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A diffusive stage-structured model with a free boundary

Jingfu Zhao^{1*} , Changming Song¹ and Hongtao Zhang¹

*Correspondence:
zhengqing1102@163.com
¹School of Science, Zhongyuan
University of Technology,
Zhengzhou, P.R. China

Abstract

In this paper we mainly consider a free boundary problem for a single-species model with stage structure in a radially symmetric setting. In our model, the individuals of a new or invasive species are classified as belonging either to the immature or to the mature cases. We firstly study the asymptotic behavior of the solution to the corresponding initial problem, then obtain a spreading–vanishing dichotomy and give sharp criteria governing spreading and vanishing for the free boundary problem.

Keywords: Stage-structured model; Free boundary; Spreading–vanishing dichotomy; Long time behavior; Criteria for spreading and vanishing

1 Introduction

Recently, Du and Lin [1] have proposed the new mathematical model

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0, 0 < x < h(t), \\ u_x(t, 0) = 0, \quad u(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ h(0) = h_0, \quad u(0, x) = u_0(x), & 0 < x < h_0, \end{cases} \quad (1)$$

to describe the expanding of a new or invasive species u , where $x = h(t)$ is the moving boundary to be determined by Stefan-like condition $h'(t) = -\mu u_x(t, h(t))$, a, b, d, μ and h_0 are given positive constants, and u_0 is a given nonnegative initial function.

Du and Lin [1] established various interesting results about the solution (u, h) of (1). One of very remarkable results is a spreading–vanishing dichotomy of the species, i.e., the solution (u, h) of (1) satisfies one of the following properties:

- (i) Spreading of the species: $h(t) \rightarrow \infty, u(t, x) \rightarrow a/b$ as $t \rightarrow \infty$;
- (ii) Vanishing of the species: $h(t) \rightarrow h_\infty \leq (\pi/2)\sqrt{d/a}$, and $u(t, x) \rightarrow 0$ as $t \rightarrow \infty$.

Another is that the spreading speed approaches to a positive constant if the spreading occurs. On the other hand, they derived the criteria for spreading and vanishing. Later on, Du and Guo [2, 3], studied a free boundary problem similar to (1) in higher dimension space, Kaneko and Yamada [4] discussed (1) and the case of bistable nonlinearity with $u_x(t, 0) = 0$ replaced by $u(t, 0) = 0$.

In problem (1), it is assumed that during the whole life histories the individual's characteristics are broadly similar to each other. In the real world, almost all animals have stage

structure of the immature and mature cases. For many animals whose babies are raised by their parents or are dependent on the nutrition from the eggs they stay in, the babies are much weaker than the mature. It is important and practical to introduce the stage structure into the model.

Stage-structured models have received much attention in recent years (see for example [5–11] and the references therein). The pioneering work of Aiello and Freedman [5] (1990) on a single-species growth model with stage structure represents a mathematically more careful and biologically meaningful formulation approach.

The ODE version on a single-species growth model with stage structure takes the form

$$\begin{cases} u_t = av - \alpha u - \beta u, & t > 0, \\ v_t = \beta u - bv^2 - cv, & t > 0, \end{cases}$$

where u and v are the population densities of immature and mature, respectively, a, α, β, b are given positive constants, and c is a nonnegative constant. In the immature stage, both the birth rate and the death rate obey the Malthus rule, while in the mature stage, the birth rate obeys the Malthus rule and the death rate logistic type.

In the present paper, we firstly consider diffusion in the above ODE model and discuss the long time behavior of the solution to PDE model

$$\begin{cases} u_t - d_1 \Delta u = av - \theta u, & t > 0, x \in \mathbb{R}^n, \\ v_t - d_2 \Delta v = \beta u - bv^2 - cv, & t > 0, x \in \mathbb{R}^n, \end{cases} \tag{2}$$

with $\theta > \beta$. This problem admits a positive steady state if and only if

$$a\beta > c\theta, \tag{3}$$

in which case the positive constant steady state is uniquely given by (\tilde{u}, \tilde{v}) , where

$$\tilde{u} = \frac{a^2\beta - ac\theta}{b\theta^2}, \quad \tilde{v} = \frac{a\beta - c\theta}{b\theta}.$$

Then, motivated by the work of [1–3] and [5–7, 9, 10], we investigate the diffusive stage-structured model (2) with a free boundary, which reads as follows:

$$\begin{cases} u_t - d_1 \Delta u = av - \theta u, & t > 0, 0 < r < h(t), \\ v_t - d_2 \Delta v = \beta u - bv^2 - cv, & t > 0, 0 < r < h(t), \\ u_r(t, 0) = v_r(t, 0) = u(t, h(t)) = v(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu[u_r(t, h(t)) + \rho v_r(t, h(t))], & t > 0, \\ h(0) = h_0, \quad u(0, r) = u_0(r), \quad v(0, r) = v_0(r), & 0 \leq r \leq h_0, \end{cases} \tag{4}$$

where $\Delta u = u_{rr} + \frac{n-1}{r}u_r$, $\Delta v = v_{rr} + \frac{n-1}{r}v_r$ ($r = |x|$, $x \in \mathbb{R}^n$), d_1, d_2, μ, ρ and h_0 are given positive constants, and the initial functions $u_0(r), v_0(r)$ satisfy

$$\begin{aligned} u_0, v_0 \in C^2([0, h_0]), \quad u_0(h_0) = v_0(h_0) = 0, \quad u_0(r) > 0, \quad v_0(r) > 0 \\ \text{in } [0, h_0]. \end{aligned} \tag{5}$$

Ecologically, this problem (4) describes the spreading of a new or invasive species with immature population density $u(t, |x|)$ and mature population density $v(t, |x|)$ over a radially symmetric setting, which exists initially in the ball $r < h_0$, disperses through random diffusion over an expanding ball $r < h(t)$, whose boundary $r = h(t)$ is the invading front, and evolves according to the free boundary condition $h'(t) = -\mu[u_r(t, h(t)) + \rho v_r(t, h(t))]$.

The well-known Stefan free boundary condition arises in many other applications. For instance, it was used to describe the melting of ice in contact with water [12], the modeling of oxygen in the muscle [13], the wound healing [14], the tumor growth [15], and so on. As far as population models are concerned, Wang and Zhao [16] used such a condition for a predator–prey system with double free boundaries in one dimension, in which the prey lives in the whole space but the predator lives in a bounded area at the initial state; in [17, 18], a Stefan condition was used for a competition system and a predator–prey system in radially symmetric setting, respectively, in which one species lives in the whole space but the other lives in a bounded area at the initial state. They established the spreading–vanishing dichotomy, long time behavior of the solution and sharp criteria for spreading and vanishing. For more biological discussion, we refer to [19–27] and the references therein.

We now describe the main results of this paper as follows. Hereafter, (3) is always assumed. First, it is proved that the positive constant steady state (\tilde{u}, \tilde{v}) of problem (2) is globally asymptotically stable.

Theorem 1.1 *Suppose that $(u_0, v_0) \in [C_b(\mathbb{R}^n)]^2$ and $(u(t, x), v(t, x))$ is the solution of the problem (2) with*

$$u(0, x) = u_0(x) \geq, \neq 0, \quad v(0, x) = v_0(x) \geq, \neq 0, \quad x \in \mathbb{R}^n,$$

then

$$\lim_{t \rightarrow \infty} (u(t, x), v(t, x)) = (\tilde{u}, \tilde{v})$$

uniformly in any bounded subset of \mathbb{R}^n .

Then, we have the following existence and uniqueness result and a priori estimates for the solution of the problem (4).

Theorem 1.2 *For any given (u_0, v_0) satisfying (5) and any $v \in (0, 1)$, the problem (4) admits a unique global solution $(u, v, h) \in C^{(1+v)/2, 1+v}(\Omega) \times C^{(1+v)/2, 1+v}(\Omega) \times C^{1+v/2}([0, \infty))$ where*

$$\Omega = \{(t, r) : t > 0, 0 \leq r \leq h(t)\},$$

such that the following inequalities hold:

$$0 < u(t, r) \leq M_1, \quad 0 < v(t, r) \leq M_1, \quad 0 < h'(t) \leq M_2 \tag{6}$$

for $t > 0, 0 < r < h(t)$ with $M_1, M_2 > 0$ depending on $d_1, d_2, a, \theta, \beta, b, c$ and $\|u_0\|_\infty, \|v_0\|_\infty$. Moreover,

$$\|u\|_{C^{(1+v)/2, 1+v}(\Omega)} + \|v\|_{C^{(1+v)/2, 1+v}(\Omega)} + \|h\|_{C^{1+v/2}([0, \infty))} \leq C, \tag{7}$$

where the constant $C > 0$ depends on v , the parameters in (4) and $\|u_0\|_{C^2([0, h_0])}, \|v_0\|_{C^2([0, h_0])}$.

Next, a spreading–vanishing dichotomy is given.

Theorem 1.3 *Assume that (3) holds and (u, v, h) is the solution of (4), then there exists $R^* > 0$ such that the following alternative holds:*

Either

(i) *vanishing: $h_\infty \leq R^*$ and*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = \lim_{t \rightarrow \infty} \|v(t, \cdot)\|_{C([0, h(t)])} = 0;$$

or

(ii) *spreading: $h_\infty = \infty$, and*

$$\lim_{t \rightarrow \infty} (u(t, r), v(t, r)) = (\tilde{u}, \tilde{v})$$

uniformly for r in any bounded set of $[0, \infty)$.

