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Two positive solutions for quasilinear elliptic equations with singularity and critical exponents

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Abstract

In this paper, we consider the quasilinear elliptic equation with singularity and critical exponents

$$\begin{cases} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = Q(x)\frac{|u|^{p^*(t)-2}u}{|x|^t} + \lambda u^{-s}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is a p-Laplace operator with $1 < p < N$. $p^*(t) := \frac{p(N-t)}{N-p}$ is a critical Sobolev–Hardy exponent. We deal with the existence of multiple solutions for the above problem by means of variational and perturbation methods.

Keywords: Quasilinear; Singularity; Critical; Sobolev–Hardy exponent

1 Introduction and preliminaries

The main goal of this paper is to consider the following singular boundary value problem:

$$\begin{cases} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = Q(x)\frac{|u|^{p^*(t)-2}u}{|x|^t} + \lambda u^{-s}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is a p-Laplace operator with $1 < p < N$. $\lambda > 0$, $0 < s < 1$, $0 \le t < p$, and $0 \le \mu < \bar{\mu} := (\frac{N-p}{p})^p$. $p^*(t) := \frac{p(N-t)}{N-p}$ is a critical Sobolev–Hardy exponent, $Q(x) \in C(\overline{\Omega})$ and $Q(x)$ is positive on $\overline{\Omega}$.

In recent years, the elliptic boundary value problems with critical exponents and singular potentials have been extensively studied [2, 6, 7, 10–23, 25, 26, 28, 30–34]. In [19], Han considered the following quasilinear elliptic problem with Hardy term and critical exponent:

$$\begin{cases} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = Q(x)|u|^{p^*-2}u + \lambda|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < p < N$. The existence of multiple positive solutions for (1.2) was established. Furthermore, Hsu [21] studied the following quasilinear equation:

$$\begin{cases} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = Q(x)|u|^{p^*-2}u + \lambda f(x)|u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $1 < q < p < N$. We should point out that the authors of [19, 21] both investigated the effect of $Q(x)$. If $p = 2$, $\mu = 0$, and $t = 0$, Liao et al. [27] proved the existence of two solutions for problem (1.1) by the constrained minimizer and perturbation methods.

Compared with [2, 4, 8, 12, 19, 21, 22, 29], problem (1.1) contains the singular term λu^{-s}. Thus, the functional corresponding to (1.1) is not differentiable on $W_0^{1,p}(\Omega)$. We will remove the singularity by the perturbation method. Our idea comes from [24, 27].

Definition 1.1 A function $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (1.1) if, for every $\varphi \in W_0^{1,p}(\Omega)$, there holds

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla\varphi - \mu \frac{|u|^{p-2}u\varphi}{|x|^p} \right) dx = \int_\Omega \left(\frac{Q(x)(u^+)^{p^*(t)-1}\varphi}{|x|^t} + \lambda \left(u^+\right)^{-s}\varphi \right) dx.$$

The energy functional corresponding to (1.1) is defined by

$$I_{\lambda,\mu}(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) dx - \frac{1}{p^*(t)} \int_\Omega Q(x) \frac{(u^+)^{p^*(t)}}{|x|^t} dx - \frac{\lambda}{1-s} \int_\Omega \left(u^+\right)^{1-s} dx.$$

Throughout this paper, Q satisfies

(Q_1) $Q(0) = Q_M = \max_{x\in\overline{\Omega}} Q(x)$ and there exists $\beta \geq p(b(\mu) - \frac{N-p}{p})$ such that

$$Q(x) - Q(0) = o\left(|x|^\beta\right), \quad \text{as } x \to 0,$$

where $b(\mu)$ is given in Sect. 1.

In this paper, we use the following notations:

(i) $\|u\|^p = \int_\Omega (|\nabla u|^p - \mu \frac{|u|^p}{|x|^p})\, dx$ is the norm in $W_0^{1,p}(\Omega)$, and the norm in $L^p(\Omega)$ is denoted by $|\cdot|_p$;
(ii) $C, C_1, C_2, C_3, \ldots$ denote various positive constants;
(iii) $u_n^+(x) = \max\{u_n, 0\}$, $u_n^-(x) = \max\{0, -u_n\}$;
(iv) We define

$$\partial B_r = \left\{u \in W_0^{1,p}(\Omega) : \|u\| = r\right\}, \qquad B_r = \left\{u \in W_0^{1,p}(\Omega) : \|u\| \leq r\right\}.$$

Let S be the best Sobolev–Hardy constant

$$S := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^p - \mu \frac{|u|^p}{|x|^p})\, dx}{(\int_\Omega \frac{|u|^{p^*(t)}}{|x|^t}\, dx)^{\frac{p}{p^*(t)}}}. \tag{1.4}$$

Our main result is the following theorem.

Theorem 1.1 *Suppose that* (Q_1) *is satisfied. Then there exists* $\Lambda > 0$ *such that, for every* $\lambda \in (0, \Lambda)$, *problem* (1.1) *has at least two positive solutions.*

The following well-known Brézis–Lieb lemma and maximum principle will play fundamental roles in the proof of our main result.

Proposition 1.1 ([3]) *Suppose that* u_n *is a bounded sequence in* $L^p(\Omega)$ $(1 \leq p < \infty)$, *and* $u_n(x) \to u(x)$ *a.e.* $x \in \Omega$, *where* $\Omega \subset \mathbb{R}^N$ *is an open set. Then*

$$\lim_{n\to\infty}\left(\int_\Omega |u_n|^p\,dx - \int_\Omega |u_n - u|^p\,dx\right) = \int_\Omega |u|^p\,dx.$$

Proposition 1.2 ([23]) *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with smooth boundary,* $0 \in \Omega$, $u \in C^1(\Omega\backslash\{0\})$, $u \geq 0$, $u \not\equiv 0$, *and*

$$-\Delta_p u \geq 0 \quad \text{in } \Omega.$$

Then $u > 0$ *in* Ω.

By [22, 23], we assume that $1 < p < N$, $0 \leq t < p$, and $0 \leq \mu < \overline{\mu}$. Then the limiting problem

$$\begin{cases} -\Delta_p u - \mu \frac{u^{p-1}}{|x|^p} = \frac{u^{p^*(t)-1}}{|x|^t}, & \text{in } \mathbb{R}^N\backslash\{0\}, \\ u > 0, \quad \text{in } \mathbb{R}^N\backslash\{0\}, & u \in D^{1,p}(\mathbb{R}^N) \end{cases}$$

has positive radial ground states

$$V_\epsilon(x) = \epsilon^{\frac{p-N}{p}} U_{p,\mu}\left(\frac{x}{\epsilon}\right) = \epsilon^{\frac{p-N}{p}} U_{p,\mu}\left(\frac{|x|}{\epsilon}\right) \quad \forall \epsilon > 0$$

that satisfy

$$\int_\Omega \left(\left|\nabla V_\epsilon(x)\right|^p - \mu \frac{|V_\epsilon(x)|^p}{|x|^p}\right) dx = \int_\Omega \left(\frac{|V_\epsilon(x)|^{p^*(t)}}{|x|^t}\right) dx = S^{\frac{N-t}{p-t}},$$

where the function $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ is the unique radial solution of the above limiting problem with

$$U_{p,\mu}(1) = \left(\frac{(N-t)(\overline{\mu} - \mu)}{N-p}\right)^{\frac{1}{p^*(t)-p}},$$

and

$$\lim_{r\to 0^+} r^{a(\mu)} U_{p,\mu}(r) = c_1 > 0, \qquad \lim_{r\to 0^+} r^{a(\mu)+1} \left|U'_{p,\mu}(r)\right| = c_1 a(\mu) \geq 0,$$
$$\lim_{r\to +\infty} r^{b(\mu)} U_{p,\mu}(r) = c_2 > 0, \qquad \lim_{r\to +\infty} r^{b(\mu)+1} \left|U'_{p,\mu}(r)\right| = c_2 b(\mu) \geq 0,$$
$$c_3 \leq U_{p,\mu}(r)\left(r^{\frac{a(\mu)}{\nu}} + r^{\frac{b(\mu)}{\nu}}\right)^\nu \leq c_4, \qquad \nu := \frac{N-p}{p},$$

where c_i ($i = 1,2,3,4$) are positive constants depending on N, μ, and p, and $a(\mu)$ and $b(\mu)$ are the zeros of the function

$$h(t) = (p-1)t^p - (N-p)t^{p-1} + \mu, \quad t \geq 0,$$

satisfying $0 \leq a(\mu) < \nu < b(\mu) \leq \frac{N-p}{p-1}$.

