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Existence and quantum calculus of weak solutions for a class of two-dimensional Schrödinger equations in $\mathbb{C}_+$

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Abstract
The aim of this paper is to investigate the existence of weak solutions for a two-dimensional Schrödinger equation with a singular potential in $\mathbb{C}_+$. Under appropriate assumptions on the nonlinearity, we introduce a new type of quantum calculus via the Morse theory and variational methods. By applying Schrödinger type inequalities and the well-known Banach fixed point theorem in conjunction with the technique of measures of weak noncompactness, the new and more accurate estimations for boundary behaviors of them are also deduced.

Keywords: Two-dimensional Schrödinger equation; Schrödinger type inequality; Boundary behavior; Singular solution

1 Introduction

In this paper, we study the following two-dimensional Schrödinger equation (see [1]):

$$\begin{aligned} &\mathfrak{L}u = \partial_t^2 u - x\partial_x^2 u + C_1\partial_x u - C_2\left(t^2 - 4x\right)^{-1} u = 0, \\ &u_t(0,x) = u_1(x), \\ &u(0,x) = u_0(x), \end{aligned} \tag{1}$$

in the upper half plane $\mathbb{C}_+ = \{z = t + ix : x > 0\}$, where the variables t and x are complex numbers in $\mathbb{C}_+$ and C_1 and C_2 are real numbers. Our first aim is to construct the solution in terms of hypergeometric functions.

Let a be a real number, $p > 1$ and $C_2 = ap$. Then Eq. (1) arises naturally by linearizing the Klein–Gordon equation variable boundary (see [2]):

$$\partial_t^2 u - x\partial_x^2 u + C_1\partial_x u = au^p,$$

in $\mathbb{C}_+$, which shows that

$$u = \left(t^2 - 4x\right)^{\frac{1}{1-p}}.$$

In general, the study of the solutions of the two-dimensional Schrödinger and their related properties are very complicated; especially if the roots of the characteristic poly-

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nomial are double and not analytic at the origin. The explicit difficulties in dealing with quadratic-type non-linearities in two dimensions are our inability to use the Strichartz inequalities.

However, many authors have showed that the solution of the two-dimensional Schrödinger equations can be expressed by using a variational inequality. In recent years, various extensions and generalizations of the classical variational inequality models and complementarity problems have emerged in quantum and fluid mechanics, nonlinear programming, physics, optimization and control, economics, transportation, finance, structural, elasticity and applied sciences (see [3–7] and the references therein for details).

The classical Schrödinger solution spaces $\mathfrak{H}^p(\mathbb{C}_+)$ (see [8]), are defined to consist of solutions of (1), holomorphic in $\mathbb{C}_+$ with the property that $\mathcal{M}_p(u,x)$ is uniformly bounded for $x > 0$, where

$$\mathcal{M}_p(u,x) = \left(\int_{-\infty}^{+\infty} \left|u(t+ix)\right|^p dt\right)^{\frac{1}{p}}.$$

Since $|u|^p$ is the weak solution of (1) for $u \in \mathfrak{H}^p(\mathbb{C}_+)$ (see [9]), the solution $\mathcal{M}_p(u,x)$ decreases in $(0,\infty)$,

$$\|u\|_{\mathfrak{H}^p(\mathbb{C}_+)} = sup\left\{\mathcal{M}_p(u,y) : 0 < x < \infty\right\} = \lim_{x\to 0} \mathcal{M}_p(u,x).$$

Define

$$\|u\|_p = \left(\int_{-\infty}^{+\infty} \left|u(t)\right|^p dt\right)^{\frac{1}{p}} = \|u\|_{\mathfrak{H}^p(\mathbb{C}_+)}.$$

We remark that $\phi^{(k)}(t) \in \mathcal{L}^p$ and $\phi(t) \in \mathcal{C}^\infty$ if and only if $\phi(t)$ belongs to the space $\mathcal{D}_{\mathcal{L}^p}$ (see [6]). Let $\mathcal{F}$ denote the space, which consists of infinitely differentiable weak solution of (1) in $\mathbb{C}_+$. Let $\mathcal{F}'_{\mathcal{L}^p}$ denote the dual of the space $\mathcal{F}_{\mathcal{L}^q}$, that is, $\mathcal{F}'_{\mathcal{L}^p} = (\mathcal{F}_{\mathcal{L}^q})'$. We also denote $q = \frac{p}{p-1}$ and by D' the dual of the space D. So we can get $D \subseteq \mathcal{F}_{\mathcal{L}^p}$ and $\mathcal{F}'_{\mathcal{L}^p} \subseteq D'$.

