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Existence of solutions for the fractional Kirchhoff equations with sign-changing potential

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Abstract

In this paper, the authors investigate the following fractional Kirchhoff boundary value problem:

$$\begin{cases} (a + b\int_0^T ({}_0D_t^{\alpha}u)^2\,dt)\,{}_tD_T^{\alpha}({}_0D_t^{\alpha}u) + \lambda V(t)u = f(t,u), & t \in [0,T], \\ u(0) = u(1) = 0, \end{cases}$$

where the parameter $\lambda > 0$ and constants $a, b > 0$. By applying the mountain pass theorem and the linking theorem, some existence results on the above fractional boundary value problem are obtained. It should be pointed out that the potential V may be sign-changing.

Keywords: Fractional equations; Kirchhoff type; Critical point; Variational method

1 Introduction

In recent ten years, the fractional differential equations have been extensively studied by many researchers due to their various applications in science and engineering [1–5]. In fact, one can find numerous applications in the modeling of various phenomena such as in neurons, viscoelasticity, biochemistry, bioengineering, porous media, electromagnetic, etc. Especially, in the last several years, the investigations on the equations including both left and right fractional derivative have received more and more attention. Due to the appearance of both left and right fractional derivatives in equations, the fixed point theory is generally not suitable for the study of the existence of solution to such problems. For the first time, Jiao and Zhou [6] showed that the variational method is a very effective tool for studying such problems. In [6], by introduction of appropriate spaces and variational structure and using some critical point theorem, the authors investigated the existence of solutions to the following equations:

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)), & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0. \end{cases} \tag{1.1}$$

Under some suitable conditions, the existence results were obtained on equation (1.1). Since then there have been many literature works investigating a variety of fractional equations with left and right derivatives via variational methods, and a lot of results on the existence of one solution, three solutions, infinite solutions, and so on, have been obtained (see [7–15]).

On the other hand, recently, more and more research has focused on the following Kirchhoff-type problem:

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2)\Delta u + V(x)u = f(x,u), & x\in\mathbb{R}^N, \\ u\in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $V:\mathbb{R}^N\to\mathbb{R}$ and constants a,b are two positive numbers. Problem (1.2) is called nonlocal because of the presence of the term $\int_{\mathbb{R}^N}|\nabla u|^2\,dx$, which means that (1.2) is no longer a pointwise identical equation. This phenomenon provokes more difficulties to overcome, which makes the study of such a class of problems particularly interesting. If the function V vanishes and $\mathbb{R}^N$ is replaced with a bounded domain $\Omega\subset\mathbb{R}^N$ in (1.2), then it reduces to the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -(a+b\int_{\Omega}|\nabla u|^2\,dx)\Delta u = f(t,u), & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases}$$

which is related to the stationary analog of equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0 \tag{1.3}$$

proposed by Kirchhoff in [16], which is an extension of the classical D'Alembert's wave equation and is used to characterize the free vibrations of elastic strings.

In recent years, Eq. (1.2) has been investigated in depth under different conditions on f and V, and a lot of existence results of the nontrivial solution to (1.2) have been obtained via variational method. In [17], Jin and Wu got existence of infinite many radial solutions by using the fountain theorem for $N=3$ and $V=1$. While f satisfies 4-superlinear condition and V admits other assumptions, Wu [18] established some existence results on nontrivial solutions. For more related research, the readers can refer to [19–30] and the references therein.

Motivated by these works mentioned above and combining the fractional equations with left-right derivatives and the Kirchhoff equations, the authors will investigate the following fractional Kirchhoff boundary value problem (BVP for short):

$$\begin{cases} (a+b\int_0^T({}_0D_t^\alpha u)^2\,dt)\,{}_tD_T^\alpha({}_0D_t^\alpha u)+\lambda Vu=f(t,u), & t\in[0,T], \\ u(0)=u(T)=0, \end{cases} \tag{1.4}$$

where the parameter $\lambda>0$ and constants $a,b>0$. By using the mountain pass theorem and the linking theorem, we establish some existence results of nontrivial solutions to BVP (1.4). It should be pointed out that in some references mentioned previously, the potential V is always assumed to be continuous and positive. Here, we consider BVP (1.4) having a more general potential V. In particular, the potential V can be sign-changing. As a result, there are more difficulties that need to be overcome and more derivation techniques need to be introduced. In addition, some critical point theorems under the $(C)_c$ condition but not the usual $(PS)_c$ condition will be applied. To the best of our knowledge, no one has studied BVP (1.4) so far.

Finally, we turn to showing the organization of the paper. In Sect. 2, we present some definitions and the variational work frame for (1.4) as well as some lemmas, which will be used later. In Sect. 3, we give the main results.

2 Preliminaries

In this section, we introduce the following definitions and lemmas.

Definition 2.1 ([5]) Let f be a function defined on $[a,b]$. The left Riemann–Liouville fractional integral of order $\gamma > 0$ for function f is defined by

$$ {}_aD_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)\,ds, \quad t \in [a,b], $$

provided the right-hand side is pointwise defined on $[a,b]$, where Γ is the gamma function.

Definition 2.2 ([5]) Let f be a function defined on $[a,b]$. The right Riemann–Liouville fractional integral of order $\gamma > 0$ for function f is defined by

$$ {}_tD_b^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s)\,ds, \quad t \in [a,b], $$

provided the right-hand side is pointwise defined on $[a,b]$.

Definition 2.3 ([5]) Let f be a function defined on $[a,b]$. The left and right Riemann–Liouville fractional derivatives of order $\gamma > 0$ for function f denoted by ${}_aD_t^{\gamma} f(t)$ and ${}_tD_b^{\gamma} f(t)$, respectively, are defined by

$$ {}_aD_t^{\gamma} f(t) = \frac{d^n}{dt^n} {}_aD_t^{\gamma-n} f(t) $$

and

$$ {}_tD_b^{\gamma} f(t) = (-1)^n \frac{d^n}{dt^n} {}_tD_b^{\gamma-n} f(t), $$

where $t \in [a,b]$, $n-1 \le \gamma < n$, $n \in N$.

Definition 2.4 ([6]) Let $0 < \alpha \le 1$ and $1 < p < \infty$. Denote the fractional derivative space $E_0^{\alpha,p}$ by the closure of $C_0^\infty([0,T],\mathbb{R})$ on the norm

$$ \|u\|_{\alpha,p} = \left(\int_0^T \big|u(t)\big|^p\,dt + \int_0^T \big|{}_0D_t^{\alpha} u(t)\big|^p\,dt \right)^{1/p}, \quad \forall u \in E_0^{\alpha,p}. $$

By [6], the space $(E_0^{\alpha,p}, \|\cdot\|_{\alpha,p})$ is a Banach space.

As usual, for $1 \le p < \infty$, we definite the norms $\|u\|_{L^p} = (\int_0^T |u(t)|^p\,dt)^{\frac{1}{2}}$ for $u \in L^p[0,T]$ and $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$ for $u \in C([0,T],\mathbb{R})$.

By [6], we have the following results.

Lemma 2.1 ([6]) *Let $0 < \alpha \le 1$, $1 < p < \infty$. Then $E_0^{\alpha,p}$ is a reflective and separable Banach space. Moreover, if $\alpha > \frac{1}{p}$, then $E_0^{\alpha,p} \subset C([0,T],\mathbb{R})$.*

Lemma 2.2 ([6]) *Let $0 < \alpha \le 1$, $1 < p < \infty$. For any $u \in E_0^{\alpha,p}$,*

(i) *if $\alpha > \frac{1}{p}$ or $\alpha \le 1 - \frac{1}{p}$, then $\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0D_t^\alpha u\|_{L^p}$;*

(ii) *if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\|u\|_\infty \le \frac{T\alpha^{-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\|{}_0D_t^\alpha u\|_{L^p}$.*

Lemma 2.3 ([6]) *Let $1 < p < \infty$ and $\alpha > \frac{1}{p}$. If $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathbb{R})$. Of course, $u_k \to u$ in $L^q([0,T])$ for any $q \ge 1$.*

In what follows, we always assume that $p = 2$ and $\frac{1}{2} < \alpha \le 1$.