Take $\rho > 0$ small enough such that $B(0,\rho) \subset \Omega$, and define the function

$$u_\epsilon(x) = \eta(x)V_\epsilon(x) = \epsilon^{\frac{p-N}{p}}\eta(x)U_{p,\mu}\left(\frac{|x|}{\epsilon}\right),$$

where $\eta \in C_0^\infty(\Omega)$ is a cutoff function

$$\eta(x) = \begin{cases} 1, & |x| \leq \frac{\rho}{2}, \\ 0, & |x| > \rho. \end{cases}$$

The following estimates hold when $\epsilon \longrightarrow 0$:

$$\|u_\epsilon\|^p = S^{\frac{N-t}{p-t}} + O\bigl(\epsilon^{b(\mu)p+p-N}\bigr),$$
$$\int_\Omega \frac{|u_\epsilon|^{p^*(t)}}{|x|^t}\,dx = S^{\frac{N-t}{p-t}} + O\bigl(\epsilon^{b(\mu)p^*(t)-N+t}\bigr).$$

2 Existence of the first solution of problem (1.1)

In this section, we will get the first solution which is a local minimizer in $W_0^{1,p}(\Omega)$ for (1.1).

Lemma 2.1 *There exist $\lambda_0 > 0$, $R, \rho > 0$ such that, for every $\lambda \in (0,\lambda_0)$, we have*

$$I_{\lambda,\mu}(u)|_{u\in\partial B_R} \geq \rho, \qquad \inf_{u\in B_R} I_{\lambda,\mu}(u) < 0.$$

Proof We can deduce from Hölder's inequality that

$$\begin{aligned} I_{\lambda,\mu}(u) &\geq \frac{1}{p}\|u\|^p - \frac{1}{p^*(t)}Q_M S^{-\frac{p^*(t)}{p}}\|u\|^{p^*(t)} - \frac{\lambda}{1-s}C_0\|u\|^{1-s} \\ &= \|u\|^{1-s}\left(\frac{1}{p}\|u\|^{-1+s+p} - \frac{1}{p^*(t)}Q_M S^{-\frac{p^*(t)}{p}}\|u\|^{-1+s+p^*(t)} - \frac{\lambda}{1-s}C_0\right), \end{aligned}$$

where C_0 is a positive constant. Put $f(x) = \frac{1}{p}x^{-1+s+p} - \frac{1}{p^*(t)}Q_M S^{-\frac{p^*(t)}{p}}x^{-1+s+p^*(t)}$, we find that there is a constant $R = [\frac{p^*(t)S^{\frac{p^*(t)}{p}}(-1+s+p)}{pQ_M(-1+s+p^*(t))}]^{\frac{1}{p^*(t)-p}} > 0$ such that $f(R) = \max_{x>0} f(x) > 0$. Letting $\lambda_0 = \frac{(1-s)f(R)}{C_0}$, we have that there is a constant $\rho > 0$ such that $I_{\lambda,\mu}(u)|_{u\in\partial B_R} \geq \rho$ for every $\lambda \in (0,\lambda_0)$.

For given R, choosing $u \in B_R$ with $u^+ \not\equiv 0$, we have

$$\begin{aligned} \lim_{r\to 0}\frac{I_{\lambda,\mu}(ru)}{r^{1-s}} &= \lim_{r\to 0}\frac{\frac{1}{p}r^p\|u\|^p - \frac{\lambda r^{1-s}}{1-s}\int_\Omega (u^+)^{1-s}\,dx - \frac{r^{p^*(t)}}{p^*(t)}\int_\Omega Q(x)\frac{(u^+)^{p^*(t)}}{|x|^t}\,dx}{r^{1-s}} \\ &= -\frac{\lambda}{1-s}\int_\Omega \left(u^+\right)^{1-s}dx < 0, \end{aligned}$$

since $p^*(t) > p > 1 > s > 0$ for $0 \le t < p$. For all $u^+ \ne 0$ such that $I_{\lambda,\mu}(ru) < 0$ as $r \to 0$, that is, $\|u\|$ sufficiently small, we have

$$\Gamma = \inf_{u\in B_R} I_{\lambda,\mu}(u) < 0. \tag{2.1}$$

The proof of Lemma 2.1 is completed. □

Theorem 2.2 *Problem* (1.1) *has a positive solution* $u_1 \in W_0^{1,p}(\Omega)$ *with* $I_{\lambda,\mu}(u_1) < 0$ *for* $\lambda \in (0,\lambda_0)$, *where* λ_0 *is defined in Lemma* 2.1.

Proof By Lemma 2.1, we have

$$\begin{aligned} &\frac{1}{p}\|u\|^p - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{(u^+)^{p^*(t)}}{|x|^t}\,dx \ge \rho, \quad \forall u \in \partial B_R, \\ &\frac{1}{p}\|u\|^p - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{(u^+)^{p^*(t)}}{|x|^t}\,dx \ge 0, \quad \forall u \in B_R. \end{aligned} \tag{2.2}$$

From (2.1) we guarantee that there exists a minimizing sequence $\{u_n\} \subset B_R$ such that $\lim_{n\to\infty} I_{\lambda,\mu}(u_n) = \Gamma < 0$. Obviously, the minimizing sequence is a closed convex set in B_R. Going if necessary to a sequence still called $\{u_n\}$, there exists $u_1 \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_1, & \text{in } W_0^{1,p}(\Omega), \\ u_n \longrightarrow u_1, & \text{in } L^{p'}(\Omega, |x|^{-t}), \quad 1 \le p' < p^*(t), \\ u_n(x) \longrightarrow u_1(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \nabla u_n(x) \longrightarrow \nabla u_1(x), & \text{a.e. in } \Omega, \\ \frac{|u_n|^{p-2}u_n}{|x|^{p-1}} \rightharpoonup \frac{|u_1|^{p-2}u_1}{|x|^{p-1}}, & \text{in } L^{\frac{p}{p-1}}(\Omega), \\ \int_\Omega \frac{|u_n|^{p^*(t)-2}u_n}{|x|^t} v\,dx \longrightarrow \int_\Omega \frac{|u_1|^{p^*(t)-2}u_1}{|x|^t} v\,dx, & \forall v \in W_0^{1,p}(\Omega). \end{cases}$$

For $s \in (0,1)$, applying Hölder's inequality, we obtain that

$$\begin{aligned} \int_\Omega (u_n^+)^{1-s}\,dx - \int_\Omega (u_1^+)^{1-s}\,dx &\le \int_\Omega \left|(u_n^+)^{1-s} - (u_1^+)^{1-s}\right|dx \\ &\le \int_\Omega \left|u_n^+ - u_1^+\right|^{1-s}dx \\ &\le \left|u_n^+ - u_1^+\right|_p^{1-s}|\Omega|^{\frac{1+s}{p}}, \end{aligned}$$

thus,

$$\int_\Omega (u_n^+)^{1-s}\,dx = \int_\Omega (u_1^+)^{1-s}\,dx + o(1). \tag{2.4}$$

Let $\omega_n = u_n - u_1$, by the Brézis–Lieb lemma, one has

$$\int_\Omega |\nabla u_n|^p\,dx = \int_\Omega |\nabla \omega_n|^p\,dx + \int_\Omega |\nabla u_1|^p\,dx + o(1), \tag{2.5}$$

$$\int_\Omega Q(x)\frac{(u_n^+)^{p^*(t)}}{|x|^t}\,dx = \int_\Omega Q(x)\frac{(\omega_n^+)^{p^*(t)}}{|x|^t}\,dx + \int_\Omega Q(x)\frac{(u_1^+)^{p^*(t)}}{|x|^t}\,dx + o(1). \tag{2.6}$$

