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The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity

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Abstract

In this paper, we focus on the convergence analysis and error estimation for the unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity. By introducing a double iterative technique, in the case of the nonlinearity with singularity at time and space variables, the unique positive solution to the problem is established. Then, from the developed iterative technique, the sequences converging uniformly to the unique solution are formulated, and the estimates of the error and the convergence rate are derived.

Keywords: Convergence analysis; Error estimation; p-Laplacian fractional differential equation; Double iterative technique; Riemann–Stieltjes integral conditions

1 Introduction

This paper is motivated by the following singular nonlocal fractional differential equation:

$$\begin{cases} -\mathcal{D}_x^{\frac{3}{2}}(\varphi_{\frac{3}{2}}(-\mathcal{D}_x^{\frac{4}{3}}z))(x) = x^{-1}z^{-\frac{1}{2}}, \quad 0 < x < 1, \\ z(0) = 0, \qquad \mathcal{D}_x^{\frac{4}{3}}z(0) = \mathcal{D}_x^{\frac{4}{3}}z(1) = 0, \qquad z(1) = \int_0^1 z(x)\,d\chi(x), \end{cases} \tag{1.1}$$

where χ is a function of bounded variation satisfying $\chi(x) = 0$, $x \in [0, \frac{1}{3})$, $\chi(x) = \frac{1}{2}$, $x \in [\frac{1}{3}, \frac{2}{3})$, $\chi(x) = 1$, $x \in [\frac{2}{3}, 1]$, which exhibits a blow-up behaviour at $x = 0$ and $z = 0$. These types of singular behaviours [1–11] as well as impulsive phenomena [12–21] often exhibit some blow-up properties [22, 23] which occur in many complex physical processes, for example, in mechanics process [1], the stress near the crack tip in elastic fracture exhibits a singularity of $r^{-0.5}$, where r is the distance measured from the crack tip.

Inspired by the above problem, this paper presents the convergence analysis and error estimation for the unique solution of the general fractional differential equation with singular decreasing nonlinearity and a p-Laplacian operator

$$\begin{cases} -\mathcal{D}_x^{\alpha}(\varphi_p(-\mathcal{D}_x^{\gamma}z))(x) = f(x, z(x)), \quad 0 < x < 1, \\ z(0) = 0, \qquad \mathcal{D}_x^{\gamma}z(0) = \mathcal{D}_x^{\gamma}z(1) = 0, \qquad z(1) = \int_0^1 z(x)\,d\chi(x), \end{cases} \tag{1.2}$$

where $\mathscr{D}_x{}^{\alpha}$, $\mathscr{D}_x{}^{\gamma}$ are the standard Riemann–Liouville derivatives with $\gamma, \alpha \in (1, 2]$, $\int_0^1 z(x)\,d\chi(x)$ is a Riemann–Stieltjes integral and χ is a function of bounded variation, $\varphi_p(x) = |x|^{p-2}x$, $p > 1$ is the p-Laplacian operator with conjugate index $q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Fractional calculus is a new research area of analytical mathematics which provides many useful tools for modelling various complex physical and biological processes with long memory [24–31]. For example, in fluid dynamics, laboratory data [24] and numerical experiment [25] show that solutes moving through a highly heterogeneous aquifer do not abide by Fick's first law, and thus in order to improve the accuracy of the model, one can adopt fractional order advection–dispersion equation to describe the convection–diffusion process in a highly heterogeneous aquifer, see [24, 32–40]. In biomedicine, Arafa et al. [41] introduced a fractional-order HIV-1 infection of CD4+ T cells dynamics model and then used the generalised Euler method to find a numerical solution of the HIV-1 infection fractional order model. Subsequently, by analytical techniques, Wang et al. [42] and Zhang et al. [43] studied the existence of positive solution for some abstract fractional dynamic systems for bioprocess, respectively.

On the other hand, the p-Laplacian equation is a second order quasilinear differential operator with the ability of modelling various fundamental nonlinear phenomena in non-Newtonian fluids, nonlinear elasticity, torsional creep problem, radiation of heat, etc. [44–56]. Thus fractional order differential equations with p-Laplacian operator not only can describe the nonlinear phenomena in non-Newtonian fluids but also can model complex processes with long memory. For example, by using the monotone iterative technique, Wu et al. [57] investigated the existence of twin iterative solutions for a fractional differential turbulent flow model

$$\begin{cases} -\mathscr{D}_x{}^{\alpha}(\varphi_p(-\mathscr{D}_x{}^{\gamma} z))(x) = g(x)h(z), \quad 0 < t < 1, \\ z(0) = 0, \qquad \mathscr{D}_x{}^{\gamma} z(0) = \mathscr{D}_x{}^{\gamma} z(1) = 0, \qquad z(1) = \int_0^1 z(x)\,d\chi(x), \end{cases}$$

where $\mathscr{D}_x{}^{\gamma}$, $\mathscr{D}_x{}^{\alpha}$ are the standard Riemann–Liouville derivatives such that $1 < \alpha, \gamma \leq 2$, and $h : [0, +\infty) \to [0, \infty)$ is a continuous and increasing function in the variable. The above work (also see [58–67]) shows that the monotone iterative technique is an effective analysis tool for obtaining iterative solutions and numerical solutions of the relative differential equations. However, to the best of our knowledge, in the application of iterative techniques, almost all works require the nonlinear term to be increasing in space variables and not to have singularity at space variables. So, even for the simplest case as Eq. (1.1), iterative solutions are difficult to construct by using classical iterative techniques. Thus in this paper, by introducing a double iterative technique, we study the convergence analysis and error estimation of the unique solution for the case where the nonlinearity in the equation is decreasing in space variables and is allowed to be singular at some time and space variables.

