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Multiplicity of solutions for a quasilinear elliptic equation with (p,q)-Laplacian and critical exponent on $\mathbb{R}^N$

Chen Huang[1], Gao Jia[2*] and Tiansi Zhang[2]

*Correspondence: gaojia89@163.com
[2]College of Science, University of Shanghai for Science and Technology, Shanghai, China
Full list of author information is available at the end of the article

Abstract

The multiplicity of solutions for a (p,q)-Laplacian equation involving critical exponent

$$-\Delta_p u - \Delta_q u = \lambda V(x)|u|^{k-2}u + K(x)|u|^{p^*-2}u, \quad x \in \mathbb{R}^N$$

is considered. By variational methods and the concentration–compactness principle, we prove that the problem possesses infinitely many weak solutions with negative energy for $\lambda \in (0,\lambda^*)$. Moreover, the existence of infinitely many solutions with positive energy is also given for all $\lambda > 0$ under suitable conditions.

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1 Introduction

In this paper, we consider multiple nontrivial weak solutions to the following nonlinear elliptic problem of (p,q)-Laplacian type involving critical Sobolev exponent:

$$-\Delta_p u - \Delta_q u = \lambda V(x)|u|^{k-2}u + K(x)|u|^{p^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\Delta_m u = \operatorname{div}(|\nabla|^{m-2}\nabla u)$ is the m-Laplacian of u, $\lambda > 0$, $1 < k < q < p < N$ and $p^* = \frac{Np}{N-p}$.

The (p,q)-Laplacian problem (1.1) comes from a general reaction–diffusion system

$$u_t = \operatorname{div}\bigl[E(u)\nabla u\bigr] + c(x,u), \tag{1.2}$$

where $E(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. The system has a wide range of applications in physics and related sciences, such as biophysics, chemical reaction and plasma physics. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.2) corresponds to the diffusion with a diffusion coefficient $E(u)$; whereas the second one is the reaction and relates to sources and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ has a polynomial form with respect to the concentration u. Specially, taking $q = 2$, we note that $(p,2)$-equations arise in many physical applications (see [2] and [5]), and recently such equations were studied by Papageorgiou et al. [10–13]. For example, in [11], they studied the existence and multiplicity of the following parametric nonlinear nonhomogeneous Dirichlet problem:

$$-\Delta_p u(z) - \Delta(z) = \lambda \left|u(z)\right|^{p-2} u(z) + f\bigl(z, u(z)\bigr) \quad \text{in } \Omega, u|\partial\Omega = 0, 2 < p < \infty,$$

where $\Omega \subset \mathbb{R}^N$ and the parameter $\lambda > 0$ is near the principal eigenvalue $\lambda_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$.

For general $q \in (1,p)$ and concave–convex nonlinearities, the stationary solution of (1.2) was studied by many authors and fruitful multiplicity results were obtained for the following problem:

$$-\operatorname{div}\bigl[E(u)\nabla u\bigr] = c(x,u). \tag{1.3}$$

For example, in [7], G. Li and G. Zhang considered problem (1.3) with the critical exponent

$$c(x,u) = |u|^{p^*-2}u + \theta|u|^{r-2}u \tag{1.4}$$

by using Lusternik–Schnirelman's theory. They proved that when $\theta > 0$, $1 < r < q < p < N$ and $\Omega \subset \mathbb{R}^N$ is bounded, there is a $\theta_0 > 0$ such that problem (1.3) possesses infinitely many weak solutions in $W_0^{1,p}(\Omega)$ for any $\theta \in (0, \theta_0)$.

Moreover, H. Yin and Z. Yang in [17] studied the equation

$$-\Delta_p u - \mu\Delta_q u = \theta V(x)|u|^{r-2}u + |u|^{p^*-2}u + \lambda f(x,u) \tag{1.5}$$

for the multiplicity of solutions on a bounded domain $\Omega \subset \mathbb{R}^N$ with $1 < r < q < p$ and $\lambda \in (0, \lambda^*)$.

But they only considered infinitely many weak solutions on a bounded domain Ω. Different from [7] and [17], our work is developed in the whole space $\mathbb{R}^N$ and the existence of infinitely many solutions with positive energy for problem (1.1) is also discussed, which are not mentioned in the references.

Our main results can be described as follows.

Theorem 1.1 *Suppose* $1 < k < q < p < N$, $N \geq 3$, $K(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $0 \leq V(x) \in C(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ *with* $r = \frac{p^*}{p^*-k}$. *Moreover,* $V(x) > 0$ *is bounded on some open subset* $\Omega \subset \mathbb{R}^N$, *with* $|\Omega| > 0$. *Then there exists a* $\lambda^* > 0$ *such that, for all* $\lambda \in (0, \lambda^*)$, *problem* (1.1) *has a sequence of weak solutions with negative energy.*

Denote the group of orthogonal linear transformations in $\mathbb{R}^N$ by $O(N)$ and let $T \subset O(N)$ be a subgroup. Set $|T| := \inf_{x \in \mathbb{R}^N, x \neq 0} |T_x|$, where $T_x := \{\tau x : \tau \in O(N)\}$ for $x \neq 0$ (see [16]). Moreover, a function $f : \mathbb{R}^N \to \mathbb{R}$ is called T-invariant if $f(\tau x) = f(x)$ for all $\tau \in T$ and $x \in \mathbb{R}^N$.

Theorem 1.2 *Suppose* $1 < k < q < p < N$, $N \geq 3$, *and assume* $V(x)$ *and* $K(x)$ *are* T-*invariant. Moreover, let* $|T| = \infty$, $K(0) = 0$, $\lim_{|x| \to \infty} K(x) = 0$, $K(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $K(x) > 0$ *a.e. in* $\mathbb{R}^N$ *and* $0 \leq V(x) \in L^r(\mathbb{R}^N) \cap L^{r'}(\mathbb{R}^N)$ *with* $r = \frac{p^*}{p^*-k}$ *and* $r' = \frac{q^*}{q^*-k}$. *Then, for all* $\lambda > 0$, *problem* (1.1) *possesses infinitely many solutions with positive energy.*

This paper is organized as follows. In Sect. 2, for the reader's convenience, we describe the main mathematical tools which we shall use. The existence theorem for $\lambda \in (0, \lambda^*)$ is proved in Sect. 3 via the application of genus. In Sect. 4, under suitable conditions, we show that problem (1.1) possesses infinitely many solutions with positive energy for every $\lambda > 0$.

2 Preliminary results

We now recall some known results and state our basic assumptions.

