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Positive solutions for a class of fractional infinite-point boundary value problems

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Abstract

In this paper, we consider a class of fractional differential equations with infinite-point boundary value conditions. Under some conditions concerning the spectral radius with respect to the relevant linear operator, both the existence of uniqueness and the nonexistence of positive solution are obtained by means of the iterative technique.

Keywords: Fractional differential equation; Positive solution; Infinite-point boundary value problem; Spectral radius

1 Introduction

In this paper, we consider the following fractional differential equations (FDE for short) with infinite-point boundary value conditions:

$$\begin{cases} D_{0+}^\alpha u(t) + a(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & D_{0+}^\beta u(1) = \sum_{i=1}^\infty \eta_i D_{0+}^\beta u(\xi_i), \end{cases} \quad (1.1)$$

where $\alpha \geq 2$, $n - 1 < \alpha \leq n$, $0 < \beta < \alpha - 1$, $\eta_i \in \mathbb{R}$, $0 < \xi_1 < \cdots < \xi_i < \xi_{i+1} < \cdots < 1$ $(i = 1, 2, \ldots)$, $\sum_{i=1}^\infty \eta_i \xi_i^{\alpha-\beta-1} \neq 1$, D_{0+}^α is the standard Riemann–Liouville derivative, $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous, $a(t) \in C((0, 1), [0, +\infty))$ may be singular at $t = 0, 1$.

FDE serve as an excellent instrument for the description of memory and hereditary properties of various materials and processes. During the last few decades, much attention has been paid to the study of boundary value problems (BVP for short) of fractional differential equation, such as the nonlocal BVP [1, 3, 7, 13, 18], singular BVP [6, 8, 11, 19, 20, 25], semipositone BVP [14–16, 23], resonant BVP [2, 12], and impulsive BVP [10, 27]. Since only positive solutions are meaningful in most practical problems, some work has been done to study the existence of positive solutions for fractional boundary value problems by using the techniques of nonlinear analysis. In [16], the authors discussed the following fractional m-point BVP:

$$\begin{cases} D_{0+}^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, 1 < \alpha \leq 2, \\ u(0) = 0, & D_{0+}^\beta u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^\beta u(\xi_i), \end{cases}$$

where $0 < \beta < \alpha - 1$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$ with $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1} < 1$. Some new positive properties of the corresponding Green function were established, and a number of theo-

rems on the existence of positive solutions were obtained. However, there are few works on the uniqueness of solution for fractional boundary value problems [4, 5, 9, 17, 21, 22, 24, 26]. To obtain the uniqueness of solution, the main tool used in most of the works in literature is the Banach contraction map principle provided the nonlinearity satisfies the Lipschitz condition. Liang and Zhang [9] studied the uniqueness of positive solution for the following fractional three-point BVP:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, 3 < \alpha \le 4, \\ u(0) = u'(0) = u''(0) = 0, & u''(1) = \beta u''(\eta), \end{cases}$$

where $0 < \eta < 1$ with $0 < \beta \eta^{\alpha-3} < 1$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and non-decreasing with respect to the second variable, and there exists

$$0 < \lambda < \left(\frac{2}{(\alpha - 2)\Gamma(\alpha + 1)} + \frac{\beta \eta^{\alpha-3}(1 - \eta)}{(\alpha - 2)(1 - \beta \eta^{\alpha-3})\Gamma(\alpha)} \right)^{-1}$$

such that, for $u, v \in [0, +\infty)$ with $u \ge v$ and $t \in [0, 1]$,

$$f(t, u) - f(t, v) \le \lambda \ln(1 + u - v). \tag{1.2}$$

It should be noted that (1.2) implies $f(t, u) - f(t, v) \le \lambda(u - v)$ since $\ln(1 + x) \le x, \forall x \ge 0$, that is, f satisfies the Lipschitz condition.

Motivated by the above works, in this paper we aim to establish the existence of uniqueness and the nonexistence of positive solution to BVP (1.1). It is well known that the cone plays a very important role in seeking positive solution. Our analysis relies on the iterative technique on the cone derived from the properties of the Green function. Our work presented in this paper has the following features. Firstly, the uniqueness results are obtained under some conditions concerning the spectral radius with respect to the relevant linear operator. Secondly, the Lipschitz condition is generalized. In addition, the error estimation of iterative sequences is also given. Thirdly, the nonexistence results of positive solution are obtained under conditions concerning the spectral radius of the relevant linear operator. Finally, we impose weaker positivity conditions on the nonlocal boundary term, that is, some of the coefficients η_i can be negative.