Definition 1.1 (see [10]) If $u \in \mathcal{F}'$, then it has the following representation:

$$\lim_{x\to 0^+} \int_{-\infty}^{+\infty} \left[g(t+ix) - g(t-ix)\right]\phi(t)\,dt = \left\langle u(t), \phi(t)\right\rangle$$

for any test function $\phi \in \mathcal{F}$ and any function $g(z)$ in $\mathbb{C}_+$, where $g(z)$ is analytic on the complement of the support of u.

Definition 1.2 (see [9]) Let Du be the Stokes operator defined by

$$\langle Du, \phi\rangle = \langle u, -D\phi\rangle$$

on $\mathcal{F}'_{\mathcal{L}^p}$ for all $\phi \in \mathcal{F}_{\mathcal{L}^q}$.

It is obvious that

$$Du \in \mathcal{F}'_{\mathcal{L}^p},$$

where $u \in \mathcal{F}'_{\mathcal{L}^p}$.

Since $u \in \mathcal{F}'_{\mathcal{L}^p}$, $D\varphi \in D_{\mathcal{L}^{p'}}$, Du defined as above is a functional on $D_{\mathcal{L}^{p'}}$. Linearity of Du is nontrivial. If $\{\varphi_v\} \to \varphi$ in $D_{\mathcal{L}^{p'}}$, then it is easy to see that

$$\langle Du, \varphi_v \rangle = \langle u, -D\varphi_v \rangle \quad \to \quad \langle u, -D\varphi \rangle = \langle Du, \varphi \rangle.$$

2 Construction of the solutions

By virtue of the weak maximum principle of superposition, it is necessary to consider the following Riemann problems:

$$\begin{aligned} &\Re\vartheta = 0, \\ &\vartheta(0,x) = x^l, \\ &\vartheta_t(0,x) = 0, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\Re\varsigma = 0, \\ &\varsigma(0,x) = 0, \\ &\varsigma_t(0,x) = x^l. \end{aligned} \tag{3}$$

First we solve (2). Put $x = \tau_1$ and $4x - t^2 = \tau_2$ in Eqs. (2) and (3). Let

$$\vartheta(t,x) = \vartheta_l(t,x) = \tau_1^l \vartheta(z),$$

where

$$4\tau_1 z = 4\tau_1 - \tau_2.$$

Substituting $\tau_1^l \vartheta(z)$ with ϑ, it follows that $\Re\vartheta = 0$, from which one concludes that

$$\begin{aligned} \Re\vartheta = {}& \frac{1}{2}\tau_1^{l-1}\vartheta' + z\tau_1^{l-1}\vartheta'' - z^2\tau_1^{l-1}\vartheta'' - 2z\tau_1^{l-1}\vartheta' + lz\tau_1^{l-1}\vartheta' \\ & + zl\tau_1^{l-1}\vartheta' - l(l-1)\tau_1^{l-1}V + C\big(l\tau_1^{l-1}V - z\tau_1^{l-1}\vartheta'\big) + \frac{B}{\tau_2}\tau_1^l V = 0. \end{aligned}$$

By a simple calculation, we know that

$$\begin{aligned} & x(1-x)\vartheta'' + \left(\frac{1}{2} - \big(C + 2(1-l)\big)z\right)\vartheta' \\ & \quad + \left(l(C-l+1) - \frac{1}{4}B\left(\frac{1}{x-1}\right)\right)\nu = 0. \end{aligned} \tag{4}$$

By replacing $\frac{\tau_1}{\tau_2}$ by $\frac{1}{4(1-x)}$, it is obvious that

$$x(1-x)\Pi'' + \left(\frac{1}{2} - \big(C + 2(1-l+\sigma)\big)z\right)\Pi' + \sigma\Pi = 0, \tag{5}$$

which is equivalent to a hyperbolic–parabolic differential equation with

$$\left(\sigma - l, C + \sigma - l + 1, \frac{1}{2}\right),$$

iff

$$\sigma\left(2l - \sigma - C - \frac{1}{2}\right) = \frac{1}{4}B. \tag{6}$$

It follows from the hypergeometric equation theory that the first and the second solutions for the hyperbolic–parabolic equation are

$$\Pi_1(z) = g(a, b; c; z) = F\left(\sigma - l, C + \sigma - l + 1; \frac{1}{2}; z\right)$$

and

$$\Pi_2(z) = z^{1/2} F\left(\sigma - l + \frac{1}{2}, C + \sigma - l + \frac{3}{2}, \frac{3}{2}; z\right),$$

respectively.