Let $V^+$and V^-be the positive part and the negative part on potential V, respectively. Then $V = V^+ - V^-$.

We give some assumptions on the potential V as follows:

(V_1) V is measurable and essentially bounded below on $[0,T]$.

(V_2) $\Omega_i \neq \emptyset$, $i = 1,2$, where $\Omega_1 = \mathrm{int}\{t \in [0,T] : V(t) \ge 0\}$, $\Omega_2 = \mathrm{int}\{t \in [0,T] : V(t) < 0\}$.

(V_3) $\lim_{R\to+\infty} \mathrm{meas}\{x \in [0,T] : V(x) \ge R\} = 0$.

In order to study the Kirchhoff-type boundary value problem with sign-changing potential V, we need the following work frame. For each fixed $\lambda > 0$, define

$$X_\lambda := \left\{u \in E_0^{\alpha,2} : \int_0^T V^+(t)u^2(t)\,dt < \infty\right\}$$

and $\langle u,v\rangle_\lambda := \int_0^T [a({}_0D_t^\alpha u)({}_0D_t^\alpha v) + \lambda V^+uv]\,dt$ for $u,v \in X_\lambda$.

In view of Lemmas 2.1–2.3, it is easy to know the following results hold.

Lemma 2.4 *Assume condition (V_1) holds. Then*

(i) *The space X_λ is a reflective and separable Hilbert space with the inner product $\langle u,v\rangle_\lambda$, and also a reflective and separable Banach space with the norm $\|u\|_\lambda = \langle u,u\rangle_\lambda^{\frac{1}{2}}$, $u \in X_\lambda$.*

(ii) *The norms $\|\cdot\|_{\alpha,2}$ and $\|\cdot\|_\lambda$ on X_λ are equivalent.*

(iii) *X_λ is continuously and compactly embedded in the spaces $C([0,T])$ and $L^2([0,T])$.*

Moreover, we need the following notation. For the fixed $\lambda > 0$, let

$$Y_\lambda = \{u \in X_\lambda : \mathrm{supp}\,u \subset V^{-1}[0,\infty)\}.$$

Then $X_\lambda = Y_\lambda \oplus Y_\lambda^\perp$. Obviously, if $V(t) \ge 0$, then $X_\lambda = Y_\lambda$. Otherwise, $Y_\lambda^\perp \neq \{0\}$.

We define a bilinear function a_λ^+ on $X_\lambda \times X_\lambda$ by

$$a_\lambda^+(u,v) = \int_0^T \left[a\left({}_0D_t^\alpha u\right)\left({}_0D_t^\alpha v\right)(t) + \lambda V^+(t)u(t)v(t)\right]dt$$

and a bilinear functional b_λ on $Y_\lambda^\perp \times Y_\lambda^\perp$ by

$$b_\lambda(u,v) = \int_0^T \lambda V^-(t)u(t)v(t)\,dt.$$

The following result will be used later.

Lemma 2.5 *Under condition* (V_1), *the function* $b_\lambda(u,u)$ *is weakly continuously on* $Y_\lambda^\perp$.

Proof By (V_1), the function V^- is essentially bounded below, that is, there exists $v_0 > 0$ such that $0 \le v(t) \le v_0$, a.e. $t \in [0,T]$. Let $\{u_n\}$ be any sequence with $u_n \rightharpoonup u$ in $Y_\lambda^\perp$. Then, by Lemma 2.4, it follows that $u_n \to u$ in $C([0,T])$, and therefore,

$$\lim_{n\to\infty} \int_0^T V^-(t)u_n^2(t)\,dt = \int_0^T V^-(t)u^2(t)\,dt$$

in terms of the dominated convergence theorem. That is, $b_\lambda(u,u)$ is weakly continuous on $Y_\lambda^\perp$. The proof is complete. □

Now, we turn to considering the following eigenvalue problem on $Y_\lambda^\perp$:

$$a_tD_T^\alpha\big({}_0D_t^\alpha u\big) + \lambda V^+ u = \beta\lambda V^- u, \quad u \in Y_\lambda^\perp, \tag{2.1}$$

where constants $a > 0$, $\lambda > 0$.

We denote an operator A_λ^+ on $Y_\lambda^\perp$ associated with (2.1) by

$$A_\lambda^+ u = a_tD_T^\alpha\big({}_0D_t^\alpha u\big) + \lambda V^+ u, \quad u \in Y_\lambda^\perp.$$

Clearly, A_λ^+ is formally self-adjoint in $L^2([0,T])$. Hence, in view of the result in [31] combined with Lemma 2.5, we have the following conclusion.

Lemma 2.6 *Under condition* (V_1), *the eigenvalue problem* (2.1) *admits a sequence of eigenvalues* $\{\beta_k(\lambda)\}$ *satisfying*

$$0 < \beta_1(\lambda) < \beta_2(\lambda) \le \cdots \le \beta_{N_0(\lambda)}(\lambda) \le \beta_{N_0(\lambda)+1}(\lambda) \le \cdots \le \beta_n(\lambda) \le \cdots,$$

with $\beta_{N_0(\lambda)}(\lambda) \le 1 < \beta_{N_0(\lambda)+1}$ *and* $\beta_k(\lambda) \to +\infty$ *as* $k \to \infty$. *In addition,* $\beta_k(\lambda)$ *is characterized with*

$$\beta_k(\lambda) = \inf_{F\perp Y_\lambda, \dim F \ge k} \sup\left\{\lambda^{-1}\|u\|_\lambda{}^2 : u \in F, \int_0^T V^{-1}(t)u^2(t)\,dt = 1\right\},$$

and eigenvector e_k *corresponding to* $\beta_k(\lambda)$ *forms a basis for* $Y_\lambda^\perp$, *which can be chosen so that* $\langle e_i, e_j\rangle_\lambda = \delta_{ij}$.

Denote the subspaces $X_{\lambda,1}$, $X_{\lambda,2}$ by $X_{\lambda,1} = \operatorname{span}\{e_k : 1 \le k \le N_0(\lambda)\}$, $X_{\lambda,2} = \overline{\operatorname{span}}\{e_k : k \ge N_0(\lambda)+1\}$, respectively. Then $Y_\lambda^\perp = X_{\lambda,1} \oplus X_{\lambda,2}$, $X_\lambda = X_{\lambda,1} \oplus X_{\lambda,2} \oplus Y_\lambda$.

Furthermore, we need to introduce a bilinear function a_λ on $X_\lambda \times X_\lambda$ as follows:

$$a_\lambda(u,v) = \int_0^T \big[a\big({}_0D_t^\alpha u\big)\big({}_0D_t^\alpha v\big)(t) + \lambda V(t)u(t)v(t)\big]\,dt, \quad u,v \in X_\lambda.$$

On the above function a_λ, we have the following conclusions.

Lemma 2.7 *Assume that condition* (V_1) *holds. For constant* $a > 0$ *and fixed* $\lambda > 0$, *we have*

(i) $a_\lambda(u,u) \leq 0$ *for any* $u \in X_{\lambda,1}$;

(ii) $a_\lambda(u,u) \geq 0$ *for any* $u \in X_{\lambda,2}$;

(iii) $a_\lambda(u,v) = 0$ *for any* $u, v \in X_\lambda$ *taken in a different subspace among* $X_{\lambda,1}, X_{\lambda,2}$, *and* Y_λ.