Noting that $\|u_1\|^p = |\nabla u_1|_p^p - \mu|u_1/x|_p^p$, we have that

$$\lim_{n\to\infty}\left(\|u_n\|^p - \|\omega_n\|^p\right) = \|u_1\|^p.$$

If $u_1 = 0$, then $\omega_n = u_n$, it follows that $\omega_n \in B_R$. If $u_1 \neq 0$, from (2.2), we derive that

$$\frac{1}{p}\|\omega_n\|^p - \frac{1}{p*(t)}\int_\Omega Q(x)\frac{(\omega_n^+)^{p^*(t)}}{|x|^t}\,dx \geq 0. \tag{2.7}$$

By (2.3)–(2.7), we have

$$\begin{aligned}
\Gamma &= I_{\lambda,\mu}(u_n) + o(1)\\
&= \frac{1}{p}\|u_n\|^p - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{(u_n^+)^{p^*(t)}}{|x|^t}\,dx - \frac{\lambda}{1-s}\int_\Omega \left(u_n^+\right)^{1-s}dx + o(1)\\
&= I_{\lambda,\mu}(u_1) + \frac{1}{p}\|\omega_n\|^p - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{(\omega_n^+)^{p^*(t)}}{|x|^t}\,dx - \frac{\lambda}{1-s}\int_\Omega \left(\omega_n^+\right)^{1-s}dx + o(1)\\
&\geq I_{\lambda,\mu}(u_1) + o(1).
\end{aligned}$$

Consequently, $\Gamma \geq I_{\lambda,\mu}(u_1)$ as $n\to\infty$. Since B_R is convex and closed, so $u_1 \in B_R$. We get that $I_{\lambda,\mu}(u_1) = \Gamma < 0$ from (2.1) and $u_1 \not\equiv 0$. It means that u_1 is a local minimizer of $I_{\lambda,\mu}$.

Now, we claim that u_1 is a solution of (1.1) and $u_1 > 0$. Letting $r > 0$ small enough, and for every $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$ such that $(u_1 + r\varphi) \in B_R$, one has

$$\begin{aligned}
0 &< I_{\lambda,\mu}(u_1 + r\varphi) - I_{\lambda,\mu}(u_1)\\
&= \frac{1}{p}\|u_1 + r\varphi\|^p - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{((u_1+r\varphi)^+)^{p^*(t)}}{|x|^t}\,dx - \frac{\lambda}{1-s}\int_\Omega \left((u_1+r\varphi)^+\right)^{1-s}dx\\
&\quad - \frac{1}{p}\|u_1\|^p + \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{(u_1^+)^{p^*(t)}}{|x|^t}\,dx + \frac{\lambda}{1-s}\int_\Omega \left(u_1^+\right)^{1-s}dx\\
&\leq \frac{1}{p}\|u_1 + r\varphi\|^p - \frac{1}{p}\|u_1\|^p.
\end{aligned} \tag{2.8}$$

Next we prove that u_1 is a solution of (1.1). According to (2.8), we have

$$\begin{aligned}
&\frac{\lambda}{1-s}\int_\Omega \left[\left((u_1+r\varphi)^+\right)^{1-s} - \left(u_1^+\right)^{1-s}\right]dx\\
&\quad \leq \frac{1}{p}\left[\|u_1+r\varphi\|^p - \|u_1\|^p\right] - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{[((u_1+r\varphi)^+)^{p^*(t)} - (u_1^+)^{p^*(t)}]}{|x|^t}\,dx.
\end{aligned}$$

Dividing by $r > 0$ and taking limit as $r \to 0^+$, we have

$$\begin{aligned}
&\frac{\lambda}{1-s}\liminf_{r\to 0^+}\int_\Omega \frac{((u_1+r\varphi)^+)^{1-s} - (u_1^+)^{1-s}}{t}\,dx\\
&\quad \leq \int_\Omega \left(|\nabla u_1|^{p-2}\nabla u_1\nabla\varphi - \mu\frac{|u_1|^{p-2}u_1\varphi}{|x|^p}\right)dx\\
&\qquad - \int_\Omega Q(x)\frac{(u_1^+)^{p^*(t)-1}\varphi}{|x|^t}\,dx.
\end{aligned} \tag{2.9}$$

However,

$$\frac{\lambda}{1-s}\frac{((u_1+r\varphi)^+)^{1-s}-(u_1^+)^{1-s}}{t}=\lambda\int_\Omega\big((u_1+\xi r\varphi)^+\big)^{-s}\varphi\,dx,$$

where $\xi\longrightarrow 0^+$ and $\lim_{r\to 0^+}((u_1+\xi r\varphi)^+)^{-s}\varphi=(u_1^+)^{-s}\varphi$ ($\xi\to 0^+$) a.e. $x\in\Omega$. Since $((u_1+\xi r\varphi)^+)^{-s}\varphi\geq 0$. By Fatou's lemma, we obtain that

$$\lambda\int_\Omega\big(u_1^+\big)^{-s}\varphi\,dx\leq\frac{\lambda}{1-s}\liminf_{r\to 0^+}\int_\Omega\frac{((u_1+r\varphi)^+)^{1-s}-(u_1^+)^{1-s}}{t}\,dx.$$

Hence, from (2.9), we obtain that

$$\begin{aligned}&\int_\Omega\left(|\nabla u_1|^{p-2}\nabla u_1\nabla\varphi-\mu\frac{|u_1|^{p-2}u_1\varphi}{|x|^p}\right)dx-\lambda\int_\Omega\big(u_1^+\big)^{-s}\varphi\,dx\\&\quad-\int_\Omega Q(x)\frac{(u_1^+)^{p^*(t)-1}\varphi}{|x|^t}\,dx\geq 0\end{aligned}\tag{2.10}$$

for $\varphi\geq 0$. Since $I_{\lambda,\mu}(u_1)<0$, combining with Lemma 2.1, we can derive that $u_1\notin\partial B_R$, thus $\|u_1\|<R$. There exists $\delta_1\in(0,1)$ such that $(1+\theta)u_1\in B_R$ ($|\theta|\leq\delta_1$). Let $h(\theta)=I_{\lambda,\mu}((1+\theta)u_1)$. Apparently, $h(\theta)$ attains its minimum at $\theta=0$. Note that

$$\begin{aligned}h'(\theta)&=\frac{d}{d\theta}\big(I_{\lambda,\mu}(1+\theta)u_1\big)\\&=(1+\theta)^{p-1}\|u_1\|^p-(1+\theta)^{p^*(t)-1}\int_\Omega Q(x)\frac{(u_1^+)^{p^*(t)}}{|x|^t}\,dx-\lambda(1+\theta)^{-s}\int_\Omega\big(u_1^+\big)^{1-s}\,dx.\end{aligned}$$

Furthermore,

$$h'(\theta)|_{\theta=0}=\|u_1\|^p-\int_\Omega Q(x)\frac{(u_1^+)^{p^*(t)}}{|x|^t}\,dx-\lambda\int_\Omega\big(u_1^+\big)^{1-s}\,dx=0.\tag{2.11}$$

Define $\Psi\in W_0^{1,p}(\Omega)$ by

$$\Psi=\big(u_1^++\varepsilon\psi\big)^+,\quad\text{for every }\psi\in W_0^{1,p}(\Omega)\text{ and }\varepsilon>0,$$

where $(u_1^++t\psi)^+=\max\{u_1^++t\psi,0\}$. We deduce from (2.10) and (2.11) that

$$\begin{aligned}0&\leq\int_\Omega\left(|\nabla u_1|^{p-2}\nabla u_1\nabla\Psi-\mu\frac{|u_1|^{p-2}u_1\Psi}{|x|^p}\right)dx-\int_\Omega Q(x)\frac{(u_1^+)^{p^*(t)-1}\Psi}{|x|^t}\,dx\\&\quad-\lambda\int_\Omega\big(u_1^+\big)^{-s}\Psi\,dx\\&=\int_{\{x|u_1^++\varepsilon\psi>0\}}\bigg[|\nabla u_1|^{p-2}\nabla u_1\nabla\big(u_1^++\varepsilon\psi\big)-\mu\frac{|u_1|^{p-2}u_1(u_1^++\varepsilon\psi)}{|x|^p}\\&\quad-Q(x)\frac{(u_1^+)^{p^*(t)-1}(u_1^++\varepsilon\psi)}{|x|^t}-\lambda\big(u_1^+\big)^{-s}\big(u_1^++\varepsilon\psi\big)\bigg]\,dx\\&=\left(\int_\Omega-\int_{\{x|u_1^++\varepsilon\psi\leq 0\}}\right)\bigg[|\nabla u_1|^{p-2}\nabla u_1\nabla\big(u_1^++\varepsilon\psi\big)-\mu\frac{|u_1|^{p-2}u_1(u_1^++\varepsilon\psi)}{|x|^p}\,dx\end{aligned}$$