In this paper $\|\cdot\|_p$ denotes the usual L^p norm and

$$D^{1,p}(\mathbb{R}^N) := \{\nabla u \in L^p(\mathbb{R}^N)\},$$

with the norm defined by

$$\|u\|_{D^{1,p}} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1/p}.$$

We deal with problem (1.1) in the reflexive Banach space [3]

$$X := D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N),$$

which is endowed with the norm

$$\|u\|_X = \|u\|_{D^{1,p}} + \|u\|_{D^{1,q}}.$$

Throughout this paper the function $K(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We consider the following functional

$$\begin{aligned} E_\lambda(u) = {} & \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \, dx \\ & - \frac{\lambda}{k}\int_{\mathbb{R}^N} V(x)|u|^k \, dx - \frac{1}{p^*}\int_{\mathbb{R}^N} K(x)|u|^{p^*} \, dx. \end{aligned} \tag{2.1}$$

From the following Lemmas 2.1–2.2 the functional E_λ is well defined in X. Obviously, a critical point of E_λ in X is a weak solution of (1.1).

The value S is the best Sobolev constant, i.e.,

$$S = \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_{p^*}^p} : u \in D^{1,p}(\mathbb{R}^N), u \neq 0\right\}. \tag{2.2}$$

Lemma 2.1 *Suppose that $V(x) \in L^r(\mathbb{R}^N)$ with $r = \frac{p^*}{p^*-k}$, then the functional*

$$J(u) = \int_{\mathbb{R}^N} V(x)|u|^k \, dx$$

is well defined and weakly continuous on $D^{1,p}(\mathbb{R}^N)$. Moreover, $J(u)$ is continuously differentiable, its derivative $J' : D^{1,p}(\mathbb{R}^N) \to (D^{1,p}(\mathbb{R}^N))^*$ is given by

$$J'(u)\psi = k\int_{\mathbb{R}^N} V(x)|u|^{k-2}u \cdot \psi \, dx, \quad \forall \psi \in D^{1,p}(\mathbb{R}^N).$$

Proof For $u \in X \subset L^{p^*}(\mathbb{R}^N)$, by Hölder inequality, we have

$$\int_{\mathbb{R}^N} V(x)|u|^k\,dx \le \|V\|_r \|u\|_{p^*}.$$

This implies that $J(u)$ is well defined.

Let $\{u_n\}$ converge weakly to u in $D^{1,p}(\mathbb{R}^N)$. Then $\{u_n\}$ is bounded in $L^{p^*}(\mathbb{R}^N)$ and $\{|u_n|^k\}$ is bounded in $L^{\frac{p^*}{k}}(\mathbb{R}^N)$. Hence, $\{|u_n|^k\}$ converges weakly to $|u|^k$ in $L^{\frac{p^*}{k}}(\mathbb{R}^N)$. Since $V(x) \in L^{\frac{p^*}{p^*-k}}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} V(x)|u_n|^k\,dx \to \int_{\mathbb{R}^N} V(x)|u|^k\,dx,$$

which implies weak continuity. The proof of the rest is similar to that of Lemma 2.6 in [15], we omit it. □

Using a similar argument as in the proof of Lemma 2.1, we have

Lemma 2.2 *Suppose that $K(x) \in L^\infty(\mathbb{R}^N)$, then the functional*

$$H(u) = \int_{\mathbb{R}^N} K(x)|u|^{p^*}\,dx$$

is well defined on $D^{1,p}(\mathbb{R}^N)$. Moreover, $H(u)$ is continuously differentiable, its derivative $H' : D^{1,p}(\mathbb{R}^N) \to (D^{1,p}(\mathbb{R}^N))^*$ is given by

$$H'(u)\psi = p^* \int_{\mathbb{R}^N} K(x)|u|^{p^*-2}u \cdot \psi\,dx, \quad \forall \psi \in D^{1,p}\big(\mathbb{R}^N\big).$$

The following lemmas and definitions are also needed in our discussion.

Lemma 2.3 ([6]) *Let $s > 1$ and Ω be an open set in $\mathbb{R}^N$. Consider u_n, $u \in W^{1,s}(\Omega)$, $n = 1, 2, 3, \dots$. Let $a(x,\xi) \in C^0(\Omega \times \mathbb{R}^N, \mathbb{R}^N)$ have, for positive numbers $\alpha, \beta > 0$, the following properties*:

(i) $\alpha|\xi|^s \le a(x,\xi)\xi$ *for all* $\xi \in \mathbb{R}^N$,
(ii) $|a(x,\xi)| \le \beta|\xi|^{s-1}$ *for all* $(x,\xi) \in \Omega \times \mathbb{R}^N$,
(iii) $(a(x,\xi) - a(x,\eta))(\xi - \eta) > 0$ *for all* $(x,\xi) \in \Omega \times \mathbb{R}^N$ *with* $\xi \ne \eta$.

Then $\nabla u_n \to \nabla u$ in $L^s(\Omega)$ if and only if

$$\lim_{n\to\infty} \int_\Omega \big(a\big(x, \nabla u_n(x)\big) - a\big(x, \nabla u(x)\big)\big)\big(\nabla u_n(x) - \nabla u(x)\big)\,dx = 0.$$

Lemma 2.4 ([8, 9]) *Let $\{u_n\}$ converge weakly to u in $D^{1,p}(\mathbb{R}^N)$ such that $\{|u_n|^{p^*}\}$ converges weakly to a nonnegative measure ν on $\mathbb{R}^N$. Then, for some at most countable set J, we have*

$$\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j} \quad \text{and} \quad \sum_{j\in J} \nu_j^{\frac{p}{p^*}} < \infty, \tag{2.3}$$

where $x_j \in \mathbb{R}^N$, δ_{x_j} denotes the Dirac measure at x_j, and ν_j are constants.

Definition 2.1

(i) *Let X be a Banach space and $E : X \to \mathbb{R}$ be a differentiable functional. A sequence $\{u_k\} \subseteq X$ is called a* $(\mathrm{PS})_{\mathrm{c}}$ *sequence for E if $E(u_k) \to c$ and $E'(u_k) \to 0$ (in X^*) as $k \to \infty$.*

(ii) *If every* $(\mathrm{PS})_{\mathrm{c}}$ *sequence for E has a converging subsequence (in X), we say that E satisfies the* $(\mathrm{PS})_{\mathrm{c}}$*-conditions.*

In the rest of this section, we introduce some preparatory work for the proof of Theorem 1.1.