From $h'(t) > 0$ for $t > 0$ and Theorem 1.3, we easily see that $h_0 \geq R^*$ implies $h_\infty = \infty$. Hence, we last only need to discuss the case $h_0 < R^*$. Whether spreading or vanishing occurs is dependent on (u_0, v_0) and coefficient μ with the other parameters fixed.

Theorem 1.4 *Suppose that $h_0 < R^*$, then there exists $\mu^* > 0$ depending on (u_0, v_0) and h_0 , such that $h_\infty \leq R^*$ if $\mu \leq \mu^*$, and $h_\infty = \infty$ if $\mu > \mu^*$.*

The rest of this paper is organized in the following way. In Sect. 2, we firstly discuss the problem (2). We study a problem corresponding to (2) with fixed boundary and then prove Theorem 1.1. Sections 3, 4 and 5 are devoted to investigating the free boundary problem (4). In Sect. 3, we show Theorem 1.2 and give a comparison principle. Section 4 is applied to the long time behavior of solution (u, v) to the problem (4). From those results we can also get the spreading–vanishing dichotomy (Theorem 1.3). In Sect. 5, the sharp criteria for spreading and vanishing (Theorem 1.4) will be given. The last section is a brief discussion.

2 Global stability

In this section, Theorem 1.1 will be proved. We firstly study the following initial-boundary value problem:

$$\begin{cases} u_t - d_1 \Delta u = av - \theta u, & t > 0, x \in B_R, \\ v_t - d_2 \Delta v = \beta u - bv^2 - cv, & t > 0, x \in B_R, \\ u(t, x) = v(t, x) = 0, & t > 0, x \in \partial B_R, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & x \in \overline{B_R}, \end{cases} \tag{8}$$

where B_R is a ball with center 0 and radius R , u_0, v_0 are positive functions satisfying

$$u_0, v_0 \in C^2(\overline{B_R}), \quad u_0 = v_0 = 0 \quad \text{on } \partial B_R. \tag{9}$$

In the sequel, we will use some characteristics of the principal eigenvalue $\lambda_1(d, R)$ of the problem

$$\begin{cases} -d\Delta\phi = \lambda\phi, & x \in B_R, \\ \phi = 0, & x \in \partial B_R, \end{cases}$$

where $d > 0$ is a constant. It is well known that $\lambda_1(d, R)$ is a strictly decreasing continuous function in R and satisfies

$$\lim_{R \rightarrow 0^+} \lambda_1(d, R) = +\infty, \quad \lim_{R \rightarrow +\infty} \lambda_1(d, R) = 0.$$

Therefore, for fixed $d > 0$ and any given $L \in (0, \infty)$, there is a unique $R_L(d)$ such that

$$\lambda_1(d, R_L(d)) = L$$

and

$$\lambda_1(d, R) < L \quad \text{for } R > R_L(d); \quad \lambda_1(d, R) > L \quad \text{for } R < R_L(d).$$

We use a squeezing method as in [28] to prove the following theorem.

Theorem 2.1 *If (3) holds, then there exists $R^* > 0$ such that the problem*

$$\begin{cases} -d_1\Delta u = \lambda(av - \theta u), & x \in B_R, \\ -d_2\Delta v = \lambda(\beta u - bv^2 - cv), & x \in B_R, \\ u = v = 0, & x \in \partial B_R, \end{cases} \tag{10}$$

has a unique positive solution (u_λ, v_λ) for every $R > R^*$ and $\lambda \geq 1$; moreover,

$$(u_\lambda, v_\lambda) \longrightarrow (\tilde{u}, \tilde{v}) \tag{11}$$

uniformly on any compact subset of B_R as $\lambda \longrightarrow \infty$.

Proof Step 1 Existence The existence follows from a upper and lower solutions argument. Clearly, (\tilde{u}, \tilde{v}) is an upper solution. Let $(\underline{u}, \underline{v}) = (\delta_1\phi, \delta_2\phi)$, where ϕ satisfying $\|\phi\|_\infty = 1$ is a positive eigenfunction corresponding to $\lambda_1(d_1, R)$, and $\lambda_1(d_1, R)$, δ_1 , δ_2 are positive constants to be determined later. By direct calculations, we have

$$\begin{cases} -d_1\Delta \underline{u} = \lambda_1(d_1, R)\underline{u} \\ \quad = a\underline{v} - \theta\underline{u} + [(\lambda_1(d_1, R) + \theta)\delta_1 - a\delta_2]\phi, \\ -d_2\Delta \underline{v} = \frac{d_2\lambda_1(d_1, R)}{d_1}\underline{v} \\ \quad = \beta\underline{u} - b\underline{v}^2 - c\underline{v} + (\frac{d_2\lambda_1(d_1, R)}{d_1}\delta_2 + c\delta_2 + b\delta_2^2\phi - \beta\delta_1)\phi. \end{cases}$$

Take R^* such that

$$\lambda_1(d_1, R^*) = \frac{-[d_2\theta + d_1c] + \sqrt{[d_2\theta + d_1c]^2 + 4d_1d_2[a\beta - c\theta]}}{2d_2}. \tag{12}$$

For every $R > R^*$, the following inequality holds:

$$\lambda_1(d_1, R) < \frac{-[d_2\theta + d_1c] + \sqrt{[d_2\theta + d_1c]^2 + 4d_1d_2[a\beta - c\theta]}}{2d_2}.$$

Hence

$$\frac{a\beta}{\lambda_1(d_1, R) + \theta} - \frac{d_2}{d_1}\lambda_1(d_1, R) - c > 0.$$

Take

$$\begin{cases} \delta_1 = \frac{\delta_2}{2\beta} \left(\frac{a\beta}{\lambda_1(d_1, R) + \theta} + \frac{d_2}{d_1}\lambda_1(d_1, R) + c + b\delta_2 \right), \\ \delta_2 = \frac{1}{Lb} \left(\frac{a\beta}{\lambda_1(d_1, R) + \theta} - \frac{d_2}{d_1}\lambda_1(d_1, R) - c \right), \end{cases} \tag{13}$$

where $L > 2$ is any constant. Then we can obtain $\delta_1 < \tilde{u}$, $\delta_2 < \tilde{v}$ and

$$(\lambda_1(d_1, R) + \theta)\delta_1 - a\delta_2 < 0, \quad \frac{d_2\lambda_1(d_1, R)}{d_1}\delta_2 + c\delta_2 + b\delta_2^2\phi - \beta\delta_1 < 0.$$

Therefore, $(\underline{u}, \underline{v})$ satisfies $\underline{u} < \tilde{u}$, $\underline{v} < \tilde{v}$ and

$$\begin{cases} -d_1\Delta \underline{u} \leq a\underline{v} - \theta\underline{u}, & x \in B_R, \\ -d_2\Delta \underline{v} \leq \beta\underline{u} - b\underline{v}^2 - c\underline{v}, & x \in B_R, \\ \underline{u} = \underline{v} = 0, & x \in \partial B_R. \end{cases}$$

It follows from $\lambda_1(d_1, R)\underline{u} \leq a\underline{v} - \theta\underline{u}$ that $a\underline{v} - \theta\underline{u} \leq \lambda(a\underline{v} - \theta\underline{u})$ for $\lambda \geq 1$. Similarly, $\beta\underline{u} - b\underline{v}^2 - c\underline{v} \leq \lambda(\beta\underline{u} - b\underline{v}^2 - c\underline{v})$ for $\lambda \geq 1$. Thus

$$\begin{cases} -d_1\Delta \underline{u} \leq \lambda(a\underline{v} - \theta\underline{u}), & x \in B_R, \\ -d_2\Delta \underline{v} \leq \lambda(\beta\underline{u} - b\underline{v}^2 - c\underline{v}), & x \in B_R, \\ \underline{u} = \underline{v} = 0, & x \in \partial B_R. \end{cases}$$

So, $(\underline{u}, \underline{v})$ is a lower solution. Thus, the problem (10) has at least one positive solution.

Step 2 Uniqueness Now we verify the uniqueness of positive solution to (10). Fix $R > R^*$ and suppose that (10) has two positive solutions (u_1, v_1) and (u_2, v_2) . Then (u_i, v_i) satisfies

$$\begin{cases} -d_1\Delta u_i + \lambda\theta u_i = \lambda a v_i > 0, & x \in B_R, \\ -d_2\Delta v_i + \lambda(bv_i + c)v_i = \lambda\beta u_i > 0, & x \in B_R, \\ u_i = v_i = 0, & x \in \partial B_R. \end{cases}$$

With the help of the maximum principle and the Hopf boundary lemma for elliptic equations, we can see that $u_i > 0$, $v_i > 0$ in B_R and $\partial_\nu u_i < 0$, $\partial_\nu v_i < 0$, $i = 1, 2$ on ∂B_R . Hence there exists $M > 1$ such that

$$(M^{-1}u_1, M^{-1}v_1) < (u_i, v_i) < (Mu_1, Mv_1) \quad \text{in } B_R \text{ for } i = 1, 2.$$

It is easily seen that (Mu_1, Mv_1) is an upper solution of (10) and $(M^{-1}u_1, M^{-1}v_1)$ is a lower solution. As a result, there exist a minimal and a maximal solution to (10) in the order interval $[M^{-1}u_1, Mu_1] \times [M^{-1}v_1, Mv_1]$ which is denoted by (u_*, v_*) and (u^*, v^*) , respectively. Thus $(u_*, v_*) \leq (u_i, v_i) \leq (u^*, v^*)$, for $i = 1, 2$. Hence it suffices to show that $(u_*, v_*) = (u^*, v^*)$. To achieve this goal, let us define

$$\sigma_* = \inf\{\sigma \in \mathbb{R} : u^* \leq \sigma u_*, v^* \leq \sigma v_*\}.$$