This paper is organised as follows. In Sect. 2, we firstly recall the definitions and properties of the Riemann–Liouville fractional derivative and integral, and then give some lemmas which will be used in the rest of this paper. In Sect. 3, we introduce a double iterative technique and establish the condition for which Eq. (1.2) has a unique positive solution, then from the developed iterative technique, the sequences converging uniformly to the unique positive solution are formulated, and the estimates of the approximation error and the convergence rate are derived.

2 Preliminaries and lemmas

In this section, we firstly recall the definitions and properties of the Riemann–Liouville fractional derivative and integral, and then give some useful lemmas.

Definition 2.1 ([68]) The Riemann–Liouville fractional integral of order $\gamma > 0$ of a function $z : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\gamma} z(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-y)^{\gamma-1} z(y)\, dy$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([68]) The Riemann–Liouville fractional derivative of order $\gamma > 0$ of a function $z : (0, +\infty) \to \mathbb{R}$ is given by

$$\boldsymbol{\mathcal{D}_x}^{\gamma} z(x) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^n \int_0^x (x-y)^{n-\gamma-1} z(y)\, dy,$$

where $n = [\gamma] + 1$, $[\gamma]$ denotes the integer part of number γ, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Property 2.1 ([68])

(1) If $z \in L^1(0, 1)$, $\gamma > \alpha > 0$, then

$$I^{\gamma} I^{\alpha} z(x) = I^{\gamma+\alpha} z(x), \qquad \boldsymbol{\mathcal{D}_x}^{\alpha} I^{\gamma} z(x) = I^{\gamma-\alpha} z(x), \qquad \boldsymbol{\mathcal{D}_x}^{\alpha} I^{\alpha} z(x) = z(x).$$

(2) If $\gamma > 0$, $\alpha > 0$, then

$$\boldsymbol{\mathcal{D}_x}^{\gamma} x^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} x^{\alpha-\gamma-1}.$$

(3) Let $\gamma > 0$, and $z(x)$ is integrable, then

$$I^{\gamma} \boldsymbol{\mathcal{D}_x}^{\gamma} z(x) = z(x) + c_1 x^{\gamma-1} + c_2 x^{\gamma-2} + \cdots + c_n x^{\gamma-n},$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \dots, n)$, n is the smallest integer greater than or equal to γ.

According to the definitions and properties of the Riemann–Liouville fractional derivative and integral and discussion in [34], we have the following lemma.

Lemma 2.1 *Given $h \in L^1(0, 1)$, the following boundary value problem*

$$\begin{cases} -\boldsymbol{\mathcal{D}_x}^{\alpha} z(x) = h(x), & 0 < x < 1, \\ z(0) = z(1) = 0, \end{cases} \tag{2.1}$$

has the unique solution

$$z(x) = \int_0^1 K_{\alpha}(x, y) h(y)\, dy,$$

where

$$K_\alpha(x,y) = \frac{1}{\Gamma(\alpha)} \begin{cases} [x(1-y)]^{\alpha-1}, & 0 \le x \le y \le 1, \\ [x(1-y)]^{\alpha-1} - (x-y)^{\alpha-1}, & 0 \le y \le x \le 1, \end{cases} \tag{2.2}$$

with an index α.

On the other hand, by using Property 2.1(3), we get that the unique solution of the equation

$$\begin{cases} -\mathcal{D}_{\boldsymbol{x}}{}^{\gamma} z(x) = 0, & 0 < x < 1, \\ z(0) = 0, & z(1) = 1, \end{cases} \tag{2.3}$$

is $x^{\gamma-1}$. Thus let

$$\mathcal{L} = \int_0^1 x^{\gamma-1}\, d\chi(x), \qquad \mathcal{K}_\chi(y) = \int_0^1 K_\gamma(x,y)\, d\chi(x),$$

and according to the strategy of [45], we have the following lemma.

Lemma 2.2 *Suppose $1 < \gamma \le 2$ and $h \in L^1(0,1)$, then the following nonlocal boundary value problem*

$$\begin{cases} -\mathcal{D}_{\boldsymbol{x}}{}^{\gamma} z(x) = h(x), & x \in (0,1), \\ z(0) = 0, & z(1) = \int_0^1 z(x)\, d\chi(x), \end{cases} \tag{2.4}$$

has the unique solution

$$z(x) = \int_0^1 H(x,y)h(y)\, dy, \tag{2.5}$$

where

$$H(x,y) = \frac{x^{\gamma-1}}{1-\mathcal{L}} \mathcal{K}_\chi(y) + K_\gamma(x,y). \tag{2.6}$$

Lemma 2.3 ([69]) *Let $0 \le \mathcal{L} < 1$ and $\mathcal{K}_\chi(y) \ge 0$ for $y \in [0,1]$, then $K_\alpha(x,y)$ and $H(x,y)$ have the following properties*:

