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Positive solution for a fractional singular boundary value problem with p-Laplacian operator

Fengli Yan, Mingyue Zuo and Xinan Hao*

*Correspondence:
haoxinan2004@163.com
School of Mathematical Sciences,
Qufu Normal University, Qufu,
P.R. China

Abstract

In this paper, we consider a fractional singular three-point boundary value problem with p-Laplacian operator. The nonlinearity $f(t,u)$ may be singular at $t=0,1$ and $u=0$. Some properties of the associated Green function are obtained. By using the upper and lower solutions method and a fixed point theorem, the existence result of positive solution is established.

Keywords: Positive solution; Fractional singular BVP; p-Laplacian operator; Upper and lower solutions method

1 Introduction

In this paper, we investigate the following fractional three-point boundary value problem (BVP) with p-Laplacian operator:

$$\begin{cases} -D_{0^+}^{\alpha}(\varphi_p(D_{0^+}^{\beta}u(t))) = f(t,u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \qquad D_{0^+}^{\beta}u(0) = 0, \qquad D_{0^+}^{\beta}u(1) = bD_{0^+}^{\beta}u(\eta), \end{cases} \tag{1.1}$$

where $\alpha \in (1,2]$, $\beta \in (3,4]$, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the standard Riemann–Liouville derivatives, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta \in (0,1)$, $b \in (0, \eta^{\frac{1-\alpha}{p-1}})$, $f(t,u) : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous and may be singular at $t = 0,1$ and $u = 0$.

The differential equations with p-Laplacian operator have deep background in physics. In recent years, boundary value problems of fractional differential equations with or without p-Laplacian operator have been widely studied. By means of nonlinear analysis theory and methods, many existence and multiplicity results of solutions or positive solutions have been obtained, see [1–29] and the references therein.

In [11], Xu and Dong considered three-point BVP (1.1), but their nonlinearity $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, the existence and uniqueness of positive solutions were obtained by using the upper and lower solutions method and Schauder's fixed point theorem.

By means of the lower and upper solutions method and monotone iterative technique, Liu et al. [12] investigated the existence of positive solutions for mixed fractional BVP with

p-Laplacian operator

$$\begin{cases} D_{0^+}^{\alpha}(\varphi_p({}^cD_{0^+}^{\beta}u(t))) = f(t,u(t),{}^cD_{0^+}^{\beta}u(t)), & 0<t<1, \\ {}^cD_{0^+}^{\beta}u(0) = u'(0) = 0, \quad u(1) = r_1u(\eta), \quad {}^cD_{0^+}^{\beta}u(1) = r_2^c D_{0^+}^{\beta}u(\xi), \end{cases}$$

where $\alpha,\beta \in (1,2]$, $D_{0^+}^{\alpha}$ and ${}^cD_{0^+}^{\beta}$ are the Riemann–Liouville fractional derivative and Caputo fractional derivative, respectively.

By using upper and lower solutions method, Wang and Xiang [13] established existence results of positive solution for a fractional BVP with p-Laplacian operator

$$\begin{cases} D_{0^+}^{\alpha}(\varphi_p(D_{0^+}^{\beta}u(t))) = f(t,u(t)), & 0<t<1, \\ u(0) = 0, \quad u(1) = au(\xi), \quad D_{0^+}^{\beta}u(0) = 0, \quad D_{0^+}^{\beta}u(1) = bD_{0^+}^{\beta}u(\eta), \end{cases}$$

where $\alpha,\beta \in (1,2]$, $a,b \in (0,1]$, $\xi,\eta \in (0,1)$, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Riemann–Liouville fractional derivatives.

