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Multiplicity for fractional differential equations with p-Laplacian

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Abstract

This paper investigates the existence of positive solution for a boundary value problem of fractional differential equations with p-Laplacian operator. Our analysis relies on the research of properties of the corresponding Green's function. By the use of Krasnosel'skii's fixed-point theorem, the multiplicity results of some positive solutions are obtained.

MSC: 34A08; 34B18; 35J05

Keywords: Fractional differential equation; Boundary value problem; p-Laplacian operator; Positive solution

1 Introduction

In this paper, we consider positive solutions for the following problem:

$$D_{0+}^{\beta}\left(\varphi_p\left(D_{0+}^{\alpha}y(x)\right)\right)=f\left(x,y(x)\right), \quad 0<x<1, \tag{1.1}$$

$$y(0)=y'(0)=y(1)=D_{0+}^{\alpha}y(0)=0, \qquad D_{0+}^{\alpha}y(1)=\lambda D_{0+}^{\alpha}y(\xi), \tag{1.2}$$

where $\alpha,\beta\in\mathbb{R}, 2<\alpha\leq 3, 1<\beta\leq 2$, and $\xi\in(0,1), \lambda\in[0,+\infty), \varphi_p(z)=|z|^{p-2}z, p>1, D_{0+}^{\alpha}$ is the Riemann–Liouville fractional derivative, and $f\in C([0,1]\times[0,+\infty),[0,+\infty))$. By using Krasnosel'skii's fixed-point theorem, we give some multiplicity results.

Differential equations of fractional order, or fractional differential equations, in which an unknown function is contained under the operation of a derivative of fractional order, have been of great interest recently. Fractional differential equation models are proved to be more adequate than integer order models for some problems in science and engineering. Many papers and books on fractional calculus and fractional differential equations have appeared recently. For an introduction of fractional calculus and fractional differential equations, we refer the reader to [17, 25] and the references therein. And there have been many results on existence and uniqueness of the solution of boundary value problems for fractional differential equations. For example, fractional boundary value problems at resonance [1, 5, 27, 39, 40], Caputo fractional derivative problems [11, 23, 37], impulsive problems [2, 15, 29, 41], multi-point problems [1, 5, 21, 22, 27–29, 31, 40], integral boundary value problems [6, 12, 13, 15], fractional p-Laplace problems [8, 10, 14, 21, 22, 35, 36], fractional lower and upper solution problems [4, 7, 30, 38], fractional delay problems, [24, 33, 34], solitons [9], singular problems [3], etc.

On the other hand, integer order differential equations with p-Laplacian operator also arise in different research areas such as physical and natural phenomena, non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology, nonlinear flow laws, and system of Monge–Kantorovich partial differential equations [8, 16, 19, 20, 32]. For example, turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problem, Leibenson [19] introduced differential equations with p-Laplacian operator

$$\bigl(\varphi_p\bigl(y'(x)\bigr)\bigr)' + f\bigl(x, y(x)\bigr) = 0.$$

In [26], by applying the fixed point index theory, Su studied the existence of positive solutions of a nonlinear four-point singular boundary value problem with a p-Laplacian operator:

$$\begin{cases} (\varphi_p(y'(x)))' + a(x)f(y(x)) = 0, & 0 < x < 1, \\ \alpha\varphi_p(y(0)) - \beta\varphi_p(y'(\xi)) = 0, & \gamma\varphi_p(y(1)) - \delta\varphi_p(y'(\eta)) = 0. \end{cases}$$

It is quite natural to study fractional differential equation relative to equation. Recently, many scholars have paid more attention to the fractional order differential equation boundary value problems with p-Laplacian operator, see [8, 10, 14, 21, 22]. Recently, Dong et al. [10] investigated the following p-Laplacian fractional differential equation boundary value problem:

$$D^\alpha\bigl(\varphi_p\bigl(D^\alpha y(x)\bigr)\bigr) = f\bigl(x, y(x)\bigr), \quad 0 < x < 1, \tag{1.3}$$

$$y(0) = y(1) = D^\alpha y(0) = D^\alpha y(1) = 0, \tag{1.4}$$

where $1 < \alpha \leq 2$ is a real number, D^α is the conformable fractional derivative. Some existence and multiplicity results of positive solutions are proved by the fixed-point theorems on cone. The purpose of this paper is to generalize some existence results of the above references to a nonlinear fractional boundary value problem with p-Laplacian.

2 Preliminaries

Definition 2.1 ([17]) The fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is defined as

$$I^\alpha_{0+} y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-z)^{\alpha-1} y(z)\,dz.$$

Definition 2.2 ([17]) The fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, +\infty) \to \mathbb{R}$ is defined as

$$D^\alpha_{0+} y(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{y(z)}{(x-z)^{\alpha-n+1}}\,dz,$$

where $n = -[-\alpha]$.

Lemma 2.1 ([17]) *Assume that $y, D^{\alpha}_{0+}y \in C(0,1) \cap L(0,1)$. Then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}y(x) = y(x) + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \cdots + c_N x^{\alpha-N}, \qquad c_i \in \mathbb{R}, i = 1, 2, \ldots, N,$$

where $N = -[-\alpha]$.