Proof (i) For any $u \in X_{\lambda,1}$ with $\sum_{i=1}^{N_0(\lambda)} t_i e_i$, by (2.1), we have

$$\begin{aligned}\delta_{ij} &= \langle e_i, e_j\rangle_\lambda = a_\lambda^+(e_i,e_j)\\ &= \int_0^T \bigl[a\bigl({}_0D_t^\alpha e_i\bigr)\bigl({}_0D_t^\alpha e_j\bigr)(t) + \lambda V^+(t)e_i(t)e_j(t)\bigr]\,dt\\ &= \beta_i\lambda \int_0^T V^-(t)e_i(t)e_j(t)\,dt, \quad 1 \leq i,j \leq N_0(\lambda).\end{aligned}$$

Thus,

$$\begin{aligned}a_\lambda(e_i,e_j) &= \int_0^T \bigl[a\bigl({}_0D_t^\alpha e_i\bigr)\bigl({}_0D_t^\alpha e_j\bigr)(t) + \lambda V(t)e_i(t)e_j(t)\bigr]\,dt\\ &= (\beta_i - 1)\lambda \int_0^T V^-(t)e_i(t)e_j(t)\,dt\\ &= \frac{\beta_i - 1}{\beta_i}\delta_{ij}, \quad 1 \leq i,j \leq N_0(\lambda),\end{aligned}$$

and therefore $a_\lambda(u,u) = \sum_{i=1}^{N_0(\lambda)} \frac{\beta_i - 1}{\beta_i} t_i^2 \leq 0$, for $u = \sum_{i=1}^{N_0(\lambda)} t_i e_i$, noting that $0 < \beta_i \leq 1$ as $1 \leq i \leq N_0(\lambda)$.

(ii) For any $u \in X_{\lambda,2}$ with $u = \sum_{i=N_0(\lambda)+1}^{m} t_i e_i$, by an argument similar to that in (i), we know that

$$a_\lambda(u,u) = \sum_{i=N_0(\lambda)+1}^{m} \frac{\beta_i - 1}{\beta_i} t_i^2 > 0, \tag{2.2}$$

noting that $\beta_i > 1$ as $i \geq N_0(\lambda) + 1$. Hence, for any $u \in X_{\lambda,2}$, taking $\{u_n\} \subset X_{\lambda,2}$ with $u_n = \sum_{i=N_0(\lambda)+1}^{m_n} t_i^{(n)} e_i$ satisfying $u_n \to u$ as $n \to \infty$. It follows from (2.2) that $a_\lambda(u_n,u_n) > 0$, $n \geq 1$. By

$$a_\lambda(u_n,u_n) = \|u_n\|_\lambda^2 - \int_0^T V^-(t)u_n^2(t)\,dt$$

and the fact that $\|u_n\|_\lambda \to \|u\|_\lambda$, $u_n \to u$ in $C([0,T])$, $0 \leq V^-(t) \leq v_0$, a.e. $t \in [0,T]$, applying the dominated convergence theorem, we know that

$$a_\lambda(u,u) = \lim_{n\to\infty} a_\lambda(u_n,u_n) \geq 0.$$

(iii) For any $u \in X_{\lambda,1}$, $v \in Y_\lambda$, because $V^-(t)v(t) = 0$, we have

$$a_\lambda(u,v) = a_\lambda^+(u,v) - \lambda \int_0^T V^-(t)u(t)v(t)\,dt = a_\lambda^+(u,v) = \langle u,v\rangle_\lambda = 0.$$

Similarly, for any $u \in X_{\lambda,2}$, $v \in Y_\lambda$, or $u \in X_{\lambda,1}$, $v \in X_{\lambda,2}$, we have $a_\lambda(u,v) = 0$. □

Now, we turn to introducing two critical point theorems. Let E be a Banach space and $I : E \to \mathbb{R}$ be a functional of class C^1. A sequence $\{u_n\} \subset E$ is a $(C)_c$ sequence of I means that if $I(u_n) \to c$ and $(1 + \|u_n\|)I'(u_n) \to 0$ as $n \to \infty$. Moreover, I satisfies the Cerami condition at level c if any $(C)_c$ sequence of I has a convergent subsequence.

Lemma 2.8 (Mountain pass theorem [32]) *Let E be a Banach space, $I \in C^1(E, \mathbb{R})$ satisfies that $\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} I(u)$ for some $\mu < \eta, \rho > 0$, and $e \in E$ with $\|e\| > \rho$. Let c be characterized by $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t))$, where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}$. Then $c \geq \eta$ and I has a $(C)_c$ sequence.*

Lemma 2.9 (Linking theorem [33]) *Let $X = Y \oplus Z$ be a Banach space with $\dim Y < \infty$. Let $\rho > r > 0$ and $e_0 \in Z$ with $\|e_0\| = r$. Set*

$$\begin{aligned}
&M := \bigl\{u = y + \lambda e_0 : \|u\| \leq \rho, \lambda \geq 0, y \in Y\bigr\}, \\
&N_r := \bigl\{u \in Z : \|u\| = r\bigr\}, \\
&M_0 := \bigl\{u = y + \lambda e_0 : y \in Y, \|u\| = \rho, \lambda \geq 0 \text{ or } \|u\| \leq \rho, \lambda = 0\bigr\}.
\end{aligned}$$

If $I \in C^1(X, \mathbb{R})$ satisfies that $b := \inf_{N_r} I > a := \max_{M_0} I$, then $c \geq b$, and there exists a $(C)_c$ sequence of I, where $c := \inf_{r\in\Gamma} \max_{u\in M} I(\gamma(u))$, $\Gamma = \{\gamma \in C(M, X) : \gamma_{|M_0} = I_d\}$.

3 Main result

In this section, we establish some existence results on solutions to BVP (1.4). First, we list some conditions on functions f and F, where $F(t, x) = \int_0^x f(t, s)\, ds, (t, x) \in [0, T] \times \mathbb{R}$.

(f_1) $f \in C([0, T] \times \mathbb{R})$.

(f_2) There exist constants $\mu > 4, 0 < \tau < 2$ and a nonnegative function $g \in L^{\frac{2}{2-\tau}}$ such that

$$F(t, x) - \frac{1}{\mu} f(t, x)x \leq g(t)|x|^\tau, \quad \text{a.e. } t \in [0, T], x \in \mathbb{R}.$$

(f_3) There exists $\sigma > 2$ such that $\lim_{|x|\to 0} \sup_{t\in[0,T]} \frac{F(t,x)}{|x|^\sigma} < \infty$.

(f_4) There exists $\theta > 2$ such that $\lim_{|x|\to\infty} \inf_{t\in[0,T]} \frac{F(t,x)}{|x|^\theta} > 0$.

(f_4') There exists $\theta > 4$ such that $\lim_{|x|\to\infty} \inf_{t\in[0,T]} \frac{F(t,x)}{|x|^\theta} > 0$.

(f_5) $f(t, x)x \geq 0$ for all $t \in [0, T]$ and $x \in \mathbb{R}$.

The energy functional associated with BVP (1.4) is expressed by

$$\begin{aligned}
I_\lambda(u) = {}& \frac{1}{2} \int_0^T \bigl[a\bigl({}_0D_t^\alpha u\bigr)^2(t) + \lambda V(t)u^2(t)\bigr]\, dt \\
&+ \frac{b}{4}\biggl(\int_0^T \bigl({}_0D_t^\alpha u\bigr)^2\, dt\biggr)^2 - \int_0^T F\bigl(t, u(t)\bigr)\, dt.
\end{aligned} \tag{3.1}$$

Furthermore, clearly, under conditions (V_1) and (f_1),

$$\begin{aligned}
I_\lambda'(u)v = {}& \int_0^T \bigl[a\bigl({}_0D_t^\alpha u\bigr)\bigl({}_0D_t^\alpha v\bigr)(t) + \lambda V(t)u(t)v(t)\bigr]\, dt \\
&+ b\int_0^T \bigl({}_0D_t^\alpha u\bigr)^2\, dt \int_0^T \bigl({}_0D_t^\alpha u\bigr)\bigl({}_0D_t^\alpha v\bigr)(t)\, dt - \int_0^T f\bigl(t, u(t)\bigr)v(t)\, dt
\end{aligned} \tag{3.2}$$

for all $u, v \in X_\lambda$.