2 Preliminaries and lemmas

In this section, we present some notations and lemmas that will be used in the proof of our main results.

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} u(s) \, ds$$

provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 2.2 The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) \, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Set

$$K(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \end{cases}$$

$$p(s) = \sum_{s \le \xi_i} \eta_i (\xi_i - s)^{\alpha-\beta-1},$$

$$G(t,s) = K(t,s) + q(s)t^{\alpha-1},$$

where

$$q(s) = \frac{p(0)(1-s)^{\alpha-\beta-1} - p(s)}{\Gamma(\alpha)[1 - p(0)]}, \qquad p(0) = \sum_{i=1}^{\infty} \eta_i \xi_i^{\alpha-\beta-1}.$$

Remark 2.1 Assume that $\sum_{i=1}^{\infty} \eta_i \xi_i^{\alpha-\beta-1}$ is convergent. Then $p \in C[0,1]$.

Remark 2.2 If $\eta_i = 0$ ($i = 1, 2, \ldots$), we have $p(s) \equiv 0$ and $q(s) \equiv 0$; if $\eta_i \ge 0$ ($i = 1, 2, \ldots$) and $p(0) < 1$, we have $q(s) \ge 0$ on $[0,1]$.

For the convenience in presentation, we here list the assumption to be used throughout the paper.

(A_1) $p(0) \ne 1$, $q(s) \ge 0$ on $[0,1]$;

(A_2) $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is continuous, $a(t) \in C((0,1), [0, +\infty))$, and

$$0 < \int_0^1 a(s) \, ds < +\infty.$$

Lemma 2.1 *Assume that $y \in L[0,1]$ and $p(0) \ne 1$. Then the unique solution of the problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & D_{0+}^{\beta} u(1) = \sum_{i=1}^{\infty} \eta_i D_{0+}^{\beta} u(\xi_i), \end{cases}$$

is

$$u(t) = \int_0^1 G(t,s)y(s) \, ds.$$

Proof The proof is similar to Lemma 2.3 in [17], so we omit it. □

Lemma 2.2 *The function $K(t,s)$ has the following properties*:

(1) $K(t, s) > 0, \forall t, s \in (0, 1)$;

(2) $K(t, s) \le \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, \forall t, s \in [0, 1]$;

(3) $\frac{t^{\alpha-1}}{\Gamma(\alpha)} h(s) \le K(t, s) \le \frac{M_1}{\Gamma(\alpha)} h(s), \forall t, s \in [0, 1]$, *where*

$$h(s) = \left[1 - (1-s)^\beta\right](1-s)^{\alpha-\beta-1}, \qquad M_1 = \max\left\{1, \beta^{-1}\right\}.$$

Proof It is obvious that (1), (2) hold. In the following, we will prove (3).

Case (i): $0 < s \le t < 1$.

If $0 < \beta < 1$, we have

$$\frac{\partial}{\partial t}\left[t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}}\right] = (\alpha - 1)t^{\alpha-2}\left[1 - \left(\frac{t-s}{t(1-s)}\right)^{\alpha-2}\right] \ge 0. \tag{2.1}$$

It follows from

$$\frac{\partial}{\partial s}\left[\beta s + (1-s)^\beta\right] \le 0, \quad \forall s \in [0, 1)$$

that

$$s \le \beta^{-1}\left[1 - (1-s)^\beta\right], \quad \forall s \in [0, 1]. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$
\begin{aligned}
t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} &= (1-s)^{\alpha-\beta-1}\left[t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-\beta-1}}\right] \\
&\le (1-s)^{\alpha-\beta-1}\left[t^{\alpha-1} - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}}\right] \\
&\le s(1-s)^{\alpha-\beta-1} \\
&\le \beta^{-1}h(s).
\end{aligned}
\tag{2.3}
$$

If $\beta \ge 1$, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}\right] \\
&\quad = (\alpha - 1)\left[t^{\alpha-2}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-2}\right] \\
&\quad \ge (\alpha - 1)\left[(t - ts)^{\alpha-2} - (t-s)^{\alpha-2}\right] \ge 0.
\end{aligned}
$$

