar{x}_2}{\partial x_1}-\partial_{\bar{x}_2}v_2\left[\mu\Delta_x\bar{x}_2+\tilde{\mu}\frac{\partial^2\bar{x}_2}{\partial x_2^2}\right]\\
&\quad-\tilde{\mu}\left[\partial_{\bar{x}_1\bar{x}_2}v_1\left(\frac{\partial\bar{x}_2}{\partial x_2}-1\right)+\partial_{\bar{x}_2}^2v_1\frac{\partial\bar{x}_2}{\partial x_1}\frac{\partial\bar{x}_2}{\partial x_2}+\partial_{\bar{x}_2}v_1\frac{\partial^2\bar{x}_2}{\partial x_1\partial_{x_2}}\right]\\
&\quad-\partial_{\bar{x}_2}^2v_2\left(\mu\left(\frac{\partial\bar{x}_2}{\partial x_1}\right)^2+(\mu+\tilde{\mu})\left[\left(\frac{\partial\bar{x}_2}{\partial x_2}\right)^2-1\right]\right),\\
\tilde{h}_1(\mathbf{B})&=-2\nu\partial_{\bar{x}_1\bar{x}_2}B_1\frac{\partial\bar{x}_2}{\partial x_1}-\nu\partial_{\bar{x}_2}B_1\Delta_x\bar{x}_2-\nu\left[\left(\frac{\partial\bar{x}_2}{\partial x_1}\right)^2+\left(\frac{\partial\bar{x}_2}{\partial x_2}\right)^2-1\right]\partial_{\bar{x}_2}^2B_1\\
&\quad+\left(\frac{\partial\bar{x}_2}{\partial x_2}-1\right)\partial_{\bar{x}_2}B_2,\\
\tilde{h}_2(\mathbf{B})&=-2\nu\partial_{\bar{x}_1\bar{x}_2}B_2\frac{\partial\bar{x}_2}{\partial x_1}-\nu\partial_{\bar{x}_2}B_2\Delta_x\bar{x}_2-\nu\left[\left(\frac{\partial\bar{x}_2}{\partial x_1}\right)^2+\left(\frac{\partial\bar{x}_2}{\partial x_2}\right)^2-1\right]\partial_{\bar{x}_2}^2B_2\\
&\quad+\frac{\partial\bar{x}_2}{\partial x_1}\partial_{\bar{x}_2}B_2,\\
\hat{h}_1(\mathbf{v})&=-\left(\frac{\partial\bar{x}_2}{\partial x_2}-1\right)\partial_{\bar{x}_2}v_2,\\
\hat{h}_2(\mathbf{v})&=-\frac{\partial\bar{x}_2}{\partial x_1}\partial_{\bar{x}_2}v_2.
\end{aligned}$$

Now the problem of the existence of solutions to the system (2.6)–(2.14) is equivalent to the problem of the solvability of system (2.21)–(2.29).

Lemma 2.5 *Assume that all the conditions in Theorem 2.4 are satisfied, then system (2.21)–(2.29) admits a weak solution* $(\mathbf{v},\rho,\mathbf{B})$ *with the following estimate*:

$$\begin{aligned}&\|\rho\|_{L^2(\Omega)}+\|\mathbf{v}\|_{1,2;\Omega}+\|\mathbf{B}\|_{1,2;\Omega}\\&\quad\le C\big(|\varrho_0|_{1,2;\Gamma_{\mathrm{in}}}+\|f\|_{L^2(\Omega)}+\|\mathbf{g}\|_{H^{-1}(\Omega)}+\|\mathbf{h}\|_{H^{-1}(\Omega)}\big),\end{aligned}\tag{2.30}$$

where C *is a constant depending on* μ, $\tilde{\mu}$, ν *and* p.

Proof To prove Lemma 2.5, we will first prove by the Leray–Schauder fixed point theorem the existence of viscous solutions to the following approximate system:

$$\partial_1\rho-\varepsilon\Delta\rho+\operatorname{div}\mathbf{v}=f+\tilde{f}(\mathbf{v})\quad\text{in }\Omega,\tag{2.31}$$

$$\partial_1 v_1-\mu\Delta v_1-\tilde{\mu}\partial_1\operatorname{div}\mathbf{v}+\tilde{g}_1(\mathbf{v})+\partial_1\rho=g+\hat{g}_1(\rho)\quad\text{in }\Omega,\tag{2.32}$$

$$\partial_1 v_2-\mu\Delta v_2-\tilde{\mu}\partial_2\operatorname{div}\mathbf{v}+\partial_2\rho-(\partial_1B_2-\partial_2B_1)+\tilde{g}_2(\mathbf{v})=g+\hat{g}_2(\rho,\mathbf{B})\quad\text{in }\Omega,\tag{2.33}$$

$$-\nu\Delta B_1-\partial_2B_2+\partial_2v_2+\tilde{h}_1(\mathbf{B})=h_1+\hat{h}_1(\mathbf{v})\quad\text{in }\Omega,\tag{2.34}$$

$$-\nu\Delta B_2+\partial_1B_2-\partial_1v_2+\tilde{h}_2(\mathbf{B})=h_2+\hat{h}_2(\mathbf{v})\quad\text{in }\Omega\tag{2.35}$$

$$\rho=\varrho_0\quad\text{on }\Gamma_{\mathrm{in}},\tag{2.36}$$

$$\frac{\partial\rho}{\partial\mathbf{n}}=0\quad\text{on }\Gamma_{\mathrm{out}}\cup\Gamma_0,\tag{2.37}$$

$$\mathbf{v}=0\quad\text{on }\Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{out}},\tag{2.38}$$

$$\mathbf{v}\cdot\mathbf{n}=\operatorname{curl}\mathbf{v}=0\quad\text{on }\Gamma_0,\tag{2.39}$$