(1) *$K_\alpha(x,y)$ and $H(x,y)$ are nonnegative and continuous for $(x,y) \in [0,1] \times [0,1]$.*

(2) *$K_\alpha(x,y)$ satisfies*

$$\frac{x^{\alpha-1}(1-x)y(1-y)^{\alpha-1}}{\Gamma(\alpha)} \le K_\alpha(x,y) \le \frac{\alpha-1}{\Gamma(\alpha)} y(1-y)^{\alpha-1}, \quad \textit{for } x,y \in [0,1]. \tag{2.7}$$

(3) *There exist two constants a, b such that*

$$a x^{\gamma-1} \mathcal{K}_\chi(y) \le H(x,y) \le b x^{\gamma-1}, \quad y,x \in [0,1]. \tag{2.8}$$

Let q be the conjugate index of p, and consider the following associated linear nonlocal boundary value problem:

$$\begin{cases} -\mathcal{D}_x{}^{\alpha}(\varphi_p(-\mathcal{D}_x{}^{\gamma} z))(x) = h(x), \quad x \in (0,1), \\ z(0) = 0, \qquad \mathcal{D}_x{}^{\gamma} z(0) = \mathcal{D}_x{}^{\gamma} z(1) = 0, \qquad z(1) = \int_0^1 z(x)\,d\chi(x), \end{cases} \tag{2.9}$$

for $h \in L^1(0,1)$ and $h \geq 0$. We have the following result.

Lemma 2.4 *The associated linear nonlocal boundary value problem* (2.9) *has a unique positive solution with the form*

$$z(x) = \int_0^1 H(x,y)\left(\int_0^1 K_\alpha(y,\tau)h(\tau)\,d\tau\right)^{q-1} dy.$$

Proof Let $w = -\mathcal{D}_x{}^{\gamma} z$, $v = \varphi_p(w) = \varphi_p(-\mathcal{D}_x{}^{\gamma} z)$, then we have

$$v(0) = \varphi_p\bigl(-\mathcal{D}_x{}^{\gamma} z(0)\bigr) = 0, \qquad v(1) = \varphi_p\bigl(-\mathcal{D}_x{}^{\gamma} z(1)\bigr) = 0, \qquad -\mathcal{D}_x{}^{\gamma} z = \varphi_p^{-1}(v). \tag{2.10}$$

Now consider the fractional Dirichlet boundary value problem

$$\begin{cases} -\mathcal{D}_x{}^{\alpha} v(x) = h(x), \quad t \in (0,1), \\ v(0) = v(1) = 0. \end{cases}$$

It follows from Lemma 2.1 that

$$v(x) = \int_0^1 K_\alpha(x,y)h(y)\,dy, \quad x \in [0,1]. \tag{2.11}$$

Thus by (2.9)–(2.11), one gets that the solution of (2.9) satisfies

$$\begin{cases} -\mathcal{D}_x{}^{\gamma} z(x) = \varphi_p^{-1}(\int_0^1 K_\alpha(x,y)h(y)\,dy), \quad x \in (0,1), \\ z(0) = 0, \qquad z(1) = \int_0^1 z(x)\,d\chi(x). \end{cases}$$

Hence, according to Lemma 2.2, the solution of the boundary value problem (2.9) can be written by

$$z(x) = \int_0^1 H(x,y)\varphi_p^{-1}\left(\int_0^1 K_\alpha(y,\tau)h(\tau)\,d\tau\right) dy, \quad x \in [0,1].$$

As $h(y) \geq 0$, $y \in [0,1]$, the solution of Eq. (2.9) is also positive. □

3 Main results

In this section, we firstly list some assumptions and then give the proof of our main results.

(K_0) χ is a function of bounded variation satisfying $\mathcal{K}_\chi(y) \geq 0$ for $y \in [0,1]$ and $0 \leq \mathcal{L} < 1$.

(F_1) $f \in C((0,1) \times (0,+\infty), [0,+\infty))$, and $f(x,z)$ is decreasing in z and for any $r \in (0,1)$, there exists a constant $0 < \mu < \frac{1}{p-1}$ such that, for any $(x,z) \in (0,1) \times (0,+\infty)$,

$$f(x,rz) \leq r^{-\mu} f(x,z). \tag{3.1}$$

Remark 3.1 Obviously, if $p = \frac{3}{2}$, then $f(x,z) = x^{-1}z^{-\frac{1}{2}}$ satisfies the assumption (F_1) which implies that f can be allowed to have singularity at $x = 0$ and $z = 0$.

Remark 3.2 If (F_1) holds, from (3.1), for any $r \geq 1$, one has the following equivalent statement:

$$f(x,rz) \geq r^{-\mu} f(x,z) \quad \text{for any } (x,z) \in (0,1) \times (0,+\infty). \tag{3.2}$$

In this paper, our work space is a Banach space $E = C[0,1]$ with the norm $\|z\| = \max_{x\in[0,1]} | z(x) |$ for any $z \in E$. Let $P = \{z \in C[0,1] : z(x) \geq 0, x \in [0,1]\}$, then P is a normal cone of E with normality constant 1. Now define a subset of P and a nonlinear integral operator $T: E \to E$ by

$$\begin{aligned} Q = \big\{ & z(x) \in P : \text{there exists a positive number } 0 < l_z < 1 \text{ such that} \\ & l_z x^{\gamma-1} \leq z(x) \leq l_z^{-1} x^{\gamma-1}, x \in [0,1] \big\}, \end{aligned}$$

and

$$(Tz)(x) = \int_0^1 H(x,y) \left(\int_0^1 K_\alpha(y,\tau) f\big(\tau, z(\tau)\big)\, d\tau \right)^{q-1} dy, \quad x \in [0,1]. \tag{3.3}$$

It follows from Lemma 2.4 that $z \in C[0,1]$ is a solution of the p-Laplacian fractional differential Eq. (1.2) if and only if $z \in C[0,1]$ is a fixed point of the nonlinear operator T.