Clearly $\sigma_* \geq 1$ and $u^* \leq \sigma_* u_*, v^* \leq \sigma_* v_*$. We next prove $\sigma_* = 1$, which will therefore yield $(u_*, v_*) = (u^*, v^*)$. Suppose for contradiction that $\sigma_* > 1$. Then, for

$$w := \sigma_* u_* - u^*, \quad z := \sigma_* v_* - v^*,$$

it is easy to check that $w, z \geq 0, \neq 0$, and (w, z) satisfies

$$\begin{cases} -d_1 \Delta w + \lambda \theta w = \lambda a z \geq 0, \neq 0, & x \in B_R, \\ -d_2 \Delta z + \lambda c z + b v^* z \geq \lambda \beta w \geq 0, \neq 0, & x \in B_R, \end{cases}$$

and $w = z = 0$ on ∂B_R . Hence, we can use the strong maximum principle and the Hopf boundary lemma for elliptic equations to deduce that $w, z > 0$ in B_R , and $\partial_\nu w, \partial_\nu z < 0$ on ∂B_R . It follows that $w \geq \varepsilon u^*$ and $z \geq \varepsilon v^*$ for some $\varepsilon > 0$ small, and hence $u^* \leq (1 + \varepsilon)^{-1} \sigma_* u_*, v^* \leq (1 + \varepsilon)^{-1} \sigma_* v_*$, which contradict the definition of σ_* . Consequently, we must have $\sigma_* = 1$, and the uniqueness conclusion is proved.

Step 3 Asymptotic behavior In what follows, let us denote by (u_λ, v_λ) the unique positive solution of (10) for $\lambda \geq 1$. We then want to show (11).

Given any compact subset K of B_R and any small $\varepsilon > 0$ such that $\varepsilon < \min\{\frac{\theta}{a} \tilde{u}, \tilde{v}\}$. Observe that $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) = (\tilde{u} + \frac{a}{\theta} \varepsilon, \tilde{v} + \varepsilon)$ is an upper solution of (10) for every $\lambda > 0$. On the other hand, we choose a small neighborhood U of ∂B_R in B_R such that $\overline{U} \cap K = \emptyset$ and $(\delta_1 \phi(x), \delta_2 \phi(x)) < (\tilde{u} - \frac{a}{\theta} \varepsilon, \tilde{v} - \varepsilon)$ for $x \in U$. Define

$$(\underline{u}_\varepsilon(x), \underline{v}_\varepsilon(x)) = \begin{cases} (\delta_1 \phi(x), \delta_2 \phi(x)), & x \in U, \\ (w_\varepsilon(x), z_\varepsilon(x)), & x \in B_R \setminus (U \cup K), \\ (\tilde{u} - \frac{a}{\theta} \varepsilon, \tilde{v} - \varepsilon), & x \in K, \end{cases}$$

where δ_1, δ_2 is defined as in (13), ϕ is a positive eigenfunction corresponding to $\lambda_1(d_1, R)$ with $\|\phi\|_\infty = 1$, $w_\varepsilon(x), z_\varepsilon(x)$ are smooth positive functions with $(w_\varepsilon(x), z_\varepsilon(x)) \leq (\tilde{u} - \frac{a}{\theta} \varepsilon, \tilde{v} - \varepsilon)$ in $B_R \setminus (U \cup K)$ such that $\underline{u}_\varepsilon, \underline{v}_\varepsilon$ are smooth functions in B_R and satisfy

$$b z^2(x) + c z(x) < \beta w(x) < \frac{a \beta}{\theta} z(x).$$

We can obtain

$$\begin{cases} -d_1 \Delta \underline{u}_\varepsilon \leq \lambda (a \underline{v}_\varepsilon - \theta \underline{u}_\varepsilon), & x \in B_R, \\ -d_2 \Delta \underline{v}_\varepsilon \leq \lambda (\beta \underline{u}_\varepsilon - b \underline{v}_\varepsilon^2 - c \underline{v}_\varepsilon), & x \in B_R, \\ \underline{u}_\varepsilon = \underline{v}_\varepsilon = 0, & x \in \partial B_R, \end{cases}$$

for all large λ . Since $(\underline{u}_\varepsilon, \underline{v}_\varepsilon) \leq (\bar{u}_\varepsilon, \bar{v}_\varepsilon)$, we deduce that $(\underline{u}_\varepsilon, \underline{v}_\varepsilon) \leq (u_\lambda, v_\lambda) \leq (\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ in B_R for all large λ . In particular,

$$\left(\tilde{u} - \frac{a}{\theta}\varepsilon, \tilde{v} - \varepsilon \right) \leq (u_\lambda, v_\lambda) \leq \left(\tilde{u} + \frac{a}{\theta}\varepsilon, \tilde{v} + \varepsilon \right)$$

on K for all large λ . This shows that $(u_\lambda, v_\lambda) \rightarrow (\tilde{u}, \tilde{v})$ as $\lambda \rightarrow \infty$ uniformly on K , as required. The proof is complete. \square

Theorem 2.2 *Assume that (u_0, v_0) satisfies (9) and $R > R^*$. If (u, v) is the solution of (8), then*

$$\lim_{t \rightarrow \infty} (u(t, x), v(t, x)) = (u^*(x), v^*(x)) \quad \text{uniformly in } \bar{B}_R,$$

where $(u^*(x), v^*(x))$ is the unique positive solution of (10) with $\lambda = 1$.

Proof Since $R > R^*$, (10) has a unique positive solution $(u^*(x), v^*(x))$. With the help of the maximum principle and the Hopf boundary lemma for elliptic equations, we can find $M > 1$ such that

$$(u_0, v_0) \leq (Mu^*, Mv^*).$$

It is easy to verify that (Mu^*, Mv^*) is an upper solution of (10) when $\lambda = 1$. One can still use the same analysis as in proof of Theorem 8 to deduce that $(\delta_1\phi, \delta_2\phi)$ is a lower solution of the problem (10), where ϕ is the positive eigenfunction corresponding to $\lambda_1(d_1, R)$ with $\|\phi\|_\infty = 1$ and δ_1, δ_2 is defined as in (13), choosing δ_1, δ_2 sufficiently small (i.e. L sufficiently large) such that $(\delta_1\phi(x), \delta_2\phi(x)) \leq (u_0(x), v_0(x))$ for $x \in \bar{B}_R$ if necessary. We consider the following two auxiliary problems:

$$\begin{cases} \bar{u}_t - d_1 \Delta \bar{u} = a\bar{v} - \theta\bar{u}, & t > 0, x \in B_R, \\ \bar{v}_t - d_2 \Delta \bar{v} = \beta\bar{u} - b\bar{v}^2 - c\bar{v}, & t > 0, x \in B_R, \\ \bar{u}(t, x) = \bar{v}(t, x) = 0, & t > 0, x \in \partial B_R, \\ \bar{u}(0, x) = Mu^*(x), \quad \bar{v}(0, x) = Mv^*(x), & x \in \bar{B}_R, \end{cases}$$

and

$$\begin{cases} \underline{u}_t - d_1 \Delta \underline{u} = a\underline{v} - \theta\underline{u}, & t > 0, x \in B_R, \\ \underline{v}_t - d_2 \Delta \underline{v} = \beta\underline{u} - b\underline{v}^2 - c\underline{v}, & t > 0, x \in B_R, \\ \underline{u}(t, x) = \underline{v}(t, x) = 0, & t > 0, x \in \partial B_R, \\ \underline{u}(0, x) = \delta_1\phi(x), \quad \underline{v}(0, x) = \delta_2\phi(x), & x \in \bar{B}_R. \end{cases}$$

We easily see that

$$(\underline{u}(t, x), \underline{v}(t, x)) \leq (u(t, x), v(t, x)) \leq (\bar{u}(t, x), \bar{v}(t, x)) \tag{14}$$

due to the comparison principle. From the theory of monotone dynamical systems (see [29, Corollary 3.6]), we see that $(\underline{u}(t, x), \underline{v}(t, x))$ is increasing while $(\bar{u}(t, x), \bar{v}(t, x))$ is decreasing in t , and

$$\lim_{t \rightarrow \infty} (\underline{u}(t, x), \underline{v}(t, x)) = (\underline{u}^*(x), \underline{v}^*(x)), \quad \lim_{t \rightarrow \infty} (\bar{u}(t, x), \bar{v}(t, x)) = (\bar{u}^*(x), \bar{v}^*(x)) \tag{15}$$

uniformly in \bar{B}_R , where $(\underline{u}^*, \underline{v}^*)$ and (\bar{u}^*, \bar{v}^*) satisfy (10) with $\lambda = 1$ and

$$(\delta_1 \phi, \delta_2 \phi) \leq (\underline{u}^*, \underline{v}^*) \leq (\bar{u}^*, \bar{v}^*).$$

So $(\underline{u}^*, \underline{v}^*) = (\bar{u}^*, \bar{v}^*) = (u^*, v^*)$, since the positive solution of (10) is unique. Due to (14) and (15), we have $\lim_{t \rightarrow \infty} (u(t, x), v(t, x)) = (u^*(x), v^*(x))$ uniformly in \bar{B}_R . \square

Proof of Theorem 1.1 First we recall that the comparison principle gives $(u(t, x), v(t, x)) \leq (\bar{u}(t), \bar{v}(t))$ for $t > 0, x \in \mathbb{R}^n$, where $(\bar{u}(t), \bar{v}(t))$ is the solution of the problem

$$\begin{cases} \bar{u}_t = a\bar{v} - \theta\bar{u}, & t > 0, \\ \bar{v}_t = \beta\bar{u} - b\bar{v}^2 - c\bar{v}, & t > 0, \\ \bar{u}(0) = \|u_0\|_\infty, \quad \bar{v}(0) = \|v_0\|_\infty. \end{cases} \tag{16}$$

Similarly to the proof of Theorem 2.1 in [30], we see that the positive constant steady state (\tilde{u}, \tilde{v}) of (16) is globally asymptotically stable. So $\lim_{t \rightarrow \infty} (\bar{u}(t), \bar{v}(t)) = (\tilde{u}, \tilde{v})$, moreover,

$$\limsup_{t \rightarrow \infty} u(t, x) \leq \tilde{u}, \quad \limsup_{t \rightarrow \infty} v(t, x) \leq \tilde{v} \quad \text{uniformly for } x \in \mathbb{R}^n.$$

Next, we show

$$\liminf_{t \rightarrow \infty} u(t, x) \geq \tilde{u}, \quad \liminf_{t \rightarrow \infty} v(t, x) \geq \tilde{v}$$

locally uniformly for $x \in \mathbb{R}^n$.