$u \in X_\lambda$ is called a weak solution of BVP (1.4) if $I'_\lambda(u)v = 0$ holds for all $v \in X_\lambda$. That is, u is a critical point of I_λ in X_λ.

Firstly, we establish several lemmas.

Lemma 3.1 *If conditions* (V_1) *and* (f_1)*–*(f_2) *hold, then any* $(C)_c$ *sequence* $\{u_n\}$ *of* I_λ *for each* $c \in \mathbb{R}$ *is bounded in* X_λ.

Proof Let $\{u_n\}$ be any $(C)_c$ sequence of I_λ. Then $I_\lambda(u_n) \to c$ and $(1 + \|u_n\|_\lambda)I'_\lambda(u_n) \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned} c + o(1) &= I_\lambda(u_n) - \frac{1}{\mu}I'_\lambda(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_0^T \left[a\left({}_0D_t^\alpha u_n(t)\right)^2 + \lambda V(t)u_n^2(t)\right]dt \\ &\quad + \left(\frac{1}{4} - \frac{1}{\mu}\right)b\left(\int_0^T \left({}_0D_t^\alpha u_n(t)\right)^2 dt\right)^2 \\ &\quad + \int_0^T \left(\frac{1}{\mu}f\left(t, u_n(t)u_n(t)\right) - F\left(t, u_n(t)\right)\right)dt \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_\lambda^2 - \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_0^T \lambda V^-(t)u_n^2(t)\,dt \\ &\quad + \left(\frac{1}{4} - \frac{1}{\mu}\right)b\left(\int_0^T \left({}_0D_t^\alpha u_n\right)^2 dt\right)^2 + \int_0^T \left(\frac{1}{\mu}f(t, u_n)u_n - F(t, u_n)\right)dt. \end{aligned} \tag{3.3}$$

By (V_1), there exists $v_0 > 0$ such that $0 \le V^-(t) \le v_0$, a.e. $t \in [0, T]$, and therefore

$$0 \le \int_0^T V^-(t)u_n^2(t)\,dt \le v_0\|u_n\|_{L^2}^2. \tag{3.4}$$

Also, by (f_1)–(f_2), one gets

$$\int_0^T \left[F\left(t, u_n(t)\right) - \frac{1}{\mu}f\left(t, u_n(t)\right)u_n(t)\right]dt \le \int_0^T g(t)|u_n|^\tau\,dt \le g_0\|u_n\|_{L^2}^\tau, \tag{3.5}$$

where $g_0 = \|g\|_{L^{\frac{2}{2-\tau}}}$. Thus, it follows from (3.3)–(3.5) that

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_\lambda^2 \le c + o(1) + \lambda\left(\frac{1}{2} - \frac{1}{\mu}\right)v_0\|u_n\|_{L^2}^2 + g_0\|u_n\|_{L^2}^\tau. \tag{3.6}$$

Inequality (3.6) shows that if the sequence $\{u_n\}$ is bounded in L^2, then so is it in X_λ.

Assume by contradiction that there exists a subsequence, still denoted by $\{u_n\}$, such that $\|u_n\|_{L^2} \to \infty$ as $n \to \infty$. Write $v_n = \frac{u_n}{\|u_n\|_{L^2}}$. Then $\|v_n\|_{L^2} = 1$. It follows from (3.6) that

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|v_n\|_\lambda^2 \le \frac{c}{\|u_n\|_{L^2}} + o(1) + \lambda\left(\frac{1}{2} - \frac{1}{\mu}\right)v_0 + g_0\|u_n\|_{L^2}^{\tau-2}.$$

The above inequality together with $0 < \tau < 2$ implies that $\{v_n\}$ is bounded in X_λ. Thus, up to a subsequence, $v_n \rightharpoonup v$ in X_λ, and then it follows from Lemma 2.4 that $v_n \to v$ in $L^q, q \ge 1$ and $v_n \to v$ in $C[0, T]$.

On the other hand, by (3.3)–(3.5), we also have

$$\left(\frac{1}{4}-\frac{1}{\mu}\right)\left(\int_0^T \left({}_0D_t^\alpha v_n(t)\right)^2 dt\right)^2 \le \frac{c}{\|u_n\|_{L^2}^4} + o(1) + \frac{g_0\nu_0}{\|u_n\|_{L^2}^2} + g_0\|u_n\|_{L^2}^{\tau-4}. \tag{3.7}$$

Denote a norm $\|\cdot\|^*$ on X_λ by $\|u\|^* = (\int_0^T ({}_0D_t^\alpha u(t))^2\,dt)^{1/2}$. Then it follows from Lemmas 2.2–2.4 that the norms $\|\cdot\|_\lambda$ and $\|\cdot\|^*$ are equivalent. Thus, the relation $v_n \rightharpoonup v$ in $(X_\lambda, \|\cdot\|_\lambda)$ implies that $v_n \rightharpoonup v$ in $(X_\lambda, \|\cdot\|^*)$, and therefore, by the weak lower semi-continuity of norm, we have $\|v\|^* \le \lim_{x\to\infty}\inf\|v_n\|^*$. That is, $\int_0^T ({}_0D_t^\alpha v)^2\,dt \le \lim_{n\to\infty}\inf\int_0^T ({}_0D_t^\alpha v_n)^2\,dt$. Hence, by (3.7), it follows that $\int_0^T ({}_0D_t^\alpha v(t))^2\,dt = 0$, and so, ${}_0D_t^\alpha v(t) = 0$, a.e. $t \in [0,T]$. Thus, $v(t) = {}_0D_t^{-\alpha}{}_0D_t^\alpha v(t) = 0, t \in [0,T]$, which contradicts $\|v\|_{L^2} = \lim_{n\to\infty}\|v_n\|_{L^2} = 1$. This means that $\{u_n\}$ is bounded in $L^2[0,T]$, and so is in X_λ. The proof is complete. □

Lemma 3.2 *Under conditions* (V_1)–(V_2), *for each fixed* $j \ge 1$, *the eigenvalue* $\beta_j(\lambda)$ *associated with* (2.1) *satisfies that* $\beta_j(\lambda) \to 0$ *as* $\lambda \to +\infty$.

Proof In terms of (V_1)–(V_2), we can choose $\phi_i \in C_0^\infty(\Omega_2)\setminus\{0\}$ with $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j = \emptyset$, $i \ne j, 1 \le i,j \le m$. Let $F = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_m\}$. Then, by Lemma 2.6,

$$0 < \beta_j(\lambda) \le \sup_{u\in F\setminus\{0\}} \frac{\|u\|_\lambda^2}{\lambda\int_0^T V^-(t)u^2(t)\,dt}. \tag{3.8}$$

Notice that $\operatorname{supp}\phi_i \subset \Omega_2, \phi_i(t)V^+(t) = 0, t \in [0,T]$. Thus $\|u\|_\lambda = \|u\|_1$.

Now, for any $u \in F$, define $\|u\|_* = (\int_0^T V^-(t)u^2(t)\,dt)^{\frac{1}{2}}$. We claim that $(F, \|\cdot\|_*)$ is a normed space. In fact, for any $u \in F$ and $k \in \mathbb{R}$, obviously, $\|u\|_* \ge 0$ and $\|ku\|_* = |k|\|u\|_*$. In addition, if $\|u\|_* = 0$, then the fact that $\operatorname{supp} u \subset \Omega_2$ implies that $u(t) = 0, t \in [0,T]$, namely $u = 0$.