Lemma 2.5 *Let $\{u_n\} \subset X$ be a* $(\mathrm{PS})_{\mathrm{c}}$ *sequence for $E_\lambda(u)$. Then $\{u_n\}$ is bounded in X.*

Proof Suppose $\{u_n\} \subset X$ is a $(\mathrm{PS})_{\mathrm{c}}$ sequence of $E_\lambda(u)$, i.e.,

$$E_\lambda(u_n) = c + o(1), \qquad E'_\lambda(u_n) = o(1), \tag{2.4}$$

where $o(1) \to 0$ as $n \to \infty$. By (2.4), for n large enough, we have

$$\begin{aligned} c + 1 + \|u_n\| &\geq E_\lambda(u_n) - \frac{1}{p^*} E'_\lambda(u_n)u_n \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} |\nabla u_n|^p \,dx + \left(\frac{1}{q} - \frac{1}{p*}\right) \int_{\mathbb{R}^N} |\nabla u_n|^q \,dx \\ &\quad - \left(\frac{\lambda}{k} - \frac{\lambda}{p^*}\right) \int_{\mathbb{R}^N} V(x)|u_n|^k \,dx \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} |\nabla u_n|^p \,dx + \left(\frac{1}{q} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} |\nabla u_n|^q \,dx \\ &\quad - \left(\frac{\lambda}{k} - \frac{\lambda}{p^*}\right) S^{-\frac{k}{p}} \|V(x)\|_{L^{\frac{p^*}{p^*-k}}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p \,dx\right)^{\frac{k}{p}}. \end{aligned}$$

That is, for all large n, we have

$$c_1\bigl(1 + \|u_n\| + \|u_n\|^k\bigr) \geq c_2 \|u_n\|^p_{D^{1,p}} + c_3 \|u_n\|^q_{D^{1,q}},$$

where c_1, c_2 and c_3 are positive constants independent of n.

Suppose $\|u_n\| \to \infty$. We distinguish the following three cases:

(1) $\|u_n\|_{D^{1,p}} \to \infty$ and $\|u_n\|_{D^{1,q}} \to \infty$;

(2) $\|u_n\|_{D^{1,p}} \to \infty$ and $\{\|u_n\|_{D^{1,q}}\}$ is bounded;

(3) $\{\|u_n\|_{D^{1,p}}\}$ is bounded and $\|u_n\|_{D^{1,q}} \to \infty$.

If case (1) occurs, for all large n, we get

$$\begin{aligned} c_1\bigl(1 + \|u_n\| + \|u_n\|^k\bigr) &\geq c_2 \|u_n\|^p_{D^{1,p}} + c_3 \|u_n\|^q_{D^{1,q}} \\ &\geq c_2 \|u_n\|^q_{D^{1,p}} + c_3 \|u_n\|^q_{D^{1,q}} \\ &\geq c_4\bigl(\|u_n\|^q_{D^{1,p}} + \|u_n\|^q_{D^{1,q}}\bigr) \\ &\geq c_5 \|u_n\|^q, \end{aligned}$$

which is a contradiction to the fact $k < q$.

If case (2) is true, for all large n, we have

$$\begin{aligned}
&c_1\bigl(1+\|u_n\|_{D^{1,p}}+\|u_n\|_{D^{1,q}}+2^{k-1}\|u_n\|_{D^{1,p}}^k+2^{k-1}\|u_n\|_{D^{1,q}}^k\bigr)\\
&\quad\geq c_2\|u_n\|_{D^{1,p}}^p+c_3\|u_n\|_{D^{1,q}}^q\\
&\quad\geq c_2\|u_n\|_{D^{1,p}}^p,
\end{aligned}$$

thus

$$0<\frac{c_2}{c_1}\leq\lim_{n\to\infty}\left(\frac{1}{\|u_n\|_{D^{1,p}}^p}+\frac{1}{\|u_n\|_{D^{1,p}}^{p-1}}+\frac{\|u_n\|_{D^{1,q}}}{\|u_n\|_{D^{1,p}}^p}+\frac{2^{k-1}}{\|u_n\|_{D^{1,p}}^{p-k}}+\frac{2^{k-1}\|u_n\|_{D^{1,q}}^k}{\|u_n\|_{D^{1,p}}^p}\right)=0.$$

This is impossible.

Proceeding as in the second case, one can also verify that the third case cannot happen. Hence, the proof is completed. □

Lemma 2.6 *If $c<0$, then there exists a $\lambda^*>0$ such that E_λ satisfies $(\mathrm{PS})_c$-conditions for all $0<\lambda<\lambda^*$.*

Proof For $\varphi\in C_0^\infty(\mathbb{R}^N)$ and $w\in X$, from (2.2) we have

$$\begin{aligned}
&S^{\frac{1}{p}}\left(\int_{\mathbb{R}^N}|w\varphi|^{p^*}\,dx\right)^{\frac{1}{p^*}}\\
&\quad\leq\left(\int_{\mathbb{R}^N}\bigl|\nabla(w\varphi)\bigr|^p\,dx\right)^{\frac{1}{p}}\\
&\quad\leq\left(\int_{\mathbb{R}^N}|w|^p|\nabla\varphi|^p\,dx\right)^{\frac{1}{p}}+\left(\int_{\mathbb{R}^N}|\nabla w|^p|\varphi|^p\,dx\right)^{\frac{1}{p}}\\
&\quad\leq\left(\int_{\mathbb{R}^N}|w|^p|\nabla\varphi|^p\,dx\right)^{\frac{1}{p}}+\left(\int_{\mathbb{R}^N}\bigl(|\nabla w|^p+|\nabla w|^q\bigr)|\varphi|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned}\tag{2.5}$$

Suppose $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence. As a consequence of the boundedness of $\{u_n\}$, given by Lemma 2.5, there exists a $u\in X$ such that, up to subsequence, $u_n\rightharpoonup u$ in X.

Let $\psi\in C_0^\infty(\mathbb{R}^N)$ satisfy $\psi(x)=0$ for $|x|>1$, $\psi(x)=1$ for $|x|\leq\frac{1}{2}$, $0\leq\psi(x)\leq1$, $x\in\mathbb{R}^N$.

Applying Lemma 2.4 gives

$$|u_n|^{p^*}\rightharpoonup|u|^{p^*}+\sum_{j\in J}\nu_j\delta_{x_j}.$$

Since $\{u_n\}$ is bounded, there exists a nonnegative measure μ such that

$$|\nabla u_n|^p+|\nabla u_n|^q\rightharpoonup\mu.\tag{2.6}$$

For each index j and each $0<\varepsilon<1$, define

$$\psi_\varepsilon(x):=\psi\left(\frac{x-x_j}{\varepsilon}\right).$$

It follows from inequality (2.5) that

$$S^{\frac{1}{p}}\left(\int_{\mathbb{R}^N}|u_n\psi_\varepsilon|^{p^*}\,dx\right)^{\frac{1}{p^*}}\le\left(\int_{\mathbb{R}^N}|u_n|^p|\nabla\psi_\varepsilon|^p\,dx\right)^{\frac{1}{p}}+\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+|\nabla u_n|^q\right)|\psi_\varepsilon|^p\,dx\right)^{\frac{1}{p}}.$$