$$\mathbf{B}=0\quad\text{on }\partial\Omega.\tag{2.40}$$

First we define the map $S:\mathbf{u}=(u_1,u_2)\in H^1(\Omega)\mapsto\rho$ by letting $\rho=\rho(\mathbf{u})$ the solution of the following boundary problem:

$$\partial_1\rho-\epsilon\Delta\rho=-\operatorname{div}\mathbf{u}+f+\tilde{f}(\mathbf{u})\quad\text{in }\Omega,\tag{2.41}$$

$$\rho=\varrho_0\quad\text{on }\Gamma_{\mathrm{in}},\tag{2.42}$$

$$\frac{\partial\rho}{\partial\mathbf{n}}=0\quad\text{on }\Gamma_{\mathrm{out}}\cup\Gamma_0.\tag{2.43}$$

By the theory of elliptic equations, there exists a solution $\rho\in H^1(\Omega)\cap H^2(D)$ satisfying

$$\|\rho\|_{1,2}\le C(\varepsilon)\big(\|\mathbf{u}\|_{1,2}+\|f\|_{L^2}+\|\rho_0\|_{1,2;\Gamma_{\mathrm{in}}}\big),\tag{2.44}$$

where $D\subset\Omega$ is any subset of Ω away from the corners. Then by the regular result in Remark 2.2, we have $\rho\in H^2(\Omega)$ and

$$\|\rho\|_{2,2}\le C(\varepsilon)\big(\|\mathbf{u}\|_{1,2}+\|f\|_{L^2}+\|\rho_0\|_{1,2;\Gamma_{\mathrm{in}}}\big).\tag{2.45}$$

Next, we define the map $\Lambda:\mathbf{u}=(u_1,u_2)\in H^1(\Omega)\mapsto\mathbf{B}=(B_1,B_2)$ by letting $\mathbf{B}=\mathbf{B}(\mathbf{u})$ be the solution to the following system:

$$-\nu\Delta B_1-\partial_2B_2+\tilde{h}_1(\mathbf{B})=\partial_2u_2+h_1+\hat{h}_1(\mathbf{u})\quad\text{in }\Omega,\tag{2.46}$$

$$-\nu\Delta B_2 + \partial_1 B_2 + \tilde{h}_2(\mathbf{B}) = \partial_1 u_2 + h_2 + \hat{h}_2(\mathbf{u}) \quad \text{in } \Omega, \tag{2.47}$$

$$\mathbf{B} = 0 \quad \text{on } \partial\Omega. \tag{2.48}$$

By the theory of elliptic systems, the solution $\mathbf{B} \in H^2(D)$, where $D \subset \Omega$ is any subset of Ω away from the corners, and the following estimate holds:

$$\|\mathbf{B}\|_{1,2} \leq C\big(\|\mathbf{u}\|_{1,2} + \|\mathbf{h}\|_{-1,2}\big). \tag{2.49}$$

To obtain the H^2 estimate of $\mathbf{B}$ in Ω, we perform the transformation $\Pi^{-1} : \Omega \to \Omega$ to system (2.46)–(2.48) to obtain

$$-\nu\Delta B_1 = \partial_2 B_2 + \partial_2 u_2 + h_1 + \hat{h}_1(\mathbf{u}) \quad \text{in } \Omega, \tag{2.50}$$

$$-\nu\Delta B_2 = -\partial_1 B_2 + \partial_1 u_2 + h_2 + \hat{h}_2(\mathbf{u}) \quad \text{in } \Omega, \tag{2.51}$$

$$\mathbf{B} = 0 \quad \text{on } \partial\Omega. \tag{2.52}$$

As the right side of (2.50) and (2.51) are all in $L^2(\Omega)$, by Lemma 2.1 and (2.49) we find that $\mathbf{B} \in H^2(\Omega)$ and

$$\|\mathbf{B}\|_{2,2} \leq C\big(\|\mathbf{u}\|_{1,2} + \|\mathbf{h}\|_{L^2}\big). \tag{2.53}$$

Finally, we define the operator $T : \mathbf{u} = (u_1, u_2) \in H^1(\Omega) \mapsto \mathbf{v} = (v_1, v_2) \in H^2(\Omega)$ by the following elliptic system:

$$\begin{aligned}
&\partial_1 v_1 - \mu\Delta v_1 - \tilde{\mu}\partial_1 \operatorname{div}\mathbf{v} + \tilde{g}_1(\mathbf{v}) = -\partial_1 S_1(\mathbf{u}) + g_1 + \hat{g}_1\big(S_1(\mathbf{u})\big) \quad \text{in } \Omega,\\
&\partial_1 v_2 - \mu\Delta v_2 - \tilde{\mu}\partial_2 \operatorname{div}\mathbf{v} + \tilde{g}_2(\mathbf{v})\\
&\quad = -\partial_2 S_1(\mathbf{u}) + \big(\partial_1 \Lambda_2(\mathbf{u}) - \partial_2 \Lambda_1(\mathbf{u})\big) + g_2 + \hat{g}_2\big(S_1(\mathbf{u}), \Lambda(\mathbf{u})\big) \quad \text{in } \Omega,\\
&\mathbf{v} = 0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}},\\
&\mathbf{v} \cdot \mathbf{n} = \operatorname{curl}\mathbf{v} = 0 \quad \text{on } \Gamma_0.
\end{aligned} \tag{2.54}$$