Therefore,

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \le (1-s)^{\alpha-\beta-1}\left[1 - (1-s)^\beta\right] = h(s). \tag{2.4}$$

On the other hand, noticing $0 < \beta < \alpha - 1$, we have

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}$$

$$\ge t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^\beta(t - ts)^{\alpha-\beta-1}$$

$$= \left[1 - \left(1 - \frac{s}{t}\right)^{\beta}\right] t^{\alpha-1}(1-s)^{\alpha-\beta-1}$$

$$\geq \left[1 - (1-s)^{\beta}\right] t^{\alpha-1}(1-s)^{\alpha-\beta-1}$$

$$= t^{\alpha-1}h(s). \tag{2.5}$$

Case (ii): $0 \leq t \leq s \leq 1$.

If $\beta \geq 1$, we have

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1} \leq s^{\alpha-1}(1-s)^{\alpha-\beta-1}$$

$$\leq \left[1 - (1-s)^{\beta}\right](1-s)^{\alpha-\beta-1} = h(s). \tag{2.6}$$

If $0 < \beta < 1$. It follows from (2.2) that

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1} \leq s^{\alpha-1}(1-s)^{\alpha-\beta-1} \leq s(1-s)^{\alpha-\beta-1} \leq \beta^{-1}h(s). \tag{2.7}$$

On the other hand, noticing $0 < \beta < \alpha - 1$, we have

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1} \geq \left[1 - (1-s)^{\beta}\right] t^{\alpha-1}(1-s)^{\alpha-\beta-1} = t^{\alpha-1}h(s). \tag{2.8}$$

It follows from (2.3)–(2.8) that (3) holds. $\qquad\square$

Lemma 2.3 *The function $G(t,s)$ has the following properties*:

(1) $G(t,s) > 0$, $\forall t, s \in (0,1)$;

(2) $G(t,s) \leq t^{\alpha-1}\Phi_1(s)$, $\forall t, s \in [0,1]$;

(3) $t^{\alpha-1}\Phi_2(s) \leq G(t,s) \leq M_1\Phi_2(s)$, $\forall t, s \in [0,1]$,

where

$$\Phi_1(s) = \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + q(s), \quad \Phi_2(s) = \frac{h(s)}{\Gamma(\alpha)} + q(s).$$

Proof It can be directly deduced from Lemma 2.2 and the definition of $G(t,s)$, so we omit the proof. $\qquad\square$

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$, $B_r = \{u \in E : \|u\| < r\}$. Define cones P, Q by

$$P = \left\{u \in E : u(t) \geq 0\right\},$$

$$Q = \left\{u \in E : u(t) \geq M_1^{-1}t^{\alpha-1}\|u\|\right\}.$$

Lemma 2.4 *Define $A : P \to E$ as follows*:

$$Au(t) = \int_0^1 G(t,s)a(s)f\big(s,u(s)\big)\,ds,$$

then $A : P \to Q$ is completely continuous.

Proof By Lemma 2.3, it is easy to check that $A : P \to Q$. By means of the Arzela–Ascoli theorem, $A : P \to Q$ is completely continuous. □

Remark 2.3 Lemma 2.4 is valid if (A_2) is replaced by (A'_2) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, $a(t) \in C((0,1),[0,+\infty))$ and

$$0 < \int_0^1 a(s)\Phi_i(s)\,ds < +\infty, \quad i = 1, 2.$$

Let

$$Tu(t) = \int_0^1 G(t,s)a(s)u(s)\,ds.$$

It is clear that T is a completely continuous linear operator and $T(P) \subset Q$. By virtue of the Krein–Rutmann theorem and Lemma 2.3, we have the following lemma.