Let $\mathcal{M} := \lambda^{p-1}\xi^{\beta-1}$ such that $\mathcal{M} \neq 1$, and

$$G(x,z) = \begin{cases} \frac{1}{\Gamma(\alpha)}[x(1-z)]^{\alpha-1}, & 0 \le x \le z \le 1; \\ \frac{1}{\Gamma(\alpha)}([x(1-z)]^{\alpha-1} - (x-z)^{\alpha-1}), & 0 \le z \le x \le 1, \end{cases} \tag{2.1}$$

$$H(x,z) = \begin{cases} \frac{[x(1-z)]^{\beta-1} - \lambda^{p-1}[x(\xi-z)]^{\beta-1} - (1-\mathcal{M})(x-z)^{\beta-1}}{(1-\mathcal{M})\Gamma(\beta)}, & 0 \le z \le x \le 1, z \le \xi; \\ \frac{[x(1-z)]^{\beta-1} - (1-\mathcal{M})(x-z)^{\beta-1}}{(1-\mathcal{M})\Gamma(\beta)}, & 0 < \xi \le z \le x \le 1; \\ \frac{[x(1-z)]^{\beta-1} - \lambda^{p-1}[x(\xi-z)]^{\beta-1}}{(1-\mathcal{M})\Gamma(\beta)}, & 0 \le x \le z \le \xi < 1; \\ \frac{[x(1-z)]^{\beta-1}}{(1-\mathcal{M})\Gamma(\beta)}, & 0 \le x \le z \le 1, \xi \le z. \end{cases} \tag{2.2}$$

Lemma 2.2 *Let $Q(x) = x(1-x)^{\alpha-1}$. Then functions $G(x,z), H(x,z) \in C([0,1] \times [0,1])$ and satisfy*:

(1) $G(x,z) = G(1-z, 1-x)$; $G(x,z) > 0$ *for* $x, z \in (0,1)$;
(2) $\frac{Q(1-x)Q(z)}{\Gamma(\alpha)} \le G(x,z) \le \frac{(\alpha-1)Q(z)}{\Gamma(\alpha)}$ *for* $x, z \in [0,1]$;
(3) *If* $\mathcal{M} < 1$, *then* $H(x,z) > 0$ *for* $x, z \in (0,1)$.

Proof It is easily seen that functions $G(x,z), H(x,z) \in C([0,1] \times [0,1])$ and (1) hold. We will only prove (2) and (3).

(2) For $0 \le z \le x \le 1$, since $0 < \alpha - 2 \le 1$, one has

$$\begin{aligned} G(x,z) &= \frac{1}{\Gamma(\alpha)}\left(\left[x(1-z)\right]^{\alpha-1} - (x-z)^{\alpha-1}\right) \\ &= \frac{\alpha-1}{\Gamma(\alpha)}\int_{x-z}^{x(1-z)} s^{\alpha-2}\,ds \\ &\le \frac{\alpha-1}{\Gamma(\alpha)}\left[x(1-z)\right]^{\alpha-2}\left[x(1-z) - (x-z)\right] \\ &\le \frac{\alpha-1}{\Gamma(\alpha)}(1-z)^{\alpha-2}z(1-x) \\ &\le \frac{(\alpha-1)Q(z)}{\Gamma(\alpha)}, \end{aligned}$$

and

$$\begin{aligned} G(x,z) &= \frac{1}{\Gamma(\alpha)}\left(\left[x(1-z)\right]^{\alpha-1} - (x-z)^{\alpha-1}\right) \\ &\ge \frac{1}{\Gamma(\alpha)}\left(\left[x(1-z)\right]^{\alpha-2}\left[x(1-z)\right] - (x-z)]\right) \\ &= \frac{1}{\Gamma(\alpha)}\left[x(1-z)\right]^{\alpha-2}z(1-x) \\ &\ge \frac{1}{\Gamma(\alpha)}x^{\alpha-1}(1-z)^{\alpha-1}z(1-x) \end{aligned}$$

$$= \frac{Q(z)Q(1-x)}{\Gamma(\alpha)}.$$

For $0 \le x \le z \le 1$, one has

$$\begin{aligned} G(x,z) &= \frac{1}{\Gamma(\alpha)} x^{\alpha-1}(1-z)^{\alpha-1} \\ &\le \frac{1}{\Gamma(\alpha)} z^{\alpha-1}(1-z)^{\alpha-1} \\ &\le \frac{1}{\Gamma(\alpha)}(\alpha-1)z(1-z)^{\alpha-1} \\ &\le \frac{(\alpha-1)Q(z)}{\Gamma(\alpha)}, \end{aligned}$$

and

$$\begin{aligned} G(x,z) &= \frac{1}{\Gamma(\alpha)} x^{\alpha-1}(1-z)^{\alpha-1} \\ &\ge \frac{1}{\Gamma(\alpha)} x^{\alpha-1}(1-z)^{\alpha-1} z(1-x) \\ &= \frac{Q(z)Q(1-x)}{\Gamma(\alpha)}. \end{aligned}$$

So, $\frac{Q(1-z)Q(z)}{\Gamma(\alpha)} \le G(x,z) \le \frac{(\alpha-1)Q(z)}{\Gamma(\alpha)}$ for $x, z \in [0,1]$.