Finally , we show that

$$\|u+v\|_* \le \|u\|_* + \|v\|_*$$

for any $u, v \in F$. Indeed, for any $u, v \in F$, since

$$\begin{aligned}\int_0^T V^-(t)|u(t)||v(t)|\,dt &= \int_0^T \left(V^-(t)\right)^{\frac{1}{2}}|u(t)|\left(V^-(t)\right)^{\frac{1}{2}}|v(t)|\,dt \\ &\le \left(\int_0^T V^-(t)u^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^T V^-(t)v^2(t)\,dt\right)^{\frac{1}{2}} \\ &= \|u\|_*\|v\|_*,\end{aligned}$$

we have

$$\begin{aligned}\|u+v\|_*^2 &= \int_0^T V^-(t)(u+v)^2\,dt \\ &\le \int_0^T V^-(t)\left(u^2+v^2+2|u||v|\right)dt\end{aligned}$$

$$= \|u\|_*^2 + \|v\|_*^2 + 2\int_0^T V^-(t)|u||v|\,dt$$
$$\le \big(\|u\|_* + \|v\|_*\big)^2.$$

That is, $\|u+v\|_* \le \|u\|_* + \|v\|_*$. Hence, $(F, \|\cdot\|_*)$ is a normed space with finite dimension.

Now, in terms of the equivalence of norms on a finite dimensional space, there exist two constants $c_1, c_2 > 0$ such that $c_1\|u\|_* \le \|u\|_1 \le c_2\|u\|_*$ for any $u \in F$. Then, it follows from (3.8) that $0 < \beta_j(\lambda) \le \frac{1}{\lambda}c_2^2 \to 0$, as $\lambda \to +\infty$, noting that $\|u\|_\lambda = \|u\|_1$. The proof is complete. □

By Lemma 3.2, there exists $\Lambda_0 > 0$ such that $X_{\lambda,1} \ne \emptyset$ as $\lambda > \Lambda_0$. In what follows, we will apply Lemma 2.9 with $Y = X_{\lambda,1}$ and $Z = X_{\lambda,2} \oplus Y_\lambda$. Of course, $Y \ne \emptyset$ and $\dim Y < \infty$.

Lemma 3.3 *Let* (V_1)–(V_2) *and* $(f_1), (f_3)$ *hold. Then, for each* $\lambda > \Lambda_0$, *there exist* $r_\lambda > 0$ *and* $k_\lambda > 0$ *such that* $I_\lambda(u) \ge k_\lambda$ *for all* $u \in X_{\lambda,2} \oplus Y_\lambda$ *with* $\|u\|_\lambda = r_\lambda$.

Proof We first show that there exists $\delta_\lambda > 0$ such that $a_\lambda(u,u) \ge \delta_\lambda\|u\|_\lambda^2$ for all $u \in X_{\lambda,2}$. The argument is similar to that in Lemma 2.7. In fact, for any $j \ge N_0(\lambda)+1$, we have

$$a_\lambda(e_j, u) = \int_0^T \big[a\big({}_0D_t^\alpha e_j(t)\big)\big({}_0D_t^\alpha u(t)\big) + \lambda V(t)e_j(t)u(t)\big]\,dt$$
$$= \lambda(\beta_j - 1)\int_0^T V^-(t)e_j(t)u(t)]\,dt,$$
$$\langle e_j, u\rangle_\lambda = \int_0^T \big[a\big({}_0D_t^\alpha e_j(t)\big)\big({}_0D_t^\alpha u(t)\big) + \lambda V^+(t)e_j(t)u(t)\big]\,dt$$
$$= \lambda\beta_j\int_0^T V^-(t)e_j(t)u(t)\,dt.$$

Thus,

$$a_\lambda(e_j, u) = \left(1 - \frac{1}{\beta_j}\right)\langle e_j, u\rangle_\lambda \ge \delta_\lambda\langle e_j, u\rangle_\lambda,$$

where $\delta_\lambda = 1 - \frac{1}{\beta_{N_0(\lambda)+1}} > 0$ noting that $\beta_{N_0(\lambda)+1} > 1$.

Since $\{e_j\}_{j=N_0(\lambda)+1}^\infty$ is a basis of $X_{\lambda,2}$, taking $\{u_n\} \subset X_{\lambda,2}$ such that $u_n \to u$ $X_{\lambda,2}$ with $u_n = \sum_{i=N_0(\lambda)+1}^{m_n} t_i^n e_i$, then

$$a_\lambda(u, u_n) = a_\lambda\left(u, \sum_{i=N_0(\lambda)+1}^{m_n} t_i^{(n)} e_i\right)$$
$$= \sum_{i=N_0(\lambda)+1}^{m_n} t_i^{(n)} a_\lambda(u, e_i)$$
$$\ge \delta_\lambda \sum_{i=N_0(\lambda)+1}^{m_n} t_i^{(n)}\langle u, e_i\rangle_\lambda$$
$$= \delta_\lambda\langle u, u_n\rangle_\lambda,$$

and so, $a_\lambda(u,u) = \lim_{n\to\infty} a_\lambda(u,u_n) \geq \lim_{n\to\infty} \delta_\lambda \langle u,u_n\rangle_\lambda = \delta_\lambda \langle u,u\rangle_\lambda = \delta_\lambda \|u\|_\lambda^2$. While for any $v \in Y_\lambda$, because $V(t)v^2(t) = V^+(t)v^2(t)$, we have $a_\lambda(v,v) = \|v\|_\lambda^2$. Hence, for any $w = u \oplus v \in X_{\lambda,2} \oplus Y_\lambda$, observing that $a_\lambda(u,v) = 0$ by Lemma 2.7, we have

$$\begin{aligned} I_\lambda(w) &= \frac{1}{2}a_\lambda(u,u) + \frac{1}{2}a_\lambda(v,v) + \frac{b}{4}\left(\int_0^T \left({}_0D_t^\alpha w(t)\right)^2 dt\right)^2 - \int_0^T F\big(t,w(t)\big)\,dt \\ &\geq \frac{1}{2}\delta_\lambda \|u\|_\lambda^2 + \frac{1}{2}\|v\|_\lambda^2 - \int_0^T F\big(t,w(t)\big)\,dt \\ &\geq \overline{\delta_\lambda}\|w\|_\lambda^2 - \int_0^T F\big(t,w(t)\big)\,dt, \end{aligned} \tag{3.9}$$

where $\overline{\delta_\lambda} = \min\{\frac{1}{2}\delta_\lambda, \frac{1}{2}\} > 0$.

On the other hand, by condition (f_3), taking $l > 0$ with $\lim_{|x|\to 0}\sup_{t\in[0,T]} \frac{F(t,x)}{|x|^\sigma} < l$, then there exists $r_1 > 0$ such that $\frac{F(t,x)}{|x|^\sigma} < l$ as $|x| < r_1$. Thus, $F(t,x) < l|x|^\sigma$ as $|x| < r_1$. So, by Lemmas 2.2 and 2.4, there is $r_2 > 0$ such that $\|u\|_\lambda \leq r_2$ ensures that $\|u\|_\infty < r_1$ for any $u \in X_\lambda$. Hence, for any $w \in X_{\lambda,2} \oplus Y_\lambda$, if $\|w\|_\lambda \leq r_2$, then

$$F\big(t,w(t)\big) \leq l\big|w(t)\big|^\sigma, \quad t \in [0,T]. \tag{3.10}$$

Thus, it follows from (3.9)–(3.10) that

$$I_\lambda(w) \geq \overline{\delta_\lambda}\|w\|_\lambda^2 - l\int_0^T \big|w(t)\big|^\sigma\,dt \geq \overline{\delta_\lambda}\|w\|_\lambda^2 - lT\|w\|_\infty^\sigma. \tag{3.11}$$

Again by Lemmas 2.2 and 2.4, there exists $c_\lambda > 0$ such that $\|w\|_\infty^\sigma \leq c_\lambda\|w\|_\lambda^\sigma$. Thus, by (3.11), we get

$$I_\lambda(w) \geq \overline{\delta_\lambda}\|w\|_\lambda^2 - \overline{c_\lambda}\|w\|_\lambda^\sigma, \tag{3.12}$$

where constant $\overline{c_\lambda} > 0$. Noting that $\sigma > 2$, by (3.12), we can take small $0 < r < r_1$ and a number $k_\lambda > 0$ such that $I_\lambda(w) \geq k_\lambda$ for $w \in X_{\lambda,2} \oplus Y_\lambda$ with $\|w\|_\lambda = r$. The proof is complete. □

By (V_2), we can take $e_0 \in C_0^\infty(\Omega_1)\backslash\{0\}$ with $e_0(t) \geq 0, t \in [0,T]$ and $\|e_0\|_\lambda = r$, then $e_0 \in Y_\lambda$. We have the following conclusion.