Since the term on the right side of (2.54) belongs to $L^2(\Omega)$, like the estimate (2.53), we first make a change of variables $\Pi^{-1} : \Omega \to \Omega$, then take the even extension of v_1 and the right side of (2.54), the odd extension of v_2 with respect to the line $x_2 = 0, 1$, respectively, and we find by (2.45) and (2.53) that $\mathbf{v} \in H^2(\Omega)$ and

$$\|\mathbf{v}\|_{2,2} \leq C(\varepsilon)\big(\|\mathbf{u}\|_{1,2} + \|f\|_{L^2} + \|\mathbf{h}\|_{L^2} + \|\mathbf{g}\|_{L^2}\big). \tag{2.55}$$

So $T : H^1 \to H^1$ is a compact mapping. Now we turn to the equation $\mathbf{v} = \sigma T\mathbf{v}$ i.e. $\mathbf{v}$ satisfies the following problem:

$$\partial_1 v_1 - \mu\Delta v_1 - \tilde{\mu}\partial_1 \operatorname{div}\mathbf{v} + \tilde{g}_1(\mathbf{v}) = \sigma\big[-\partial_1 S_1(\mathbf{v}) + g_1 + \hat{g}_1\big(S_1(\mathbf{v})\big)\big] \quad \text{in } \Omega, \tag{2.56}$$

$$\begin{aligned}
&\partial_1 v_2 - \mu\Delta v_2 - \tilde{\mu}\partial_2 \operatorname{div}\mathbf{v} + \tilde{g}_2(\mathbf{v})\\
&\quad = \sigma\big[-\partial_2 S_1(\mathbf{v}) + \big(\partial_1 \Lambda_2(\mathbf{v}) - \partial_2 \Lambda_1(\mathbf{v})\big) + g_2 + \hat{g}_2\big(S_1(\mathbf{v}), \Lambda(\mathbf{v})\big)\big] \quad \text{in } \Omega,
\end{aligned} \tag{2.57}$$

$$\mathbf{v} = 0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \tag{2.58}$$

$$\mathbf{v} \cdot \mathbf{n} = \operatorname{curl} \mathbf{v} = 0 \quad \text{on } \Gamma_0. \tag{2.59}$$

By the Leray–Schauder fixed point theorem, to prove the existence of fixed point to the system (2.54), one only needs to obtain the uniform-in-σ estimate of system (2.56)–(2.59) in $H^1(\Omega)$. To this end, we first multiply ρ, v_1, v_2, B_1, B_2 to (2.31), (2.34)–(2.35), (2.56)–(2.57), respectively, and integrate by parts to obtain

$$\begin{aligned}
&\sigma \frac{1}{2} \int_0^1 \rho^2(1,x_2)\,dx_2 - \sigma\varepsilon \int_0^1 \rho_0(x_2)\partial_1\rho(0,x_2)\,dx_2 \\
&\quad + \varepsilon\sigma \int_\Omega |\nabla\rho|^2\,dx + \int_\Omega \big[\mu|\nabla\mathbf{v}|^2 + \tilde{\mu}(\operatorname{div}\mathbf{v})^2\big]\,dx \\
&\quad + \sigma\nu \int_\Omega |\nabla\mathbf{B}|^2\,dx - \sigma \int_\Omega v_2[\partial_1 B_2 - \partial_2 B_1]\,dx \\
&\quad + \sigma \int_\Omega B_1[-\partial_2 B_2 + \partial_2 v_2]\,dx + \sigma \int_\Omega B_2[\partial_1 B_2 - \partial_1 v_2]\,dx \\
&= \sigma \frac{1}{2} \int_0^1 \rho_0^2(x_2)\,dx_2 + \sigma \int_\Omega \big\{\rho\big[f + \tilde{f}(\mathbf{v})\big] + v_1\big[g + \hat{g}_1(\rho)\big] + v_2\big[g + \hat{g}_2(\rho,\mathbf{B})\big] \\
&\quad + B_1\big[-\tilde{h}_1(\mathbf{B}) + h_1 + \hat{h}_1(\mathbf{v})\big] + B_2\big[-\tilde{h}_2(\mathbf{B}) + h_2 + \hat{h}_2(\mathbf{v})\big]\big\}\,dx \\
&\quad - \int_\Omega \big[v_1\tilde{g}_1(\mathbf{v}) + v_2\tilde{g}_2(\mathbf{v})\big] \\
&\le C\big(\|\rho\|_{L^2}\|f\|_{L^2} + \|\hat{\mathbf{v}}\|_{2,p}\big[\|\mathbf{v}\|_{1,2}\|\rho\|_{L^2} + \|\mathbf{v}\|_{1,2}^2 + \|\mathbf{B}\|_{1,2}^2 + \|\mathbf{B}\|_{1,2}\|\mathbf{v}\|_{1,2}\big] \\
&\quad + \|\mathbf{g}\|_{H^{-1}} + \|\mathbf{h}\|_{H^{-1}}\big).
\end{aligned} \tag{2.60}$$

Using the boundary condition and the divergence free condition of $\mathbf{B}$, we have

$$\begin{aligned}
&-\int_\Omega v_2[\partial_1 B_2 - \partial_2 B_1]\,dx + \int_\Omega B_1[-\partial_2 B_2 + \partial_2 v_2]\,dx + \int_\Omega B_2[\partial_1 B_2 - \partial_1 v_2]\,dx \\
&\quad = \int_\Omega [-B_1\partial_2 B_2 + B_2\partial_1 B_2]\,dx = \int_\Omega (B_1\partial_1 B_1 + B_2\partial_2 B_2)\,dx = 0.
\end{aligned} \tag{2.61}$$