$$
\begin{aligned}
&- Q(x)\frac{(u_1^+)^{p^*(t)-1}(u_1^+ + \varepsilon\psi)}{|x|^t} - \lambda\bigl(u_1^+\bigr)^{-s}\bigl(u_1^+ + \varepsilon\psi\bigr)\biggr] dx \\
\le{}& \|u_1\|^p - \int_\Omega Q(x)\frac{(u_1^+)^{p^*(t)}}{|x|^t}\,dx - \lambda\int_\Omega \bigl(u_1^+\bigr)^{1-s}\,dx + \varepsilon\int_\Omega\biggl[|\nabla u_1|^{p-2}\nabla u_1\nabla\psi \\
&- \mu\frac{|u_1|^{p-2}u_1\psi}{|x|^p} - Q(x)\frac{(u_1^+)^{p^*(t)-1}\psi}{|x|^t} - \lambda\bigl(u_1^+\bigr)^{-s}\psi\biggr] dx \\
&- \int_{\{x|u_1^+ + \varepsilon\psi\le 0\}}\biggl[|\nabla u_1|^{p-2}\nabla u_1\nabla\bigl(u_1^+ + \varepsilon\psi\bigr) - \mu\frac{|u_1|^{p-2}u_1(u_1^+ + \varepsilon\psi)}{|x|^p}\biggr] dx \\
&+ \int_{\{x|u_1^+ + \varepsilon\psi\le 0\}}\biggl[Q(x)\frac{(u_1^+)^{p^*(t)-1}(u_1^+ + \varepsilon\psi)}{|x|^t} + \lambda\bigl(u_1^+\bigr)^{-s}\bigl(u_1^+ + \varepsilon\psi\bigr)\biggr] dx \\
\le{}& \varepsilon\int_\Omega\biggl[|\nabla u_1|^{p-2}\nabla u_1\nabla\psi - \mu\frac{|u_1|^{p-2}u_1\psi}{|x|^p} - Q(x)\frac{(u_1^+)^{p^*(t)-1}\psi}{|x|^t} - \lambda\bigl(u_1^+\bigr)^{-s}\psi\biggr] dx \\
&- \varepsilon\int_{\{x|u_1^+ + \varepsilon\psi\le 0\}}\biggl[|\nabla u_1|^{p-2}\nabla u_1\nabla\psi - \mu\frac{|u_1|^{p-1}u_1\psi}{|x|^p}\biggr] dx. \qquad (2.12)
\end{aligned}
$$

Since the measure of $\{x \mid u_1^+ + \varepsilon\psi \le 0\} \to 0$ as $\varepsilon \to 0$, we have

$$
\lim_{\varepsilon\to 0}\int_{\{x|u_1^+ + \varepsilon\psi\le 0\}}\biggl[|\nabla u_1|^{p-2}\nabla u_1\nabla\psi - \mu\frac{|u_1|^{p-2}u_1\psi}{|x|^p}\biggr] dx = 0.
$$

Dividing by ε and letting $\varepsilon \to 0^+$ in (2.12), we deduce that

$$
\int_\Omega\biggl[|\nabla u_1|^{p-2}\nabla u_1\nabla\psi - \mu\frac{|u_1|^{p-2}u_1\psi}{|x|^p} - Q(x)\frac{(u_1^+)^{p^*(t)-1}}{|x|^t}\psi - \lambda\bigl(u_1^+\bigr)^{-s}\psi\biggr] dx \ge 0.
$$

Since $\psi \in W_0^{1,p}(\Omega)$ is arbitrary, replacing ψ with $-\psi$, we have

$$
\begin{aligned}
&\int_\Omega\biggl[|\nabla u_1|^{p-2}\nabla u_1\nabla\psi - \mu\frac{|u_1|^{p-2}u_1\psi}{|x|^p} \\
&- Q(x)\frac{(u_1^+)^{p^*(t)-1}\psi}{|x|^t} - \lambda\bigl(u_1^+\bigr)^{-s}\psi\biggr] dx = 0, \quad \forall\psi \in W_0^{1,p}(\Omega), \qquad (2.13)
\end{aligned}
$$

which implies that u_1 is a weak solution of problem (1.1). Putting the test function $\psi = u_1^-$ in (2.13), we obtain that $u_1 \ge 0$. Noting that $I_{\lambda,\mu}(u_1) = \Gamma < 0$, then $u_1 \not\equiv 0$. In terms of the maximum principle, we have that $u_1 > 0$, a.e. $x \in \Omega$.

The proof of Theorem 2.2 is completed. □

3 Existence of a solution of the perturbation problem

In order to find another solution, we consider the following problem:

$$
\begin{cases}
-\Delta_p u - \mu\frac{|u|^{p-2}u}{|x|^p} = Q(x)\frac{(u^+)^{p^*(t)-1}}{|x|^t} + \lambda(u^+ + \gamma)^{-s}, & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
\end{cases}
\qquad (3.1)
$$

where $\gamma > 0$ is small. The solution of (3.1) is equivalent to the critical point of the following C^1-functional on $W_0^{1,p}(\Omega)$:

$$I_\gamma(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{(u^+)^{p^*(t)}}{|x|^t}\,dx - \frac{\lambda}{1-s}\int_\Omega \bigl[(u^+ + \gamma)^{1-s} - \gamma^{1-s}\bigr]\,dx.$$

For every $\varphi \in W_0^{1,p}(\Omega)$, the definition of weak solution $u \in W_0^{1,p}(\Omega)$ gives that

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u\nabla\varphi - \mu\frac{|u|^{p-2}u\varphi}{|x|^p}\right) - \lambda\int_\Omega (u^+ + \gamma)^{-s}\varphi - \int_\Omega Q(x)\frac{(u^+)^{p^*(t)-1}\varphi}{|x|^t} = 0. \quad (3.2)$$

Lemma 3.1 *For $R, \rho > 0$, suppose that $\lambda < \lambda_0$, then I_γ satisfies the following properties*:

(i) *$I_\gamma(u) \geq \rho > 0$ for $u \in \partial B_R$;*

(ii) *There exists $u_2 \in W_0^{1,p}(\Omega)$ such that $\|u_2\| > R$ and $I_\gamma(u_2) < \rho$,*

where R, ρ, and λ_0 are given in Lemma 2.1.

Proof (i) By the subadditivity of t^{1-s}, we have

$$(u^+ + \gamma)^{1-s} - \gamma^{1-s} \leq (u^+)^{1-s}, \quad \forall u \in W_0^{1,p}(\Omega), \quad (3.3)$$

which leads to

$$I_\gamma(u) \geq I_{\lambda,\mu}(u), \quad \forall u \in W_0^{1,p}(\Omega).$$

Hence, if $\lambda < \lambda_0$ for $\rho, \lambda_0 > 0$, we can obtain the conclusion from Lemma 2.1.

(ii) $\forall u^+ \in W_0^{1,p}(\Omega)$, $u^+ \not\equiv 0$ and $r > 0$, which yields

$$\begin{aligned} I_\gamma(ru) &= \frac{r^p}{p}\|u\|^p - r^{p^*(t)}\int_\Omega Q(x)\frac{(u^+)^{p^*(t)}}{|x|^t}\,dx - \frac{\lambda}{1-s}\int_\Omega \bigl[(ru^+ + \gamma)^{1-s} - \gamma^{1-s}\bigr]\,dx \\ &\leq \frac{r^p}{p}\|u\|^p - r^{p^*(t)}\int_\Omega Q(x)\frac{(u^+)^{p^*(t)}}{|x|^t}\,dx \\ &\to -\infty \quad (r \to +\infty). \end{aligned}$$

Therefore, there exists u_2 such that $\|u_2\| > R$ and $I_\gamma(u_2) < \rho$.