(3) For $0 < z \le x < 1, z \le \xi$, set

$$h(x,z) = \frac{[x(1-z)]^{\beta-1} - (x-z)^{\beta-1}}{\Gamma(\beta)}.$$

It is obvious that $h(x,z) > 0$ for $0 < z \le x < 1$. Hence, we get

$$\begin{aligned} H(x,z) &= \frac{[x(1-z)]^{\beta-1} - \lambda^{p-1}[x(\xi-z)]^{\beta-1} - (1-\mathcal{M})(x-z)^{\beta-1}}{(1-\mathcal{M})\Gamma(\beta)} \\ &= \left(1 + \frac{\mathcal{M}}{1-\mathcal{M}}\right)\frac{[x(1-z)]^{\beta-1}}{\Gamma(\beta)} - \frac{(x-z)^{\beta-1}}{\Gamma(\beta)} - \frac{\lambda^{p-1}[x(\xi-z)]^{\beta-1}}{(1-\mathcal{M})\Gamma(\beta)} \\ &= \frac{[x(1-z)]^{\beta-1} - (x-z)^{\beta-1}}{\Gamma(\beta)} + \frac{\lambda^{p-1}x^{\beta-1}[\xi^{\beta-1}(1-z)^{\beta-1} - (\xi-z)^{\beta-1}]}{(1-\mathcal{M})\Gamma(\beta)} \\ &= h(x,z) + \frac{\lambda^{p-1}x^{\beta-1}}{1-\mathcal{M}} h(\xi,z) \\ &> 0. \end{aligned}$$

Similarly, there holds $H(x,z) > 0$ for $0 < \eta \le z \le x < 1$ or $0 < x \le z \le \eta < 1$ or $0 < x \le z < 1, \eta \le z$.

Thus, $H(x,z) > 0$ for $x, z \in (0,1)$. The proof is completed. □

Lemma 2.3 *Suppose that* $h \in C[0,1], p > 1, \alpha, \beta \in \mathbb{R}, 2 < \alpha \leq 3, 1 < \beta \leq 2$ *and* $\xi \in (0,1), \lambda \in [0,+\infty)$. *Then the following problem*

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}y(x))) = y(x), & 0 < x < 1, \\ y(0) = y'(0) = y(1) = D_{0+}^{\alpha}y(0) = 0, & D_{0+}^{\alpha}y(1) = \lambda D_{0+}^{\alpha}y(\xi), \end{cases} \tag{2.3}$$

has a unique solution

$$y(x) = \int_0^1 G(x,z)\varphi_q\left(\int_0^1 H(z,\tau)y(\tau)\,d\tau\right)dz,$$

where $\varphi_q = (\varphi_p)^{-1}, \frac{1}{p} + \frac{1}{q} = 1$.

Proof From the equation of (2.2), Lemma 2.1, and the fact that $D_{0+}^{\alpha}y(0) = 0$, there is

$$\varphi_p\big(D_{0+}^{\alpha}y(x)\big) = \frac{1}{\Gamma(\beta)}\int_0^x (x-z)^{\beta-1}y(z)\,dz + c_1x^{\beta-1} \tag{2.4}$$

for some $c_1 \in \mathbb{R}$. Thus,

$$\varphi_p\big(D_{0+}^{\alpha}y(1)\big) = \frac{1}{\Gamma(\beta)}\int_0^1 (1-z)^{\beta-1}y(z)\,dz + c_1, \tag{2.5}$$

$$\varphi_p\big(D_{0+}^{\alpha}y(\xi)\big) = \frac{1}{\Gamma(\beta)}\int_0^{\xi} (\xi-z)^{\beta-1}y(z)\,dz + c_1\xi^{\beta-1}. \tag{2.6}$$

Taking into account that $D_{0+}^{\alpha}y(1) = \lambda D_{0+}^{\alpha}y(\xi)$, combining with (2.5) and (2.6), we obtain

$$c_1 = -\int_0^1 \frac{(1-z)^{\beta-1}}{\Gamma(\beta)(1-\mathcal{M})}y(z)\,dz + \int_0^{\xi}\frac{\lambda^{p-1}(\xi-z)^{\beta-1}}{\Gamma(\beta)(1-\mathcal{M})}y(z)\,dz.$$

Thus,

$$\begin{aligned}\varphi_p\big(D_{0+}^{\alpha}y(x)\big) &= \int_0^x \frac{(x-z)^{\beta-1}}{\Gamma(\beta)}y(z)\,dz - \int_0^1 \frac{x^{\beta-1}(1-z)^{\beta-1}}{\Gamma(\beta)(1-\mathcal{M})}y(z)\,dz \\ &\quad + \int_0^{\xi}\frac{\lambda^{p-1}x^{\beta-1}(\xi-z)^{\beta-1}}{\Gamma(\beta)(1-\mathcal{M})}y(z)\,dz \\ &= -\int_0^1 H(x,z)y(z)\,dz,\end{aligned}$$

then

$$D_{0+}^{\alpha}y(x) + \varphi_q\left(\int_0^1 H(x,z)y(z)\,dz\right) = 0. \tag{2.7}$$

By the use of Lemma 2.1, Eq. (2.7) is equivalent to the integral equation

$$y(x) = -I_{0+}^{\alpha}\varphi_q\left(\int_0^1 H(x,z)y(z)\,dz\right) + d_1x^{\alpha-1} + d_2x^{\alpha-2} + d_3x^{\alpha-3} \tag{2.8}$$

for some $d_1, d_2, d_3 \in \mathbb{R}$.