Theorem 3.1 *Suppose* (K_0) *and* (F_1) *hold. If*

$$0 < \int_0^1 x(1-x)^{\alpha-1} f\big(x, x^{\gamma-1}\big)\, dx < +\infty, \tag{3.4}$$

then

(i) *the p-Laplacian fractional differential Eq.* (1.2) *has a unique positive solution* $z^* \in C[0,1]$;

(ii) *for any initial value* $z_0 \in Q$, *the sequence of functions* $\{z_n\}_{n\geq 1}$ *defined by*

$$z_n = \int_0^1 H(x,y) \left(\int_0^1 K_\alpha(y,\tau) f\big(\tau, z_{n-1}(\tau)\big)\, d\tau \right)^{q-1} dy, \quad n = 1,2,3,\ldots, \tag{3.5}$$

converge uniformly to the unique positive solution z^* *of Eq.* (1.2) *on* [0,1];

(iii) *the error between the iterative value* z_n *and the exact solution* z^* *can be estimated by*

$$\big\| z_n - z^* \big\| \leq \big(1 - \epsilon^{[\mu(q-1)]^{2n}}\big) \epsilon^{-\frac{1}{2}},$$

with an exact convergence rate

$$\big\| z_n - z^* \big\| = o\big(1 - \epsilon^{[\mu(q-1)]^{2n}}\big),$$

where $0 < \epsilon < 1$ *is a positive constant.*

(iv) *there exists a constant* $0 < l < 1$ *such that the exact solution* z^* *of Eq.* (1.2) *intervenes between two known curves* $lx^{\gamma-1}$ *and* $l^{-1}x^{\gamma-1}$, *i.e.*,

$$lx^{\gamma-1} \leq z^*(x) \leq l^{-1}x^{\gamma-1}, \quad x \in [0,1].$$

Proof Step 1. We show that $T: Q \to Q$ is a compact operator.

In fact, for any $z \in Q$, it follows from the definition of the set Q that there exists a constant $0 < l_z < 1$ such that

$$l_z x^{\gamma-1} \leq z(x) \leq l_z^{-1} x^{\gamma-1}, \quad x \in [0,1]. \tag{3.6}$$

Notice that $f(x,z)$ is decreasing in z, by Lemma 2.3, (3.1), (3.4) and (3.6), one has

$$\begin{aligned}
(Tz)(x) &= \int_0^1 H(x,y)\left(\int_0^1 K_\alpha(y,\tau)f\big(\tau,z(\tau)\big)\,d\tau\right)^{q-1} dy \\
&\leq \int_0^1 bx^{\gamma-1}\left(\int_0^1 \frac{\alpha-1}{\Gamma(\alpha)}\tau(1-\tau)^{\alpha-1}f\big(\tau,l_z\tau^{\gamma-1}\big)\,d\tau\right)^{q-1} dy \\
&\leq bl_z^{-\mu(q-1)}\left(\frac{\alpha-1}{\Gamma(\alpha)}\right)^{q-1}\left(\int_0^1 \tau(1-\tau)^{\alpha-1}f\big(\tau,\tau^{\gamma-1}\big)\,d\tau\right)^{q-1} \\
&< +\infty.
\end{aligned}$$

So T is well defined and uniformly bounded.

On the other hand, since $H(x,y)$ is uniformly continuous on $[0,1] \times [0,1]$, let $0 \leq x_1 < x_2 \leq 1$, for all $z \in Q$, one has

$$\begin{aligned}
&\big|(Tz)(x_1) - (Tz)(x_2)\big| \\
&\quad\leq \int_0^1 \big|H(x_1,y) - H(x_2,y)\big|\left(\int_0^1 K_\alpha(y,\tau)f\big(\tau,z(\tau)\big)\,d\tau\right)^{q-1} dy \\
&\quad\leq \int_0^1 \big|H(x_1,y) - H(x_2,y)\big|\left(\int_0^1 K_\alpha(y,\tau)f\big(\tau,l_z\tau^{\gamma-1}\big)\,d\tau\right)^{q-1} dy \\
&\quad\leq l_z^{-\mu(q-1)}\left(\frac{\alpha-1}{\Gamma(\alpha)}\right)^{q-1}\left(\int_0^1 \tau(1-\tau)^{\alpha-1}f\big(\tau,\tau^{\gamma-1}\big)\,d\tau\right)^{q-1}\int_0^1 \big|H(x_1,y) - H(x_2,y)\big|\,dy,
\end{aligned}$$

which implies that $T(Q)$ is equicontinuous, and then T is a compact operator in Q.