This completes the proof of Lemma 3.1. □

Lemma 3.2 *Assume that $0 < \gamma < 1$. Then I_γ satisfies the $(PS)_c$ condition with $c < \frac{(p-t)}{p(N-t)}\frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} - D\lambda^{\frac{p}{p+s-1}}$, where*

$$D = \frac{p+s-1}{p}\left\{\left(\frac{1}{1-s} + \frac{N-p}{p(N-t)}\right)C_2\left[\frac{p}{(N-t)(1-s)}\right]^{\frac{s-1}{p}}\right\}^{\frac{p}{p+s-1}}.$$

Proof Choose $\{\tau_n\} \subset W_0^{1,p}(\Omega)$ satisfying

$$I_\gamma(\tau_n) \to c, \quad \text{and} \quad I'_\gamma(\tau_n) \to 0 \quad (n \to \infty). \quad (3.4)$$

We assert that $\{\tau_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Otherwise, we assume that $\lim_{n\to\infty}\|\tau_n\|\to\infty$. By (3.4), we have

$$\begin{aligned}
c &= I_\gamma(\tau_n) - \frac{1}{p^*(t)}\langle I'_\gamma(\tau_n),\tau_n\rangle + o(1)\\
&= \frac{1}{p}\|\tau_n\|^p - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{(\tau_n^+)^{p^*(t)}}{|x|^t}\,dx - \frac{\lambda}{1-s}\int_\Omega\big[(\tau_n^+ + \gamma)^{1-s} - \gamma^{1-s}\big]\,dx\\
&\quad - \frac{1}{p^*(t)}\|\tau_n\|^p + \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{(\tau_n^+)^{p^*(t)-1}\tau_n}{|x|^t}\,dx + \frac{\lambda}{p^*(t)}\int_\Omega(\tau_n^+ + \gamma)^{-s}\tau_n\,dx + o(1)\\
&= \left(\frac{1}{p} - \frac{1}{p^*(t)}\right)\|\tau_n\|^p - \frac{\lambda}{1-s}\int_\Omega\big[(\tau_n^+ + \gamma)^{1-s} - \gamma^{-s}\big]\,dx\\
&\quad + \frac{\lambda}{p^*(t)}\int_\Omega(\tau_n^+ + \gamma)^{-s}\tau_n\,dx + o(1)\\
&\ge \frac{p-t}{p(N-t)}\|\tau_n\|^p - \lambda\left(\frac{1}{1-s} + \frac{1}{p^*(t)}\right)\int_\Omega|\tau_n|^{1-s}\,dx + o(1)\\
&\ge \frac{p-t}{p(N-t)}\|\tau_n\|^p - \lambda\left(\frac{1}{1-s} + \frac{1}{p^*(t)}\right)C_1\|\tau_n\|^{1-s} + o(1).
\end{aligned}$$

The last inequality is absurd thanks to $0 < 1-s < 1$. That is, $\{\tau_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, up to a sequence, there exists a subsequence, still called $\{\tau_n\}$. We assume that there exists $\{\tau_1\} \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases}
\tau_n \rightharpoonup \tau_1, & \text{in } W_0^{1,p}(\Omega),\\
\tau_n \longrightarrow \tau_1, & \text{in } L^p(\Omega,|x|^{-t}),\\
\tau_n(x) \longrightarrow \tau_1(x), & \text{a.e. in } \Omega,\\
|\tau_n(x)| \le h(x), & \text{a.e. in } \Omega \text{ for all } n \text{ with } h(x)\in L^1(\Omega).
\end{cases}\qquad 1 \le p < p^*(t),$$

Since

$$\big|(\tau_n - \tau_1)(\tau_n^+ + \gamma)^{-s}\big| \le \gamma^{-s}\big(h + |\tau_1|\big),$$

it follows from the dominated convergence theorem that

$$\lim_{n\to\infty}\int_\Omega(\tau_n - \tau_1)(\tau_n^+ + \gamma)^{-s}\,dx = 0.$$

Furthermore, by $|\tau_1|(\tau_n^+ + \gamma)^{-s} \le |\tau_1|\gamma^{-s}$, and applying the dominated convergence theorem again, we have

$$\lim_{n\to\infty}\int_\Omega(\tau_n^+ + \gamma)^{-s}\tau_1\,dx = \int_\Omega(\tau_1^+ + \gamma)^{-s}\tau_1\,dx.$$

Thus, we deduce that

$$\lim_{n\to\infty}\int_\Omega(\tau_n^+ + \gamma)^{-s}\tau_n\,dx = \int_\Omega(\tau_1^+ + \gamma)^{-s}\tau_1\,dx.$$

Now we prove that $\tau_n \to \tau_1$ strongly in $W_0^{1,p}(\Omega)$. Set $\omega_n = \tau_n - \tau_1$. Since $I'_{\lambda,\mu}(\tau_n) \to 0$ in $(W_0^{1,p}(\Omega))^*$, we have

$$\|\tau_n\|^p - \int_\Omega Q(x)\frac{(\tau_n^+)^{p^*(t)-1}\tau_n}{|x|^t}\,dx - \lambda\int_\Omega \left(\tau_n^+ + \gamma\right)^{-s}\tau_n\,dx = o(1).$$

According to the Brézis–Lieb lemma, together with (3.4), we have

$$\begin{aligned}&\|\omega_n\|^p + \|\tau_1\|^p - \int_\Omega Q(x)\frac{(\omega_n^+)^{p^*(t)-1}\omega_n}{|x|^t}\,dx - \int_\Omega Q(x)\frac{(\tau_1^+)^{p^*(t)-1}\tau_1}{|x|^t}\,dx \\ &\quad - \lambda\int_\Omega \left(\tau_1^+ + \gamma\right)^{-s}\tau_1\,dx = o(1),\end{aligned}$$

and

$$\lim_{n\to\infty}\left\langle I'_\gamma(\tau_n), \tau_1\right\rangle = \|\tau_1\|^p - \int_\Omega Q(x)\frac{(\tau_1^+)^{p^*(t)-1}\tau_1}{|x|^t}\,dx - \lambda\int_\Omega \left(\tau_1^+ + \gamma\right)^{-s}\tau_1\,dx = 0.$$

Thus

$$\begin{aligned}&\lim_{n\to\infty}\|\omega_n\|^p = \lim_{n\to\infty}\int_\Omega Q(x)\frac{(\omega_n^+)^{p^*(t)-1}\omega_n}{|x|^t}\,dx = l, \\ &\int_\Omega \frac{|\omega_n|^{p^*(t)}}{|x|^t}\,dx \geq \int_\Omega \frac{Q(x)}{Q_M}\frac{|\omega_n|^{p^*(t)}}{|x|^t}\,dx \geq \int_\Omega \frac{Q(x)}{Q_M}\frac{(\omega_n^+)^{p^*(t)-1}\omega_n}{|x|^t}\,dx.\end{aligned}$$

Sobolev's inequality implies that

$$\|\omega_n\|^p \geq S\left(\int_\Omega \frac{|\omega_n|^{p^*(t)}}{|x|^t}\,dx\right)^{\frac{p}{p^*(t)}}.$$

Consequently, $l \geq S(\frac{l}{Q_M})^{\frac{p}{p^*(t)}}$. We guarantee that $l = 0$. Otherwise, we suppose that

$$l \geq \frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}}.$$

It follows that

$$\begin{aligned}c &= I_\gamma(\tau_n) - \frac{1}{p^*(t)}\left\langle I'_\gamma(\tau_n), \tau_n\right\rangle + o(1) \\ &= \frac{(p-t)}{p(N-t)}\|\tau_n\|^p - \frac{\lambda}{1-s}\int_\Omega \left[\left(\tau_n^+ + \gamma\right)^{1-s} - \gamma^{-s}\right]dx + \frac{\lambda}{p^*(t)}\int_\Omega \left(\tau_n^+ + \gamma\right)^{-s}\tau_n\,dx + o(1) \\ &\geq \frac{(p-t)}{p(N-t)}\frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} + \frac{p-t}{p(N-t)}\|\tau_1\|^p - \lambda\left(\frac{1}{1-s} + \frac{1}{p^*(t)}\right)\int_\Omega |\tau_n|^{1-s}\,dx + o(1) \\ &\geq \frac{(p-t)}{p(N-t)}\frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} + \frac{p-t}{p(N-t)}\|\tau_1\|^p - \lambda\left(\frac{1}{1-s} + \frac{1}{p^*(t)}\right)C_2\|\tau_1\|^{1-s} + o(1)\end{aligned}$$

$$\geq \frac{(p-t)}{p(N-t)} \frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} - D\lambda^{\frac{p}{p+s-1}},$$

which contradicts the condition of Lemma 3.2. Hence $l = 0$. Therefore $\tau_n \to \tau_1$.