By $y(0) = y'(0) = 0$, there are $d_2 = d_3 = 0$. Thus

$$y(x) = -I_{0+}^{\alpha}\varphi_q\left(\int_0^1 H(x,z)y(z)\,dz\right) + d_1 x^{\alpha-1}$$
$$= -\frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1}\varphi_q\left(\int_0^1 H(z,\tau)y(\tau)\,d\tau\right)dz + d_1 x^{\alpha-1}.$$

By $y(1) = 0$, there is

$$d_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-z)^{\alpha-1}\varphi_q\left(\int_0^1 H(z,\tau)y(\tau)\,d\tau\right)dz.$$

Therefore, the unique solution of problem (2.2) is

$$y(x) = -\frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1}\varphi_q\left(\int_0^1 H(z,\tau)y(\tau)\,d\tau\right)dz$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^1 \left[x(1-z)\right]^{\alpha-1}\varphi_q\left(\int_0^1 H(z,\tau)y(\tau)\,d\tau\right)dz$$
$$= \int_0^1 G(x,z)\varphi_q\left(\int_0^1 H(z,\tau)y(\tau)\,d\tau\right)dz.$$

The proof is completed. □

Let $E = C[0,1]$ be a Banach space with the maximum norm $\|y\| = \max_{0\le x\le 1}|y(x)|$. Define a cone $P \subset E$ by

$$P = \left\{y \in E \mid y(x) \ge \frac{Q(1-x)}{\alpha-1}\|y\|, 0 \le x \le 1\right\}.$$

Lemma 2.4 *Define $\mathcal{T} : P \to E$ as*

$$(\mathcal{T}y)(x) = \int_0^1 G(x,z)\varphi_q\left(\int_0^1 H(z,\tau)f\bigl(\tau,y(\tau)\bigr)\,d\tau\right)dz.$$

Then $\mathcal{T} : P \to P$ is a completely continuous operator.

Proof By the use of relation (2) of Lemma 2.2, for any $y \in P$, there hold

$$(\mathcal{T}y)(x) \le \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)f\bigl(\tau,y(\tau)\bigr)\,d\tau\right)dz$$

and

$$(\mathcal{T}y)(x) \ge \frac{Q(1-x)}{\Gamma(\alpha)}\int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)f\bigl(\tau,y(\tau)\bigr)\,d\tau\right)dz.$$

Then $(\mathcal{T}y)(x) \ge \frac{q(1-x)}{\alpha-1}\|\mathcal{T}y\|$, which implies $\mathcal{T} : P \to P$. By the use of the Arzela–Ascoli theorem, a standard proof shows that $\mathcal{T} : P \to P$ is completely continuous. □

Lemma 2.5 ([18]) *Let E be an ordered Banach space, $P \subset E$ be a cone. Suppose that Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and $\mathcal{T} : P \to P$ is a completely continuous operator such that either*

(A_1) $\|\mathcal{T}y\| \le \|y\|, y \in P \cap \partial\Omega_1$ *and* $\|\mathcal{T}y\| \ge \|y\|, y \in P \cap \partial\Omega_2$, *or*

(A_2) $\|\mathcal{T}y\| \ge \|y\|, y \in P \cap \partial\Omega_1$ *and* $\|\mathcal{T}y\| \le \|y\|, y \in P \cap \partial\Omega_2$.

Then $\mathcal{T}$ has a fixed point in $P \cap \overline{\Omega}_2 \backslash \Omega_1$.

3 Existence of positive solutions

For notational convenience, denote

$$f_0 = \liminf_{y \to +0} \min_{x \in [1/4,3/4]} \frac{f(x,y)}{y^{p-1}}, \qquad f^0 = \limsup_{y \to +0} \max_{x \in [0,1]} \frac{f(x,y)}{y^{p-1}},$$

$$f_\infty = \liminf_{y \to +\infty} \min_{x \in [1/4,3/4]} \frac{f(x,y)}{y^{p-1}}, \qquad f^\infty = \limsup_{y \to +\infty} \max_{x \in [0,1]} \frac{f(x,y)}{y^{p-1}},$$

$$\rho_* = \left(\frac{\alpha - 1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q \left(\int_0^1 H(z,\tau)\, d\tau \right) dz \right)^{-1},$$

$$\rho^* = \left(\frac{Q(\frac{1}{2})}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q \left(\int_{\frac{1}{4}}^{\frac{3}{4}} H(z,\tau)\, d\tau \right) dz \right)^{-1}, \qquad \sigma = \min_{1/4 \le x \le 3/4} \frac{Q(1-x)}{\alpha - 1}.$$

From now on we will use the following assumptions:

(C_1) $f_0 \in ((\frac{\rho^*}{\sigma})^{p-1}, \infty], f_\infty \in ((\frac{\rho^*}{\sigma})^{p-1}, \infty]$.