To control the term $\varepsilon\sigma \int_0^1 \rho_0(x_2)\partial_1\rho(0,x_2)\,dx_2$ on the left side of (2.60), we multiply (2.31) with $\partial_1\rho$ to obtain

$$\begin{aligned}
&\int_\Omega \partial_1\rho(\partial_1\rho - \varepsilon\Delta\rho + \operatorname{div}\mathbf{v}) \\
&\quad = \int_\Omega (\partial_1\rho)^2\,dx + \frac{\varepsilon}{2}\int_0^1 (\partial_1\rho)^2(0,x_2)\,dx_2 + \frac{\varepsilon}{2}\int_0^1 (\partial_2\rho)^2(1,x_2)\,dx_2 \\
&\qquad - \frac{\varepsilon}{2}\int_0^1 (\partial_2\rho_0)^2\,dx_2 + \int_\Omega \partial_1\rho \operatorname{div}\mathbf{v} \\
&\quad = \int_\Omega \partial_1\rho\big(f + \tilde{f}(\mathbf{v})\big)\,dx,
\end{aligned} \tag{2.62}$$

which implies by Young's inequality that

$$\int_\Omega (\partial_1\rho)^2\,dx + \varepsilon\int_0^1 (\partial_1\rho)^2(0,x_2)\,dx_2 + \varepsilon\int_0^1 (\partial_2\rho)^2(1,x_2)\,dx_2 \\ \le C\big(\|\mathbf{v}\|_{1,2}^2 + \|f\|_{L^2}^2 + |\varrho_0|_{1,2;\Gamma_{\text{in}}}\big). \tag{2.63}$$

Moreover, since

$$\rho(x_1,x_2) = \varrho_0 + \int_0^{x_1} \partial_1\rho(s,x_2)\,ds, \tag{2.64}$$

we have from (2.63)

$$\|\rho\|_{L^2} \le C\big(|\varrho_0|_{1,2;\Gamma_{\text{in}}} + \|\mathbf{v}\|_{1,2} + \|f\|_{L^2}\big). \tag{2.65}$$

Combining (2.60)–(2.65) and the smallness assumption of $\|\hat{\mathbf{v}}\|_{2,p}$, we arrive at the following estimate:

$$\sigma\int_0^1 \rho^2(1,x_2)\,dx_2 + \varepsilon\sigma\int_0^1 (\partial_1\rho)^2(0,x_2)\,dx_2 \\ + \sigma\varepsilon\int_0^1 (\partial_2\rho)^2(1,x_2)\,dx_2 + \sigma\varepsilon\int_\Omega |\nabla\rho|^2\,dx \\ + \int_\Omega \big[\mu|\nabla\mathbf{v}|^2 + \tilde{\mu}(\operatorname{div}\mathbf{v})^2\big]\,dx + \nu\sigma\int_\Omega |\nabla\mathbf{B}|^2\,dx \\ \le C\big(\|f\|_{L^2}^2 + \|\mathbf{g}\|_{H^{-1}}^2 + \|\mathbf{h}\|_{H^{-1}}^2 + |\varrho_0|_{1,2;\Gamma_{\text{in}}}^2\big). \tag{2.66}$$

Consequently, we have

$$\|\mathbf{v}\|_{1,2} \le C\big(\|f\|_{L^2} + \|\mathbf{g}\|_{H^{-1}} + \|\mathbf{h}\|_{H^{-1}} + \|\varrho_0\|_{L^2}\big) \tag{2.67}$$

and

$$\|\rho\|_{L^2(\Omega)} + \sqrt{\varepsilon}\big\|\partial_1\rho(0,\cdot)\big\|_{L^2} + \sqrt{\varepsilon}\|\nabla\rho\|_{L^2} + \|\mathbf{v}\|_{1,2;\Omega} + \|\mathbf{B}\|_{1,2;\Omega} \\ \le C\big(|\varrho_0|_{1,2;\Gamma_{\text{in}}} + \|f\|_{L^2(\Omega)} + \|\mathbf{g}\|_{H^{-1}} + \|\mathbf{h}\|_{H^{-1}}\big), \tag{2.68}$$

where C is a constant depending on μ, $\tilde{\mu}$, ν and p.

Thus by the Leray–Schauder fixed point theorem, for any given $\varepsilon > 0$, there is a sequence of solutions $(\rho_\varepsilon, \mathbf{v}_\varepsilon, \mathbf{B}_\epsilon) \in H^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)$ to the system (2.54). Taking a limit with respect to ε, we obtain $\rho \in L^2(\Omega)$, $\mathbf{v} \in \Xi$ and $\mathbf{B} \in H^1(\Omega)$, such that

$$\rho_\varepsilon \rightharpoonup \rho \quad \text{weakly in } L^2(\Omega);$$
$$\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v} \quad \text{weakly in } H^1(\Omega);$$
$$\mathbf{B}_\varepsilon \rightharpoonup \mathbf{B} \quad \text{weakly in } H^1(\Omega),$$

and for $\forall\varphi \in H^1(\Omega)$

$$(\varepsilon\Delta\rho_\varepsilon, \varphi) = \varepsilon\int_0^1 \partial_1\rho(0,x_2)\varphi(0,x_2)\,dx_2 - \int_\Omega \varepsilon\nabla\rho_\varepsilon\nabla\varphi\,dx \to 0, \quad \text{as } \varepsilon \to 0. \tag{2.69}$$

Thus $(\rho, \mathbf{v}, \mathbf{B}) \in L^2(\Omega) \times \Xi \times H^1(\Omega)$ is a weak solution to system (2.21)–(2.29) satisfying

$$\|\rho\|_{L^2(\Omega)} + \|\mathbf{v}\|_{1,2;\Omega} + \|\mathbf{B}\|_{1,2;\Omega} \leq C\big(|\varrho_0|_{1,2;\hat{\Gamma}_{\mathrm{in}}} + \|f\|_{L^2(\hat{\Omega})} + \|\mathbf{g}\|_{H^{-1}} + \|\mathbf{h}\|_{H^{-1}}\big). \tag{2.70}$$

□

Taking the transformation Π^{-1} to the system (2.21)–(2.29), we can easily have the following corollary.