Let $z = \frac{t^2}{4x}$, where $|z| < 1$. It is easy to see that a complete solution of the hyperbolic–parabolic equation is

$$\begin{aligned}\Pi = {} & F\left(\sigma - l, C + \sigma - l + 1, \frac{1}{2}; z\right) \\ & + E z^{1/2} F\left(\sigma - l + \frac{1}{2}, C + \sigma - l + \frac{3}{2}, \frac{3}{2}; z\right).\end{aligned} \tag{7}$$

So $\vartheta = x^l (1 - x)^\sigma x$ is a solution of $\Re\vartheta = 0$. Notice that

$$\vartheta(0, z) = z^l \quad \text{and} \quad \vartheta_t(0, z) = 0,$$

which immediately shows that

$$\vartheta = \vartheta_{l\sigma} = z^l \left(1 - \frac{t^2}{4z}\right)^\sigma F\left(\sigma - l, C + \sigma - l + 1, \frac{1}{2}; \frac{t^2}{4z}\right). \tag{8}$$

Similarly, we can solve the problem (2) by letting

$$\varsigma(t, x) = t x^l \left(4x - 4xt^2\right)^{\sigma'} y,$$

which shows that

$$\begin{aligned}\varsigma(t, x) &= \varsigma_{l,\sigma'}(t, x) \\ &= t(4)^{-\sigma'} x^{l-\sigma'} \left(4x - t^2\right)^{\sigma'} F\left(\sigma' - l, C + \sigma' - l + 1, \frac{3}{2}; \frac{t^2}{4x}\right).\end{aligned} \tag{9}$$

3 Boundary behaviors

Theorem 3.1 *If* $u \in \mathcal{F}'_{\mathcal{L}^p}$, *then*

$$2\pi i g(z) = \langle u(t), (t-z)^{-1} \rangle$$

is one of the representations of the solution u such that

$$\sup_{t\in\mathbb{R}, t\geq\delta>0} \| g(t+ix) \| = A_\delta < \infty$$

and

$$\sup_{t\in\mathbb{R}} \| g(t+ix) \| = O\bigl(t^{-\frac{1}{p}}\bigr),$$

where $x \to \infty$ *and there exists a function* $G_k(z) \in \mathfrak{H}^p(\mathbb{C}_+)$ *such that*

$$g(z) = \sum_{l=1}^{r} \frac{\partial^{l-1} G_k(z)}{\partial z^{l-1}} \tag{10}$$

and

$$G^{(j)}(z) = \sum_{l=1}^{r} \frac{\partial^{j+l-1} G_k(z)}{\partial z^{j+l-1}}.$$

Theorem 3.2 *If g is defined in* (10) *and* $G_k \in \mathfrak{H}^p(\mathbb{C}_+)$, *then there exists a Schrödinger distributional solution* $u(t) \in \mathcal{F}'_{\mathcal{L}^p}$ *such that* $g(z)$ *is one of the analytic representations of u.*

Corollary 1 *If* $u(t) \in \mathcal{F}'_{\mathcal{L}^p}$, *then*

$$2\pi i g(z) = \langle u(t), (t-z)^{-1} \rangle$$

satisfies

$$\sup_{t\in\mathbb{R}, t\geq\delta>0} \| g(t+ix) \| = C_\delta < \infty$$

and

$$\sup_{t\in\mathbb{R}} \| g(t+ix) \| = O\bigl(t^{-\frac{1}{p}}\bigr),$$

as $x \to \infty$ *and there exists a function* G_k *in* $\mathfrak{H}^p(\mathbb{C}_+)$ *such that*

$$g = \sum_{l=1}^{r} \frac{\partial^{j+l-1} G_k}{\partial z^{j+l-1}}.$$

4 Lemmas

The following lemmas are required in this section.

Lemma 4.1 (see [11, p. 69]) *If $u \in \mathcal{L}^p(\mathbb{R})$ and G is defined by*

$$2\pi i G(u)(t) = \int_{-\infty}^{\infty} u(t)(t-z)^{-1}\,dt,$$

then

$$G(u) \in \mathfrak{H}^p(\mathbb{C}_+).$$

Lemma 4.2 (see [11, p. 77]) *Let $g(z)$ be any weak solution of Eq.* (1) *such that the following properties hold.*

(I) $g(t+ix) \in \mathcal{L}^p$ *for any fixed* $x > 0$;

(II)

$$\lim_{x\to 0^+} g(t+ix) = g^+(t)$$

in $\mathcal{F}'_{\mathcal{L}^p}$ (weakly),

$$\sup_{t\in\mathbb{R}} \bigl\| g(t+ix) \bigr\| \to O,$$

as $x \to \infty$ and

$$\sup_{t\in\mathbb{R}, t\geq\delta>0} \bigl\| g(t+ix) \bigr\| = C_\delta < \infty.$$

Then

$$2\pi i g(z) = \bigl\langle g^+(t), (t-z)^{-1} \bigr\rangle,$$

where $\operatorname{Im} z > 0$.