In [17], Zhang et al. studied the integral BVP of fractional differential equations with parameter and p-Laplacian operator

$$\begin{cases} -D_{0^+}^{\alpha}(\varphi_p(D_{0^+}^{\beta}u(t))) = \lambda f(t,u(t), & 0<t<1, \\ u(0) = 0, \quad D_{0^+}^{\beta}u(0) = 0, \quad u(1) = \int_0^1 u(s)dA(s), \end{cases}$$

where $\alpha \in (0,1]$, $\beta \in (1,2]$, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Riemann–Liouville fractional derivatives, $\int_0^1 u(s)dA(s)$ is the Riemann–Stieltjes integral, $f(t,u) : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous.

Motivated by the papers mentioned above, in this paper, we study the p-Laplacian fractional differential equation three-point BVP (1.1). The existence of positive solution is obtained by using the upper and lower solutions method and a fixed point theorem. It is worth pointing out that $f(t,u)$ may be singular at $t = 0,1$ and $u = 0$.

2 Preliminaries and lemmas

Let $\varphi_p(D_{0^+}^{\beta}u(t)) = v(t)$, then $v(0) = 0$, $v(1) = b^{p-1}v(\eta)$. We now consider the following BVP:

$$\begin{cases} -D_{0^+}^{\alpha}v(t) = y(t), & 0<t<1, \\ v(0) = 0, \quad v(1) = b^{p-1}v(\eta). \end{cases} \tag{2.1}$$

Lemma 2.1 ([11]) *If $y \in C[0,1]$, then BVP* (2.1) *has a unique solution*

$$v(t) = \int_0^1 H(t,s)y(s)\,ds,$$

where

$$H(t,s) = h(t,s) + \frac{b^{p-1}t^{\alpha-1}}{1-b^{p-1}\eta^{\alpha-1}}h(\eta,s),$$

$$h(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}, & 0\le t\le s\le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1. \end{cases}$$

From the above analysis, the BVP

$$\begin{cases} -D_{0^+}^{\alpha}(\varphi_p(D_{0^+}^{\beta}u(t))) = y(t), \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \qquad D_{0^+}^{\beta}u(0) = 0, \qquad D_{0^+}^{\beta}u(1) = bD_{0^+}^{\beta}u(\eta) \end{cases}$$

is equal to

$$\begin{cases} D_{0^+}^{\beta}u(t) = \varphi_q(\int_0^1 H(t,s)y(s)\,ds), \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \tag{2.2}$$

Lemma 2.2 ([30]) *If $y \in C[0,1]$, BVP* (2.2) *has a unique solution*

$$u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right) ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-2}(1-s)^{\beta-2}[(s-t) + (\beta-2)(1-t)s], & 0 \le t \le s \le 1, \\ t^{\beta-2}(1-s)^{\beta-2}[(s-t) + (\beta-2)(1-t)s] + (t-s)^{\beta-1}, & 0 \le s \le t \le 1. \end{cases}$$

Lemma 2.3 *The functions $H, G \in C([0,1] \times [0,1], [0,+\infty))$ have the following properties*:

(1)

$$H(t,s) \le d_1(1-s)^{\alpha-1}, \quad t,s \in (0,1),$$

where $d_1 = \frac{1}{(1-b^{p-1}\eta^{\alpha-1})\Gamma(\alpha)}$.

(2)

$$(\beta-2)k(t)q(s) \le \Gamma(\beta)G(t,s) \le M_0 q(s), \quad t,s \in (0,1),$$

where

$$k(t) = t^{\beta-2}(1-t)^2, \qquad q(s) = s^2(1-s)^{\beta-2}, \qquad M_0 = \max\{\beta-1, (\beta-2)^2\}.$$

(3)

$$G(t,s) \ge \frac{\beta-2}{M_0}t^{\beta-2}(1-t)^2 G(t_0,s), \quad t,s,t_0 \in (0,1).$$

Proof

(1) For any $t,s \in (0,1)$,

$$h(t,s) \le \frac{1}{\Gamma(\alpha)}\big[t(1-s)\big]^{\alpha-1} \le \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)},$$

then

$$h(\eta,s) \le \frac{1}{\Gamma(\alpha)}\big[\eta(1-s)\big]^{\alpha-1}.$$

Therefore

$$\begin{aligned}H(t,s) &\leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b^{p-1}}{1-b^{p-1}\eta^{\alpha-1}} \frac{\eta^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-b^{p-1}\eta^{\alpha-1})} = d_1(1-s)^{\alpha-1}.\end{aligned}$$

(2) See Lemma 2.4(2) of [30].