Corollary 2.6 *Assume that all the conditions in Theorem 2.4 are satisfied, then there exists a weak solution* $(\mathbf{v}, \rho, \mathbf{B})$ *to system* (2.6)–(2.14), *with the following estimate*:

$$\begin{aligned} &\|\rho\|_{L^2(\Omega)} + \|\mathbf{v}\|_{1,2;\Omega} + \|\mathbf{B}\|_{1,2;\Omega} \\ &\quad \leq C\big(|\varrho_0|_{1,2;\Gamma_{\mathrm{in}}} + \|f\|_{L^2(\Omega)} + \|\mathbf{g}\|_{H^{-1}(\Omega)} + \|\mathbf{h}\|_{H^{-1}(\Omega)}\big), \end{aligned} \tag{2.71}$$

where C is a constant depending on μ, $\tilde{\mu}$, ν and p.

Proof of Theorem 2.4 Let $(\mathbf{v}, \rho, \mathbf{B})$ be a weak solution to the system (2.6)–(2.14), then $\mathbf{B}$ satisfies the following elliptic system in the weak sense:

$$\begin{cases} -\nu\Delta B_1 = \partial_2 B_2 - \partial_2 v_2 + h_1 & \text{in } \Omega, \\ -\nu\Delta B_2 = -\partial_1 B_2 + \partial_1 v_2 + h_2 & \text{in } \Omega, \\ \mathbf{B} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.72}$$

As $\mathbf{v}, \mathbf{B} \in H^1$, the right side of equations in system (2.72) are in L^2. By Lemma 2.1, we have $\mathbf{B} \in H^2(\Omega)$ and

$$\|\mathbf{B}\|_{2,2;\Omega} \leq C\big(\|\mathbf{v}\|_{1,2;\Omega} + \|\mathbf{B}\|_{1,2;\Omega} + \|\mathbf{h}\|_{L^2(\Omega)}\big). \tag{2.73}$$

Then Sobolev imbedding inequality implies that $\forall 1 < p < \infty$,

$$\|\mathbf{B}\|_{1,p;\Omega} \leq C\big(\|\mathbf{v}\|_{1,2;\Omega} + \|\mathbf{B}\|_{1,2;\Omega} + \|\mathbf{h}\|_{L^2(\Omega)}\big). \tag{2.74}$$

By (2.74) we have $\partial_1 B_2 - \partial_2 B_1 \in L^p(\Omega)$. Then the regularity result of the linearized system of compressible Navier–Stokes system in Theorem 2.5 in [4] implies that

$$\|\rho\|_{1,p;\Omega} + \|\mathbf{v}\|_{2,p;\Omega} \leq C\big(\|f\|_{1,p;\Omega} + \|\mathbf{g}\|_{p;\Omega} + \|\mathbf{B}\|_{1,p;\Omega} + |\varrho_0|_{1,p;\Gamma_{\mathrm{in}}}\big). \tag{2.75}$$

Now we back to the system (2.72), the right side of equations in system (2.72) are in $L^p(\Omega)$. By Lemma 2.1, we have $\mathbf{B} \in W^{2,p}(\Omega)$ and

$$\|\mathbf{B}\|_{2,p;\Omega} \leq C\big(\|\mathbf{v}\|_{1,p;\Omega} + \|\mathbf{B}\|_{1,p;\Omega} + \|\mathbf{h}\|_{L^p(\Omega)}\big). \tag{2.76}$$

Finally, estimate (2.18) follows from (2.71), (2.74), (2.75) and (2.76) immediately. □

3 Proof of Theorem 1.1

In this section, we shall prove the existence of solutions to the nonlinear system (2.4). As the compressible condition results in a loss of regularity in the mass equation (the term $\mathbf{v}\cdot\nabla\rho$), we will first prove the weak solvability to the nonlinear system from a solution sequence $\{(\rho_n,\mathbf{v}_n,\mathbf{B}_n)\}_{n=1}^{\infty}$ which are the solutions of the following system:

$$\begin{aligned}
&\partial_1\bar{\rho}^{n+1}+\left(\bar{\mathbf{v}}^n+\tilde{\mathbf{v}}\right)\cdot\nabla\bar{\rho}^{n+1}+\operatorname{div}\bar{\mathbf{v}}^{n+1}=R\left(\bar{\mathbf{v}}^n,\bar{\rho}^n\right)\quad\text{in }\Omega,\\
&\partial_1\bar{v}_1^{n+1}-\mu\Delta\bar{v}_1^{n+1}-\tilde{\mu}\partial_1\operatorname{div}\bar{\mathbf{v}}^{n+1}+\partial_1\bar{\rho}^{n+1}=K_1\left(\bar{\mathbf{v}}^n,\bar{\rho}^n,\bar{\mathbf{B}}^n\right)\quad\text{in }\Omega,\\
&\partial_1\bar{v}_2^{n+1}-\mu\Delta\bar{v}_2^{n+1}-\tilde{\mu}\partial_2\operatorname{div}\bar{\mathbf{v}}^{n+1}+\partial_2\bar{\rho}^{n+1}-\left(\partial_1\bar{B}_2^{n+1}-\partial_2\bar{B}_1^{n+1}\right)\\
&\quad=K_2\left(\bar{\mathbf{v}}^n,\bar{\rho}^n,\bar{\mathbf{B}}^n\right)\quad\text{in }\Omega,\\
&-\nu\Delta\bar{B}_1^{n+1}-\partial_2\bar{B}_2^{n+1}+\partial_2\bar{v}_2^{n+1}=H_1\left(\bar{\mathbf{v}}^n,\bar{\rho}^n,\bar{\mathbf{B}}^n\right)\quad\text{in }\Omega,\\
&-\nu\Delta\bar{B}_2^{n+1}+\partial_1\bar{B}_2^{n+1}-\partial_1\bar{v}_2^{n+1}=H_2\left(\bar{\mathbf{v}}^n,\bar{\rho}^n,\bar{\mathbf{B}}^n\right)\quad\text{in }\Omega,\\
&\bar{\mathbf{v}}^{n+1}=0\quad\text{on }\Gamma_{\text{in}}\cup\Gamma_{\text{out}},\\
&\bar{\mathbf{v}}^{n+1}\cdot\mathbf{n}=\operatorname{curl}\bar{\mathbf{v}}^{n+1}=0\quad\text{on }\Gamma_0,\\
&\bar{\mathbf{B}}^{n+1}=0\quad\text{on }\partial\Omega,\\
&\bar{\rho}^{n+1}=\rho_0-1\quad\text{on }\Gamma_0,
\end{aligned}\tag{3.1}$$