Let $(u_R(t, x), v_R(t, x))$ be the unique solution of (8) with any $R > R^*$. It follows from comparison principle that $(u_R(t, x), v_R(t, x)) \leq (u(t, x), v(t, x))$ for any R . By using of Theorem 2.2, we easily see that $(u_R(t, x), v_R(t, x)) \rightarrow (u_R^*(x), v_R^*(x))$ as $t \rightarrow \infty$ uniformly in B_R , where $(u_R^*(x), v_R^*(x))$ is the unique positive solution of (8) with $\lambda = 1$. It follows that

$$\liminf_{t \rightarrow \infty} u(t, x) \geq u_R^*(x), \quad \liminf_{t \rightarrow \infty} v(t, x) \geq v_R^*(x) \tag{17}$$

uniformly in B_R . Let

$$(w_R(y), z_R(y)) = \left(u_R^* \left(\frac{R}{2R^*} y \right), v_R^* \left(\frac{R}{2R^*} y \right) \right),$$

then $(w_R(y), z_R(y))$ satisfies

$$\begin{cases} -d_1 \Delta_y w_R = \left(\frac{R}{2R^*} \right)^2 (a z_R - \theta w_R), & y \in B_{2R^*}, \\ -d_2 \Delta_y z_R = \left(\frac{R}{2R^*} \right)^2 (\beta w_R - b z_R^2 - c z_R), & y \in B_{2R^*}, \\ w = z = 0, & y \in \partial B_{2R^*}. \end{cases}$$

Using Theorem 2.1, we easily see that $(w_R(y), z_R(y)) \rightarrow (\tilde{u}, \tilde{v})$ as $R \rightarrow \infty$ uniformly in any compact subset of B_{2R^*} . Therefore $(u_R^*(x), v_R^*(x)) \rightarrow (\tilde{u}, \tilde{v})$ as $R \rightarrow \infty$ uniformly in any compact subset of \mathbb{R}^n and then

$$\liminf_{t \rightarrow \infty} u(t, x) \geq \tilde{u}, \quad \liminf_{t \rightarrow \infty} v(t, x) \geq \tilde{v}$$

locally uniformly for $x \in \mathbb{R}^n$, due to (17). This completes the proof of the desired result. \square

3 Existence and uniqueness

For the existence and uniqueness of local solution, in previous work, for example in the references [17, 18, 24], the embedding theorem

$$\|u\|_{C^{(1+\nu)/2, 1+\nu}(Q_T)} \leq C \|u\|_{W^{1,2}(Q_T)}, \quad p > n + 2,$$

was used, where $Q_T = \{(t, x) | 0 \leq t \leq T, x \in Q\}$, Q is a bounded domain of \mathbb{R}^n . This is not appropriate because the embedding constant $C = C(T^{-1})$ depends on T^{-1} and $C(T^{-1}) \rightarrow \infty$ as $T \rightarrow 0$. For example, for the function $u \equiv 1$, $\|u\|_{C^{(1+\nu)/2, 1+\nu}(Q_T)} = 1$ and $\|u\|_{W^{1,2}(Q_T)} = |Q|^{1/p} T^{1/p} \rightarrow 0$ as $T \rightarrow 0$. Very recently, Wang has overcome this loophole and given a strict proof in [31]. So the existence and uniqueness of local solution can be obtained from the proof of Theorem 1.1 in [31], and the local solution can be extended to Ω by Theorem 2.4 in [17], we omit the details. Therefore, we will only show the inequalities (6) and (7). Then some comparison results for (4) are given.

Theorem 3.1 *Let (u, v, h) be a solution to the problem (4) defined for $t \in (0, \infty)$. Then there exist constants M_1 and M_2 such that*

$$0 < u(t, r) \leq M_1, \quad 0 < v(t, r) \leq M_1, \quad 0 < h'(t) \leq M_2$$

for all $t > 0$ and $0 \leq r \leq h(t)$.

Proof Using the strong maximum principle, we are easy to see that $u > 0$ in $(0, \infty) \times [0, h(t))$ and $v > 0$ in $(0, \infty) \times [0, h(t))$ as long as the solution exists. It follows from the comparison principle that $(u(t, r), v(t, r)) \leq (\bar{u}(t), \bar{v}(t))$ for $t > 0$ and $r \in [0, h(t)]$, where $(\bar{u}(t), \bar{v}(t))$ solves the initial value problem (16). Let $U(t) = \bar{u}(t) + \bar{v}(t)$, then U satisfies $U(0) = \|u_0\|_\infty + \|v_0\|_\infty$ and

$$\begin{aligned} U_t &= a\bar{v} - (\theta - \beta)\bar{u} - b\bar{v}^2 - c\bar{v} \\ &= -(\theta - \beta)(\bar{u} + \bar{v}) - b\left(v - \frac{a + \theta - \beta - c}{2b}\right)^2 + \frac{(a + \theta - \beta - c)^2}{4b} \\ &\leq -(\theta - \beta)U + \frac{(a + \theta - \beta - c)^2}{4b}. \end{aligned}$$

By using of the comparison principle again, we have

$$\bar{u}(t) + \bar{v}(t) = U(t) \leq M_1 := \max \left\{ \|u_0\|_\infty + \|v_0\|_\infty, \frac{(a + \theta - \beta - c)^2}{4b(\theta - \beta)} \right\}.$$

Thus

$$u(t, r) \leq M_1, \quad v(t, r) \leq M_1.$$

By straightening the free boundaries with $s = \frac{h_0 r}{h(t)}$, we can obtain a fixed boundary problem. Then the strong maximum principle yields the inequalities $u_r(t, h(t)) < 0$ and $v_r(t, h(t)) < 0$, and therefore $h'(t) > 0$ in $(0, \infty)$. It remains to show that $h'(t) \leq M_2$ for some positive constant M_2 . To this end, let M be a positive constant, define

$$\Omega_M := \{(t, r) : 0 < t < \infty, h(t) - 1/M < r < h(t)\}$$

and construct an auxiliary function

$$w(t, r) = M_1 [2M(h(t) - r) - M^2(h(t) - r)^2].$$

We will choose M so that $(w(t, r), w(t, r)) \geq (u(t, r), v(t, r))$ holds over Ω_M .

Direct calculations show that, for $(t, r) \in \Omega_M$,

$$\begin{aligned} w_t &= 2M_1 M h'(t) [1 - M(h(t) - r)] \geq 0, \\ -w_r &= 2M_1 M [1 - M(h(t) - r)] \geq 0, \\ -\Delta w &= -w_{rr} - \frac{n-1}{r} w_r \geq 2M_1 M^2, \\ av - \theta u &\leq aM_1, \quad \beta u - bv^2 - cv \leq \beta M_1. \end{aligned}$$

It follows that

$$\begin{cases} w_t - d_1 \Delta w \geq 2d_1 M_1 M^2 \geq av - \theta u, & (t, r) \in \Omega_M, \\ w_t - d_2 \Delta w \geq 2d_2 M_1 M^2 \geq \beta u - bv^2 - cv, & (t, r) \in \Omega_M, \end{cases}$$

if $M^2 \geq \max\{\frac{a}{2d_1}, \frac{\beta}{2d_2}\}$. On the other hand,

$$\begin{aligned} w(t, h(t) - M^{-1}) &= M_1 \geq \max\{u(t, h(t) - M^{-1}), v(t, h(t) - M^{-1})\}, \\ w(t, h(t)) &= 0 = u(t, h(t)) = v(t, h(t)). \end{aligned}$$

Note that

$$\begin{aligned} u_0(r) &= - \int_r^{h_0} u'_0(s) ds \leq (h_0 - r) \|u'_0\|_{C([0, h_0])} \quad \text{in } [h_0 - M^{-1}, h_0], \\ w(0, r) &= M_1 [2M(h_0 - r) - M^2(h_0 - r)^2] \geq M_1 M (h_0 - r) \quad \text{in } [h_0 - M^{-1}, h_0], \end{aligned}$$

we know that if $MM_1 \geq \|u'_0\|_{C([0, h_0])}$, then

$$u_0(r) \leq (h_0 - r) \|u'_0\|_{C([0, h_0])} \leq w(0, r) \quad \text{in } [h_0 - M^{-1}, h_0].$$