(3) For any $t, s, t_0 \in (0,1)$, we have

$$\begin{aligned}G(t,s) &\geq \frac{(\beta-2)k(t)q(s)}{\Gamma(\beta)} \\ &= \frac{(\beta-2)k(t)}{M_0\Gamma(\beta)} M_0 q(s) \\ &\geq \frac{(\beta-2)k(t)}{M_0\Gamma(\beta)} \Gamma(\beta) G(t_0,s) \\ &= \frac{\beta-2}{M_0} t^{\beta-2}(1-t)^2 G(t_0,s).\end{aligned}$$

This completes the proof. □

Remark 2.1 By Lemmas 2.2 and 2.3, if $D_{0^+}^{\beta} u \geq 0$ and $u(0) = u(1) = u'(0) = u'(1) = 0$, we conclude that $u(t) \geq 0$, $t \in [0,1]$.

u is said to be a lower solution for BVP (1.1) if u satisfies the following inequality system:

$$\begin{cases} -D_{0^+}^{\alpha}(\varphi_p(D_{0^+}^{\beta} u(t))) \leq f(t,u(t)), & 0 < t < 1, \\ u(0) \leq 0, \quad u(1) \leq 0, \quad u'(0) \leq 0, \quad u'(1) \leq 0, \\ -D_{0^+}^{\beta} u(0) \leq 0, \quad -D_{0^+}^{\beta} u(1) \leq -bD_{0^+}^{\beta} u(\eta). \end{cases}$$

Similarly, we define the upper solution for BVP (1.1) by replacing "least or equal" by "greater or equal".

3 Main result

Theorem 3.1 *Assume that the following conditions* (H_1)–(H_3) *are satisfied*:

(H_1) $f(t,u) \in C((0,1) \times (0,+\infty), [0,+\infty))$ *and* $f(t,u)$ *is nonincreasing relative to* u.

(H_2) *For any constant* $\lambda > 0$,

$$0 < \int_0^1 (1-s)^{\alpha-1} f\bigl(s, \lambda s^{\beta-2}(1-s)^2\bigr)\,ds < +\infty.$$

(H_3) *There exist a function* $a \in C[0,1]$ *and a constant* $k > 0$ *such that* $a(t) \geq kt^{\beta-2}(1-t)^2$, $t \in [0,1]$, *and*

$$\int_0^1 G(t,r)\varphi_q\biggl(\int_0^1 H(r,s)f\bigl(s,a(s)\bigr)\,ds\biggr)\,dr = b(t) \geq a(t),$$

$$\int_0^1 G(t,r)\varphi_q\biggl(\int_0^1 H(r,s)f\bigl(s,b(s)\bigr)\,ds\biggr)\,dr \geq a(t).$$

Then BVP (1.1) *has at least one positive solution w which satisfies $w(t) \geq mt^{\beta-2} \times (1-t)^2$ for some $m > 0$.*

Proof Let

$$P = \{u \in C[0,1] : \text{there exists } k_u > 0 \text{ such that } u(t) \geq k_u t^{\beta-2}(1-t)^2, t \in [0,1]\}.$$

Define an operator T by

$$Tu(t) = \int_0^1 G(t,r)\varphi_q\left(\int_0^1 H(r,s)f\big(s,u(s)\big)\,ds\right)dr, \quad u \in P.$$

For $u \in P$, there exists $k_u > 0$ such that $u(t) \geq k_u t^{\beta-2}(1-t)^2$, $t \in [0,1]$. By (H_1) and (H_2), we have

$$\int_0^1 H(t,s)f\big(s,u(s)\big)\,ds \leq d_1 \int_0^1 (1-s)^{\alpha-1}f\big(s,k_u s^{\beta-2}(1-s)^2\big)\,ds < +\infty.$$