where R, K and H are defined in (2.5) and $(\mathbf{v}^0,\rho^0,\mathbf{B}^0)=(0,0,0)$. By Theorem 2.4, there is a solution sequence $\{(\mathbf{v}^n,\rho^n,\mathbf{B}^n)\}$ in $W^{2,p}(\Omega)\times W^{1,p}(\Omega)\times W^{2,p}(\Omega)$ to the system (3.1). More precisely, one has the following.

Lemma 3.1 *Let $\{(\mathbf{v}^n,\rho^n,\mathbf{B}^n)\}$ be the solution sequence to system (3.1), then it is a Cauchy sequence in $H^1(\Omega)\times L^2(\Omega)\times H^1(\Omega)$ satisfying*

$$\left\|\mathbf{v}^n\right\|_{2,p;\Omega}+\left\|\rho^n\right\|_{1,p;\Omega}+\left\|\mathbf{B}^n\right\|_{2,p;\Omega}\le M\left(\|\tilde{\mathbf{v}}\|_{2,p;\Omega}+|\rho_0-1|_{1,p;\Gamma_0}+\|\tilde{\mathbf{B}}\|_{2,p;\Omega}\right),\tag{3.2}$$

where $M>0$ is a constant depending only on p.

Proof Let $\{(\mathbf{v}^n,\rho^n,\mathbf{B}^n)\}$ be the solution sequence to system (3.1), then Theorem 2.4 implies that $\{(\mathbf{v}^n,\rho^n,\mathbf{B}^n)\}\in W^{2,p}(\Omega)\times W^{1,p}(\Omega)\times W^{2,p}(\Omega)$ and

$$\begin{aligned}
&\left\|\mathbf{v}^{n+1}\right\|_{2,p;\Omega}+\left\|\rho^{n+1}\right\|_{1,p;\Omega}+\left\|\mathbf{B}^{n+1}\right\|_{2,p;\Omega}\\
&\quad\le C_1\left(\left\|R\left(\mathbf{v}^n,\rho^n\right)\right\|_{1,p;\Omega}+\left\|K\left(\mathbf{v}^n,\rho^n,\mathbf{B}^n\right)\right\|_{p;\Omega}\right.\\
&\qquad\left.+\left\|H\left(\mathbf{v}^n,\rho^n,\mathbf{B}^n\right)\right\|_{p;\Omega}+|\rho_0-1|_{1,p;\Gamma_{\text{in}}}\right),
\end{aligned}\tag{3.3}$$

where the constant $C_1=C_1(\mu,\lambda,p,\kappa)$.

To prove (3.2), we will use the method of induction.

First of all, let

$$L_k=\left\|\mathbf{v}^k\right\|_{2,p;\Omega}+\left\|\rho^k\right\|_{1,p;\Omega}+\left\|\mathbf{B}^k\right\|_{2,p;\Omega}$$

and

$$\Upsilon = \|\tilde{\mathbf{v}}\|_{2,p;\Omega} + \|\tilde{\mathbf{B}}\|_{2,p;\Omega} + |\rho_0 - 1|_{1,p;\Gamma_{\text{in}}}.$$

When $n = 1$, since $(\mathbf{u}^0, \rho^0, \theta^0) = (0,0,0)$, from (3.3) and the definition of R, K, H we obviously have

$$L_1 \le C_1 \Upsilon.$$

Now taking $M = 3C_1$, and assuming that, for any $1 \le k \le n$,

$$L_k \le M\Upsilon, \tag{3.4}$$

then by (3.3), one has

$$\begin{aligned} L_{n+1} &\le C_1 \Big(\big\| R\big(\mathbf{v}^n, \rho^n\big) \big\|_{1,p;\Omega} + \big\| K\big(\mathbf{v}^n, \rho^n, \mathbf{B}^n\big) \big\|_{p;\Omega} \\ &\quad + \big\| H\big(\mathbf{v}^n, \rho^n, \mathbf{B}^n\big) \big\|_{p;\Omega} + |\rho_0 - 1|_{1,p;\Gamma_{\text{in}}} \Big) \\ &\le C_1 \big(10 L_n^2 + 10 L_n \Upsilon + \Upsilon \big) \\ &\le C_1 \big(10 M^2 \Upsilon^2 + 10 M \Upsilon^2 + \Upsilon \big); \end{aligned} \tag{3.5}$$

by the smallness assumption of Υ, we obtain

$$L_{n+1} \le M\Upsilon, \tag{3.6}$$

consequently we can obtain (3.2).