Furthermore, letting $n\to\infty$, Lemma 2.4 and (2.6) together imply that

$$S^{\frac{1}{p}}\left(\int_{\mathbb{R}^N}|\psi_\varepsilon|^{p^*}\,d\nu\right)^{\frac{1}{p^*}}\le\left(\int_{\mathbb{R}^N}|u|^p|\nabla\psi_\varepsilon|^p\,dx\right)^{\frac{1}{p}}+\left(\int_{\mathbb{R}^N}|\psi_\varepsilon|^p\,d\mu\right)^{\frac{1}{p}},$$

and then, by taking $\varepsilon\to 0$,

$$S^{\frac{1}{p}}\left(\int_{x_j}d\nu\right)^{\frac{1}{p^*}}\le\left(\int_{x_j}d\mu\right)^{\frac{1}{p}},$$

which yields

$$S\nu_j^{\frac{p}{p^*}}\le\mu_j:=\int_{x_j}d\mu. \tag{2.7}$$

On the other hand, from the fact that $E'_\lambda(u_n)\psi_\varepsilon u_n\to 0$ we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}u_n\nabla\psi_\varepsilon\nabla u_n\,dx+\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}u_n\nabla\psi_\varepsilon\nabla u_n\,dx\\&\quad=\lambda\int_{\mathbb{R}^N}V(x)|u_n|^k\psi_\varepsilon\,dx+\int_{\mathbb{R}^N}K(x)|u_n|^{p^*}\psi_\varepsilon\,dx\\&\qquad-\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+|\nabla u_n|^q\right)\psi_\varepsilon\,dx+o(1),\quad\text{as }n\to\infty,\end{aligned} \tag{2.8}$$

and since $V(x)\psi_\varepsilon\in L^r(\mathbb{R}^N)$, Lemma 2.1 and (2.8) show that

$$\begin{aligned}&\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}u_n\nabla\psi_\varepsilon\nabla u_n\,dx+\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}u_n\nabla\psi_\varepsilon\nabla u_n\,dx\right)\\&\quad=\lambda\int_{\mathbb{R}^N}V(x)|u|^k\psi_\varepsilon\,dx+\int_{\mathbb{R}^N}K(x)\psi_\varepsilon\,d\nu-\int_{\mathbb{R}^N}\psi_\varepsilon\,d\mu.\end{aligned} \tag{2.9}$$

From Hölder inequality with p, $p/(p-1)$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}u_n\nabla\psi_\varepsilon\nabla u_n\,dx\right|&\le\int_{\mathbb{R}^N}|\nabla u_n|^{p-1}|u_n\nabla\psi_\varepsilon|\,dx\\&\le\|u_n\|^{p-1}\left(\int_{B_\varepsilon(x_j)}|u_n|^p|\nabla\psi_\varepsilon|^p\,dx\right)^{1/p}.\end{aligned}$$

Furthermore, since $|u_n\nabla\psi_\varepsilon| \to |u\nabla\psi_\varepsilon|$ in $L^p(\mathbb{R}^N)$,

$$\begin{aligned}
&\lim_{n\to\infty}\left(\|u_n\|^{p-1}\left(\int_{B_\varepsilon(x_j)}|u_n|^p|\nabla\psi_\varepsilon|^p\,dx\right)^{1/p}\right)\\
&\quad\le C\left(\int_{B_\varepsilon(x_j)}|u|^p|\nabla\psi_\varepsilon|^p\,dx\right)^{1/p}\\
&\quad\le C\left(\int_{B_\varepsilon(x_j)}|u|^{p^*}\,dx\right)^{1/p^*}\left(\int_{B_\varepsilon(x_j)}|\nabla\psi_\varepsilon|^N\,dx\right)^{1/N}\\
&\quad\le C\left(\int_{B_\varepsilon(x_j)}|u|^{p^*}\,dx\right)^{1/p^*}. \qquad (2.10)
\end{aligned}$$

Now by replacing p with q, (2.10) reveals

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}u_n\nabla\psi_\varepsilon\nabla u_n\,dx \le C\left(\int_{B_\varepsilon(x_j)}|u|^{q^*}\,dx\right)^{1/q^*}.$$

In (2.9),

$$K(x_j)\nu_j = \mu_j \qquad (2.11)$$

is valid if $\varepsilon\to 0$. Besides, if $K(x_j)\le 0$, one gets $\mu_j=\nu_j=0$; while if $K(x_j)>0$, by (2.7), we have

(i) $\nu_j = 0$;

(ii) $\nu_j \ge (\frac{S}{K(x_j)})^{\frac{N}{p}}$.

Define

$$\begin{aligned}
&\nu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|u_n|^{p^*}\,dx;\\
&\mu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|\nabla u_n|^p + |\nabla u_n|^q\,dx.
\end{aligned}$$

By the concentration–compactness principle at infinity, ν_∞ and μ_∞ exist and satisfy:

(a_1) $\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\,dx = \int_{\mathbb{R}^N}d\nu + \nu_\infty$;

(a_2) $\limsup_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla u_n|^p + |\nabla u_n|^q)\,dx = \int_{\mathbb{R}^N}d\mu + \mu_\infty$;

(a_3) $S\nu_\infty^{\frac{p}{p^*}} \le \mu_\infty$.

Let $\psi_R\in C^\infty(\mathbb{R}^N)$ satisfy $\psi_R(x)=0$ for $|x|<R$, $\psi_R(x)=1$ for $|x|>2R$, $0\le\psi_R(x)\le 1$, $x\in\mathbb{R}^N$. Then we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}u_n\nabla\psi_R\nabla u_n\,dx + \int_{\mathbb{R}^N}|\nabla u_n|^{q-2}u_n\nabla\psi_R\nabla u_n\,dx\\
&\quad= \lambda\int_{\mathbb{R}^N}V(x)|u_n|^k\psi_R\,dx + \int_{\mathbb{R}^N}K(x)|u_n|^{p^*}\psi_R\,dx\\
&\qquad-\int_{\mathbb{R}^N}\left(|\nabla u_n|^p + |\nabla u_n|^q\right)\psi_R\,dx + o(1). \qquad (2.12)
\end{aligned}$$

Similar to the proof of (2.10), we have

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}u_n\nabla\psi_R\nabla u_n\,dx\right|\le C\left(\int_{R<|x|<2R}|u|^{p^*}\,dx\right)^{1/p^*}\to 0,\quad \text{as } R\to\infty.$$

Let $R\to\infty$ in (2.12), then

$$\|K\|_\infty\nu_\infty=\mu_\infty, \tag{2.13}$$

which in turn means, by (a_3),

(iii) $\nu_\infty=0$;

(iv) $\nu_\infty\ge(\frac{S}{\|K\|_\infty})^{\frac{N}{p}}$.