Hence

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,r)\varphi_q\left(\int_0^1 H(r,s)f\big(s,u(s)\big)\,ds\right)dr \\ &\leq \int_0^1 \frac{M_0}{\Gamma(\beta)}q(r)\varphi_q\left(\int_0^1 d_1(1-s)^{\alpha-1}f\big(s,k_u s^{\beta-2}(1-s)^2\big)\,ds\right)dr \\ &= \frac{M_0}{\Gamma(\beta)}d_1^{q-1}\int_0^1 q(r)\,dr\varphi_q\left(\int_0^1 (1-s)^{\alpha-1}f\big(s,k_u s^{\beta-2}(1-s)^2\big)\,ds\right) < +\infty. \end{aligned} \tag{3.1}$$

On the other hand, choose $t_0 \in (0,1)$ such that $Tu(t_0) = k_{T_u} > 0$. It follows from Lemma 2.3 that

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,r)\varphi_q\left(\int_0^1 H(r,s)f\big(s,u(s)\big)\,ds\right)dr \\ &\geq \frac{\beta-2}{\Gamma(\beta)}k(t)\int_0^1 q(r)\varphi_q\left(\int_0^1 H(r,s)f\big(s,u(s)\big)\,ds\right)dr \\ &\geq \frac{\beta-2}{M_0}k(t)Tu(t_0) = \frac{\beta-2}{M_0}k_{T_u}t^{\beta-2}(1-t)^2, \quad t \in [0,1]. \end{aligned} \tag{3.2}$$

It follows from (3.1) and (3.2) that T is well defined and $T(P) \subset P$.

Next, we determine upper and lower solutions of BVP (1.1). In fact, by simple computations, we have

$$-D_{0^+}^{\alpha}\big(\varphi_p\big(D_{0^+}^{\beta}\big(Tu(t)\big)\big)\big) = f\big(t,u(t)\big), \quad t \in (0,1), \tag{3.3}$$

$$\begin{cases} (Tu)(0) = (Tu)(1) = (Tu)'(0) = (Tu)'(1) = 0, \\ D_{0^+}^{\beta}(Tu)(0) = 0, \qquad D_{0^+}^{\beta}(Tu)(1) = bD_{0^+}^{\beta}(Tu)(\eta). \end{cases} \tag{3.4}$$

Let $b(t) = Ta(t)$, then by (H_1) and (H_3), we have

$$a(t) \leq Ta(t) = b(t), \qquad b(t) = Ta(t) \geq Tb(t), \quad t \in [0,1]. \tag{3.5}$$

Since $a(t) \in P$, from (3.2), we obtain $Ta(t), Tb(t) \in P$. Thus, by (3.3) and (3.5),

$$-D_{0^+}^{\alpha}\left(\varphi_p\left(D_{0^+}^{\beta}(Tb)(t)\right)\right) - f\left(t,(Tb)(t)\right) \le -D_{0^+}^{\alpha}\left(\varphi_p\left(D_{0^+}^{\beta}(Tb)(t)\right)\right) - f\left(t,b(t)\right) = 0, \quad (3.6)$$

$$-D_{0^+}^{\alpha}\left(\varphi_p\left(D_{0^+}^{\beta}(Ta)(t)\right)\right) - f\left(t,(Ta)(t)\right) \ge -D_{0^+}^{\alpha}\left(\varphi_p\left(D_{0^+}^{\beta}(Ta)(t)\right)\right) - f\left(t,a(t)\right) = 0. \quad (3.7)$$

Meanwhile, (3.4) implies that $Ta(t), Tb(t)$ satisfy the boundary conditions of BVP (1.1). Then, from (3.5)–(3.7), $\varphi(t) = Tb(t)$ and $\psi(t) = Ta(t)$ are lower and upper solutions of BVP (1.1), respectively.