(C_2) $f^0 \in [0, \rho_*^{p-1}), f^\infty \in [0, \rho_*^{p-1})$.

(C_3) There exist constants $d \in (0, \rho_*)$ and $\lambda_1 > 0$ such that

$$f(x,y) \le (d\lambda_1)^{p-1}, \quad 0 \le x \le 1, 0 \le y \le \lambda_1.$$

(C_4) There exist constants $D \in (\rho^*, \infty)$ and $\lambda_2 > 0$ such that

$$f(x,y) \ge (D\lambda_2)^{p-1}, \quad 1/4 \le x \le 3/4, \sigma\lambda_2 \le y \le \lambda_2.$$

Theorem 3.1 *Assume that conditions (C_1), (C_3) hold, then problem (1.1) has at least two solutions y_1 and y_2 such that $0 < \|y_1\| < \lambda_1 < \|y_2\|$.*

Proof Firstly, by condition (C_3), there exist constants $d \in (0, \rho_*)$ and $\lambda_1 > 0$ such that

$$f(x,y) \le (d\lambda_1)^{p-1}, \quad 0 \le x \le 1, 0 \le y \le \lambda_1.$$

Set $\Omega_{\lambda_1} = \{y \in P \mid \|y\| < \lambda_1\}$. Taking into account the monotonicity of $\varphi(z)$ and relation (2) of Lemma 2.2, for $y \in \partial\Omega_{\lambda_1}$, we have

$$\begin{aligned} \|\mathcal{T}y\| &= \max_{0 \le x \le 1} \left| \mathcal{T}y(x) \right| \\ &\le \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q \left(\int_0^1 H(z,\tau) f\bigl(\tau, y(\tau)\bigr)\, d\tau \right) dz \\ &\le \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q \left(\int_0^1 H(z,\tau)(d\lambda_1)^{(p-1)}\, d\tau \right) dz \end{aligned}$$

$$
\begin{aligned}
&\le \rho_* \lambda_1 \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^1 Q(z) \varphi_q \left(\int_0^1 H(z, \tau) \, d\tau \right) dz \\
&= \lambda_1 = \|y\|.
\end{aligned}
$$

Thus, $\|\mathcal{T} y\| \le \|y\|$ for all $y \in \partial \Omega_{\lambda_1}$.

Secondly, with the first relation of condition (C_1), $f_0 \in ((\frac{\rho^*}{\sigma})^{p-1}, \infty)$, there exists a real number $r_1 \in (0, \lambda_1)$ such that

$$
f(x, y) \ge y^{p-1} \left(\frac{\rho^*}{\sigma} \right)^{p-1}, \quad \text{for } \frac{1}{4} \le x \le \frac{3}{4}, 0 < y \le r_1.
$$

Set $\Omega_{r_1} = \{y \in P \mid \|y\| < r_1\}$. For $y \in \partial \Omega_{r_1}$, we have

$$
r_1 = \|y\| \ge y(x) \ge \frac{Q(1-x)}{\alpha - 1} \|y\| \ge \sigma \|y\| = \sigma r_1, \quad x \in \left[\frac{1}{4}, \frac{3}{4} \right].
$$

Thus, with relation (2) of Lemma 2.2, there is

$$
\begin{aligned}
\|\mathcal{T} y\| &= \max_{0 \le x \le 1} \left| \mathcal{T} y(x) \right| \\
&= \max_{0 \le x \le 1} \int_0^1 G(x, z) \varphi_q \left(\int_0^1 H(z, \tau) f\bigl(\tau, y(\tau)\bigr) \, d\tau \right) dz \\
&\ge \int_0^1 G\left(\frac{1}{2}, z \right) \varphi_q \left(\int_{\frac{1}{4}}^{\frac{3}{4}} H(z, \tau) y^{p-1}(\tau) \left(\frac{\rho^*}{\sigma} \right)^{p-1} d\tau \right) dz \\
&\ge \frac{r_1 \rho^*}{\Gamma(\alpha)} \int_0^1 Q\left(\frac{1}{2} \right) Q(z) \varphi_q \left(\int_{\frac{1}{4}}^{\frac{3}{4}} H(z, \tau) \, d\tau \right) dz \\
&= r_1 = \|y\|.
\end{aligned}
$$

So, $\|\mathcal{T} y\| \ge \|y\|$ for all $y \in \partial \Omega_{r_1}$.