This proof of Lemma 3.2 is finished. □

Lemma 3.3 *For $0 < s < 1$ and $\lambda > 0$ small enough, there exists $u_2 \in W_0^{1,p}(\Omega)$ such that*

$$\sup_{t\geq 0} I_{\lambda,\mu}(tu_2) \leq \frac{(p-t)}{p(N-t)} \frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} - D\lambda^{\frac{p}{p-1+s}}, \tag{3.5}$$

where D is defined in Lemma 3.2.

Proof For every $r \geq 0$, we have

$$I_\gamma(ru_\epsilon) = \frac{r^p}{p}\|u_\epsilon\|^p - \frac{r^{p^*(t)}}{p^*(t)} \int_\Omega Q(x)\frac{(u_\epsilon^+)^{p^*(t)}}{|x|^t}\,dx - \frac{\lambda}{1-s}\int_\Omega \bigl[\bigl(ru_\epsilon^+ + \gamma\bigr)^{1-s} - \gamma^{1-s}\bigr]\,dx,$$

which implies that there exists a positive constant ϵ_0 such that

$$\lim_{r\to 0} I_\gamma(ru_\epsilon) = 0, \quad \forall \epsilon \in (0,\epsilon_0),$$

and

$$\lim_{r\to +\infty} I_\gamma(ru_\epsilon) = -\infty, \quad \forall \epsilon \in (0,\epsilon_0),$$

where u_ϵ is defined in Sect. 1. Let

$$A_\epsilon(r) = \frac{r^p}{p}\|u_\epsilon\|^p - \frac{r^{p^*(t)}}{p^*(t)} \int_\Omega Q(x)\frac{(u_\epsilon^+)^{p^*(t)}}{|x|^t}\,dx;$$
$$B_\epsilon(r) = -\frac{1}{1-s}\int_\Omega \bigl[\bigl(ru_\epsilon^+ + \gamma\bigr)^{1-s} - \gamma^{1-s}\bigr]\,dx,$$

because of $\lim_{r\to\infty} A_\epsilon(r) = -\infty$, $A_\epsilon(0) = 0$, and $\lim_{r\to 0^+} A_\epsilon(r) > 0$, so $A_\epsilon(r)$ attains its maximum at some positive number. In fact, we let

$$A'_\epsilon(r) = r^{p-1}\|u_\epsilon\|^p - r^{p^*(t)-1}\int_\Omega Q(x)\frac{(u_\epsilon^+)^{p^*(t)}}{|x|^t}\,dx = 0,$$

therefore

$$r = \left(\frac{\|u_\epsilon\|^p}{\int_\Omega Q(x)\frac{(u_\epsilon^+)^{p^*(t)}}{|x|^t}\,dx}\right)^{\frac{1}{p^*(t)-p}} := T_\epsilon.$$

Noting that $A'_\epsilon(r) > 0$ for every $0 < r < T_\epsilon$ and $A'_\epsilon(r) < 0$ for every $r > T_\epsilon$, our claim is proved. Thus, the properties of $I_\gamma(ru_\epsilon)$ at $r = 0$ and $r = +\infty$ tell us that $\sup_{r\geq 0} I_\gamma(ru_\epsilon)$ is attained for some $r_\epsilon > 0$.

From condition (Q_1), we have

$$\left|\int_\Omega Q(x)\frac{u_\epsilon^{p^*(t)}}{|x|^t}\,dx - \int_\Omega Q_M\frac{u_\epsilon^{p^*(t)}}{|x|^t}\,dx\right| \le \int_\Omega \bigl|Q(x)-Q(0)\bigr|\frac{u_\epsilon^{p^*(t)}}{|x|^t}\,dx = O\bigl(\epsilon^\beta\bigr).$$

It follows that

$$\int_\Omega Q(x)\frac{u_\epsilon^{p^*(t)}}{|x|^t}\,dx = Q(0)S^{\frac{N-t}{p-t}} + O\bigl(\epsilon^{b(\mu)p^*(t)-N+t}\bigr) + O\bigl(\epsilon^\beta\bigr). \tag{3.6}$$

By (3.6), we deduce that

$$\begin{aligned} A_\epsilon(T_\epsilon) &= \frac{1}{p}\left[\frac{\|u_\epsilon\|^p}{\int_\Omega Q(x)\frac{u_\epsilon^{p^*(t)}}{|x|^t}\,dx}\right]^{\frac{p}{p^*(t)-p}}\|u_\epsilon\|^p \\ &\quad - \frac{1}{p^*(t)}\left[\frac{\|u_\epsilon\|^p}{\int_\Omega Q(x)\frac{u_\epsilon^{p^*(t)}}{|x|^t}\,dx}\right]^{\frac{p^*(t)}{p^*(t)-p}}\int_\Omega Q(x)\frac{u_\epsilon^{p^*(t)}}{|x|^t}\,dx \\ &= \frac{p-t}{p(N-t)}\left[\frac{\|u_\epsilon\|^p}{\int_\Omega Q(x)\frac{u_\epsilon^{p^*(t)}}{|x|^t}\,dx}\right]^{\frac{p}{p^*(t)-p}}\|u_\epsilon\|^p \\ &\le \frac{p-t}{p(N-t)}\frac{S^{\frac{N-t}{p-t}}}{(Q(0))^{\frac{N-p}{p-t}}} + O\bigl(\epsilon^{b(\mu)p+p-N}\bigr) + O\bigl(\epsilon^\beta\bigr). \end{aligned} \tag{3.7}$$

Next, we will estimate B_ϵ. Here, we use the following inequality from [24, 27]:

$$x^{1-s} - (x+y)^{1-s} \le -(1-s)y^{\frac{1-s}{4}}x^{\frac{3(1-s)}{4}}, \quad 0 < x < y. \tag{3.8}$$

Observe from (3.8) that

$$\begin{aligned} B_\epsilon(r_\epsilon) &\le \frac{1}{1-s}\int_{\{x||x|\le\epsilon^{\frac{1-s}{2p}}\}}\bigl[\gamma^{1-s} - (r_\epsilon u_\epsilon + \gamma)^{1-s}\bigr]\,dx \\ &\le -C_3\int_{\{x||x|\le\epsilon^{\frac{1-s}{2p}}\}}(r_\epsilon u_\epsilon)^{\frac{1-s}{4}}\,dx \\ &\le -C_3\int_{\{x||x|\le\epsilon^{\frac{1-s}{2p}}\}\cap\{\eta(x)=1\}}\left[r_\epsilon\epsilon^{-\frac{N-p}{p}}U_{p,\mu}\left(\frac{|x|}{\epsilon}\right)\right]^{\frac{1-s}{4}}\,dx \\ &\le -C_4\int_0^{\epsilon^{\frac{1-s-2p}{2p}}}\bigl[\epsilon^{-\frac{N-p}{p}}U_{p,\mu}(y)\bigr]^{\frac{1-s}{4}}y^{N-1}\epsilon^N\,dy \\ &\le -C_5\epsilon^{-\frac{(N-p)(1-s)}{4p}+N}\int_0^{\epsilon^{\frac{1-s-2p}{2p}}}y^{-b(\mu)p+N-1}\,dy \\ &\le -C_5\begin{cases}\epsilon^{-\frac{(N-p)(1-s)}{4p}+N}, & b(\mu) > \frac{N}{p}, \\ \epsilon^{-\frac{(N-p)(1-s)}{4p}+N}|\ln\epsilon|, & b(\mu) = \frac{N}{p}, \\ \epsilon^{-\frac{(N-p)(1-s)}{4p}+N+\frac{(1-s-2p)(-b(\mu)p+N)}{2p}}, & b(\mu) < \frac{N}{p}.\end{cases} \end{aligned} \tag{3.9}$$