We now claim that (ii) and (iv) are impossible if λ is chosen small enough. Indeed, since $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence, for n large enough, we have

$$\begin{aligned}0>c+o(1)\|u_n\|&=E_\lambda(u_n)-\frac{1}{p^*}E'_\lambda(u_n)u_n\\&\ge\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx-\frac{(p^*-k)\lambda}{kp^*}\int_{\mathbb{R}^N}V(x)|u_n|^k\,dx\\&\ge\frac{S}{N}\|u_n\|_{p*}^p-\frac{(p^*-k)\lambda}{kp^*}\|V(x)\|_r\|u_n\|_{p*}^k.\end{aligned} \tag{2.14}$$

This yields that

$$\|u_n\|_{p*}\le C\lambda^{\frac{1}{p-k}}. \tag{2.15}$$

On the other hand, for n and R large enough and if (iv) occurs, we get

$$\begin{aligned}0&>c+o(1)\|u_n\|\\&\ge\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx+\left(\frac{1}{q}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}|\nabla u_n|^q\,dx\\&\quad-\frac{(p^*-k)\lambda}{kp^*}\int_{\mathbb{R}^N}V(x)|u_n|^k\,dx\\&\ge\frac{1}{N}\int_{\mathbb{R}^N}\bigl(|\nabla u_n|^p+|\nabla u_n|^q\bigr)\psi_R\,dx-\frac{(p^*-k)\lambda}{kp^*}\|V(x)\|_r\|u_n\|_{p*}^k\\&\ge\frac{1}{N}\mu_\infty+o(1)-\frac{(p^*-k)\lambda}{kp^*}\|V(x)\|_r\|u_n\|_{p*}^k\\&\ge\frac{1}{N}S^{\frac{N}{p}}\|K\|_\infty^{\frac{p-N}{p}}-C\lambda^{\frac{p}{p-k}},\end{aligned} \tag{2.16}$$

where we use (2.15) and (a_3). Therefore we can choose $\lambda^*>0$ such that for every $\lambda\in(0,\lambda^*)$

$$\frac{1}{N}S^{\frac{N}{p}}\|K\|_\infty^{\frac{p-N}{p}}-C\lambda^{\frac{p}{p-k}}>0,$$

which is a contradiction to (2.16).

A similar argument shows that (ii) cannot occur if λ^* is chosen properly. Thus, $\mu_i = \nu_i = \mu_\infty = \nu_\infty = 0$. From (a_1) and (2.3),

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\,dx = \int_{\mathbb{R}^N}|u|^{p^*}\,dx. \tag{2.17}$$

And Brezis–Lieb Lemma [16] implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|^{p^*}\,dx = 0. \tag{2.18}$$

Since $K(x)\in L^\infty(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N}K(x)|u_n\right|^{p^*-1}|u_n-u|\,dx| \le \|K\|_\infty\|u_n\|_{p^*}^{p^*-1}\left(\int_{\mathbb{R}^N}|u_n-u|^{p^*}\,dx\right)^{\frac{1}{p^*}}. \tag{2.19}$$

Then from (2.18) and (2.19), one gets

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}K(x)|u_n|^{p^*-1}|u_n-u|\,dx = 0. \tag{2.20}$$

A similar argument shows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}V(x)|u_n|^{k-1}|u_n-u|\,dx = 0. \tag{2.21}$$

Now we define

$$\langle A_r(u),\varphi\rangle := \int_{\mathbb{R}^N}|\nabla u|^{r-2}\langle\nabla u,\nabla\varphi\rangle_{\mathbb{R}^N}\,dx, \quad \forall u,\varphi\in X.$$

Considering $\langle E'_\lambda(u_n), u_n-u\rangle$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\Big[\langle A_p(u_n),u_n-u\rangle + \langle A_q(u_n),u_n-u\rangle - \lambda\int_{\mathbb{R}^N}V(x)|u_n|^{k-2}u_n(u_n-u)\,dx - \int_{\mathbb{R}^N}K(x)|u_n|^{p^*-2}u_n(u_n-u)\,dx\Big] = 0.$$

It means

$$\lim_{n\to\infty}\big[\langle A_p(u_n),u_n-u\rangle + \langle A_q(u_n),u_n-u\rangle\big] = 0.$$

From the monotonicity of $A_q(u)$ (see [4]), the following is true:

$$\limsup_{n\to\infty}\big[\langle A_p(u_n),u_n-u\rangle + \langle A_q(u),u_n-u\rangle\big] \le 0.$$

Notice that $u_n \rightharpoonup u$ in $D^{1,q}(\mathbb{R}^N)$,

$$\lim_{n\to\infty}\langle A_q(u),u_n-u\rangle = 0.$$

Therefore

$$\limsup_{n\to\infty}\langle A_p(u_n), u_n - u\rangle \leq 0. \tag{2.22}$$

Consequently,

$$\lim_{n\to\infty}\langle A_p(u_n) - A_p(u), u_n - u\rangle = 0.$$

Finally, the following two results can be obtained by taking $a(x,\xi) = |\xi|^{p-2}\xi$ and using Lemma 2.3:

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left|\nabla(u_n - u)\right|^p dx = 0,$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left|\nabla(u_n - u)\right|^q dx = 0.$$

The proof is complete. □

Now truncate the energy functional of problem (1.1). By Sobolev embedding theorem, for all $u \in X$, we have

$$\begin{aligned} E_\lambda(u) &= \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,dx \\ &\quad - \frac{\lambda}{k}\int_{\mathbb{R}^N}V(x)|u|^k\,dx - \frac{1}{p^*}\int_{\mathbb{R}^N}K(x)|u|^{p^*}\,dx \\ &\geq \frac{1}{p}\|u\|_{D^{1,p}}^p - \lambda c_1\|u\|_{D^{1,p}}^k - c_2\|u\|_{D^{1,p}}^{p^*}. \end{aligned} \tag{2.23}$$

Let $h(t) = c_3t^p - \lambda c_4t^k - c_5t^{p^*}$, we need to discuss the further properties of $h(t)$. Firstly, it is easy to see that there exist λ^*, T_0 and T_1, with $0 < T_0 < T_1$, such that

$$\begin{aligned} &h(T_0) = h(T_1) = 0, \\ &h(t) \leq 0, \quad \forall 0 \leq t \leq T_0, \\ &h(t) > 0, \quad \forall T_0 < t < T_1, \\ &h(t) \leq 0, \quad \forall t \geq T_1. \end{aligned}$$

for every $\lambda \in (0,\lambda^*)$.

Secondly, let $\tau : R^+ \to [0,1]$ be a C^∞ non-increasing function such that

$$\tau(t) = 1, \quad \text{if } t \leq T_0; \qquad \tau(t) = 0, \quad \text{if } t \geq T_1.$$

We consider the truncated functional

$$\begin{aligned} E_\infty(u) &= \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,dx - \frac{\lambda}{k}\int_{\mathbb{R}^N}V(x)|u|^k\,dx \\ &\quad - \frac{\tau(\|u\|_{D^{1,p}})}{p^*}\int_{\mathbb{R}^N}K(x)|u|^{p^*}\,dx \end{aligned}$$

and suppose

$$\overline{h}(t) = c_3 t^p - \lambda c_4 t^k - c_5 t^{p^*} \tau(t).$$

Then

$$E_\infty(u) \geq \overline{h}\big(\|u\|_{D^{1,p}}\big).$$

At the same time, we notice that $\overline{h}(t) \geq h(t)$, if $t > 0$; $\overline{h}(t) = h(t)$ if $0 \leq t \leq T_0$; $0 \leq h(t) \leq \overline{h}(t)$, if $T_0 < t < T_1$; $\overline{h}(t) > 0$, if $t > T_1$. Thus we get that $E_\lambda(u) = E_\infty(u)$ when $0 \leq \|u\|_{D^{1,p}} \leq T_0$. Furthermore, for $\tau \in C^\infty$ we have $E_\infty(u) \in C^1(X, \mathbb{R})$ and obtain the following lemma.