Thirdly, with the second relation of condition (C_1), $f_\infty \in ((\frac{\rho^*}{\sigma})^{p-1}, \infty)$, there exists a real number $R^* > 0$ such that

$$
f(x, y) \ge y^{p-1} \left(\frac{\rho^*}{\sigma} \right)^{p-1}, \quad \text{for } \frac{1}{4} \le x \le \frac{3}{4}, y \ge R^*.
$$

Choose $R_1 = \max\{2\lambda_1, \frac{R^*}{\sigma}\}$, set $\Omega_{R_1} = \{y \in P \mid \|y\| < R_1\}$. For $y \in \partial \Omega_{R_1}$, we get

$$
R_1 = \|y\| \ge y(x) \ge \frac{Q(1-x)}{\alpha - 1} \|y\| \ge \sigma \|y\| = \sigma R_1 \ge R^*, \quad x \in \left[\frac{1}{4}, \frac{3}{4} \right].
$$

Thus, with relation (2) of Lemma 2.2, there is

$$
\begin{aligned}
\|\mathcal{T} y\| &= \max_{0 \le x \le 1} \left| \mathcal{T} y(x) \right| \\
&= \max_{0 \le x \le 1} \int_0^1 G(x, z) \varphi_q \left(\int_0^1 H(z, \tau) f\bigl(\tau, y(\tau)\bigr) \, d\tau \right) dz \\
&\ge \int_0^1 G\left(\frac{1}{2}, z \right) \varphi_q \left(\int_{\frac{1}{4}}^{\frac{3}{4}} H(z, \tau) u^{p-1}(\tau) \left(\frac{\rho^*}{\sigma} \right)^{p-1} d\tau \right) dz
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{R_1\rho^*}{\Gamma(\alpha)}\int_0^1 Q\left(\frac{1}{2}\right)Q(z)\varphi_q\left(\int_{\frac{1}{4}}^{\frac{3}{4}} H(z,\tau)\,d\tau\right)dz \\
&= R_1 = \|y\|.
\end{aligned}
$$

So, $\|\mathcal{T}y\| \geq \|y\|$ for all $y \in \partial\Omega_{R_1}$.

By Lemma 2.5, $\mathcal{T}$ has a fixed point $y_1 \in (\Omega_{\lambda_1} \setminus \overline{\Omega}_{r_1})$ and a fixed point $y_2 \in (\Omega_{R_1} \setminus \overline{\Omega}_{\lambda_1})$. That is to say, y_1, y_2 are both positive solutions of problem (1.1) such that $0 < \|y_1\| < \lambda_1 < \|y_2\|$. □

Theorem 3.2 *Assume that conditions* (C_2), (C_4) *hold, then problem* (1.1) *has at least two solutions* y_1 *and* y_2 *satisfying* $0 < \|y_1\| < \lambda_2 < \|y_2\|$.

Proof Firstly, by condition (C_4), there exist two constants $D \in (\rho^*, \infty)$ and $\lambda_2 > 0$ such that

$$
f(x,y) \geq (D\lambda_2)^{p-1}, \quad 1/4 \leq x \leq 3/4, \sigma\lambda_2 \leq y \leq \lambda_2.
$$

Set $\Omega_{\lambda_2} = \{y \in P \mid \|y\| < \lambda_2\}$. For $y \in \partial\Omega_{\lambda_2}$, one has

$$
\lambda_2 = \|y\| \geq y(x) \geq \frac{Q(1-x)}{\alpha - 1}\|y\| \geq \sigma\|y\| = \sigma\lambda_2, \quad x \in \left[\frac{1}{4}, \frac{3}{4}\right].
$$

Thus, we get

$$
\begin{aligned}
\|\mathcal{T}y\| &= \max_{0\leq x\leq 1}\left|Ty(x)\right| \\
&= \max_{0\leq x\leq 1}\int_0^1 G(x,z)\varphi_q\left(\int_0^1 H(z,\tau)f\big(\tau,y(\tau)\big)\,d\tau\right)dz \\
&\geq \int_0^1 G\left(\frac{1}{2},z\right)\varphi_q\left(\int_{\frac{1}{4}}^{\frac{3}{4}} H(z,\tau)(D\lambda_2)^{(p-1)}\,d\tau\right)dz \\
&\geq \frac{\lambda_2\rho^*}{\Gamma(\alpha)}\int_0^1 Q\left(\frac{1}{2}\right)Q(z)\varphi_q\left(\int_{\frac{1}{4}}^{\frac{3}{4}} H(z,\tau)\,d\tau\right)dz \\
&= \lambda_2 = \|y\|.
\end{aligned}
$$

So, $\|\mathcal{T}y\| \geq \|y\|$ for all $y \in \partial\Omega_{\lambda_2}$.

Secondly, with the first relation of condition (C_2), $f^0 \in [0, \rho_*^{p-1})$, there exists a real number $r_2 \in (0, \lambda_2)$ such that

$$
f(x,y) \leq y^{p-1}\rho_*^{p-1} \leq (r_2\rho_*)^{p-1}, \quad \text{for } 0 \leq x \leq 1, 0 < y \leq r_2.
$$

Set $\Omega_{r_2} = \{y \in P \mid \|y\| < r_2\}$. For $y \in \partial\Omega_{r_2}$, one has

$$
\begin{aligned}
\|\mathcal{T}y\| &= \max_{0\leq x\leq 1}\left|Ty(x)\right| \\
&\leq \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)f\big(\tau,y(\tau)\big)\,d\tau\right)dz \\
&\leq \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)(r_2\rho_*)^{p-1}\,d\tau\right)dz
\end{aligned}
$$

$$
\begin{aligned}
&= r_2\rho_* \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)\,d\tau\right) dz \\
&= r_2 = \|y\|.
\end{aligned}
$$

So, $\|\mathcal{T}y\| \le \|y\|$ for all $y \in \partial\Omega_{r_2}$.