From (3.7) and (3.9), we find that there exists a positive constant $\widetilde{\lambda}_0$ such that, for every $\lambda \in (0, \widetilde{\lambda}_0)$, one has

$$\begin{aligned} I_\gamma(r_\epsilon u_\epsilon) &= A_\epsilon(r_\epsilon) + \lambda B_\epsilon(r_\epsilon) \\ &\le \frac{p-t}{p(N-p)} \frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} + O\bigl(\epsilon^{b(\mu)p-N+p}\bigr) + O\bigl(\epsilon^{\beta}\bigr) \\ &\quad - C_5 \begin{cases} \epsilon^{-\frac{(N-p)(1-s)}{4p}+N}, & b(\mu) > \frac{N}{p}, \\ \epsilon^{-\frac{(N-p)(1-s)}{4p}+N} |\ln \epsilon|, & b(\mu) = \frac{N}{p}, \\ \epsilon^{-\frac{(N-p)(1-s)}{4p}+N+\frac{(1-s-2p)(-b(\mu)p+N)}{2p}}, & b(\mu) < \frac{N}{p}, \end{cases} \\ &< \frac{p-t}{p(N-p)} \frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} - D\lambda^{\frac{p}{p+s-1}}. \end{aligned}$$

This completes the proof of Lemma 3.3. □

Theorem 3.4 *For* $0 < \gamma < 1$, *there is* $\lambda^* > 0$ *such that* $\lambda \in (0, \lambda^*)$, *problem* (3.1) *admits a positive solution* $\tau_\gamma \in W_0^{1,p}(\Omega)$ *satisfying* $I_\gamma(\tau_\gamma) > \rho$, *where* ρ *is given in Lemma* 2.1.

Proof Let $\lambda^* = \min\{\lambda_0, \widetilde{\lambda}_0\}$, then Lemmas 3.1–3.3 hold for $0 < \lambda < \lambda^*$. Based on Lemma 3.1, we know that I_γ satisfies the geometry of the mountain pass lemma [1]. Therefore, there is a sequence $\{\tau_n\} \subset W_0^{1,p}(\Omega)$ such that

$$I_\gamma(\tau_n) \to c_\gamma > \rho > 0, \qquad I'_\gamma(\tau_n) \to 0, \tag{3.10}$$

where

$$\begin{aligned} c_\gamma &= \inf_{\phi\in\Phi} \max_{r\in[0,1]} I_\gamma\bigl(\phi(r)\bigr), \\ \Phi &= \bigl\{\phi \in C\bigl([0,1], W_0^{1,p}(\Omega)\bigr) : \phi(0) = 0, \phi(1) = u_2\bigr\}. \end{aligned}$$

So, according to Lemmas 3.1 and 3.3, one has

$$\begin{aligned} 0 < \rho < c_\gamma &\le \max_{r\in[0,1]} I_\gamma(ru_2) \le \sup_{r\ge 0} I_\gamma(ru_2) \\ &< \frac{p-t}{p(N-p)} \frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} - D\lambda^{\frac{p}{p+s-1}}. \end{aligned} \tag{3.11}$$

From Lemma 3.2, note that $\{\tau_n\}$ has a convergent subsequence, still denoted by $\{\tau_n\}$ ($\{\tau_n\} \subset W_0^{1,p}(\Omega)$). Assume that $\lim_{n\to\infty} \tau_n = \tau_\gamma$ in $W_0^{1,p}(\Omega)$. Hence, combining (3.10) and (3.11), we have

$$I_\gamma(\tau_\gamma) = \lim_{n\to\infty} I_\gamma(\tau_n) = c_\gamma > \rho > 0,$$

which implies that $\tau_\gamma \not\equiv 0$. By the continuity of I'_γ, we know that τ_γ is a solution of (3.1). Furthermore, $\tau_\gamma \ge 0$. Hence, applying the strong maximum principle, we obtain that τ_γ is a positive solution of (3.1). □

4 Existence of the second solution of problem (1.1)

Theorem 4.1 *For* $\lambda \in (0, \lambda^*)$, *problem* (1.1) *possesses a positive solution* τ_1 *satisfying* $I_{\lambda,\mu}(\tau_1) > 0$, *where* λ^* *is given in Theorem* 3.4.

Proof Let $\{\tau_\gamma\}$ be a family of positive solutions of (1.1), we will show that $\{\tau_\gamma\}$ has a uniform lower bound. Indeed, we denote

$$d(r) = r^{p^*(t)-1} + \frac{\lambda}{(r+p-1)^s};$$

$$\text{case (i)} \quad 0 < r < 1, \quad d(r) \geq \frac{\lambda}{(1+p-1)^s} = \frac{\lambda}{p^s};$$

$$\text{case (ii)} \quad r \geq 1, \quad d(r) \geq 1.$$

Therefore, for every $\gamma \in (0,1)$, $r \geq 0$, we get

$$r^{p^*(t)-1} + \frac{\lambda}{(r+\gamma)^s} \geq r^{p^*(t)-1} + \frac{\lambda}{(r+p-1)^s} \geq \min\left\{1, \frac{\lambda}{p^s}\right\}.$$

Recall that e is a weak solution of the following problem:

$$\begin{cases} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

so $e(x) > 0$ in Ω. According to the comparison principle, we have

$$\tau_\gamma \geq \min\{1, Q_m\} \min\left\{1, \frac{\lambda}{p^s}\right\} e > 0, \tag{4.1}$$

where $Q_m = \min_{x \in Q} Q(x) > 0$. Since $\{\tau_\gamma\}$ are solutions of problem (3.1), one has

$$\|\tau_\gamma\|^p - \int_\Omega Q(x) \frac{\tau_\gamma^{p^*(t)}}{|x|^t}\,dx - \lambda \int_\Omega (\tau_\gamma + \gamma)^{-s} \tau_\gamma\,dx = 0. \tag{4.2}$$

Combining with (3.3), (4.2), and Theorem 3.4, we have

$$\begin{aligned}
&\frac{p-t}{p(N-p)} \frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} - D\lambda^{\frac{p}{p+s-1}} \\
&\quad > I_\gamma(\tau_\gamma) - \frac{1}{p^*(t)} \langle I_\gamma'(\tau_\gamma), \tau_\gamma \rangle \\
&\quad = \frac{p-t}{p(N-t)} \|\tau_\gamma\|^p + \frac{\lambda}{p^*(t)} \int_\Omega (\tau_\gamma + \gamma)^{-s} \tau_\gamma\,dx - \frac{\lambda}{1-s} \int_\Omega \left[(\tau_\gamma + \gamma)^{1-s} - \gamma^{1-s}\right] dx \\
&\quad \geq \frac{p-t}{p(N-t)} \|\tau_\gamma\|^p - \frac{\lambda}{1-s} \int_\Omega \left[(\tau_\gamma + \gamma)^{1-s} - \gamma^{1-s}\right] dx \\
&\quad = \frac{p-t}{p(N-t)} \|\tau_\gamma\|^p - \frac{\lambda C_6}{1-s} \|\tau_\gamma\|^{1-s},
\end{aligned}$$

since $s \in (0,1)$, so $\{\tau_\gamma\}$ is bounded in $W_0^{1,p}(\Omega)$. Going if necessary to a subsequence, also called $\{\tau_\gamma\}$, there exists $\tau_1 \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} \tau_\gamma \rightharpoonup \tau_1, & \text{in } W_0^{1,p}(\Omega), \\ \tau_\gamma \longrightarrow \tau_1, & \text{in } L^{p'}(\Omega, |x|^{-t}), \quad 1 \le p' < p^*(t), \\ \tau_\gamma(x) \longrightarrow \tau_1(x), & \text{a.e. in } \Omega. \end{cases} \tag{4.3}$$