Lemma 2.7

(a) *If $E_\infty(u) < 0$, then $\|u\|_{D^{1,p}} < T_0$, and $E_\lambda(v) = E_\infty(v)$ for all v in a small enough neighborhood of u.*

(b) *For all $\lambda \in (0, \lambda^*)$, $E_\infty(u)$ satisfies the* $(\mathrm{PS})_{\mathrm{c}}$*-conditions for $c < 0$.*

Proof We prove (a) by contradiction. If $\|u\|_{D^{1,p}} \in [T_0, +\infty)$, by the above analysis we see that

$$E_\infty(u) \geq \overline{h}\big(\|u\|_{D^{1,p}}\big) \geq 0.$$

This is a contradiction to $E_\infty(u) < 0$, thus $\|u\|_{D^{1,p}} < T_0$ and (a) holds.

Claim (b) can be proved by the $(\mathrm{PS})_{\mathrm{c}}$-conditions for E_λ as $\lambda \in (0, \lambda^*)$ (see Lemma 2.6). □

The following is the classical Deformation Lemma (see [14]):

Lemma 2.8 *Let Y be a Banach space and consider an $f \in C^1(Y, \mathbb{R})$, satisfying the (PS)-conditions. If $c \in \mathbb{R}$ and N is any neighborhood of $K_c \triangleq \{u \in Y : f(u) = c, f'(u) = 0\}$, there exist $\eta(t,u) \equiv \eta_t(u) \in C([0,1] \times Y, Y)$ and constants $\overline{\epsilon} > \epsilon > 0$ such that*

(1) $\eta_0(u) = u \ \forall u \in Y$;

(2) $\eta_t(u) = u \ \forall u \notin f^{-1}[c - \overline{\epsilon}, c + \overline{\epsilon}]$;

(3) *$\eta_t(u) = u$ is a homeomorphism of Y onto Y $\forall t \in [0,1]$;*

(4) $f(\eta_t(u)) \leq f(u) \ \forall u \in Y \ \forall t \in [0,1]$;

(5) *$\eta_1(f^{c+\epsilon} \setminus N) \subset f^{c-\epsilon}$, where $f^c = \{u \in Y : f(u) \leq c\} \ \forall c \in \mathbb{R}$;*

(6) *If $K_c = \emptyset$, $\eta_1(f^{c+\epsilon}) \subset f^{c-\epsilon}$;*

(7) *If f is even, η_t is odd in u.*

We end up this section by pointing out some concepts and results about Z_2 index theory.

Let Y be a Banach space and set

$$\Sigma = \big\{A \subset Y \setminus \{0\} : A \text{ is closed}, -A = A\big\}.$$

For $A \in \Sigma$, we define the Z_2 genus of A by

$$\gamma(A) = \min\big\{n \in \mathbb{N} : \text{there exists an odd, continuous } \phi : A \to \mathbb{R}^n \setminus \{0\}\big\},$$

if such a minimum does not exist, then $\gamma(A) = +\infty$.

The main properties of genus are given in the following lemma (see [14]).

Lemma 2.9 *Let $A, B \in \Sigma$. Then*

(1) *If there exists $f \in C(A,B)$, odd, then $\gamma(A) \le \gamma(B)$;*
(2) *If $A \subset B$, then $\gamma(A) \le \gamma(B)$;*
(3) *If there exists an odd homeomorphism between A and B, then $\gamma(A) = \gamma(B)$;*
(4) *If S^{N-1} is the unit sphere in $\mathbb{R}^N$, then $\gamma(S^{N-1}) = N$;*
(5) *$\gamma(A \cup B) \le \gamma(A) + \gamma(B)$;*
(6) *If $\gamma(A) < \infty$, then $\gamma(\overline{A - B}) \ge \gamma(A) - \gamma(B)$;*
(7) *If A is compact, then $\gamma(A) < \infty$, and there exists a $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$, where $N_\delta(A) = \{x \in Y : d(x,A) \le \delta\}$;*
(8) *If Y_0 is a subspace of Y with codimension k, and $\gamma(A) > k$, then $A \cap Y_0 \ne \emptyset$.*

3 Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1 via genus argument.

For $1 \le j \le n$, we define

$$c_j = \inf_{A \in \sum_j} \sup_{u \in A} E_\infty(u),$$

where

$$\Sigma_j = \bigl\{A \subset X \setminus \{0\} : A \text{ is closed in } X, -A = A, \gamma(A) \ge j\bigr\}.$$

Let $K_c = \{u \in X : E_\infty(u) = c, E'_\infty(u) = 0\}$ and suppose that $\lambda \in (0, \lambda^*)$, where λ^* is given by Lemma 2.6.

Firstly, we claim that if $j \in \mathbb{N}$, there is an $\varepsilon_j = \varepsilon(j) > 0$ such that

$$\gamma\bigl(E_\infty^{-\varepsilon_j}\bigr) \ge j,$$

where $E_\infty^{-\varepsilon} = \{u \in X : E_\infty(u) \le -\varepsilon\}$.

Here $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with $\|u\|_{W_0^{1,p}(\Omega)} = (\int_\Omega |\nabla u|^p)^{\frac{1}{p}}$, and $\Omega \subset \mathbb{R}^N$ is an open bounded subset with $|\Omega| > 0$ and C^1-boundary, $V(x) > 0$ in Ω. Extending functions in $W_0^{1,p}(\Omega)$ by 0 outside Ω, we can assume that $W_0^{1,p}(\Omega) \subset X$.

Let W_j be a j-dimensional subspace of $W_0^{1,p}(\Omega)$. For every $v \in W_j$ with $\|v\|_{W_0^{1,p}(\Omega)} = 1$, from the assumptions of $V(x)$, it is easy to see that there exists a $d_j > 0$ such that

$$\int_\Omega V(x)|v|^k \, dx \ge d_j.$$

Since W_j is a finite-dimensional space, all the norms in W_j are equivalent. Thus we can define

$$\begin{aligned} a_j &= \sup\bigl\{|\nabla v|_q^q : v \in W_j, \|v\|_{W_0^{1,p}(\Omega)} = 1\bigr\} < \infty, \\ b_j &= \sup\bigl\{|v|_{p^*}^{p^*} : v \in W_j, \|v\|_{W_0^{1,p}(\Omega)} = 1\bigr\} < \infty. \end{aligned} \tag{3.1}$$