5 Proofs of main results

5.1 Proof of Theorem 3.1

By virtue of the fixed point theorem with respect to the stationary Schrödinger operator in [8], we have

$$2\pi i g(z) = \bigl\langle u^+(t), (t-z)^{-1} \bigr\rangle$$

for any $u \in \mathcal{F}'_{\mathcal{L}^p}$, which shows that

$$\begin{aligned} 2\pi i g(z) &= \sum_{l=1}^{r} \int_{\mathbb{R}} u_l(t) \left(-\frac{\partial}{\partial t}\right)^{(l-1)} (t-z)^{-1}\,dt \\ &= \sum_{l=1}^{r} \int_{\mathbb{R}} u_l(t)(-1)^{l-1}(l-1)!(t-z)^{-l}\,dt, \end{aligned}$$

where r is a nonnegative integer and $u_l \in L_p$.

So

$$2\pi \left|g(t+ix)\right| \leq \sum_{l=1}^{r} \int_{\mathbb{R}} \left|u_l(t)\right| (l-1)!(t-z)^{-l}\,dt,$$

which shows that

$$\left|g(t+ix)\right| \leq \frac{1}{2\pi} \sum_{l=1}^{r} (l-1)! \left(\int_{\mathbb{R}} \left|u_l(t)\right|^p dt \right)^{\frac{1}{p}} \left(\int_{R} (t-z)^{-lq}\,dt \right)^{\frac{1}{q}}$$

from the Hölder inequality.

Put

$$I = \int_{\mathbb{R}} (t-z)^{-lq}\,dt.$$

So

$$\begin{aligned} I &= \int_{\mathbb{R}} \frac{1}{[(t-x)^2+t^2])^{\frac{lq}{2}}}\,dt \\ &= \int_{\mathbb{R}} \frac{1}{t^{lq}[(\frac{t}{y})^2+1]^{\frac{lq}{2}}}\,dt \\ &= \frac{1}{t^{lq-1}} \int_{\mathbb{R}} \frac{1}{(1+t^2)^{\frac{lq}{y}}}\,dt, \end{aligned}$$

which yields

$$I \leq \frac{C}{\delta^{lq-1}} < \infty,$$

where C is a positive constant.

Since $u_l \in \mathcal{L}^p$, then

$$\sup_{t\in\mathbb{R}, t\geq\delta>0} \left\|g(t+ix)\right\| = C_\delta < \infty,$$

where M is a positive constant,

$$\begin{aligned} 2\pi A_\delta &= \sum_{l=1}^{r} (l-1)! \frac{MC^{\frac{1}{q}}}{\frac{k\prime-1}{\delta q}}, \\ \left|t^{\frac{1}{p}} g(t+ix)\right| &\leq \sum_{l=1}^{r} (l-1)! \|u_l\|_L^p \frac{1}{t^{lq-1-\frac{1}{p}}} \int_{\mathbb{R}} \frac{1}{(1+t^2)^{\frac{lq}{x}}}\,dt \end{aligned}$$

and

$$lq - 1 - \frac{1}{p} = p^2(1-l) - 1 < 0.$$

So

$$\lim_{x\to\infty}\sup_{t\in\mathbb{R}}\bigl|g(t+ix)\bigr| = O\bigl(t^{-\frac{1}{p}}\bigr).$$

By virtue of the structure formula, we have

$$\begin{aligned}
2\pi ig(z) &= \sum_{l=1}^{r}\int_{\mathbb{R}} u_l(t)\biggl(-\frac{\partial}{\partial t}\biggr)^{(l-1)}(t-z)^{-1}\,dt \\
&= \sum_{l=1}^{r}\int_{\mathbb{R}} u_l(t)\biggl(\frac{\partial}{\partial z}\biggr)^{(l-1)}(t-z)^{-1}\,dt \\
&= \sum_{l=1}^{r}\biggl(\frac{\partial}{\partial z}\biggr)^{(l-1)}\int_{\mathbb{R}}(t-z)^{-1}u_l(t)\,dt \\
&= \sum_{l=1}^{r}\biggl(\frac{\partial}{\partial z}\biggr)^{(l-1)}G_k(z),
\end{aligned}$$

where

$$2\pi iG_k(z) = \int_{\mathbb{R}} u_l(t)(t-z)^{-1}\,dt.$$

So we obtain $G_k(z)\in\mathfrak{H}^p(C_+)$ from Lemma 4.1, which shows that

$$G^{(j)} = \sum_{l=1}^{r}\biggl(\frac{\partial}{\partial z}\biggr)^{(k+j-1)}G_k.$$

5.2 Proof of Theorem 3.2

Since $G_k(z)\in\mathfrak{H}^p(C_+)$, where $G_k(t+ix)\in\mathcal{L}^p$ for fixed x, there exists the solution $u_l(t)\in\mathcal{L}^p$, where u_l is the nontangential limit of $g(z)$.