Similarly, if $MM_1 \geq \|v'_0\|_{C([0,h_0])}$ then

$$v_0(r) \leq (h_0 - r) \|v'_0\|_{C([0,h_0])} \leq w(0, r) \quad \text{in } [h_0 - M^{-1}, h_0].$$

Let

$$M = \max \left\{ \sqrt{\frac{a}{2d_1}}, \sqrt{\frac{\beta}{2d_2}}, \frac{\|u'_0\|_{C([0,h_0])}}{M_1}, \frac{\|v'_0\|_{C([0,h_0])}}{M_1} \right\}.$$

Applying the maximum principle to $w - u$ and $w - v$ over Ω_M we can deduce that $u(t, r), v(t, r) \leq w(t, r)$ for $(t, r) \in \Omega_M$. Thanks to $w(t, h(t)) = 0 = u(t, h(t)) = v(t, h(t))$, it would then follow that

$$\begin{aligned} h'(t) &= -\mu [u_r(t, h(t)) + \rho v_r(t, h(t))] \leq -\mu [w_r(t, h(t)) + \rho w_r(t, h(t))] \\ &\leq 2\mu(1 + \rho)M_1M := M_2. \end{aligned}$$

The proof is complete. □

Theorem 3.2 *If (u, v, h) is a solution to problem (4) defined for $t \in (0, \infty)$, then, for any $\nu \in (0, 1)$, the estimate (7) holds, i.e.*

$$\|u\|_{C^{(1+\nu)/2, 1+\nu}(\Omega)} + \|v\|_{C^{(1+\nu)/2, 1+\nu}(\Omega)} + \|h\|_{C^{1+\nu/2}([0, \infty))} \leq C,$$

where

$$\Omega = \{(t, r) : t > 0, 0 \leq r \leq h(t)\},$$

the constant $C > 0$ depends on ν , the parameters in (4) and $\|u_0\|_{C^2([0, h_0])}, \|v_0\|_{C^2([0, h_0])}$.

Proof We first consider the case $h_\infty < \infty$. Define

$$s = \frac{h_0 r}{h(t)}, \quad (w(t, s), z(t, s)) = (u(t, r), v(t, r)).$$

Hence $w(t, s)$ satisfies

$$\begin{cases} w_t - \frac{d_1 h_0^2}{h^2(t)} \Delta_s w - \frac{h'(t)s}{h(t)} w_s = az - \theta w, & t > 0, 0 \leq s < h_0, \\ w_s(t, 0) = w(t, h_0) = 0, & t > 0, \\ w(0, s) = u_0(s), & 0 \leq s \leq h_0. \end{cases}$$

This is an initial-boundary value problem over a fixed ball $\{s < h_0\}$. Since $h_0 \leq h(t) < h_\infty < \infty$, the differential operator is uniformly parabolic. It follows from Theorem 3.1 that

$$\|az - \theta w\|_\infty \leq (a + \theta)M_1, \quad \left\| \frac{h'(t)s}{h(t)} \right\|_\infty \leq M_2.$$

Therefore, we can apply the standard L^p theory and the Sobolev embedding theorem ([32, 33]) to obtain, for any $\nu \in (0, 1)$,

$$\|w\|_{C^{(1+\nu)/2, 1+\nu}([0,2] \times [0,h_0])} \leq C_1.$$

Furthermore, by virtue of a similar analysis to Proposition A.1 of [16] we can conclude that

$$\|w\|_{C^{(1+\nu)/2, 1+\nu}([1,\infty) \times [0,h_0])} \leq C_2,$$

where C_1 and C_2 are constants depending on $\nu, h_0, h_\infty, M_1, M_2$ and $\|u_0\|_{C^{1+\nu}([0,h_0])}$. Thus

$$\|w\|_{C^{(1+\nu)/2, 1+\nu}([0,\infty) \times [0,h_0])} \leq \max\{C_1, C_2\}.$$

Similarly, we may obtain

$$\|z\|_{C^{(1+\nu)/2, 1+\nu}([0,\infty) \times [0,h_0])} \leq C_3,$$

where C_3 is a positive constant depending on $\nu, h_0, h_\infty, M_1, M_2$ and $\|v_0\|_{C^{1+\nu}([0,h_0])}$.

It follows that there exists a constant C depending on $\nu, h_0, h_\infty, M_1, M_2, \|u_0\|_{C^{1+\nu}([0,h_0])}$ and $\|v_0\|_{C^{1+\nu}([0,h_0])}$ such that

$$\|u\|_{C^{(1+\nu)/2, 1+\nu}(\Omega)} + \|v\|_{C^{(1+\nu)/2, 1+\nu}(\Omega)} + \|h\|_{C^{1+\nu/2}([0,\infty))} \leq C.$$

When $h_\infty = \infty$, similarly to the arguments in Theorem 2.2 of [34] we can obtain (7). So we omit the details.

The proof is complete. □

We now present some comparison principles which will be used in the following sections to estimate the solution $(u(t, r), v(t, r))$ and the free boundary $r = h(t)$ of (4).

Theorem 3.3 (The comparison principle) *Assume that $T \in (0, \infty)$, $\bar{u}, \bar{v} \in C(\bar{D}_T) \cap C^{1,2}(D_T)$ with $D_T = \{(t, r) \in \mathbb{R}^2 : t \in (0, T], r \in [0, \bar{h}(t)]\}$, and $(\bar{u}, \bar{v}, \bar{h})$ satisfies*

$$\begin{cases} \bar{u}_t - d_1 \Delta \bar{u} \geq a\bar{v} - \theta \bar{u}, & 0 < t < T, 0 < r < \bar{h}(t), \\ \bar{v}_t - d_2 \Delta \bar{v} \geq \beta \bar{u} - b\bar{v}^2 - c\bar{v}, & 0 < t < T, 0 < r < \bar{h}(t), \\ \bar{u}_r(t, 0) = \bar{v}_r(t, 0) = 0, & 0 < t < T, \\ \bar{u}(t, \bar{h}(t)) = \bar{v}(t, \bar{h}(t)) = 0, & 0 < t < T, \\ \bar{h}'(t) \geq -\mu[\bar{u}_r(t, \bar{h}(t)) + \rho \bar{v}_r(t, \bar{h}(t))], & 0 < t < T, \\ \bar{h}(0) \geq h_0, \quad \bar{u}(0, r) \geq u_0(r), \quad \bar{v}(0, r) \geq v_0(r), & 0 \leq r < h_0. \end{cases}$$

Let (u, v, h) be the unique positive solution of (4). Then $h(t) \leq \bar{h}(t)$ in $(0, T]$ and

$$u(t, r) \leq \bar{u}(t, r), \quad v(t, r) \leq \bar{v}(t, r) \quad \text{for } (t, r) \in (0, T] \times [0, h(t)]. \tag{18}$$

Proof First assume that $h_0 < \bar{h}(0)$. We claim that $h(t) < \bar{h}(t)$ for all $t \in (0, T]$. Clearly, this is true for small $t > 0$. If our claim does not hold, then we can find a first $\tau \leq T$ such that $h(t) < \bar{h}(t)$ for all $t \in (0, \tau)$ and $h(\tau) = \bar{h}(\tau)$. It follows that

$$h'(\tau) \geq \bar{h}'(\tau). \tag{19}$$

We now show that

$$u(t, r) \leq \bar{u}(t, r), \quad v(t, r) \leq \bar{v}(t, r) \quad \text{for } (t, r) \in [0, \tau] \times [0, \infty).$$

Let $U = \bar{u} - u, V = \bar{v} - v$, we obtain

$$\begin{cases} U_t - d_1 \Delta U \geq aV - \theta U, & 0 < t \leq \tau, 0 < r < h(t), \\ V_t - d_2 \Delta V \geq \beta U - (b(\bar{v} + v) + c)V, & 0 < t \leq \tau, 0 < r < h(t), \\ U_r(t, 0) = V_r(t, 0) = 0, & 0 < t \leq \tau, \\ U(t, h(t)) \geq 0, \quad V(t, h(t)) \geq 0, & 0 < t \leq \tau, \\ U(0, r) \geq 0, \quad U(0, r) \geq 0, & 0 \leq r \leq h_0. \end{cases}$$

The strong maximum principle yields $U(t, r) > 0$ and $V(t, r) > 0$ for $(t, r) \in (0, \tau) \times [0, h(t))$. It follows from the Hopf boundary lemma that $U_r(\tau, h(\tau)) < 0$ and $V_r(\tau, h(\tau)) < 0$, i.e.

$$\bar{u}_r(\tau, h(\tau)) < u_r(\tau, h(\tau)), \quad \bar{v}_r(\tau, h(\tau)) < v_r(\tau, h(\tau)).$$

We then deduce that $h'(\tau) < \bar{h}'(\tau)$. But this contradicts (19). This proves our claim that $h(t) < \bar{h}(t)$ for all $t \in (0, T]$. We may now apply the above procedure over $[0, T] \times [0, h(t))$ to conclude (18). Moreover, $(u, v) < (\bar{u}, \bar{v})$ for $t \in (0, T]$ and $r \in [0, h(t))$.