In the following, we shall show that $T(Q) \subset Q$. In fact, by (2.7), (2.8), (3.6) and (3.1), for any $z \in Q$, we have

$$\begin{aligned}
(Tz)(x) &= \int_0^1 H(x,y)\left(\int_0^1 K_\alpha(y,\tau)f\big(\tau,z(\tau)\big)\,d\tau\right)^{q-1} dy \\
&\leq \frac{bx^{\gamma-1}}{\Gamma^{q-1}(\alpha)}\left(\int_0^1 \tau(1-\tau)^{\alpha-1}f\big(\tau,l_z\tau^{\gamma-1}\big)\,d\tau\right)^{q-1} \\
&\leq l_z^{-\mu(q-1)}\left(\frac{\alpha-1}{\Gamma(\alpha)}\right)^{q-1}x^{\gamma-1}\left(\int_0^1 \tau(1-\tau)^{\alpha-1}f\big(\tau,\tau^{\gamma-1}\big)\,d\tau\right)^{q-1} \\
&\leq \widetilde{l}_{T_z}^{-1}x^{\gamma-1},
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
(Tz)(x) &= \int_0^1 H(x,y)\left(\int_0^1 K_\alpha(y,\tau)f\big(\tau,z(\tau)\big)\,d\tau\right)^{q-1} dy \\
&\geq a x^{\gamma-1}\int_0^1 \mathcal{K}_\chi(y)\left(\int_0^1 K_\alpha(y,\tau)f\big(\tau,l_z^{-1}\tau^{\gamma-1}\big)\,d\tau\right)^{q-1} dy \\
&\geq a x^{\gamma-1} l_z^{\mu(q-1)}\int_0^1 \mathcal{K}_\chi(y)\left(\int_0^1 K_\alpha(y,\tau)f\big(\tau,\tau^{\gamma-1}\big)\,d\tau\right)^{q-1} dy \\
&\geq a x^{\gamma-1}\left(\frac{l_z^\mu}{\Gamma(\alpha)}\right)^{q-1}\int_0^1 \mathcal{K}_\chi(y)y^{(\alpha-1)(q-1)}(1-y)^{q-1}\,dy \\
&\quad\times\left(\int_0^1 \tau(1-\tau)^{\alpha-1}f\big(\tau,\tau^{\gamma-1}\big)\,d\tau\right)^{q-1} \\
&\geq \widetilde{l}_{T_z} x^{\gamma-1},
\end{aligned} \tag{3.8}$$

where $\widetilde{l}_{T_z}$ satisfies

$$\begin{aligned}
0 &< \widetilde{l}_{T_z} \\
&< \min\left\{\frac{1}{2}, \left\{l_z^{-\mu(q-1)}\left(\frac{\alpha-1}{\Gamma(\alpha)}\right)^{q-1}\left(\int_0^1 \tau(1-\tau)^{\alpha-1}f\big(\tau,\tau^{\gamma-1}\big)\,d\tau\right)^{q-1}\right\}^{-1},\right. \\
&\quad a\left(\frac{l_z^\mu}{\Gamma(\alpha)}\right)^{q-1}\int_0^1 \mathcal{K}_\chi(y)y^{(\alpha-1)(q-1)}(1-y)^{q-1}\,dy \\
&\quad\left.\times\left(\int_0^1 \tau(1-\tau)^{\alpha-1}f\big(\tau,\tau^{\gamma-1}\big)\,d\tau\right)^{q-1}\right\}.
\end{aligned} \tag{3.9}$$

Hence we have $T(Q) \subset Q$.

Step 2. In this step, we prove that Eq. (1.2) has a unique positive solution $z^* \in C[0,1]$.

In fact, let $\eta(x) = x^{\gamma-1}$, then $\eta \in Q$. By Step 1, we have $T\eta \in Q$. Thus there exists a constant l_{T_η} such that $0 < l_{T_\eta} < 1$ and

$$l_{T_\eta}\eta(x) \leq T\eta(x) \leq l_{T_\eta}^{-1}\eta(x), \tag{3.10}$$

where l_{T_η} can be chosen as in (3.9). Notice that $0 < \mu(q-1) < 1$, for some $\kappa \in (0,1)$, we can choose a sufficiently large positive constant σ such that

$$\left[\kappa^{(-\mu(q-1)+1)}\right]^\sigma \leq l_{T_\eta}. \tag{3.11}$$

Now fix the initial value $z_0 = \kappa^\sigma\eta(x)$ and let

$$z_n = Tz_{n-1}, \quad n = 1,2,\ldots. \tag{3.12}$$

We firstly show

$$z_0 \leq z_2 \leq \cdots \leq z_{2n} \leq \cdots \leq z_{2n+1} \leq \cdots \leq z_3 \leq z_1. \tag{3.13}$$