Lemma 3.4 *Suppose that* (V_1)–(V_2) *and* (f_1), (f_4)–(f_5) *hold. Then, for each* $\lambda > \Lambda_0$, *there exist* $b_\lambda > 0$ *and* $\rho_\lambda(> r_\lambda)$ *such that* $\sup_{u\in\partial\Phi} I_\lambda(u) < k_\lambda$ *as* $b < b_\lambda$, *where*

$$\Phi = \big\{u = v + se_0 : v \in X_{\lambda,1}, \|u\|_\lambda \leq \rho_\lambda, s \geq 0\big\}.$$

Proof By (f_4), take $d_0 > 0$ with $\lim_{|x|\to\infty}\inf_{t\in[0,T]} \frac{F(t,x)}{|x|^\theta} > d_0$. Then $\exists M_0 > 0$ such that $\frac{F(t,x)}{|x|^\theta} > d_0$ as $|x| \geq M_0$. That is, $F(t,x) > d_0|x|^\theta$, as $|x| \geq M_0$. By (f_1), let $m_0 = \min_{t\in[0,T],|x|\leq M_0}(F(t,x) - d_0|x|^\theta)$. Thus $F(t,x) \geq d_0|x|^\theta - |m_0|, t \in [0,T], x \in \mathbb{R}$.

The following argument is divided into two parts.

(i) We show that $\exists \rho_\lambda(> r_\lambda)$ and $\overline{b_\lambda} > 0$ such that $I_\lambda(u) < 0$ as $u \in X_{\lambda,1} \oplus \mathbb{R}e_0$ with $\|u\|_\lambda = \rho_\lambda$ and $b < \overline{b_\lambda}$.

In fact, for any $u = v + w \in X_{\lambda,1} \oplus \mathbb{R}e_0$, we already know that $a_\lambda(v,w) = 0, a_\lambda(v,v) \leq 0$ by Lemma 2.7. Moreover, owing to the fact that $e_0 \in C_0^\infty(\Omega_1)$ and $V(t)e_0^2(t) = V^+(t)e_0^2(t)$, we have $a_\lambda(w,w) = \|w\|_\lambda^2$. Thus

$$\begin{aligned} I_\lambda(u) &\leq \frac{1}{2}\|w\|_\lambda^2 + \frac{b}{4}\left(\int_0^T \left({}_0D_t^\alpha u\right)^2(t)\,dt\right)^2 - \int_0^T F\bigl(t,u(t)\bigr)\,dt \\ &\leq \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4a^2}\|u\|_\lambda^4 - \int_0^T \left[d_0\bigl|u(t)\bigr|^\theta - |m_0|\right]dt \\ &= \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4a^2}\|u\|_\lambda^4 + |m_0|T - d_0\|u\|_{L^\theta}^\theta. \end{aligned}$$

In terms of equivalence of the norms on a finite dimensional space, there exists $d_1 > 0$ such that $d_0\|u\|_{L^\theta}^\theta \geq d_1\|u\|_\lambda^\theta$. Thus,

$$I_\lambda(u) \leq \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4a^2}\|u\|_\lambda^4 + |m_0|T - d_1\|u\|_\lambda^\theta. \tag{3.13}$$

Let $h(t) = \frac{1}{2}t^2 + |m_0|T - d_1t^\theta$. The assumption $\theta > 2$ yields that $h(t) \to -\infty$ as $t \to +\infty$. Thus, we can take $\rho_\lambda(> r_\lambda)$ such that $h(\rho_\lambda) < 0$, and then, choose small $\overline{b_\lambda} > 0$ so that $h(\rho_\lambda) + \frac{1}{4a^2}\overline{b_\lambda}\rho_\lambda^4 < 0$. Hence, it follows from (3.13) that $I_\lambda(u) < 0$ as $u \in X_{\lambda,1} \oplus \mathbb{R}e_0$ with $\|u\|_\lambda = \rho_\lambda$ and $b \leq \overline{b_\lambda}$.

(ii) We show that $\exists b_\lambda \in (0, \overline{b_\lambda}]$ such that $I_\lambda(u) < k_\lambda$ for $u \in X_{\lambda,1}$ with $\|u\|_\lambda \leq \rho_\lambda$, and $b < b_\lambda$.

In fact, by (f_5), $F(t,x) \geq 0$, $t \in [0,T]$ and $x \in \mathbb{R}$. For any $u \in X_{\lambda,1}$ with $\|u\|_\lambda \leq \rho_\lambda$, by Lemma 2.7, $a_\lambda(u,u) \leq 0$, and therefore

$$I_\lambda(u) \leq \frac{b}{4}\left(\int_0^T \left({}_0D_t^\alpha u\right)^2(t)\,dt\right)^2 \leq \frac{b}{4a^2}\|u\|_\lambda^4 \leq \frac{b}{4a^2}\rho_\lambda^4.$$

For $0 < \overline{b_\lambda}$ taken previously in (i), choose small $0 < b_\lambda \leq \overline{b_\lambda}$ so that $\frac{b_\lambda}{4a^2}\rho_\lambda^4 < k_\lambda$. Then $I_\lambda(u) < k_\lambda$ as $\|u\|_\lambda \leq \rho_\lambda$.

By the above arguments on (i)–(ii), we conclude that $\sup_{u\in\partial\Phi} I_\lambda(u) \leq k_\lambda$. The proof is complete. □

Lemma 3.5 *Assume that* (V_1), (V_3) *and* (f_1)–(f_2) *hold. Then any* $(C)_c$ *sequence* $\{u_n\}$ *of* I_λ *satisfies the Cerami condition at level* c *for each* $\lambda > 0$ *for any* $c \in \mathbb{R}$.

Proof Let $\{u_n\}$ be any $(C)_c$ sequence of I_λ. Then, by Lemma 3.1, $\{u_n\}$ is bounded in X_λ. Thus, up to a subsequence, $u_n \rightharpoonup u$ in X_λ, and therefore, $(I'_\lambda(u_n) - I'_\lambda(u))(u_n - u) \to 0$ as $n \to \infty$. Let $v_n = u_n - u$, then by (3.2)

$$\begin{aligned} o(1) &= \bigl(I'_\lambda(u_n) - I'_\lambda(u)\bigr)(v_n) \\ &= \|v_n\|_\lambda^2 - \lambda\int_0^T V^-(t)v_n^2(t)\,dt + bA_n - \int_0^T \bigl(f(t,u_n) - f(t,u)\bigr)v_n\,dt, \end{aligned} \tag{3.14}$$

where

$$A_n = \int_0^T \left({}_0D_t^\alpha u_n\right)^2 dt \int_0^T \left({}_0D_t^\alpha u_n\right)\left({}_0D_t^\alpha v_n\right)dt - \int_0^T \left({}_0D_t^\alpha u\right)^2 dt \int_0^T \left({}_0D_t^\alpha u\right)\left({}_0D_t^\alpha v_n\right)dt$$

$$
\begin{aligned}
&= \int_0^T \left({}_0D_t^\alpha u_n\right)^2 dt \int_0^T \left({}_0D_t^\alpha v_n\right)^2 dt \\
&\quad + \left[\int_0^T \left({}_0D_t^\alpha u_n\right)^2 dt - \int_0^T \left({}_0D_t^\alpha u\right)^2 dt\right] \int_0^T \left({}_0D_t^\alpha u\right)\left({}_0D_t^\alpha v_n\right) dt \\
&\geq \left[\int_0^T \left({}_0D_t^\alpha u_n\right)^2 dt - \int_0^T \left({}_0D_t^\alpha u\right)^2 dt\right] \int_0^T \left({}_0D_t^\alpha u\right)\left({}_0D_t^\alpha v_n\right) dt.
\end{aligned}
$$