Thirdly, with the second relation of condition (C_2), $f^\infty \in [0, \rho_*^{p-1})$, there exists a positive number R^* such that

$$
f(x,y) \le y^{p-1}\rho_*^{p-1}, \quad \text{for } 0 \le x \le 1, y \ge R^*.
$$

We now consider two situations.

Case 1. The function f is bounded on $[0,\infty)$. We can choose a positive number $G > 0$ such that $f(x,y) \le G^{p-1}\rho_*^{p-1}$ for $x \in [0,1]$, $y \in [0,\infty)$. Let $R_2 = \max\{2\lambda_2, G\}$ and $\Omega_{R_2} = \{y \in P \mid \|y\| < R_2\}$. For $y \in \partial\Omega_{R_2}$, one has

$$
\begin{aligned}
\|\mathcal{T}y\| &= \max_{0\le x\le 1}\left|\mathcal{T}y(x)\right| \\
&\le \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)f\big(\tau,y(\tau)\big)\,d\tau\right) dz \\
&\le \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)G^{p-1}\rho_*^{p-1}\,d\tau\right) dz \\
&\le R_2\rho_* \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)\,d\tau\right) dz \\
&= R_2 = \|y\|.
\end{aligned}
$$

So, $\|\mathcal{T}y\| \le \|y\|$ for all $y \in \partial\Omega_{R_2}$.

Case 2. The function f is unbounded on $[0,\infty)$. We can choose a positive number $R_2 > \max\{2\lambda_2, R^*\}$ such that $f(x,y) \le f(x,R_2)$ for $x \in [0,1], y \in (0,R_2)$. Set $\Omega_{R_2} = \{y \in P \mid \|y\| < R_2\}$. For $y \in \partial\Omega_{R_2}$, one has

$$
\begin{aligned}
\|\mathcal{T}y\| &= \max_{0\le x\le 1}\left|\mathcal{T}y(x)\right| \\
&\le \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)f\big(\tau,y(\tau)\big)\,d\tau\right) dz \\
&\le \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)f(\tau,R_2)\,d\tau\right) dz \\
&\le \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)R_2^{p-1}\rho_*^{p-1}\,d\tau\right) dz \\
&\le R_2\rho_* \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 Q(z)\varphi_q\left(\int_0^1 H(z,\tau)\,d\tau\right) dz \\
&= R_2 = \|y\|.
\end{aligned}
$$

So, $\|\mathcal{T}y\| \le \|y\|$ for all $y \in \partial\Omega_{R_2}$.

By Lemma 2.5, $\mathcal{T}$ has a fixed point $y_1 \in (\Omega_{\lambda_2} \setminus \overline{\Omega}_{r_2})$ and a fixed point $y_2 \in (\Omega_{R_2} \setminus \overline{\Omega}_{\lambda_2})$. That is to say, y_1, y_2 are both positive solutions of problem (1.1) and $0 < \|y_1\| < \lambda_2 < \|y_2\|$. □

By Theorems 3.1 and 3.2, we can obtain the following corollary.

Corollary 3.1 *Problem* (1.1) *has at least one positive solution if one of the following assumptions is satisfied*:

($A1$) *Conditions* (C_3) *and* (C_4) *hold; or*

($A2$) *Conditions* $f^0 \in [0, \rho_*^{p-1})$ *and* $f_\infty \in ((\frac{\rho^*}{\sigma})^{p-1}, \infty)$ *hold; or*

($A3$) *Conditions* $f_0 \in ((\frac{\rho^*}{\sigma})^{p-1}, \infty)$ *and* $f^\infty \in [0, \rho_*^{p-1})$ *hold; or*

($A4$) *Conditions* (C_3) *and* $f_\infty \in ((\frac{\rho^*}{\sigma})^{p-1}, \infty)$ (*or* $f_0 \in ((\frac{\rho^*}{\sigma})^{p-1}, \infty)$) *hold; or*

($A5$) *Conditions* (C_4) *and* $f^0 \in [0, \rho_*^{p-1})$ (*or* $f^\infty \in [0, \rho_*^{p-1})$) *hold.*