Lemma 2.5 *The spectral radius $r(T) > 0$ and T has a positive eigenfunction φ_1 corresponding to its first eigenvalue $(r(T))^{-1}$, that is, $T\varphi_1 = r(T)\varphi_1$.*

3 Main results

3.1 Uniqueness results

Theorem 3.1 *Suppose that $f(t,0) \not\equiv 0$ on $[0,1]$, and there exists $\lambda \in C(0,1) \cap L[0,1]$ satisfying*

$$0 < \int_0^1 a(s)\lambda(s)\,ds < +\infty$$

such that

$$\left| f(t,x) - f(t,y) \right| \le \lambda(t)|x - y|, \quad t \in [0,1], x, y \in [0, \infty).$$

Then (1.1) has a unique positive solution if the spectral radius $r(L_\lambda) \in (0,1)$, where

$$L_\lambda u(t) = \int_0^1 G(t,s)a(s)\lambda(s)u(s)\,ds.$$

Proof It follows from $f(t,0) \not\equiv 0$ that θ is not a fixed point of A. Then we only need to prove that A has a unique fixed point in Q.

Firstly, we will prove that A has a fixed point in Q.

Set

$$Q_1 = \left\{ u \in P : \exists l_1, l_2 > 0 \text{ such that } l_2 t^{\alpha-1} \le u(t) \le l_1 t^{\alpha-1} \right\}.$$

For any $u \in Q \setminus \{\theta\}$, let

$$l_i(u) = \int_0^1 \Phi_i(s)a(s)\lambda(s)u(s)\,ds, \quad i = 1, 2.$$

It follows from Lemma 2.3 that $l_1(u)$, $l_2(u) > 0$ and

$$l_2(u)t^{\alpha-1} \le (L_\lambda u)(t) \le l_1(u)t^{\alpha-1}. \tag{3.1}$$

Therefore,

$$L_\lambda : Q \setminus \{\theta\} \to Q_1.$$

Similar to Lemma 2.5, we have that the spectral radius $r(L_\lambda) > 0$ and L_λ has a positive eigenfunction ψ_1, that is, $L_\lambda \psi_1 = r(L_\lambda)\psi_1$. It is easy to see that

$$\left(r(L_\lambda)\right)^{-1}l_2(\psi_1)t^{\alpha-1} \le \psi_1(t) \le \left(r(L_\lambda)\right)^{-1}l_1(\psi_1)t^{\alpha-1}. \tag{3.2}$$

For any $u_0 \in Q \setminus \{\theta\}$, let

$$u_n = A(u_{n-1}), \quad n = 1, 2, \dots.$$

We may suppose that $u_1 - u_0 \ne \theta$ (otherwise, the proof is finished). It follows from (3.1) and (3.2) that

$$L_\lambda\left(|u_1 - u_0|\right) \le \frac{r(L_\lambda)l_1(|u_1 - u_0|)}{l_2(\psi_1)}\psi_1. \tag{3.3}$$

Then

$$|u_2 - u_1| = \left|\int_0^1 G(t,s)a(s)\big[f\big(s, u_1(s)\big) - f\big(s, u_0(s)\big)\big]\,ds\right|$$

$$\le \int_0^1 G(t,s)a(s)\lambda(s)\big|u_1(s) - u_0(s)\big|\,ds$$

$$= L_\lambda\big(|u_1 - u_0|\big)$$

$$\le \frac{r(L_\lambda)l_1(|u_1 - u_0|)}{l_2(\psi_1)}\psi_1, \tag{3.4}$$

$$|u_3 - u_2| = \left|\int_0^1 G(t,s)a(s)\big[f\big(s, u_2(s)\big) - f\big(s, u_1(s)\big)\big]\,ds\right|$$

$$\le \int_0^1 G(t,s)a(s)\lambda(s)\big|u_2(s) - u_1(s)\big|\,ds$$

$$= L_\lambda\big(|u_2 - u_1|\big)$$

$$\le \frac{r(L_\lambda)l_1(|u_1 - u_0|)}{l_2(\psi_1)}L_\lambda\psi_1$$

$$= \frac{[r(L_\lambda)]^2l_1(|u_1 - u_0|)}{l_2(\psi_1)}\psi_1,$$

$$\cdots \tag{3.5}$$

$$|u_{m+1} - u_m| = \left|\int_0^1 G(t,s)a(s)\big[f\big(s, u_m(s)\big) - f\big(s, u_{m-1}(s)\big)\big]\,ds\right|$$

$$\leq \int_0^1 G(t,s)a(s)\lambda(s)\big|u_m(s) - u_{m-1}(s)\big|\,ds$$

$$= L_\lambda\big(|u_m - u_{m-1}|\big)$$

$$\leq \frac{[r(L_\lambda)]^{m-1}l_1(|u_1 - u_0|)}{l_2(\psi_1)}L_\lambda\psi_1$$

$$= \frac{[r(L_\lambda)]^m l_1(|u_1 - u_0|)}{l_2(\psi_1)}\psi_1. \tag{3.6}$$

By induction, we can get

$$|u_{n+1} - u_n| \leq \frac{[r(L_\lambda)]^n l_1(|u_1 - u_0|)}{l_2(\psi_1)}\psi_1, \quad n = 1, 2, \dots. \tag{3.7}$$