Again, owing to the fact that $v_n \rightharpoonup 0$ in X_λ, we have

$$
o(1) = \langle v_n, u\rangle_\lambda = \int_0^T a\left({}_0D_t^\alpha v_n\right)\left({}_0D_t^\alpha u\right) dt + \lambda \int_0^T V^+(t) v_n(t) u(t)\, dt. \tag{3.15}
$$

We turn to showing that $\int_0^T V^+(t)v_n(t)u(t)\,dt \to 0$ as $n \to \infty$. Set

$$
V_R = \left\{x \in [0,T] : V^+(t) \geq R\right\}, \qquad V_R^c = [0,T]\backslash V_R.
$$

By (V_3), $\lim_{R\to+\infty} \operatorname{meas} V_R = 0$. Because $\{v_n\}$ is bounded in X_λ, there exists $M_0 > 0$ such that $\|v_n\|_\lambda \leq \sqrt{\lambda} M_0$, and so $(\int_0^T V^+(t)v_n^2\,dt)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\lambda}}\|v_n\|_\lambda \leq M_0$. Thus

$$
\begin{aligned}
\int_{V_R} V^+|v_n||u|\,dt &= \int_{V_R} \left(V^+\right)^{\frac{1}{2}}|v_n|\left(V^+\right)^{\frac{1}{2}}|u|\,dt \\
&\leq \left(\int_{V_R} V^+v_n^2\,dt\right)^{\frac{1}{2}} \left(\int_{V_R} V^+u^2\,dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^T V^+v_n^2\,dt\right)^{\frac{1}{2}} \left(\int_{V_R} V^+u^2\,dt\right)^{\frac{1}{2}} \\
&\leq M_0\left(\int_{V_R} V^+u^2\,dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.16}
$$

Now, since $\int_0^T V^+(t)u^2(t)\,dt < \infty$, by the absolute continuity of integral combined with the fact $\lim_{R\to+\infty} \operatorname{meas} V_R = 0$, there exists large $R_0 > 0$ so that $(\int_{V_{R_0}} V^+(t)u^2(t)\,dt)^{\frac{1}{2}} < \frac{\varepsilon}{2M_0}$. Then it follows from (3.16) that

$$
\int_{V_{R_0}} V^+(t)\left|v_n(t)\right|\left|u(t)\right| dt < \frac{\varepsilon}{2}. \tag{3.17}
$$

On the other hand, taking into account that $u_n \rightharpoonup 0$ implies that $v_n \to 0$ in $C([0,T])$ and $v_n \to 0$ in $L^2[0,T]$, observing that $\int_{V_{R_0}^c} V^+(t)|v_n||u|\,dt \leq R_0\|v_n\|_\infty\|u\|_\infty$, we know that there exists $N_0 \geq 1$ such that

$$
\int_{V_{R_0}^c} V^+(t)|v_n||u|\,dt < \frac{\varepsilon}{2}, \tag{3.18}
$$

as $n \geq N_0$.

Then, by (3.17)–(3.18), one has

$$
\int_0^T V^+|v_n||u|\,dt \leq \int_{V_{R_0}} V^+|v_n||u|\,dt + \int_{V_{R_0}^c} V^+|v_n||u|\,dt < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
$$

as $n \geq N_0$. Namely, $\int_0^T V^+|v_n||u|\,dt \to 0$ as $n \to \infty$, and so $\int_0^T V^+ v_n u\,dt \to 0$, as $n \to \infty$. Hence, it follows from (3.15) that

$$\int_0^T \big({}_0D_t^\alpha v_n\big)\big({}_0D_t^\alpha u\big)\,dt \to 0 \tag{3.19}$$

as $n \to \infty$. Hence, by (3.19) it is easy to see that

$$\int_0^T \big({}_0D_t^\alpha u_n\big)^2\,dt - \int_0^T ({}_0D_t u)^2\,dt = \int_0^T \big({}_0D_t^\alpha v_n\big)^2\,dt + o(1). \tag{3.20}$$

Thus

$$A_n \geq \int_0^T \big({}_0D_t^\alpha v_n\big)^2\,dt \int_0^T \big({}_0D_t^\alpha u\big)\big({}_0D_t^\alpha v_n\big)\,dt + o(1) \int_0^T \big({}_0D_t^\alpha u\big)\big({}_0D_t^\alpha v_n\big)\,dt.$$

Again, from (3.19), it follows that there exists $N_1 \geq 1$ such that

$$A_n \geq -\frac{1}{3ba} \int_0^T \big({}_0D_t^\alpha v_n\big)^2\,dt + o(1) \geq -\frac{1}{3b}\|v_n\|_\lambda^2 + o(1) \tag{3.21}$$

as $n \geq N_1$.

Finally, owing to the fact that $f \in C([0,T],\mathbb{R})$ and $v_n \to 0$ in $C([0,T])$, $u \in C([0,T])$, $0 \leq V^-(t) \leq v_0$, a.e., $t \in [0,T]$, it is easy to see that

$$\int_0^T \big(f(t,u_n) - f\big(t,u(t)\big)v_n\big)\,dt = o(1), \qquad \lambda \int_0^T V^-(t)v_n^2(t)\,dt = o(1).$$

Combining (3.14) with (3.21), we get

$$o(1) \geq \frac{2}{3}\|v_n\|_\lambda^2 + o(1).$$

This means that $v_n \to 0$ in X_λ and the proof is complete. □

Now, we are in a position to show our first result on the existence of solution to BVP (1.4).

Theorem 3.1 *Assume that conditions* (V_1)–(V_3) *and* (f_1)–(f_5) *hold. Then there exist constants* $\Lambda_0 > 0$ *and* $b_\lambda > 0$ *such that BVP* (1.4) *has at least one nontrivial weak solution for* $\lambda > \Lambda_0$ *and* $b < b_\lambda$.

Proof Firstly, we show that I_λ is of class C^1.

In fact, let $\{u_n\}$ be any sequence with $u_n \to u$ in X_λ. Then $u_n \to u$ in $C([0,T])$ and in $L^2[0,T]$ by Lemma 2.4. Set $L_u\varphi = \langle u,\varphi\rangle_\lambda$, $\phi_u\varphi = \int_0^T V^-(t)u\varphi\,dt$, $\psi_u\varphi = \int_0^T ({}_0D_t u)^2\,dt \times \int_0^T ({}_0D_t^\alpha u)({}_0D_t^\alpha \varphi)\,dt$ and $G_u\varphi = \int_0^T f(t,u)\varphi\,dt$ for any $\varphi \in X_\lambda$. Then, by (3.2), $I_\lambda'(u)\varphi = L_u\varphi - \lambda\phi_u\varphi + b\psi_u - G_u\varphi, \varphi \in X_\lambda$.

It is well known that L_u is continuous in X_λ^*. Next, we show that ϕ_u, ψ_u, and G_u are also continuous in X_λ^*.