Next, straightforward calculation shows that $(\rho^{n+1} - \rho^n, \mathbf{v}^{n+1} - \mathbf{v}^n, \mathbf{B}^{n+1} - \mathbf{B}^n)$ satisfy the following system:

$$\begin{cases} \partial_1\big(\rho^{n+1} - \rho^n\big) + \big(\mathbf{v}^n + \tilde{\mathbf{v}}\big) \cdot \nabla\big(\rho^{n+1} - \rho^n\big) + \operatorname{div}\big(\mathbf{v}^{n+1} - \mathbf{v}^n\big) \\ \quad = R\big(\mathbf{v}^n, \rho^n\big) - R\big(\mathbf{v}^{n-1}, \rho^{n-1}\big) - \big(\mathbf{v}^n - \mathbf{v}^{n-1}\big) \cdot \nabla \rho^n \quad \text{in } \Omega, \\ \partial_1\big(v_1^{n+1} - v_1^n\big) + \partial_1\big(\rho^{n+1} - \rho^n\big) - \mu \Delta\big(v_1^{n+1} - v_1^n\big) - \tilde{\mu} \partial_1 \operatorname{div}\big(\mathbf{v}^{n+1} - \mathbf{v}^n\big) \\ \quad = K_1\big(\mathbf{v}^n, \rho^n, \mathbf{B}^n\big) - K_1\big(\mathbf{v}^{n-1}, \rho^{n-1}, \mathbf{B}^{n-1}\big) \quad \text{in } \Omega, \\ \partial_1\big(v_2^{n+1} - v_2^n\big) + \partial_2\big(\rho^{n+1} - \rho^n\big) - \big[\partial_1\big(B_2^{n+1} - B_2^n\big) - \partial_2\big(B_1^{n+1} - B_1^n\big)\big] - \mu \Delta\big(v_2^{n+1} - v_2^n\big) \\ \quad - \tilde{\mu} \partial_2 \operatorname{div}\big(\mathbf{v}^{n+1} - \mathbf{v}^n\big) = K_2\big(\mathbf{v}^n, \rho^n, \mathbf{B}^n\big) - K_2\big(\mathbf{v}^{n-1}, \rho^{n-1}, \mathbf{B}^{n-1}\big) \quad \text{in } \Omega, \\ -\nu \Delta\big(B_1^{n+1} - B_1^n\big) - \partial_2\big(B_2^{n+1} - B_2^n\big) + \partial_2\big(v_2^{n+1} - v_2^n\big) \\ \quad = H_1\big(\mathbf{v}^n, \rho^n, \mathbf{B}^n\big) - H_1\big(\mathbf{v}^{n-1}, \rho^{n-1}, \mathbf{B}^{n-1}\big) \quad \text{in } \Omega, \\ -\nu \Delta\big(B_2^{n+1} - B_2^n\big) + \partial_1\big(B_2^{n+1} - B_2^n\big) - \partial_1\big(v_2^{n+1} - v_2^n\big) \\ \quad = H_2\big(\mathbf{v}^n, \rho^n, \mathbf{B}^n\big) - H_2\big(\mathbf{v}^{n-1}, \rho^{n-1}, \mathbf{B}^{n-1}\big) \quad \text{in } \Omega, \\ \big(\rho^{n+1} - \rho^n\big)\big|_{\Gamma_{\text{in}}} = 0, \\ \big(\mathbf{v}^{n+1} - \mathbf{v}^n\big)\big|_{\Gamma_{\text{in}} \cup \Gamma_{\text{out}}} = 0, \\ \big(\mathbf{v}^{n+1} - \mathbf{v}^n\big) \cdot \mathbf{n}\big|_{\Gamma_0} = \operatorname{curl}\big(\mathbf{v}^{n+1} - \mathbf{v}^n\big)|_{\Gamma_0} = 0, \\ \big(\mathbf{B}^{n+1} - \mathbf{B}^n\big)\big|_{\partial\Omega} = 0. \end{cases} \tag{3.7}$$

By Corollary 2.6 and (3.2) we can see that

$$\begin{aligned}
&\left\|\mathbf{v}^{n+1}-\mathbf{v}^{n}\right\|_{1,2;\Omega}+\left\|\rho^{n+1}-\rho^{n}\right\|_{2;\Omega}+\left\|\mathbf{B}^{n+1}-\mathbf{B}^{n}\right\|_{1,2;\Omega}\\
&\quad\le C\Big[\left\|R(\mathbf{v}^{n},\rho^{n})-R(\mathbf{v}^{n-1},\rho^{n-1})\right\|_{L^2}+\left\|\mathbf{K}(\mathbf{v}^{n},\rho^{n},\mathbf{B}^{n})-\mathbf{K}(\mathbf{v}^{n-1},\rho^{n-1},\mathbf{B}^{n-1})\right\|_{H^{-1}}\\
&\qquad+\left\|\mathbf{H}(\mathbf{v}^{n},\rho^{n},\mathbf{B}^{n})-\mathbf{H}(\mathbf{v}^{n-1},\rho^{n-1},\mathbf{B}^{n-1})\right\|_{H^{-1}}+\left\|(\mathbf{v}^{n}-\mathbf{v}^{n-1})\cdot\nabla\rho^{n}\right\|_{L^2}\Big]\\
&\quad\le C(L_n+L_{n-1}+\Upsilon)\left(\left\|\mathbf{v}^{n}-\mathbf{v}^{n-1}\right\|_{1,2;\Omega}+\left\|\rho^{n}-\rho^{n-1}\right\|_{2;\Omega}+\left\|\mathbf{B}^{n}-\mathbf{B}^{n-1}\right\|_{1,2;\Omega}\right)\\
&\quad\le\frac{1}{2}\left(\left\|\mathbf{v}^{n}-\mathbf{v}^{n-1}\right\|_{1,2;\Omega}+\left\|\rho^{n}-\rho^{n-1}\right\|_{2;\Omega}+\left\|\mathbf{B}^{n}-\mathbf{B}^{n-1}\right\|_{1,2;\Omega}\right),
\end{aligned}$$

consequently we find that $\{(\mathbf{v}^n,\rho^n,\mathbf{B}^n)\}$ is a Cauchy sequence in $H^1(\Omega)\times L^2(\Omega)\times H^1(\Omega)$. □