On the other hand, for $0 < t < T_0$, since

$$E_\infty(tv) = E_\lambda(tv) = \frac{1}{p}t^p + \frac{t^q}{q}|\nabla v|_q^q - \frac{\lambda t^k}{k}\int_\Omega V(x)|v|^k\,dx - \frac{t^{p^*}}{p^*}\int_\Omega K(x)|v|^{p^*}\,dx,$$

for every $v \in W_j$ with $\|v\|_{W_0^{1,p}(\Omega)} = 1$, we obtain

$$\begin{aligned} E_\infty(tv) &\le \frac{1}{p}t^p + \frac{a_j}{q}t^q - \frac{\lambda d_j}{k}t^k - \frac{t^{p^*}}{p^*}\int_\Omega K(x)|v|^{p^*}\,dx \\ &\le \frac{1}{p}t^p + \frac{a_j}{q}t^q - \frac{\lambda d_j}{k}t^k + \frac{b_j|K|_\infty}{p^*}t^{p^*}. \end{aligned} \tag{3.2}$$

Therefore for $\lambda \in (0, \lambda^*)$, there must be a $t_0 \in (0, T_0)$ sufficiently small such that $E_\infty(t_0 v) \le -\varepsilon_j < 0$, where $\varepsilon_j = -\frac{1}{p}t_0^p - \frac{a_j}{q}t_0^q + \frac{\lambda d_j}{k}t_0^k - \frac{b_j|K|_\infty}{p^*}t_0^{p^*}$. Denote $S_{t_0} = \{v \in X : \|v\|_{W_0^{1,p}(\Omega)} = t_0\}$, then $S_{t_0} \cap W_j \subset E_\infty^{-\varepsilon_j}$. By Lemma 2.9,

$$\gamma\left(E_\infty^{-\varepsilon_j}\right) \ge \gamma(S_{t_0} \cap X_j) \ge j.$$

As E_∞ is continuous and even, $E_\infty^{-\varepsilon_j} \in \Sigma_j$ and $c_j \le -\varepsilon_j < 0$. Since E_∞ is bounded from below, $c_j > -\infty$ (that is why we consider E_∞ instead of E_λ). Then from Lemma 2.6 we see that E_∞ satisfies the $(\mathrm{PS})_c$-conditions (for $c < 0$) and this implies that K_c is a compact set.

Secondly, we claim that if for some $j \in \mathbb{N}$ there is an $i \ge 0$ such that $c = c_j = c_{j+1} = \cdots = c_{j+i}$, then $\gamma(K_c) \ge i + 1$.

We now prove the main claim by contradiction. If $\gamma(K_c) \le i$, there exists a closed and symmetric set U with $K_c \subset U$ and $\gamma(U) \le i$. Since $c < 0$, we can also assume that the closed set $U \subset E_\infty^0$. Using Lemma 2.8, there is an odd homeomorphism

$$\eta : [0,1] \times X \to X$$

such that $\eta(E_\infty^{c+\delta} \setminus U) \subset E_\infty^{c-\delta}$ for some $\delta \in (0, -c)$.

From the hypothesis of $c = c_{j+i}$, there exists an $A \in \Sigma_{j+i}$ such that

$$\sup_{u \in A} E_\infty(u) < c + \delta.$$

Thus

$$\eta(A \setminus U) \subset \eta\left(E_\infty^{c+\delta} \setminus U\right) \subset E_\infty^{c-\delta},$$

which means

$$\sup_{u \in \eta(A \setminus U)} E_\infty(u) \le c - \delta.$$

But Lemma 2.9 reveals

$$\gamma\left(\overline{\eta(A \setminus U)}\right) \ge \gamma(\overline{A \setminus U}) \ge \gamma(A) - \gamma(U) \ge j.$$

Hence $\overline{\eta(A\setminus U)} \in \Sigma_j$ and

$$c = c_j \leq \sup_{u\in\overline{\eta(A\setminus U)}} E_\infty(u) = \sup_{u\in\eta(A\setminus U)} E_\infty(u) \leq c - \delta.$$

So we have proved the main claim.

We now complete the proof of Theorem 1.1. For all $j \in \mathbb{N}$, we have $\Sigma_{j+1} \subset \Sigma_j$ and $c_j \leq c_{j+1} < 0$. If all c_js are distinct, then $\gamma(K_{c_j}) \geq 1$, and we know that $\{c_j\}$ is a sequence of distinct negative critical values of E_∞. If for some j_0, there exists an $i \geq 1$ such that

$$c = c_{j_0} = c_{j_0+1} = \cdots = c_{j_0+i},$$

from the main claim, we have

$$\gamma(K_{c_{j_0}}) \geq i + 1,$$

which shows that $K_{c_{j_0}}$ has infinitely many distinct elements.

By Lemma 2.7, we know $E_\lambda(u) = E_\infty(u)$ when $E_\infty(u) < 0$, and we see that there exist $2n$ critical points of $E_\lambda(u)$ with negative critical values. Therefore problem (1.1) has $2n$ weak solutions with negative critical energy.

4 Proof of Theorem 1.2

We denote $X_T = \{u \in X : u(\tau x) = u(x), \tau \in O(N)\}$ and $L_T^{p^*} = \{u \in L^{p^*}(\mathbb{R}^N) : u(\tau x) = u(x), \tau \in O(N)\}$. By the principle of symmetric criticality, we have

Lemma 4.1 ([15]) *If $E'_\lambda(u) = 0$ in X_T^*, then $E'_\lambda(u) = 0$ in X^*.*

Lemma 4.2 *If $1 < k < q < p < N$, $|T| = \infty$, $K(0) = 0$ and $\lim_{|x|\to\infty} K(x) = 0$, then E_λ in X_T satisfies the $(\mathrm{PS})_c$-conditions for all $c \in \mathbb{R}$.*

Proof We only give a sketch of the proof because it is analogous to that of Lemma 2.6. Let $\{u_n\} \subset X_T$ be a $(\mathrm{PS})_c$ sequence of E_λ. An argument similar to the one used in proving Lemma 2.5 shows that $\{u_n\}$ is bounded. Using Lemma 2.4, there exists a measure ν such that (2.3) holds. We claim that the concentration of ν cannot occur at any $x \neq 0$. Assuming that $x_k \neq 0$ is a singular point of ν, we have $\nu_k = \nu(x_k) > 0$ and since ν is T-invariant, $\nu(\tau x_k) = \nu_k > 0$ for all $\tau \in T$. Since $|T| = \infty$, the sum in (2.3) (see Lemma 2.4) is infinite, which is a contradiction. On the other hand, by (2.11) and since $K(0) = 0$, we get $\nu_0 = 0$.