Since $D_{\mathcal{L}^q}\in\mathcal{L}^q$, we see that $u_l(t)\in D'_{\mathcal{L}^p}$ and

$$\begin{aligned}
\bigl|G_k(t+ix)\bigr|^p &\le \frac{1}{\pi t^2}\int_{D(t+ix,x)}\bigl|G_k(\tau+i\eta)\bigr|^p\,d\lambda \\
&\le \frac{1}{\pi t^2}\int_{t-x}^{t+x}\int_0^{2x}\bigl|G_k(\tau+i\eta)\bigr|^p\,d\eta\,d\zeta \\
&\le \frac{2}{\pi x}\|G_k\|_{\mathfrak{H}^p}^p.
\end{aligned}$$

So

$$\bigl|G_k(t+ix)\bigr| \le \biggl(\frac{1}{x}\|G_k\|_{\mathfrak{H}^p}^p\biggr)^{\frac{1}{p}} = t^{\frac{1}{p}}\bigl(\|G_k\|_{\mathfrak{H}^p}^p\bigr)^{\frac{1}{p}},$$

which shows that

$$G_k(t+ix) = O\bigl(t^{-\frac{1}{p}}\bigr),$$

where $x > 0$ and

$$\sup_{t\in\mathbb{R},t\geq\delta>0}\|g(t+ix)\| \leq \frac{1}{\delta}\|G_k\|^p_{\mathfrak{H}^p} = C_\delta < \infty.$$

We know that $G_k(z)$ can be represented as follows:

$$2\pi iG_k(t) = \langle u_l(t),(t-z)^{-1}\rangle,$$

from Lemma 4.2, which yields

$$\begin{aligned}2\pi ig(z) &= \sum_{l=1}^{r}\left(\frac{\partial}{\partial z}\right)^{l-1}G_k(z)\\ &= \sum_{l=1}^{r}\left(\frac{\partial}{\partial z}\right)^{l-1}\langle u_l(t),(t-z)^{-1}\rangle\\ &= \sum_{l=1}^{r}\left\langle u_l(t),\left(\frac{\partial}{\partial z}\right)^{l-1}(t-z)^{-1}\right\rangle\\ &= \sum_{l=1}^{r}\langle D^{(l-1)}u_l(t),(t-z)^{-1}\rangle\\ &= \left\langle\sum_{l=1}^{r}D^{(l-1)}u_l(t),(t-z)^{-1}\right\rangle.\end{aligned}$$

Put

$$u = \sum_{l=1}^{r}D^{(l-1)}u_l,$$

where $u \in D'_{\mathcal{L}^p}$, which shows that $g(z)$ is one of the analytic representations of u.

5.3 Proof of Corollary 1

By virtue of the fixed point theorem with respect to the stationary Schrödinger operator in [8], we know that

$$\begin{aligned}2\pi ig(z) &= \langle u(t),(t-z)^{-j}\rangle\\ &= \sum_{l=1}^{r}\int_{\mathbb{R}}u_l(t)\left(-\frac{\partial}{\partial t}\right)^{(l-1)}(t-z)^{-j}\,dt\\ &= \sum_{l=1}^{r}\int_{\mathbb{R}}u_l(t)\frac{(l+j-2)!}{(j-1)!(t-z)^{l+j-1}}\,dt.\end{aligned}$$

The rest of the proof of the corollary is similar to the proof of Theorem 3.1. So we omit the details here for the sake of brevity.

The proof of Corollary 1 is complete.

6 Conclusions

In this paper, we investigated the existence of weak solutions for a two-dimensional Schrödinger equations with a singular potential in $\mathbb{C}_+$. Under appropriate assumptions on the nonlinearity, we introduced a new type of quantum calculus via the Morse theory and variational methods. By applying the well-known Banach fixed point theorem in conjunction with the technique of measures of weak noncompactness, new and more accurate estimations for boundary behaviors of them were also deduced. We significantly extended and complemented some results from the current literature.

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