Proof of Theorem 1.1 Taking a limit in the solution sequence $\{(\mathbf{v}^n,\rho^n,\mathbf{B}^n)\}$, by Lemma 3.1, there exists a solution $(\mathbf{v},\rho,\mathbf{B})\in W^{2,p}(\Omega)\times W^{1,p}(\Omega)\times W^{2,p}(\Omega)$ to the system (2.4) satisfying the estimate (3.2).

On the other hand, assume that $(\mathbf{v},\rho,\mathbf{B})$ and $(\hat{\mathbf{v}},\hat{\rho},\hat{\mathbf{B}})$ are two solutions to the system (1.5)–(1.14), then similar to Lemma 3.1, we have

$$\begin{aligned}
&\|\mathbf{v}-\hat{\mathbf{v}}\|_{1,2;\Omega}+\|\rho-\hat{\rho}\|_{2;\Omega}+\|\mathbf{B}-\hat{\mathbf{B}}\|_{1,2;\Omega}\\
&\quad\le\frac{1}{2}\left(\|\mathbf{v}-\hat{\mathbf{v}}\|_{1,2;\Omega}+\|\rho-\hat{\rho}\|_{2;\Omega}+\|\mathbf{B}-\hat{\mathbf{B}}\|_{1,2;\Omega}\right),
\end{aligned}$$

i.e. $(\mathbf{v},\rho,\mathbf{B})=(\hat{\mathbf{v}},\hat{\rho},\hat{\mathbf{B}})$. Theorem 1.1 is thus proved. □

Acknowledgements
Not applicable.

Funding
This work is supported by NNSFC 11871147.

Abbreviation
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The author declares that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
All authors read and approved the final manuscript.

Publisher's Note

Received: 13 September 2018 Accepted: 2 December 2018 Published online: 06 December 2018

References

1. Chen, G.-Q., Wang, D.: Global solution of nonlinear magnetohydrodynamics with large initial data. J. Differ. Equ. **182**, 344–376 (2002)

2. Fan, J., Jiang, S., Nakamura, G.: Vanishing shear viscosity limit in the magnetohydrodynamic equations. Commun. Math. Phys. **270**(3), 691–708 (2007)
3. Grisvard, P.: Elliptica Problems in Nonsmooth Domains. Pitman, Boston (1985)
4. Guo, Y., Jiang, S., Zhou, C.: Steady viscous compressible channel flows. SIAM J. Math. Anal. **47**(5), 3648–3670 (2015)
5. Hu, X., Wang, D.: Global solutions to the three-dimensional full compressible magnetohydrodynamic flows. Commun. Math. Phys. **283**, 255–284 (2008)
6. Hu, X., Wang, D.: Low Mach number limit of viscous compressible magnetohydrodynamic flows. SIAM J. Math. Anal. **41**, 1272–1294 (2009)
7. Hu, X., Wang, D.: Global existence and large-time behavior of solutions to the three dimensional equations of compressible magnetohydrodynamic flows. Arch. Ration. Mech. Anal. **197**, 203–238 (2010)
8. Jiang, S., Ju, Q., Li, F.: Incompressible limit of the compressible magnetohydrodynamic equations with vanishing viscosity coefficients. SIAM J. Math. Anal. **42**(6), 2539–2553 (2010)
9. Jiang, S., Ju, Q., Li, F.: Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions. Commun. Math. Phys. **297**(2), 371–400 (2010)
10. Kawashima, S.: Smooth global solutions for two-dimensional equations of electromagnetofluid dynamics. Jpn. J. Appl. Math. **1**, 207–222 (1984)
11. Li, F., Yu, H.: Optimal decay rate of classical solutions to the compressible magnetohydrodynamic equation. Proc. R. Soc. Edinb. A **141**, 109–126 (2011)
12. Piasecki, T.: On an inhomogeneous slip-inflow boundary value problem for a steady flow of a viscous compressible fluid in a cylindrical domain. J. Differ. Equ. **248**, 2171–2198 (2010)
13. Piasecki, T., Pokorný, M.: Strong solutions to the Navier–Stokes–Fourier system with slip-inflow boundary conditions. Z. Angew. Math. Mech. **94**, 1035–1057 (2014)
14. Plotnikov, P.I., Ruban, E.V., Sokolowski, J.: Inhomogeneous boundary value problems for compressible Navier–Stokes equations: well-posedness and sensitivity analysis. SIAM J. Math. Anal. **40**, 1152–1200 (2008)
15. Plotnikov, P.I., Ruban, E.V., Sokolowski, J.: Inhomogeneous boundary value problems for compressible Navier–Stokes an transport equations. J. Math. Pures Appl. **92**, 113–162 (2009)
16. Wang, D.: Large solutions to the initial-boundary value problem for planar magnetohydrodynamics. SIAM J. Appl. Math. **63**, 1424–1441 (2003)
17. Yang, Y.-F., Gu, X., Dou, C.: Existence and uniqueness of steady solutions to the magnetohydrodynamic equations of compressible flows. J. Math. Anal. Appl. **409**(2), 846–850 (2014)
18. Zhou, C.H.: Existence of weak solutions to the three-dimensional steady compressible magnetohydrodynamic equations. Acta Math. Sin. Engl. Ser. **28**(12), 2431–2442 (2012)