In fact, since T is a decreasing operator in z, it follows from (3.10)–(3.12) that

$$\begin{aligned} &z_0(x) \leq \eta(x), \\ &z_1 = Tz_0 \geq T\eta \geq l_{T_\eta}\eta(x) \geq \left(\kappa^{-\mu(q-1)+1}\right)^{\sigma}\eta(x) = \left(\kappa^{\mu(q-1)}\right)^{-\sigma}\kappa^{\sigma}\eta(x) \\ &\quad = \left(\kappa^{\mu(q-1)}\right)^{-\sigma}z_0 \geq z_0, \end{aligned} \tag{3.14}$$

and then

$$z_2 = Tz_1(x) \leq Tz_0(x) = z_1. \tag{3.15}$$

On the other hand, it follows from (3.1) and (3.10) that

$$\begin{aligned} z_1 = T(z_0) &= \int_0^1 H(x,y)\left(\int_0^1 K_\alpha(y,\tau)f\left(\tau,\kappa^{\sigma}\eta(\tau)\right)d\tau\right)^{q-1}dy \\ &\leq \kappa^{-\mu\sigma(q-1)}\int_0^1 H(x,y)\left(\int_0^1 K_\alpha(y,\tau)f\left(\tau,\eta(\tau)\right)d\tau\right)^{q-1}dy \\ &= \kappa^{-\mu\sigma(q-1)}T\eta \leq \kappa^{-\mu\sigma(q-1)}l_{T_\eta}^{-1}\eta(x) \leq \kappa^{-\sigma}\eta(x), \end{aligned} \tag{3.16}$$

and then by (3.2), (3.10), (3.16) and the monotonicity of T, one gets

$$\begin{aligned} z_2 = Tz_1(x) \geq T\left(\kappa^{-\sigma}\eta(x)\right) &= \int_0^1 H(x,y)\left(\int_0^1 K_\alpha(y,\tau)f\left(\tau,\kappa^{-\sigma}\eta(\tau)\right)d\tau\right)^{q-1}dy \\ &\geq \kappa^{\sigma\mu(q-1)}T\eta(x) \geq \kappa^{\sigma\mu(q-1)}l_{T_\eta}\eta(x) \geq \kappa^{\sigma}\eta(x) = z_0. \end{aligned} \tag{3.17}$$

Equation (3.14), (3.15) and (3.17) yield

$$z_0 \leq z_2 \leq z_1. \tag{3.18}$$

Consequently, by applying induction for (3.18), we obtain (3.13).

Now, for any $c \in (0,1)$, from (3.1) and (3.3) we have

$$T^2(cz) \geq c^{\mu^2(q-1)^2}T^2z. \tag{3.19}$$

Noticing that T^2 is a nondecreasing operator with respect to z, by using (3.19) repeatedly, we obtain

$$\begin{aligned} z_{2n} &= Tz_{2n-1}(x) = T^{2n}z_0 = T^{2n}\left(\kappa^{\sigma}\eta(x)\right) = T^{2n}\left(\kappa^{2\sigma}\kappa^{-\sigma}\eta(x)\right) \\ &\geq T^{2n-2}\left(T^2\left(\kappa^{2\sigma}z_1(x)\right)\right) \geq T^{2n-2}\left(\left(\kappa^{2\sigma}\right)^{\mu^2(q-1)^2}T^2z_1(x)\right) \\ &= T^{2n-4}T^2\left(\left(\kappa^{2\sigma}\right)^{\mu^2(q-1)^2}T^2z_1(x)\right) \geq T^{2n-4}\left(\left(\kappa^{2\sigma}\right)^{\mu^4(q-1)^4}T^4z_1(x)\right) \\ &\geq \cdots \geq \left(\kappa^{2\sigma}\right)^{\mu^{2n}(q-1)^{2n}}T^{2n}z_1(x) = \left(\kappa^{2\sigma}\right)^{\mu^{2n}(q-1)^{2n}}T^{2n+1}z_0(x) \\ &= \left(\kappa^{2\sigma}\right)^{\mu^{2n}(q-1)^{2n}}z_{2n+1}, \end{aligned} \tag{3.20}$$

that is,

$$\left(\kappa^{2\sigma}\right)^{\mu^{2n}(q-1)^{2n}} z_{2n+1} \leq z_{2n} \leq z_{2n+1}. \tag{3.21}$$

Consequently, for all natural numbers n and p, one has

$$\begin{aligned} 0 &\leq z_{2(n+p)}(x) - z_{2n}(x) \leq z_{2n+1}(x) - z_{2n}(x) \leq \left(1 - \left(\kappa^{2\sigma}\right)^{\mu^{2n}(q-1)^{2n}}\right) z_{2n+1} \\ &\leq \left(1 - \left(\kappa^{2\sigma}\right)^{\mu^{2n}(q-1)^{2n}}\right) z_1 \leq \left(1 - \left(\kappa^{2\sigma}\right)^{\mu^{2n}(q-1)^{2n}}\right) \kappa^{-\sigma} \eta(x), \end{aligned} \tag{3.22}$$

and

$$0 \leq z_{2n+1}(x) - z_{2(n+p)+1}(x) \leq z_{2n+1}(x) - z_{2n}(x) \leq \left(1 - \left(\kappa^{2\sigma}\right)^{\mu^{2n}(q-1)^{2n}}\right) \kappa^{-\sigma} \eta(x). \tag{3.23}$$

It follows from (3.22), (3.23) and the fact that P is a normal cone with normality constant 1 that

$$\|z_{n+p} - z_n\| \leq \left(1 - \left(\kappa^{2\sigma}\right)^{\mu^{2n}(q-1)^{2n}}\right) \kappa^{-\sigma} \to 0, \quad n \to +\infty. \tag{3.24}$$

Since $\{z_n\} \in Q$ and $T(Q) \subset Q$ is compact, $\{z_n\}$ is a Cauchy sequence of compact set, and then $\{z_n\}$ converges to some $z^* \in Q$ with

$$z_{2n} \leq z^* \leq z_{2n+1}.$$

So

$$z_{2n+2} = Tz_{2n+1} \leq Tz^* \leq Tz_{2n} = z_{2n+1}. \tag{3.25}$$

Let $n \longrightarrow \infty$ in (3.25), we get $z^*(x) = Tz^*(x)$, which implies that z^* is a positive solution of Eq. (1.2).