If $\bar{h}(0) = h_0$, we use approximation. For small $\varepsilon > 0$, let $(u^\varepsilon, v^\varepsilon, h^\varepsilon)$ denote the unique solution of (4) with h_0 replaced by $h_0(1 - \varepsilon)$. Since the unique solution of (4) depends continuously on the parameters in (4), as $\varepsilon \rightarrow 0$, $(u^\varepsilon, v^\varepsilon, h^\varepsilon)$ converges to (u, v, h) , the unique solution of (4). The desired result then follows by letting $\varepsilon \rightarrow 0$ in the inequalities $(u^\varepsilon, v^\varepsilon) < (\bar{u}, \bar{v})$ and $h^\varepsilon < \bar{h}$. □

Remark 3.1 The pair $(\bar{u}, \bar{v}, \bar{h})$ in Theorem 3.3 is called an upper solution of the problem (4). We can define a lower solution by reversing all the inequalities in the above places. Moreover, one can easily prove an analog of Theorem 3.3 for lower solutions.

We next fix $u_0, v_0, d_1, d_2, a, \theta, \beta, b, c$ and h_0 to examine the dependence of the solution on μ . The solution is denoted as (u^μ, v^μ, h^μ) to emphasize this dependence. As a consequence of Theorem 3.3, we have the following result.

Corollary 3.1 *For fixed $u_0, v_0, d_1, d_2, a, \theta, \beta, b, c$ and h_0 . If $\mu_1 \leq \mu_2$, then $h^{\mu_1}(t) \leq h^{\mu_2}(t)$ in $(0, \infty)$ and*

$$u^{\mu_1}(t, r) \leq u^{\mu_2}(t, r), \quad v^{\mu_1}(t, r) \leq v^{\mu_2}(t, r)$$

for $t \in (0, \infty), r \in [0, h^{\mu_1}(t)]$.

4 Spreading and vanishing

In this section, we will prove Theorem 1.3. Precisely, we can deduce Theorem 1.3 directly from the following three theorems. To discuss the asymptotic behavior of u and v for the vanishing case ($s_\infty < \infty$), we first give the following proposition.

Proposition 4.1 *Let D, η, σ and s_0 be positive constants and C be any real number. Suppose*

$$w_0 \in C^2([0, s_0]), \quad w_{0r}(0) = w_0(s_0) = 0, \quad w_0(r) > 0 \quad \text{in } (0, s_0).$$

Assume that $s(t) \in C^{1+\frac{\sigma}{2}}([0, \infty))$, $s(t) > 0$ for $0 \leq t < \infty$, $\lim_{t \rightarrow \infty} s(t) = s_\infty < \infty$, $\lim_{t \rightarrow \infty} s'(t) = 0$; and that $w \in C^{\frac{1+\sigma}{2}, 1+\sigma}([0, \infty) \times [0, s(t)])$, $w(t, r) > 0$ for $0 \leq t < \infty$ and $0 < r < s(t)$, $\|w(t, \cdot)\|_{C^1[0, r(t)]} \leq M$ for any $t \geq 1$ and some $M > 0$. If (w, r) satisfies

$$\begin{cases} w_t - D\Delta w \geq Cw, & t > 0, 0 < r < s(t), \\ w_r(t, 0) = w(t, s(t)) = 0, & t > 0, \\ s'(t) \geq -\eta w_r, & t > 0, r = s(t), \\ w(0, r) = w_0(r), & 0 \leq r \leq s_0 = s(0), \end{cases}$$

then $\lim_{t \rightarrow \infty} \max_{0 \leq x \leq r(t)} w(t, x) = 0$.

Proof The proof is identical to that of Proposition 3.1 in [23], so we leave out the details. \square

Theorem 4.1 *Assume that (u, v, s) is the solution of problem (2). If $h_\infty < \infty$, then*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = \lim_{t \rightarrow \infty} \|v(t, \cdot)\|_{C([0, h(t)])} = 0. \tag{20}$$

Proof By the estimate of (7) we know that $\|h'\|_{C^{\frac{\nu}{2}}([1, \infty))} \leq C$. Combining this with $h'(t) > 0$ and $h_\infty < \infty$ implies $h'(t) \rightarrow 0$ as $t \rightarrow \infty$.

From the proof of Theorem 3.1 we can find that $u, v > 0$ for $t > 0, 0 < r < h(t)$ and $u_r(t, h(t)) < 0, v_r(t, h(t)) < 0$. Thus it follows that

$$\begin{cases} u_t - d_1 \Delta u \geq -\theta u, & t > 0, 0 < r < h(t), \\ u_r(t, 0) = u(t, h(t)) = 0, & t > 0, \\ h'(t) \geq -\mu u_r(t, h(t)), & t > 0, \\ h(0) = h_0, \quad u(0, r) = u_0(r), & 0 \leq r \leq h_0, \end{cases}$$

By virtue of (5), (7) and Proposition 4.1 it is derived that

$$\lim_{t \rightarrow \infty} \max_{0 \leq x \leq s(t)} u(t, x) = 0.$$

In the same way we immediately get $\lim_{t \rightarrow \infty} \max_{0 \leq x \leq s(t)} v(t, x) = 0$.

This proof is completed. \square

The result of Theorem 4.1 shows that if the new or invasive species cannot spread into the whole space, then it will die out eventually. In the following theorem, we will give a necessary condition for vanishing.

Theorem 4.2 *Let (u, v, h) be any solution of (4). If $h_\infty < \infty$, then $h_\infty \leq R^*$, where R^* is defined as in (12).*

Proof Theorem 3.1 implies that, if $h_\infty < \infty$, then (20) holds. We assume $h_\infty > R^*$ to get a contradiction. It is easy to see that there exists $\tau \gg 1$ such that $R^* < h(\tau) := R$. Let (w, z) be the positive solution of the following initial-boundary value problem with fixed boundary:

$$\begin{cases} w_t - d_1 \Delta w = az - \theta w, & t > \tau, 0 < r < R, \\ z_t - d_2 \Delta z = \beta w - bz^2 - cz, & t > \tau, 0 < r < R, \\ w_r(t, 0) = z_r(t, 0) = w(t, R) = z(t, R) = 0, & t > \tau, \\ w(\tau, r) = u(\tau, r), \quad z(\tau, r) = v(\tau, r), & 0 \leq r \leq R. \end{cases}$$

By the comparison principle,

$$(w(t, r), z(t, r)) \leq (u(t, r), v(t, r)), \quad \forall t \geq \tau, 0 \leq r \leq R.$$

Since $R > R^*$ and Theorem 2.2, one can easily see that $(w(t, r), z(t, r)) \rightarrow (u_R^*(r), v_R^*(r))$ as $t \rightarrow \infty$ uniformly in $[0, R]$, where $(u_R^*(r), v_R^*(r))$ is the unique positive solution of (10) with $\lambda = 1$. Therefore,

$$\liminf_{t \rightarrow \infty} u(t, r) \geq u_R^*(r), \quad \liminf_{t \rightarrow \infty} v(t, r) \geq v_R^*(r) \quad \text{for } r \in [0, R].$$

This is a contradiction to Eq. (20). Therefore, $h_\infty \leq R^*$. □

Theorem 4.3 *If $h_\infty = \infty$, then*

$$\lim_{t \rightarrow \infty} (u(t, r), v(t, r)) = (\tilde{u}, \tilde{v})$$

uniformly in any bounded subset of $[0, \infty)$.

Proof First, we consider the ODE problem (16). By the comparison principle, we can easily see that $(u(t, r), v(t, r)) \leq (\bar{u}(t), \bar{v}(t))$ for $t > 0$ and $r \in [0, h(t)]$. Similarly to the proof of Theorem 2.1 in [30], we see that the positive constant steady state (\tilde{u}, \tilde{v}) of (16) is globally asymptotically stable. So $\lim_{t \rightarrow \infty} (\bar{u}(t), \bar{v}(t)) = (\tilde{u}, \tilde{v})$, moreover,

$$\limsup_{t \rightarrow \infty} u(t, r) \leq \tilde{u}, \quad \limsup_{t \rightarrow \infty} v(t, r) \leq \tilde{v} \quad \text{uniformly for } r \in [0, \infty).$$

Next, we show

$$\liminf_{t \rightarrow \infty} u(t, r) \geq \tilde{u}, \quad \liminf_{t \rightarrow \infty} v(t, r) \geq \tilde{v}$$

locally uniformly for $r \in [0, \infty)$. For any $R > \max\{h_0, R^*\}$, there exists $t_R > 0$ such that $h(t_R) = R$. By use of the comparison principle we have $(u(t, r), v(t, r)) \geq (\underline{u}_R(t, r), \underline{v}_R(t, r))$ in $(t_R, \infty) \times$

$(0, R)$, where $(\underline{u}_R(t, r), \underline{v}_R(t, r))$ is the solution of the following problem with fixed boundary:

$$\begin{cases} (\underline{u}_R)_t - d_1 \Delta \underline{u}_R = a \underline{v}_R - \theta \underline{u}_R, & t > t_R, 0 < r < R, \\ (\underline{v}_R)_t - d_2 \Delta \underline{v}_R = \beta \underline{u}_R - b \underline{v}_R^2 - c \underline{v}_R, & t > t_R, 0 < r < R, \\ (\underline{u}_R)_r(t, 0) = (\underline{v}_R)_r(t, 0) = 0, & t > t_R, \\ \underline{u}_R(t, R) = \underline{v}_R(t, R) = 0, & t > t_R, \\ \underline{u}_R(t_R, r) = u(t_R, r), \quad \underline{v}_R(t_R, r) = v(t_R, r), & 0 \leq r \leq R. \end{cases}$$

By virtue of $R > R^*$ and Theorem 2.2, we easily see that $(\underline{u}_R(t, r), \underline{v}_R(t, r)) \rightarrow (u_R^*(r), v_R^*(r))$ as $t \rightarrow \infty$ uniformly in $[0, R]$, where $(u_R^*(r), v_R^*(r))$ is the unique positive solution of (10) with $\lambda = 1$. It follows that

$$\liminf_{t \rightarrow \infty} u(t, r) \geq u_R^*(r), \quad \liminf_{t \rightarrow \infty} v(t, r) \geq v_R^*(r) \tag{21}$$

uniformly in compact subsets of $[0, R)$. Similarly to the process of the last half proof of Theorem 1.1, we see that $(u_R^*(r), v_R^*(r)) \rightarrow (\tilde{u}, \tilde{v})$ as $R \rightarrow \infty$ uniformly in any compact subset of $[0, \infty)$ and then

$$\liminf_{t \rightarrow \infty} u(t, r) \geq \tilde{u}, \quad \liminf_{t \rightarrow \infty} v(t, r) \geq \tilde{v}$$

locally uniformly for $r \in [0, \infty)$, due to (21). This completes the proof of the desired result. □

Combining Theorems 4.1, 4.2 and 4.3, we have Theorem 1.3.