4 Examples

In this section, we present some examples to check our results. Let

$$\alpha = \frac{5}{2}, \qquad \beta = \frac{3}{2}, \qquad q = 2, \qquad p = 2, \qquad \xi = \frac{1}{2}, \qquad \lambda = \frac{\sqrt{2}}{2},$$

then there are

$$H(x,z) = \begin{cases} \frac{1}{\sqrt{\pi}}\{4[x(1-z)]^{\frac{1}{2}} - 2\sqrt{2}[x(\frac{1}{2}-z)]^{\frac{1}{2}} - 2(x-z)^{\frac{1}{2}}\}, & 0 \le z \le x \le 1, z \le \frac{1}{2}; \\ \frac{1}{\sqrt{\pi}}\{4[x(1-z)]^{\frac{1}{2}} - 2(x-z)^{\frac{1}{2}}\}, & 0 < \frac{1}{2} \le z \le x \le 1; \\ \frac{1}{\sqrt{\pi}}\{4[x(1-z)]^{\frac{1}{2}} - 2\sqrt{2}[x(\frac{1}{2}-z)]^{\frac{1}{2}}\}, & 0 \le x \le z \le \frac{1}{2} < 1; \\ \frac{4}{\sqrt{\pi}}[x(1-z)]^{\frac{1}{2}}, & 0 \le x \le z \le 1, \frac{1}{2} \le z, \end{cases}$$

and

$$M = \lambda^{p-1}\xi^{\beta-1} = \frac{1}{2}, \qquad Q(x) = x(1-x)^{\alpha-1} = x(1-x)^{\frac{3}{2}},$$

$$\rho^* = 22.5, \qquad \rho_* = 16, \qquad \sigma = \frac{1}{3}.$$

Example 4.1 Consider the boundary value problem

$$D_{0+}^{\beta}\left(\varphi_p\left(D_{0+}^{\alpha}y(x)\right)\right) = f\left(x, y(x)\right), \quad 0 < x < 1, \tag{4.1}$$

$$y(0) = y'(0) = y(1) = D_{0+}^{\alpha}y(0) = 0, \qquad D_{0+}^{\alpha}y(1) = \lambda D_{0+}^{\alpha}y(\xi), \tag{4.2}$$

where

$$f(x,y) = \frac{y^2}{2}\cos x + 70\sin y.$$

Direct computations show that

$$f_0 = \liminf_{u\to 0+} \min_{x\in[1/4,3/4]} \frac{f(x,y)}{y} = \liminf_{y\to 0+}\left(\frac{y\cos\frac{3}{4}}{2} + 70\frac{\sin y}{y}\right) = 70 > 68 = \left(\frac{\rho^*}{\sigma}\right)^{p-1},$$

$$f_\infty = \liminf_{y\to +\infty} \min_{x\in[1/4,3/4]} \frac{f(x,y)}{y} = \liminf_{y\to +\infty}\left(\frac{y\cos\frac{3}{4}}{2} + 70\frac{\sin y}{y}\right) = \infty > 68 = \left(\frac{\rho^*}{\sigma}\right)^{p-1},$$

so condition (C_1) holds. Choose $\lambda_1 = 6$, $d = 15 \in (0, \rho_*)$, one has

$$f(x,y) \le 88 < 90 = d\lambda_1, \quad \text{when } 0 \le x \le 1, 0 \le y \le 6,$$

so condition (C_3) holds. By the use of Theorem 3.1, problem (1.1) has at least two solutions y_1 and y_2 satisfying $0 < \|y_1\| < 6 < \|y_2\|$.

Example 4.2 Consider the boundary value problem (4.1), (4.2), where

$$f(x,y) = \frac{500y \sin y}{y+1} + 15xy.$$

Direct computations show that

$$f^0 = \limsup_{y\to 0+} \max_{x\in[0,1]} \frac{f(x,y)}{y} = \limsup_{y\to 0+}\left(\frac{500\sin y}{y+1} + 15\right) = 15 < 16 = \rho_*^{p-1},$$

$$f^\infty = \limsup_{y\to +\infty} \max_{x\in[0,1]} \frac{f(x,y)}{y} = \limsup_{y\to +\infty}\left(\frac{500\sin y}{y+1} + 15\right) = 15 < 16 = \rho_*^{p-1},$$

so condition (C_2) holds. Choose $\lambda_2 = \frac{1}{2}, D = 23 \in (\rho^*, \infty)$, one has

$$f(x,y) \ge 12.4 > D\lambda_2 = \frac{23}{2}, \quad \text{for } \frac{1}{4} \le x \le \frac{3}{4}, \frac{1}{6} = \sigma\lambda_2 \le y \le \lambda_2 = \frac{1}{2},$$

so condition (C_4) holds. By the use of Theorem 3.2, problem (1.1) has at least two solutions y_1 and y_2 satisfying $0 < \|y_1\| < \frac{1}{2} < \|y_2\|$.

Example 4.3 Consider the boundary value problem (4.1), (4.2), where

$$f(x,y) = 5 + xy^2.$$

Let $\lambda_1 = 2, d = 5 \in (0, \rho_*)$. Direct computations show that

$$f(x,y) \le 9 < d\lambda_1 = 10, \quad \text{for } 0 \le x \le 1, 0 \le y \le 2,$$

and

$$f_\infty = \liminf_{y\to +\infty} \min_{x\in[1/4,3/4]} \frac{f(x,y)}{y} = \liminf_{y\to +\infty}\left(\frac{5}{y} + \frac{y}{4}\right) = \infty > 68 = \left(\frac{\rho^*}{\sigma}\right)^{p-1}.$$

So condition $(A4)$ was satisfied. By the use of Corollary 3.1, problem (1.1) has at least one positive solution.

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