Then, for any $n, m \in \mathbb{N}$, we have

$$|u_{n+m} - u_n| \leq |u_{n+m} - u_{n+m-1}| + \cdots + |u_{n+1} - u_n|$$

$$\leq \big([r(L_\lambda)]^{n+m-1} + \cdots + [r(L_\lambda)]^n\big)\frac{l_1(|u_1 - u_0|)}{l_2(\psi_1)}\psi_1$$

$$\leq \frac{[r(L_\lambda)]^n l_1(|u_1 - u_0|)}{[1 - r(L_\lambda)]l_2(\psi_1)}\psi_1.$$

It follows from $r(L_\lambda) < 1$ that

$$\|u_{n+m} - u_m\| \to 0 \quad (n \to \infty),$$

which implies $\{u_n\}$ is a Cauchy sequence. Therefore, there exists $u^* \in Q$ such that $\{u_n\}$ converges to u^*. Clearly u^* is a fixed point of A.

In the following, we will prove that the fixed point of A is unique.

Suppose that $v \neq u^*$ is a positive fixed point of A. Then there exists $l_1(|u^* - v|) > 0$ such that

$$L_\lambda\big(|u^* - v|\big) \leq \frac{r(L_\lambda)l_1(|u^* - v|)}{l_2(\psi_1)}\psi_1.$$

Therefore,

$$\big|Au^* - Av\big| = \left|\int_0^1 G(t,s)a(s)\big[f\big(s, u^*(s)\big) - f\big(s, v(s)\big)\big]\,ds\right|$$

$$\leq \int_0^1 G(t,s)a(s)\lambda(s)\big|u^*(s) - v(s)\big|\,ds$$

$$\leq \frac{r(L_\lambda)l_1(|u^* - v|)}{l_2(\psi_1)}\psi_1.$$

Similar to the proof of (3.7), we can get

$$\big|A^n u^* - A^n v\big| \leq \frac{[r(L_\lambda)]^n l_1(|u^* - v|)}{l_2(\psi_1)}\psi_1. \tag{3.8}$$

It follows from $r(L_\lambda) < 1$ that

$$\|u^* - v\| = \|A^n u^* - A^n v\| \to 0 \quad (n \to \infty),$$

which implies that the positive fixed point of A is unique. $\qquad\square$

Remark 3.1 The unique positive solution u^* of (1.1) can be approximated by the iterative schemes: for any $u_0 \in Q \setminus \{\theta\}$, let

$$u_n = A(u_{n-1}), \quad n = 1, 2, \ldots,$$

then $u_n \to u^*$. Furthermore, we have error estimation

$$|u_n - u^*| \le \frac{[r(L_\lambda)]^n l_1(|u_1 - u_0|)}{[1 - r(L_\lambda)] l_2(\psi_1)} \psi_1,$$

and with the rate of convergence

$$\|u_n - u^*\| = O([r(L_\lambda)]^n).$$

Remark 3.2 The spectral radius satisfies $r(L_\lambda) = \lim_{n \to \infty} \|L_\lambda^n\|^{\frac{1}{n}}$ and $r(L_\lambda) \le \|L_\lambda^n\|^{\frac{1}{n}}$. Particularly,

$$r(L_\lambda) \le \|L_\lambda\| = \sup_{0 \le t \le 1} \int_0^1 G(t,s) a(s) \lambda(s) \, ds.$$