Now, we show that $\tau_\gamma \to \tau_1$ in $W_0^{1,p}(\Omega)$ as $\gamma \to 0$. Set $w_\gamma = \tau_\gamma - \tau_1$, then $\|w_\gamma\| \to 0$ as $\gamma \to 0$; otherwise, there exists a subsequence (still denoted by w_γ) such that $\lim_{\gamma\to 0}\|w_\gamma\| = l > 0$. Since $0 \le \frac{\tau_\gamma}{(\tau_\gamma+\gamma)^s} \le \tau_\gamma^{1-s}$, applying Hölder's inequality and (4.3), we have

$$\begin{aligned} \int_\Omega \tau_\gamma(\tau_\gamma+\gamma)^{-s}\,dx &\le \int_\Omega \tau_\gamma^{1-s}\,dx \le \int_\Omega |w_\gamma|^{1-s}\,dx + \int_\Omega |\tau_1|^{1-s}\,dx \\ &= |w_\gamma|_p^{1-s}|\Omega|^{\frac{1+s}{p}} + \int_\Omega |\tau_1|^{1-s}\,dx \\ &\le \int_\Omega |\tau_1|^{1-s}\,dx + o(1). \end{aligned}$$

Similarly,

$$\int_\Omega |\tau_1|^{1-s}\,dx \le \int_\Omega \tau_\gamma(\tau_\gamma+\gamma)^{-s}\,dx + o(1).$$

Therefore

$$\lim_{\gamma\to 0}\int_\Omega \tau_\gamma(\tau_\gamma+\gamma)^{-s}\,dx = \int_\Omega \tau_1^{1-s}\,dx.$$

It follows from $\langle I'_\gamma(\tau_\gamma), \tau_\gamma\rangle = 0$ and the Brézis–Lieb lemma that

$$\|w_\gamma\|^p + \|\tau_1\|^p - \int_\Omega Q(x)\frac{w_\gamma^{p^*(t)}}{|x|^t}\,dx - \int_\Omega Q(x)\frac{\tau_1^{p^*(t)}}{|x|^t}\,dx - \lambda\int_\Omega \tau_1^{1-s}\,dx = o(1). \tag{4.4}$$

Note that $\tau_\gamma \rightharpoonup \tau_1$ as $\gamma \to 0^+$. Choose the test function $\varphi = \phi \in W_0^{1,p}(\Omega) \cap C_0(\Omega)$ in (3.2). Letting $\gamma \to 0^+$ and using (4.1), we deduce that $\tau_1 \ge \min\{1, Q_m\}\min\{1, \frac{\lambda}{p^s}\}e > 0$, and

$$\int_\Omega \left(|\nabla\tau_1|^{p-2}\nabla\tau_1\nabla\phi - \mu\frac{|\tau_1|^{p-2}\tau_1\phi}{|x|^p}\right)dx = \int_\Omega Q(x)\frac{\tau_1^{p^*(t)-1}}{|x|^t}\phi\,dx + \lambda\int_\Omega \tau_1^{-s}\phi\,dx. \tag{4.5}$$

We show that (4.5) holds for every $\phi \in W_0^{1,p}(\Omega)$. In fact, since $W_0^{1,p}(\Omega) \cap C_0(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, then for every $\phi \in W_0^{1,p}(\Omega)$, there exists a sequence $\{\phi_n\} \subset W_0^{1,p}(\Omega) \cap C_0(\Omega)$ such that $\lim_{n\to\infty}\phi_n = \phi$. For $m, n \in \mathbb{N}^+$ large enough, replacing ϕ with $\phi_n - \phi_m$ in (4.5) yields

$$\begin{aligned} &\int_\Omega \left(|\nabla\tau_1|^{p-2}\nabla\tau_1\nabla(\phi_n-\phi_m) - \mu\frac{|\tau_1|^{p-2}\tau_1|\phi_n-\phi_m|}{|x|^p}\right)dx \\ &\quad = \int_\Omega Q(x)\frac{\tau_1^{p^*(t)}}{|x|^t}|\phi_n-\phi_m|\,dx + \lambda\int_\Omega \tau_1^{-s}|\phi_n-\phi_m|\,dx. \end{aligned} \tag{4.6}$$

On the one hand, using $\phi_n \to \phi$ and (4.6), we have that $\{\frac{\phi_n}{\tau_1}\}$ is a Cauchy sequence in $L^p(\Omega)$, hence there exists $\nu \in L^p(\Omega)$ such that $\lim_{n\to\infty} \frac{\phi_n}{\tau_0^s} = \nu$, which implies that $\lim_{n\to\infty} \frac{\phi_n}{\tau_0^s} = \nu$ in measure. By Riesz's theorem, without loss of generality, choose a subsequence of $\{\frac{\phi_n}{\tau_0^s}\}$, still denoted by $\{\frac{\phi_n}{\tau_0^s}\}$, such that

$$\lim_{n\to\infty} \frac{\phi_n}{\tau_0^s} = \nu(x), \quad \text{a.e. } x \in \Omega. \tag{4.7}$$

On the other hand, from (4.7), we have that $\nu = \frac{\phi}{\tau_0^s}$, which leads to

$$\lim_{n\to\infty} \int_\Omega \frac{\phi_n(x)}{\tau_0^s}\,dx = \int_\Omega \frac{\phi(x)}{\tau_0^s}\,dx.$$

Therefore, we deduce that (4.5) holds for $\phi \in W_0^{1,p}(\Omega)$. Setting $\phi = \tau_1$ in (4.5), we have

$$\|\tau_1\|^p - \int_\Omega Q(x)\frac{\tau_1^{p^*(t)}}{|x|^t}\,dx - \lambda \int_\Omega \tau_1^{1-s}\,dx = 0. \tag{4.8}$$

Together with (4.4), we obtain that

$$\|w_\gamma\|^p - \int_\Omega Q(x)\frac{w_\gamma^{p^*(t)}}{|x|^t}\,dx = o(1). \tag{4.9}$$

Hence

$$\lim_{\gamma\to 0^+} \|w_\gamma\|^p = \lim_{\gamma\to 0^+} \int_\Omega Q(x)\frac{w_\gamma^{p^*(t)}}{|x|^t}\,dx = l > 0.$$

Since

$$\int_\Omega \frac{|w_\gamma|^{p^*(t)}}{|x|^t}\,dx \geq \int_\Omega \frac{Q(x)}{Q_M}\frac{|w_\gamma|^{p^*(t)}}{|x|^t}\,dx \geq \int_\Omega \frac{Q(x)}{Q_M}\frac{(w_\gamma^+)^{p^*(t)}}{|x|^t}\,dx.$$

Then $l \geq \frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}}$. By (4.8), we have

$$\begin{aligned}
I_{\lambda,\mu}(\tau_1) &= \frac{1}{p}\|\tau_1\|^p - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{\tau_1^{p^*(t)}}{|x|^t}\,dx - \frac{\lambda}{1-s}\int_\Omega \tau_1^{1-s}\,dx \\
&= \frac{p-t}{p(N-t)}\|\tau_1\|^p - \lambda\left(\frac{1}{1-s} - \frac{1}{p^*(t)}\right)\int_\Omega \tau_1^{1-s}\,dx \\
&\geq \frac{p-t}{p(N-t)}\|\tau_1\|^p - \lambda\left(\frac{1}{1-s} + \frac{1}{p^*(t)}\right)C_2\|\tau_1\|^{1-s} \\
&> -D\lambda^{\frac{p}{p+s-1}}.
\end{aligned} \tag{4.10}$$

At the same time, it follows from (4.4) and (4.9) that

$$I_{\lambda,\mu}(\tau_1) = I_\gamma(\tau_\gamma) - \frac{p-t}{p(N-t)}\|w_\gamma\|^p + o(1)$$

$$< \frac{p-t}{p(N-t)}\left(\frac{S^{\frac{N-t}{p-t}}}{Q_M^{\frac{N-p}{p-t}}} - l\right) - D\lambda^{\frac{p}{p-1+s}}$$
$$\leq -D\lambda^{\frac{p}{p-1+s}},$$

which contradicts (4.10). Therefore, we deduce that

$$I_{\lambda,\mu}(\tau_1) = \lim_{\gamma\to 0} I_\gamma(\tau_\gamma) > \rho > 0.$$

Consequently, problem (1.1) has two different solutions u_1 and τ_1. Furthermore, $\tau_1 \not\equiv 0$, together with the maximum principle, we conclude that $\tau_1 > 0$ a.e. $x \in \Omega$. That is, τ_1 is a positive solution of problem (1.1).

The proof of Theorem 4.1 is completed. □

Remark 4.1 In order to apply the Brézis–Lieb lemma, we need to establish the convergence results for the sequences with gradient terms [5, 9]. Furthermore, the strong maximum principle for a p-Laplace operator is also used.

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