Now we prove $z^* \in Q$ is unique. Let $\tilde{z}$ be another positive solution of Eq. (1.2). Take $r_1 = \sup\{r > 0 \mid \tilde{z} \geq rz^*\}$. Obviously, $0 < r_1 < +\infty$. We assert $r_1 \geq 1$. If not, we have $0 < r_1 < 1$, which leads to

$$\tilde{z} = T\tilde{z} = T^2\tilde{z} \geq T^2\left(r_1 z^*\right) \geq r_1^{\mu^2(q-1)^2} T^2 z^* = r_1^{\mu^2(q-1)^2} z^*.$$

Since $r_1^{\mu^2(q-1)^2} > r_1$, this contradicts with the definition of r_1. Hence $r_1 \geq 1$ and $\tilde{z} \geq z^*$. Similarly, we also have $\tilde{z} \leq z^*$. Therefore $\tilde{z} = z^*$, which implies that the positive solution of Eq. (1.2) is unique.

Step 3. At the end, we give the convergence analysis and error estimation for the unique solution of Eq. (1.2).

For any initial value $\omega_0 \in Q$, there exists a constant $l_{\omega_0} \in (0,1)$ such that

$$l_{\omega_0} \eta(x) \leq \omega_0(x) \leq l_{\omega_0}^{-1} \eta(x), \quad x \in [0,1].$$

Since $T(Q) \subset Q$, there still exists a constant $l_{\omega_1} \in (0,1)$ such that

$$l_{\omega_1} \eta(x) \leq \omega_1 = T\omega_0 \leq l_{\omega_1}^{-1} \eta(x), \quad x \in [0,1].$$

Choose sufficiently large $\widetilde{\sigma} > 2\sigma$ such that

$$\kappa^{\widetilde{\sigma}-\sigma} \leq \min\{l_{\omega_0}, l_{\omega_1}\},$$

where $\kappa \in (0,1)$ and $\sigma > 0$ are defined by (3.11). Thus

$$z_0 = \kappa^{\sigma}\eta(x) \leq \kappa^{\widetilde{\sigma}-\sigma}\eta(x) \leq l_{\omega_0}\eta(x) \leq \omega_0, \qquad z_0 = \kappa^{\sigma}\eta(x) \leq \kappa^{\widetilde{\sigma}-\sigma}\eta(x) \leq l_{\omega_1}\eta(x) \leq \omega_1,$$

which implies that $\omega_1 = T\omega_0 \leq Tz_0 = z_1$, and then

$$z_0 \leq \omega_1 \leq z_1. \tag{3.26}$$

Let

$$\omega_n = \int_0^1 H(x,y)\left(\int_0^1 K_\alpha(y,\tau)f\big(\tau,\omega_{n-1}(\tau)\big)\,d\tau\right)^{q-1} dy, \quad n = 1,2,3,\ldots, \tag{3.27}$$

it follows from (3.26) and (3.27) that

$$z_{2n}(x) \leq \omega_{2n}(x) \leq z_{2n+1}(x), \qquad z_{2n+2}(x) \leq \omega_{2n+1}(x) \leq z_{2n+1}(x). \tag{3.28}$$

Letting $n \to \infty$ in (3.28) and using (3.25), we get that ω_n uniformly converges to the unique positive solution z^* of Eq. (1.2).

Moreover, by (3.23) and (3.28), we have the following estimate of error:

$$\|\omega_n - z^*\| \leq \big(1 - (\kappa^{2\sigma})^{\mu^{2n}(q-1)^{2n}}\big)\kappa^{-\sigma} = \big(1 - \epsilon^{\mu^{2n}(q-1)^{2n}}\big)\epsilon^{-\frac{1}{2}}, \tag{3.29}$$

with an exact rate of convergence

$$\|\omega_n - z^*\| = o\big(1 - (\kappa^{2\sigma})^{\mu^{2n}(q-1)^{2n}}\big) = o\big(1 - \epsilon^{\mu^{2n}(q-1)^{2n}}\big),$$

where $0 < \epsilon = \kappa^{2\sigma} < 1$ is a positive constant which is determined by $z_0 = \kappa^{\sigma}\eta(x)$, that is, it is independent of the initial value ω_0.

At the end, it follows from $z^* \in Q$ that there exists a constant $0 < l_1 < 1$ such that

$$l_1 x^{\gamma-1} \leq z^*(x) \leq l_1^{-1} x^{\gamma-1}.$$

The proof is completed. □

4 Example

Now we recall the singular nonlocal fractional differential Eq. (1.1). By simple computation, we get that Eq. (1.1) is equivalent to the following 4-point boundary value problem:

$$\begin{cases} -\mathcal{D}_x^{\frac{3}{2}}(\varphi_{\frac{3}{2}}(-\mathcal{D}_x^{\frac{4}{3}}z))(x) = x^{-1}z^{-\frac{1}{2}}, \quad 0 < x < 1, \\ z(0) = 0, \qquad \mathcal{D}_x^{\frac{4}{3}}z(0) = \mathcal{D}_x^{\frac{4}{3}}z(1) = 0, \qquad z(1) = \frac{1}{2}z(\frac{1}{3}) + \frac{1}{2}z(\frac{2}{3}). \end{cases} \tag{4.1}$$

In the following, we shall verify that Eq. (1.1) satisfies all conditions of Theorem 3.1. Let $\alpha = \frac{3}{2}$, $\gamma = \frac{4}{3}$, $p = \frac{3}{2}$ and

$$f(x,z) = x^{-1}z^{-\frac{1}{2}},$$

then $f \in C((0,1) \times [0,\infty), [0,+\infty))$, and for any fixed $x \in (0,1)$, $f(x,z)$ is nondecreasing in z.