(i) On ϕ_u, for any $\varphi \in X_\lambda$ with $\|\varphi\|_\lambda \leq 1$, noting that $0 \leq V^-(t) \leq \nu_0$, we have

$$\begin{aligned} \|\phi_{u_n}\varphi - \phi_u\varphi\| &\leq \int_0^T V^-|u_n - u||\varphi|\,dt \\ &\leq \nu_0 \int_0^T |u_n - u||\varphi|\,dt \\ &\leq \nu_0 \|u_n - u\|_{L^2}\|\varphi\|_{L^2} \\ &\leq c_0\|u_n - u\|_{L^2}\|\varphi\|_\lambda \end{aligned}$$

for some $c_0 > 0$, because $\|\varphi\|_{L^2} \leq c_1\|\varphi\|_\lambda$ by Lemma 2.4. Hence

$$\|\phi_{u_n} - \phi_u\|_{X_\lambda^*} = \sup_{\|\varphi\|_\lambda \leq 1} \|\phi_{u_n}\varphi - \phi_u\varphi\| \leq c_0\|u_n - u\|_{L^2} \to 0$$

as $n \to \infty$. That is, ϕ_u is continuous in X_λ^*.

(ii) On ψ_u, for any $\varphi \in X_\lambda$ with $\|\varphi\|_\lambda \leq 1$, by an argument similar to (3.20), we have

$$\begin{aligned} &|\psi_{u_n}\varphi - \psi_u\varphi| \\ &\quad\leq \left|\int_0^T ({}_0D_t^\alpha u_n)^2\,dt - \int_0^T ({}_0D_t^\alpha u)^2\,dt\right|\left|\int_0^T ({}_0D_t^\alpha u_n)({}_0D_t^\alpha \varphi)\,dt\right| \\ &\qquad + \int_0^T ({}_0D_t^\alpha u)^2\,dt\left|\int_0^T {}_0D_t^\alpha (u_n - u)({}_0D_t^\alpha \varphi)\,dt\right| \\ &\quad\leq \left(\int_0^T ({}_0D_t^\alpha (u_n - u))^2\,dt + |o(1)|\right)\int_0^T |{}_0D_t^\alpha u_n||{}_0D_t^\alpha \varphi|\,dt \\ &\qquad + \int_0^T ({}_0D_t^\alpha u)^2\,dt\int_0^T |{}_0D_t^\alpha (u_n - u)||{}_0D_t^\alpha \varphi|\,dt \\ &\quad\leq \left(\int_0^T ({}_0D_t^\alpha (u_n - u))^2\,dt + |o(1)|\right)\left(\int_0^T ({}_0D_t^\alpha u_n)^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T ({}_0D_t^\alpha \varphi)^2\,dt\right)^{\frac{1}{2}} \\ &\qquad + \int_0^T ({}_0D_t^\alpha u)^2\,dt\left(\int_0^T ({}_0D_t^\alpha (u_n - u))^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T ({}_0D_t^\alpha \varphi)^2\,dt\right)^{\frac{1}{2}} \\ &\quad\leq \left(\frac{1}{a}\|u_n - u\|_\lambda^2 + |o(1)|\right)\frac{1}{a}\|u_n\|_\lambda\|\varphi\|_\lambda + \frac{1}{a^2}\|u_n - u\|_\lambda\|u\|_\lambda^2\|\varphi\|_\lambda \\ &\quad\leq c_1\|u_n - u\|_\lambda^2 + |o(1)| + c_2\|u_n - u\|_\lambda \end{aligned}$$

for some $c_1 > 0$, $c_2 > 0$ observing that $\{\|u_n\|_\lambda\}$ is bounded, $\|\varphi\|_\lambda \leq 1$ and $c_2 = \frac{1}{a^2}\|u\|_\lambda^2$.

Thus $\|\psi_{u_n} - \psi_u\|_{X_\lambda^*} = \sup_{\|\varphi\|_\lambda \leq 1} |\psi_{u_n}\varphi - \psi_u\varphi| \to 0$ as $n \to \infty$. That is, ψ_u is continuous in X_λ^*.

(iii) On G_u, by $f \in C([0,T],\mathbb{R})$, it is easy to see that G_u is also continuous in X_λ^*, we omit it.

Summing up the above arguments (i)–(iii), we know that I_λ is of class C^1. Now, by Lemmas 3.3 and 3.4 and applying Lemma 2.9, for $\lambda > \Lambda_0$, there exists a $(C)_c$ sequence $\{u_n\}$ of I_λ in X_λ with $c \geq k_\lambda > 0$. Then, by Lemma 3.5, up to a subsequence, $u_n \to u$ in X_λ. Hence, $\forall \varphi \in X_\lambda$, thanks to the fact that I_λ is of class C^1, $0 = \lim_{n\to\infty} I'_\lambda(u_n)\varphi = I'_\lambda(u)\varphi$ and

$0 < k_\lambda \le c = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u)$. Thus, u is a nontrivial solution of BVP (1.4). The proof is complete. □

Now, we give the second existence result.

Theorem 3.2 *Assume that conditions* (V_1), (V_3), (f_1)*–*(f_3), *and* (f_4') *hold. Furthermore,* $V(t) \ge 0$, *a.e.* $t \in [0,T]$. *Then BVP* (1.4) *has at least one nontrivial weak solution for each* $\lambda > 0$.

Proof We already know that I_λ is of class C^1 by the proof of Theorem 3.1. Moreover, under conditions (f_1) and (f_3), making an argument similar to (3.12), we know that the following inequality also holds:

$$I_\lambda(u) \ge \overline{\delta_\lambda}\|u\|_\lambda^2 - \bar{c}_\lambda\|u\|_\lambda^\sigma, \quad u \in X_\lambda$$

for some $\overline{\delta_\lambda} > 0$, $\overline{c_\lambda} > 0$. Thus, observing that $\sigma > 2$, there exist constants $\alpha_0 > 0$, $\rho > 0$ such that $I_\lambda(u) \ge \alpha_0$ as $\|u\|_\lambda = \rho$ small enough.

On the other hand, by (f_4') and making an argument similar to (3.13), we also have

$$I_\lambda(u) \le \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4a^2}\|u\|_\lambda^4 + |m_0|T - d_0\|u\|_{L^\theta}^\theta, \quad u \in X_\lambda.$$

Then taking $u_0 \in X_\lambda$ with $\|u_0\|_\lambda > \rho$, we have

$$I_\lambda(tu_0) \le \frac{1}{2}t\|u_0\|_\lambda^2 + \frac{b}{4}t^4\|u_0\|_\lambda^4 + |m_0|T - d_0t^\theta\|u_0\|_{L^\theta}^\theta \to -\infty$$

as $t \to +\infty$, noting that $\theta > 4$. So, we choose $t_0 > 0$ large so that $I_\lambda(t_0u_0) < 0$ and $\|t_0u_0\| > \rho$. Write $e_0 = t_0u_0$. Then $I_\lambda(e_0) < 0$ and $\|e_0\|_\lambda > \rho$. By Lemma 2.8, there is a $(C)_c$ sequence $\{u_n\}$ of I_λ with $c = c_\lambda$, where

$$0 < \alpha_0 \le c_\lambda := \inf_{\gamma\in T}\max_{t\in[0,1]} I_\lambda\bigl(\gamma(t)\bigr), \qquad \Gamma = \bigl\{\gamma \in \bigl([0,1], X_\lambda\bigr) : \gamma(0) = 0, \gamma(1) = e\bigr\}.$$

Then, by Lemma 3.5, $\{u_n\}$ satisfies the Cerami condition at level c_λ for each $\lambda > 0$. Thus, passing to a subsequence, $u_n \to u$ in X_λ. By an argument as before, we know that u is a nontrivial weak solution to BVP(1.4). This completes the proof. □

4 Conclusion

In this paper, by applying the mountain pass theorem and the linking theorem, some existence results of the nontrivial solutions to BVP (1.4) were obtained. Here, problem (1.4) is a nonlocal problem as the appearance of the term $\int_0^T ({}_0D_t^\alpha u)^2\,dt$ and has a general potential V, which can be sign-changing. As a result, there are more difficulties that need to be overcome and more derivation techniques need to be introduced.

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