3.2 Nonexistence results

Theorem 3.2 *Suppose that there exists $b_1 \in C(0,1) \cap L[0,1]$ satisfying*

$$0 < \int_0^1 a(s) b_1(s) \, ds < +\infty$$

such that

$$f(t,x) \le b_1(t)x, \quad t \in [0,1], x \in [0,\infty).$$

Then (1.1) *has no positive solution if the spectral radius $r(L_{b_1}) \in (0,1)$, where*

$$L_{b_1} u(t) = \int_0^1 G(t,s) a(s) b_1(s) u(s) \, ds.$$

Proof We only need to prove that A has no fixed point in Q_1. If otherwise, there exists $v \in Q_1$ such that $Av = v$. Similar to Lemma 2.5, we have that the spectral radius $r(L_{b_1}) > 0$ and L_{b_1} has a positive eigenfunction φ_{b_1} satisfying

$$L_{b_1} \varphi_{b_1} = r(L_{b_1}) \varphi_{b_1}.$$

It is clear that $v, \varphi_{b_1} \in Q_1$. Therefore, there exists $c_1 > 0$ such that

$$v \le c_1 \varphi_{b_1}.$$

By $f(t,x) \leq b_1(t)x$, we have $v = Av \leq L_{b_1}v$. It is obvious that L_{b_1} is increasing in Q_1. By induction, we can get $v \leq L_{b_1}^n v, \forall n = 1, 2, 3, \ldots$. Then

$$v \leq L_{b_1}^n v \leq L_{b_1}^n c_1 \varphi_{b_1} = c_1 \left[r(L_{b_1}) \right]^n \varphi_{b_1}, \quad \forall n = 1, 2, 3, \ldots.$$

Noticing $r(L_{b_1}) < 1$, we have $v = \theta$, which contradicts $v \in Q_1$. □

Theorem 3.3 *Suppose that there exists $b_2 \in C(0,1) \cap L[0,1]$ satisfying*

$$0 < \int_0^1 a(s)b_2(s)\,ds < +\infty$$

such that

$$f(t,x) \geq b_2(t)x, \quad t \in [0,1], x \in [0,\infty).$$

Then (1.1) *has no positive solution if the spectral radius $r(L_{b_2}) > 1$, where*

$$L_{b_2}u(t) = \int_0^1 G(t,s)a(s)b_2(s)u(s)\,ds.$$

Proof The proof is just similar to Theorem 3.2, so we omit it. □

4 Example

Example 4.1 Consider the following problem:

$$\begin{cases} D_{0+}^{\frac{5}{2}}u(t) + a(t)f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & D_{0+}^{\frac{1}{2}}u(1) = \sum_{i=1}^{\infty} \eta_i D_{0+}^{\frac{1}{2}}u(\xi_i), \end{cases} \tag{4.1}$$

where

$$\eta_i = \frac{1}{2^{4+i}}, \qquad \xi_i = 1 - \frac{1}{2^i}, \quad i = 1, 2, \ldots,$$

$$a(t) = \sqrt{\frac{100\pi}{3t}}, \qquad f(t,x) = 1 + \sqrt{t}|\sin x|.$$

Let $\lambda(t) = \sqrt{t}$, we have

$$\left| f(t,x) - f(t,y) \right| \leq \lambda(t)|x - y|, \quad t \in [0,1], x, y \in [0,\infty).$$

It is clear that

$$K(t,s) = \frac{1}{\Gamma(\frac{5}{2})} \begin{cases} t^{\frac{3}{2}}(1-s), & 0 \leq t \leq s \leq 1, \\ t^{\frac{3}{2}}(1-s) - (t-s)^{\frac{3}{2}}, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$p(0) = \sum_{i=1}^{\infty} \eta_i \xi_i = \frac{1}{24},$$

$$p(s) = \begin{cases} \sum_{i=1}^{\infty} \eta_i(\xi_i - s), & s \in [0, \xi_1], \\ \sum_{i=2}^{\infty} \eta_i(\xi_i - s), s \in (\xi_1, \xi_2], \\ \cdots \\ \sum_{i=n+1}^{\infty} \eta_i(\xi_i - s), & s \in (\xi_n, \xi_{n+1}], \\ \cdots, \end{cases}$$

$$q(s) = \frac{p(0)(1 - s) - p(s)}{\Delta},$$

$$G(t, s) = K(t, s) + q(s)t^{\frac{3}{2}},$$

where $\Delta := \Gamma(\frac{5}{2})[1 - p(0)]$.