5 The criteria governing spreading and vanishing

Now we decide exactly when each of the two alternative occurs. The discussion will be divided into two cases:

- (a) $h_0 \geq R^*$, (b) $h_0 < R^*$.

For the case (a), due to $h'(t) > 0$ for $t > 0$, we must have $h_\infty > R^*$. Hence Theorem 4.2 implies that if $h_0 \geq R^*$, then $h_\infty = \infty$.

Next we discuss the case (b).

Theorem 5.1 *If $h_0 < R^*$, then there exists $\mu^0 > 0$ depending on (u_0, v_0) such that $h_\infty = \infty$ if $\mu \geq \mu^0$.*

Proof This proof is similar to [24, Lemma 3.6]. We give the details below for completeness.

We see from (6) that there exists a constant $\delta^* > 0$ such that the solution (u, v, h) of problem (4) satisfies

$$av - \theta u \geq -\delta^* u, \quad \beta u - bv^2 - cv \geq -\delta^* v$$

for all $(t, r) \in \Omega$. We next consider the auxiliary free boundary problem

$$\begin{cases} w_t - d_1 \Delta w = -\delta^* w, & t > 0, 0 < r < g(t), \\ z_t - d_2 \Delta z = -\delta^* z, & t > 0, 0 < r < g(t), \\ z_r(t, 0) = w_r(t, 0) = z(t, g(t)) = w(t, g(t)) = 0, & t > 0, \\ g'(t) = -\mu(w_r + \rho z_r), & t > 0, r = g(t), \\ g(0) = h_0, \quad z(0, r) = u_0(r), \quad w(0, r) = v_0(r), & 0 \leq r \leq h_0. \end{cases} \tag{22}$$

Arguing as in proving the existence and uniqueness of the solution to (4), one will easily see that (22) also admits a unique solution (w, z, g) which is well defined for all $t > 0$. Moreover, due to the Hopf boundary lemma, $g'(t) > 0$ for $t > 0$. To stress the dependence of the solutions on the parameter μ , in the sequel, we always write (u^μ, v^μ, h^μ) and (w^μ, z^μ, g^μ) instead of (u, v, h) and (w, z, g) . By Lemma 11, we have

$$\begin{aligned} u^\mu(t, r) &\geq w^\mu(t, x), & v^\mu(t, r) &\geq z^\mu(t, r), & h^\mu(t) &\geq g^\mu(t), \\ \forall t \geq 0, r &\in [0, g^\mu(t)]. \end{aligned} \tag{23}$$

In what follows, we are going to prove that, for all large μ ,

$$g^\mu(2) \geq 2R^*. \tag{24}$$

To the end, we first choose a smooth function $\underline{g}(t)$ with $\underline{g}(0) = h_0/2$, $\underline{g}'(t) > 0$ and $\underline{g}(2) = 2R^*$. We then consider the following initial-boundary value problem:

$$\begin{cases} \underline{w}_t - d_1 \Delta \underline{w} = -\delta^* \underline{w}, & t > 0, 0 < r < \underline{g}(t), \\ \underline{z}_t - d_2 \Delta \underline{z} = -\delta^* \underline{z}, & t > 0, 0 < r < \underline{g}(t), \\ \underline{w}_r(t, 0) = \underline{z}_r(t, 0) = 0, & t > 0, \\ \underline{w}(t, \underline{g}(t)) = \underline{z}(t, \underline{g}(t)) = 0, & t > 0, \\ \underline{w}(0, r) = \underline{w}_0(r), \quad \underline{z}(0, r) = \underline{z}_0(r), & 0 \leq r \leq h_0/2. \end{cases} \tag{25}$$

Here, for the smooth initial value $(\underline{w}_0, \underline{z}_0)$, we require

$$\begin{cases} \underline{w}_0 \leq u_0 & \text{on } [0, h_0/2], & (\underline{w}_0)_r(0) = \underline{w}_0(h_0/2) = 0, & (\underline{w}_0)_r(h_0/2) < 0, \\ \underline{z}_0 \leq v_0 & \text{on } [0, h_0/2], & (\underline{z}_0)_r(0) = \underline{z}_0(h_0/2) = 0, & (\underline{z}_0)_r(h_0/2) < 0. \end{cases} \tag{26}$$

The standard theory for parabolic equations ensures that (25) has a unique positive solution $(\underline{w}, \underline{z})$, and $\underline{w}_r(t, \underline{g}(t)) < 0$, $\underline{z}_r(t, \underline{g}(t)) < 0$ for all $t \in [0, 2]$ due to the Hopf boundary lemma. According to our choice of $\underline{g}(t)$ and $(\underline{w}_0, \underline{z}_0)$, there is a constant $\mu^0 > 0$ such that, for all $\mu \geq \mu^0$,

$$\underline{g}'(t) \leq -\mu[\underline{w}_r(t, \underline{g}(t)) + \rho \underline{z}_r(t, \underline{g}(t))], \quad 0 \leq t \leq 2. \tag{27}$$

On the other hand, for system (25), we can establish the comparison principle analogous with lower solution to Theorem 3.3 by the same argument. Thus, note that $\underline{g}(0) = h_0/2 < h_0$,

it follows from (25), (26) and (27) that

$$w^\mu(t, r) \geq \underline{w}(t, r), \quad z^\mu(t, r) \geq \underline{z}(t, r), \quad g^\mu(t) \geq \underline{g}(t), \quad \forall t \in [0, 2], r \in [0, \underline{g}(t)].$$

This particularly implies $g^\mu(2) \geq \underline{g}(2) = 2R^*$, and so (24) holds. Hence, in view of (23) and (24), we find

$$h_\infty = \lim_{t \rightarrow \infty} h^\mu(t) > h^\mu(2) \geq 2R^*.$$

This, together with Theorem 4.2, yields the desired result. □

Theorem 5.2 *If $h_0 < R^*$, then there exists $\mu_0 > 0$ depending on (u_0, v_0) such that $h_\infty < \infty$ if $\mu \leq \mu_0$.*

Proof We are going to construct a suitable supper solution to (4) and then apply Theorem 3.3.

Inspired by [2], for $t > 0$ and $r \in [0, \eta(t)]$, we define

$$\eta(t) = h_0 \left(1 + \delta - \frac{\delta}{2} e^{-\gamma t} \right), \quad \bar{u}(t, r) = K_1 e^{-\gamma t} \phi \left(\frac{h_0 r}{\eta(t)} \right), \quad \bar{v}(t, r) = K_2 e^{-\gamma t} \phi \left(\frac{h_0 r}{\eta(t)} \right),$$

where δ, γ, K_1 and K_2 are positive constants to be chosen later and $\phi(|x|)$ is the first eigenfunction of the problem

$$\begin{cases} -d_1 \Delta \phi = \lambda_1(d_1, h_0) \phi, & x \in B_{h_0}, \\ \phi = 0, & x \in \partial B_{h_0}, \end{cases}$$

with $\phi > 0$ in B_{h_0} and $\|\phi\|_\infty = 1$. Since $h_0 < R^*$, we have

$$\lambda_1(d_1, h_0) > \frac{-[d_2\theta + d_1c] + \sqrt{[d_2\theta + d_1c]^2 + 4d_1d_2[a\beta - c\theta]}}{2d_2}. \tag{28}$$

We also observe that $\phi'(0) = 0$ and

$$-(r^{n-1} \phi')' = r^{n-1} \lambda_1(d_1, h_0) \phi > 0, \quad \forall 0 < r < h_0.$$