The next step in the proof is showing that the concentration of ν cannot occur at infinity. Since $\lim_{|x|\to\infty} K(x) = 0$, we have

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} K(x)|u_n|^{p^*}\,dx = 0.$$

By the same arguments as when proving (2.13), we have $\mu_\infty = 0$, then from (a_3) (see Lemma 2.6), we obtain $\nu_\infty = 0$. Thus $u_n \to u$ in $L_T^{p^*}(\mathbb{R}^N)$, and the argument at the end of the proof of Lemma 2.6 implies that $u_n \to u$ in X_T. □

Since X_T is a separable Banach space (see [1]), there is a linearly independent sequence $\{e_j\}$ such that

$$X_T = \overline{\bigoplus_{j\geq 1} X_j}, X_j := \operatorname{span}\{e_j\}.$$

Denote $Y_k = \bigoplus_{j\leq k} X_j$ and $Z_k = \overline{\bigoplus_{j\geq k} X_j}$.

Lemma 4.3 ([15]) *Let $E \in C^1(X_T, \mathbb{R})$ be an even functional satisfying the $(\mathrm{PS})_c$-conditions for every $c > 0$. If for every $k \in \mathbb{N}$ there exist $\rho_k > r_k > 0$ such that*

(a) $\alpha_k := \max_{u\in Y_k, \|u\|=\rho_k} E(u) \leq 0$,

(b) $\beta_k := \inf_{u\in Z_k, \|u\|=r_k} E(u) \to \infty$, *as* $k \to \infty$,

then E has a sequence of critical values tending to ∞.

Proof of Theorem 1.2 Obviously, E_λ is even and $E_\lambda \in C^1(X_T, \mathbb{R})$. By Lemma 4.2, E_λ satisfies the $(\mathrm{PS})_c$ conditions for every $c \in \mathbb{R}$. Since Y_k is a finite-dimensional subspace of X_T for each $k \in \mathbb{N}$ and $K(x) > 0$ a.e. in $\mathbb{R}^N$, this implies that there exists a constant $\varepsilon_k > 0$ such that for all $v \in Y_k$ with $\|v\| = 1$ we have

$$\int_{\mathbb{R}^N} K(x)|v|^{p^*}\,dx \geq \varepsilon_k. \tag{4.1}$$

On the other hand,

$$\begin{aligned} E_\lambda(u) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\,dx \\ &\quad - \frac{\lambda}{k}\int_{\mathbb{R}^N} V(x)|u|^k\,dx - \frac{1}{p^*}\int_{\mathbb{R}^N} K(x)|u|^{p^*}\,dx \\ &\leq \frac{1}{p}\|u\|^p + \frac{1}{q}\|u\|^q - \frac{1}{p^*}\int_{\mathbb{R}^N} K(x)|u|^{p^*}\,dx. \end{aligned} \tag{4.2}$$

Therefore, if $u \in Y_k$, $u \neq 0$, and writing $u = t_k v$ with $\|v\| = 1$, from (4.1) and (4.2) we get

$$E_\lambda(u) \leq \frac{1}{p}t_k^p + \frac{1}{q}t_k^q - \frac{\varepsilon_k}{p^*}t_k^{p^*} \leq 0$$

for large t_k. This proves (a) of Lemma 4.3.

Define

$$\beta_k := \sup_{u\in Z_k, \|u\|=1}\left(\int_{\mathbb{R}^N} K(x)|u|^{p^*}\,dx\right)^{\frac{1}{p^*}}. \tag{4.3}$$

It is clear that $0 \leq \beta_{k+1} \leq \beta_k$ and $\beta_k \to \beta_0 \geq 0$. Then for every $k \geq 1$ there exists a $u_k \in Z_k$ such that $\|u_k\| = 1$ and

$$\left(\int_{\mathbb{R}^N} K(x)|u_k|^{p^*}\,dx\right)^{\frac{1}{p^*}} \geq \frac{\beta_0}{2}. \tag{4.4}$$

By the definition of Z_k, we get $u_k \rightharpoonup 0$ in X_T. Thus, there exists a ν such that (2.3) holds. Combining the arguments proving Lemma 4.2 and the fact that $|T| = \infty$, we see that a concentration of the measure ν can only occur at 0 and ∞. Thus, $u_k \to 0$ in $L^{p^*}(\Omega)$, where $\Omega = \{x \in \mathbb{R}^N : r < |x| < R\}$ for each $0 < r < R$. Due to $K(x)$ being continuous, $K(0) = 0$ and $\lim_{|x|\to\infty} K(x) = 0$, for any $\varepsilon > 0$, we can choose small r and large R such that

$$\left(\int_{\{x\in\mathbb{R}^N:|x|<r\}} K(x)|u_k|^{p^*}\,dx\right)^{\frac{1}{p^*}} < \frac{\varepsilon}{2}, \qquad \left(\int_{\{x\in\mathbb{R}^N:|x|>R\}} K(x)|u_k|^{p^*}\,dx\right)^{\frac{1}{p^*}} < \frac{\varepsilon}{2}.$$

Therefore, from $K(x) \in L^\infty(\mathbb{R}^N)$, we have

$$\left(\int_{\mathbb{R}^N} K(x)|u_k|^{p^*}\,dx\right)^{\frac{1}{p^*}} \to 0,$$

as $k \to \infty$. Hence, by (4.4), we get $\beta_0 = 0$.

If we take $\|u\| = r_k$, by the definition of $\|\cdot\|$, either $\|u\|_{D^{1,p}}$ or $\|u\|_{D^{1,q}}$ is not less than $r_k/2$. Without loss of generality, we let $\|u\|_{D^{1,p}} \ge r_k/2$. Since $V(x) \ge 0$ and $K(x) > 0$ a.e. in $\mathbb{R}^N$ and $\lambda > 0$, for $u \in Z_k$, by Sobolev inequality and (4.3), we have

$$E_\lambda(u) \ge \frac{1}{p}\|u\|^p_{D^{1,p}} - \frac{\lambda C}{k}\|u\|^k_{D^{1,p}} - \frac{\beta_k^{p^*}}{p^*}\|u\|^{p^*}.$$

On the other hand, there exists an $R > 0$ such that for all $\|u\|_{D^{1,p}} \ge R$, we have

$$\frac{1}{2p}\|u\|^p_{D^{1,p}} \ge \frac{\lambda C}{k}\|u\|^k_{D^{1,p}}.$$

Hence, taking $\|u\| = r_k := (\frac{p^*}{p2^{p+2}\beta_k^{p^*}})^{\frac{1}{p^*-p}}$, since $\beta_k \to 0$, we get $r_k \to \infty$ and

$$\begin{aligned} E_\lambda(u) &\ge \frac{1}{2p}\|u\|^p_{D^{1,p}} - \frac{\beta_k^{p^*}}{p^*}\|u\|^{p^*} \\ &\ge \frac{1}{p2^{p+1}}\|u\|^p - \frac{\beta_k^{p^*}}{p^*}\|u\|^{p^*} \\ &= \frac{1}{p2^{p+2}} r_k^p \to \infty, \quad \text{as } k \to \infty. \end{aligned}$$

This concludes the proof of Theorem 1.2. □

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Authors' contributions
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Author details
[1]Business School, University of Shanghai for Science and Technology, Shanghai, China. [2]College of Science, University of Shanghai for Science and Technology, Shanghai, China.

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