Take $\mu = \frac{2}{3}$, then $0 < \mu < \frac{1}{p-1} = 2$. For any $r \in (0,1)$ and $(x,z) \in (0,1) \times (0,+\infty)$, we have

$$f(x,rz) = r^{-\frac{1}{2}}x^{-1}z^{-\frac{1}{2}} \leq r^{-\frac{2}{3}}x^{-1}z^{-\frac{1}{2}} = r^{-\frac{2}{3}}f(x,z). \tag{4.2}$$

Thus condition (F_1) is satisfied.

Next we verify condition (K_0). In fact, since

$$K_{\frac{4}{3}}(x,y) = \frac{1}{\Gamma(\frac{4}{3})}\begin{cases} [x(1-y)]^{\frac{1}{3}}, & 0 \leq x \leq y \leq 1, \\ [x(1-y)]^{\frac{1}{3}} - (x-y)^{\frac{1}{3}}, & 0 \leq y \leq x \leq 1, \end{cases} \tag{4.3}$$

we have

$$\mathcal{K}_\chi(y) = \int_0^1 K_{\frac{4}{3}}(x,y)\,d\chi(x) = \frac{1}{2}K_{\frac{4}{3}}\left(\frac{1}{3},y\right) + \frac{1}{2}K_{\frac{4}{3}}\left(\frac{2}{3},y\right) \geq 0, \quad \text{for all } y \in [0,1],$$

and

$$\mathcal{L} = \int_0^1 x^{\gamma-1}\,d\chi(x) = \frac{1}{2}\left(\frac{2}{3}\right)^{\frac{1}{3}} + \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{3}} = 0.7835 < 1.$$

So condition (K_0) is also satisfied.

Now we check condition (3.1). In fact, substituting $f(x,z) = x^{-1}z^{-\frac{1}{2}}$ into (3.1), we get

$$0 < \int_0^1 x(1-x)^{\alpha-1}f\left(x,x^{\gamma-1}\right)dx = \int_0^1 x^{-\frac{1}{6}}(1-x)^{\frac{1}{2}}\,dx < +\infty,$$

which implies that (3.1) holds. Thus, according to Theorem 3.1, we have the following conclusions:

(i) the p-Laplacian fractional differential Eq. (1.1) has a unique positive solution $z^* \in C[0,1]$;

(ii) for any initial value $z_0 \in Q$, the sequence of functions $\{z_n\}_{n\geq 1}$ defined by

$$\begin{aligned} z_n = {} & \int_0^1 \left[\frac{x^{\frac{1}{3}}}{0.2635}\left(\frac{1}{2}K_{\frac{4}{3}}\left(\frac{1}{3},y\right) + \frac{1}{2}K_{\frac{4}{3}}\left(\frac{2}{3},y\right)\right) + K_{\frac{4}{3}}(x,y)\right] \\ & \times \left(\int_0^1 K_{\frac{3}{2}}(y,\tau)\tau^{-1}z_n^{-\frac{1}{2}}(\tau)\,d\tau\right)^2 dy, \\ & n = 1,2,3,\ldots, \end{aligned}$$

converges uniformly to the unique positive solution z^* of Eq. (1.1) on [0,1];

(iii) the error between the iterative value z_n and the exact solution z^* can be estimated by

$$\left\|z_n - z^*\right\| \leq \left(1 - \epsilon^{[\frac{4}{3}]^{2n}}\right)\epsilon^{-\frac{1}{2}},$$

and the convergence rate can be formulated by

$$\left\|z_n - z^*\right\| = o\left(1 - \epsilon^{[\frac{4}{3}]^{2n}}\right),$$

where $0 < \epsilon < 1$ is a positive constant which is determined by the fixed function $\kappa^{\sigma} x^{\frac{4}{3}}$;

(iv) there exists a constant $0 < l < 1$ such that the exact solution z^* of Eq. (1.1) intervenes between two known curves $lx^{\frac{2}{3}}$ and $l^{-1}x^{\frac{2}{3}}$, i.e.,

$$lx^{\frac{2}{3}} \leq z^*(x) \leq l^{-1}x^{\frac{2}{3}}, \quad x \in [0,1].$$

5 Conclusion

In this paper, by introducing a double iterative technique, we established the convergence analysis and error estimation for the unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity. The equation we studied in the present paper exhibits a blow-up behaviour at time and space variables, which occurs in many complex physical processes, such as mechanics processes, the convection-diffusion process and the bioprocess with long memory. The developed double iterative technique can be applied for solving the case where the nonlinear term is decreasing and has singularity at time and space variables.

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