It follows that

$$\phi'(r) < 0, \quad \forall 0 < r \leq h_0.$$

Set $\sigma(t) = \eta(t)/h_0$, then $\eta(t) = h_0\sigma(t)$. Direct computations yield

$$\begin{aligned} & \bar{u}_t - d_1 \Delta \bar{u} - a\bar{v} + \theta \bar{u} \\ &= K_1 e^{-\gamma t} \left[-\gamma \phi - r \sigma^{-2} \sigma' \phi' + \left(-d_1 \Delta \phi \left(\frac{r}{\sigma} \right) \right) - \frac{aK_2}{K_1} \phi + \theta \phi \right] \\ &\geq K_1 e^{-\gamma t} \left[-\gamma \phi + \sigma^{-2} \lambda_1(d_1, h_0) \phi - \frac{aK_2}{K_1} \phi + \theta \phi \right] \end{aligned}$$

$$\begin{aligned} &\geq K_1 e^{-\gamma t} \phi \left[-\gamma + \sigma^{-2} \lambda_1(d_1, h_0) - \frac{aK_2}{K_1} + \theta \right], \\ \bar{v}_t - d_2 \Delta \bar{v} - \beta \bar{u} + b \bar{v}^2 + c \bar{v} \\ &= K_2 e^{-\gamma t} \left[-\gamma \phi - r \sigma^{-2} \sigma' \phi' + \left(-d_2 \Delta \phi \left(\frac{r}{\sigma} \right) \right) - \frac{\beta K_1}{K_2} \phi + b K_2 e^{-\gamma t} \phi^2 + c \phi \right] \\ &\geq K_2 e^{-\gamma t} \left[-\gamma \phi + \frac{d_2}{d_1 \sigma^2} \lambda_1(d_1, h_0) \phi - \frac{\beta K_1}{K_2} \phi + b K_2 e^{-\gamma t} \phi^2 + c \phi \right] \\ &\geq K_2 e^{-\gamma t} \phi \left[-\gamma + \frac{d_2}{d_1} \frac{\lambda_1(d_1, h_0)}{(1 + \delta)^2} - \frac{\beta K_1}{K_2} + b K_2 e^{-\gamma t} \phi + c \right]. \end{aligned}$$

We easily see that

$$h_0(1 + \delta/2) \leq \eta(t) \leq h_0(1 + \delta), \quad 1 + \delta/2 \leq \sigma(t) \leq 1 + \delta.$$

Hence, due to (28), we can choose $\gamma, \delta > 0$ sufficiently small such that

$$\begin{aligned} \frac{\lambda_1(d_1, h_0)}{(1 + \delta)^2} &> \frac{1}{2d_2} \left\{ -[d_2(\theta - \gamma) + d_1(c - \gamma)] \right. \\ &\quad \left. + \sqrt{[d_2(\theta - \gamma) + d_1(c - \gamma)]^2 + 4d_1 d_2 [a\beta - (c - \gamma)(\theta - \gamma)]} \right\}. \end{aligned}$$

Then

$$\frac{d_2 \lambda_1(d_1, h_0)}{d_1(1 + \delta)^2} - \frac{a\beta}{\theta - \gamma + (1 + \delta)^{-2} \lambda_1(d_1, h_0)} + c - \gamma > 0.$$

Let $\frac{K_1}{K_2}$ satisfy

$$\frac{a}{\theta - \gamma + (1 + \delta)^{-2} \lambda_1(d_1, h_0)} \leq \frac{K_1}{K_2} \leq \frac{1}{\beta} \left(\frac{d_2 \lambda_1(d_1, h_0)}{d_1(1 + \delta)^2} + c - \gamma \right). \tag{29}$$

It can be derived that

$$\begin{aligned} \bar{u}_t - d_1 \Delta \bar{u} - a \bar{v} + \theta \bar{u} &\geq 0, \quad t > 0, r \in [0, \eta(t)], \\ \bar{v}_t - d_2 \Delta \bar{v} - \beta \bar{u} + b \bar{v}^2 + c \bar{v} &\geq 0, \quad t > 0, r \in [0, \eta(t)]. \end{aligned}$$

We now choose $K_1, K_2 > 0$ satisfying (29) and sufficiently large such that

$$\begin{aligned} u_0(r) &\leq K_1 \phi \left(\frac{r}{1 + \delta/2} \right) = \bar{u}(0, r), \quad r \in [0, h_0], \\ v_0(r) &\leq K_2 \phi \left(\frac{r}{1 + \delta/2} \right) = \bar{v}(0, r), \quad r \in [0, h_0]. \end{aligned}$$

The direct calculation yields

$$\eta'(t) = \frac{1}{2} h_0 \gamma \delta e^{-\gamma t},$$

$$\begin{aligned}
 -\mu[\bar{u}_r(t, \eta(t)) + \rho\bar{v}_r(t, \eta(t))] &= \mu(K_1 + \rho K_2)e^{-\gamma t} \frac{h_0|\phi_r(h_0)|}{\eta(t)} \\
 &\leq \mu(K_1 + \rho K_2)e^{-\gamma t} \frac{h_0|\phi_r(h_0)|}{1 + \delta/2}.
 \end{aligned}$$

Therefore, if we take

$$\mu_0 = \frac{h_0\gamma\delta(1 + \delta/2)}{2(K_1 + \rho K_2)|\phi_r(h_0)|},$$

then, for any $0 < \mu \leq \mu_0$, we have $\eta'(t) \geq -\mu[\bar{u}_r(t, \eta(t)) + \rho\bar{v}_r(t, \eta(t))]$. Thus, (\bar{u}, \bar{v}, η) satisfies

$$\begin{cases}
 \bar{u}_t - d_1\Delta\bar{u} \geq a\bar{v} - \theta\bar{u}, & t > 0, 0 < r < \eta(t), \\
 \bar{v}_t - d_2\Delta\bar{v} \geq \beta\bar{u} - b\bar{v}^2 - c\bar{v}, & t > 0, 0 < r < \eta(t), \\
 \bar{u}_r(t, 0) = \bar{v}_r(t, 0) = 0, & t > 0, \\
 \bar{u}(t, \eta(t)) = \bar{v}(t, \eta(t)) = 0, & t > 0, \\
 \eta'(t) \geq -\mu[\bar{u}_r(t, \eta(t)) + \rho\bar{v}_r(t, \eta(t))], & t > 0, \\
 \bar{u}(0, r) \geq u_0(r), \quad \bar{v}(0, r) \geq v_0(r), & 0 \leq r \leq h_0, \\
 \eta(0) = h_0(1 + \delta/2).
 \end{cases}$$

We can apply Theorem 3.3 to conclude that $h(t) \leq \eta(t)$, $u(t, r) \leq \bar{u}(t, r)$ and $v(t, r) \leq \bar{v}(t, r)$ for $(t, r) \in \Omega$. Therefore, $h_\infty \leq \lim_{t \rightarrow \infty} \eta(t) \leq h_0(1 + \delta) < \infty$. □

Proof of Theorem 1.4 The proof is similar to that of Theorem 5.2 of [16] and Theorem 4.11 of [4]. For convenience of the reader we shall give the details. Denote (u_μ, v_μ, h_μ) in place of (u, v, h) to clarify the dependence of the solution of (4) on μ . Set

$$\Sigma = \{ \mu > 0 : h_{\mu, \infty} \leq R^* \}.$$

By Theorem 5.2 and Theorem 3.3, $(0, \mu_0) \subset \Sigma$. In view of Theorem 5.1, $\Sigma \cap [\mu^0, \infty) = \emptyset$. Therefore, $\mu^* := \sup \Sigma \in [\mu_0, \mu^0]$. By this definition and Corollary 3.1 we find that $h_{\mu, \infty} \leq R^*$ when $\mu < \mu^*$ and $h_{\mu, \infty} = \infty$ when $\mu > \mu^*$.

We will show that $\mu^* \in \Sigma$. Otherwise, $h_{\mu^*, \infty} = \infty$. There exists $T > 0$ such that $h_{\mu^*}(T) > R^*$. By the continuous dependence of (u_μ, v_μ, h_μ) on μ , there is $\varepsilon > 0$ such that $h_\mu(T) > R^*$ for $\mu \in (\mu^* - \varepsilon, \mu^* + \varepsilon)$. It follows that, for all such μ ,

$$\lim_{t \rightarrow \infty} h_\mu(t) > h_\mu(T) > R^*.$$

This implies that $[\mu^* - \varepsilon, \mu^* + \varepsilon] \cap \Sigma = \emptyset$, and $\sup \Sigma \leq \mu^* - \varepsilon$. This contradicts the definition of μ^* . The proof is completed. □

6 Discussion

We have examined a free boundary problem of a single-species stage-structured model with higher space dimensions and heterogeneous environment for the special case that the environment and solution are radially symmetric. If the environment or solution is not radially symmetric, then the boundary of the spreading domain would not still be a sphere

and the Stefan condition $h'(t) = -\mu[u_r(t, h(t)) + \rho v_r(t, h(t))]$ would become very complicated. Similar to the classical Stefan problem, smooth solutions to these free boundary problems need not exist even if the initial data are smooth. It is necessary to make use of other methods to discuss these problems.

In this paper, we firstly discuss the model on \mathbb{R}^n , prove that the positive constant steady state is globally asymptotically stable (Theorem 1.1). Then we investigate a free boundary problem of the single-species stage-structured model. Our results about vanishing and spreading of the model generalize and unify the previous Theorems 1.2–1.4, which are the existence and uniqueness of solution, the spreading–vanishing dichotomy, the long time behavior of the solution and sharp criteria for spreading and vanishing.

Biologically, the model with stage structure is more realistic than the model without stage structure. From our results, one can control vanishing and spreading of the species more flexibly by the introduction of stage structure. Note that with (12) and Theorem 1.3, if (u_0, v_0) and h_0 are fixed, then R^* is a direct factor determining the vanishing and spreading. To help the species spreading to infinity, it can be realized by enlarging the birth rate a of the immature or the birth rate β of the mature.

There are some problems left unsolved in our work. When spreading happens, can we find the spreading speed? Can we extend system (4) into the two-species competitive system with staged structure? We leave these problems to our future work.

Acknowledgements

The authors would like to deeply thank all the reviewers for their insightful and constructive comments.

Funding

This work was supported by the National Natural Science Foundation of China (No. 11601542, 11671367) and Key Research Projects of Henan Higher Education Institutions (No. 18A110036, 18A110038).

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 10 May 2018 Accepted: 2 September 2018 Published online: 14 September 2018

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