Let

$$h_1(t) = \int_0^1 K(t, s) \, ds,$$

$$h_2(t) = \int_0^1 K(t, s) \int_0^1 K(s, \tau) \, d\tau \, ds.$$

Then

$$h_1(t) = \frac{1}{\Gamma(\frac{5}{2})} \left(\frac{t^{\frac{3}{2}}}{2} - \frac{2t^{\frac{5}{2}}}{5} \right),$$

$$h_2(t) = \int_0^1 K(t, s) h_1(s) \, ds = \frac{32 t^{\frac{3}{2}}}{567\pi} - \frac{t^4}{48} + \frac{t^5}{120}.$$

By direct calculations, we have

$$\max_{0 \le t \le 1} h_1(t) = h_1\left(\frac{3}{4} \right) = \frac{\sqrt{3}}{10\sqrt{\pi}},$$

$$\max_{0 \le t \le 1} h_2(t) \approx h_2(0.7744) \approx 0.0070708134,$$

and

$$\int_0^1 q(s) \, ds = \frac{p(0)}{2\Delta} - \frac{1}{\Delta} \sum_{i=1}^{\infty} \frac{\eta_i \xi_i^2}{2}$$

$$= \frac{1}{2\Delta} \sum_{i=1}^{\infty} \eta_i \xi_i (1 - \xi_i)$$

$$\approx 0.00467238.$$

It is clear that

$$\|L_\lambda\| = \sup_{0 \le t \le 1} \int_0^1 \sqrt{\frac{100\pi}{3}} G(t, s) \, ds$$

$$\ge \int_0^1 \sqrt{\frac{100\pi}{3}} G\left(\frac{3}{4}, s \right) ds$$

$$= \sqrt{\frac{100\pi}{3}} h_1\left(\frac{3}{4}\right) + \sqrt{\frac{100\pi}{3}} \left(\frac{3}{4}\right)^{\frac{3}{2}} \int_0^1 q(s)\,ds$$

$$> \sqrt{\frac{100\pi}{3}} h_1\left(\frac{3}{4}\right) = 1.$$

On the other hand, we have

$$\left\| L_\lambda^2 \right\| = \sup_{0 \le t \le 1} \int_0^1 \sqrt{\frac{100\pi}{3}} G(t,s) \int_0^1 \sqrt{\frac{100\pi}{3}} G(s,\tau)\,d\tau\,ds.$$

Noticing that

$$G(t,s) = K(t,s) + q(s)t^{\frac{3}{2}} \le K(t,s) + q(s),$$

we have

$$\int_0^1 \sqrt{\frac{100\pi}{3}} G(t,s) \int_0^1 \sqrt{\frac{100\pi}{3}} G(s,\tau)\,d\tau\,ds$$

$$= \frac{100\pi}{3} \int_0^1 G(t,s) \int_0^1 G(s,\tau)\,d\tau\,ds$$

$$\le \frac{100\pi}{3} \int_0^1 \left[K(t,s) + q(s) \right] \int_0^1 \left[K(s,\tau) + q(\tau) \right] d\tau\,ds$$

$$= \frac{100\pi}{3} \left[h_2(t) + h_1(t) \int_0^1 q(\tau)\,d\tau + \int_0^1 q(s)h_1(s)\,ds + \left(\int_0^1 q(s)\,ds \right)^2 \right]$$

$$\le \frac{100\pi}{3} \left[\|h_2\| + 2\|h_1\| \int_0^1 q(s)\,ds + \left(\int_0^1 q(s)\,ds \right)^2 \right]$$

$$\approx 0.83837,$$

which implies that

$$\left\| L_\lambda^2 \right\|^{\frac{1}{2}} < 1.$$

It follows from Lemma 2.5 and Remark 3.2 that

$$0 < r(L_\lambda) \le \left\| L_\lambda^2 \right\|^{\frac{1}{2}} < 1 < \|L_\lambda\|.$$

Therefore the assumptions of Theorem 3.1 are satisfied. Thus Theorem 3.1 ensures that BVP (4.1) has a unique positive solution.

5 Conclusions

In this paper, we consider both the existence of uniqueness and the nonexistence of positive solution for fractional differential equations with infinite-point boundary value conditions. The interesting point lies in that both the existence of uniqueness and the nonexistence of positive solution are closely associated with the spectral radius with respect to the relevant linear operator.

Abbreviations
FDE, Fractional differential equations; BVP, Boundary value problems.

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Authors' contributions
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