Next, we shall show that the BVP

$$\begin{cases} -D_{0^+}^{\alpha}(\varphi_p(D_{0^+}^{\beta}u(t))) = g(t,u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \qquad D_{0^+}^{\beta}u(0) = 0, \qquad D_{0^+}^{\beta}u(1) = bD_{0^+}^{\beta}u(\eta) \end{cases} \quad (3.8)$$

has a positive solution, where

$$g\left(t,u(t)\right) = \begin{cases} f(t,\varphi(t)), & u(t) < \varphi(t), \\ f(t,u(t)), & \varphi(t) \le u(t) \le \psi(t), \\ f(t,\psi(t)), & u(t) > \psi(t). \end{cases} \quad (3.9)$$

To see this, we consider the operator $A : C[0,1] \to C[0,1]$ defined as follows:

$$Au(t) = \int_0^1 G(t,r)\varphi_q\left(\int_0^1 H(r,s)g\left(s,u(s)\right)ds\right)dr.$$

It is well known that a fixed point of the operator A is a solution of BVP (3.8).

It is clear that A is continuous. Since $\varphi(t) \in P$, there exists $k_\varphi > 0$ such that $\varphi(t) \ge k_\varphi t^{\beta-2}(1-t)^2$, $t \in [0,1]$. It follows from (H_2) that

$$\begin{aligned} \int_0^1 H(t,s)g\left(s,u(s)\right)ds &\le d_1 \int_0^1 (1-s)^{\alpha-1}f\left(s,\varphi(s)\right)ds \\ &\le d_1 \int_0^1 (1-s)^{\alpha-1}f\left(s,k_\varphi s^{\beta-2}(1-s)^2\right)ds < +\infty. \end{aligned} \quad (3.10)$$

Consequently, for $u \in C[0,1]$ and $t \in [0,1]$, by (3.9) and (3.10), we have

$$\begin{aligned} Au(t) &= \int_0^1 G(t,r)\varphi_q\left(\int_0^1 H(r,s)g\left(s,u(s)\right)ds\right)dr \\ &\le M_0 \int_0^1 q(r)\varphi_q\left(\int_0^1 d_1(1-s)^{\alpha-1}g\left(s,u(s)\right)ds\right)dr \\ &\le M_0 d_1^{q-1} \int_0^1 q(r)\,dr\varphi_q\left(\int_0^1 (1-s)^{\alpha-1}f\left(s,k_\varphi s^{\beta-2}(1-s)^2\right)ds\right) < +\infty, \end{aligned}$$

which implies that A is uniformly bounded.

On the other hand, since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. Thus, for $s\in[0,1]$ and for each $\varepsilon>0$, there exists $\delta>0$ such that $|t_1-t_2|<\delta$ implies

$$\left|G(t_1,s)-G(t_2,s)\right|<\frac{\varepsilon}{\varphi_q(d_1\int_0^1(1-s)^{\alpha-1}f(s,k_\varphi s^{\beta-2}(1-s)^2)\,ds)}.$$

Furthermore, for $u\in C[0,1]$,

$$\begin{aligned}
&\left|Au(t_1)-Au(t_2)\right|\\
&\quad\leq\int_0^1\left|G(t_1,r)-G(t_2,r)\right|\varphi_q\left(\int_0^1 H(r,s)g\bigl(s,u(s)\bigr)\,ds\right)dr\\
&\quad\leq\int_0^1\left|G(t_1,r)-G(t_2,r)\right|\varphi_q\left(\int_0^1 d_1(1-s)^{\alpha-1}f\bigl(s,\varphi(s)\bigr)\,ds\right)dr\\
&\quad\leq\int_0^1\left|G(t_1,r)-G(t_2,r)\right|dr\varphi_q\left(d_1\int_0^1(1-s)^{\alpha-1}f\bigl(s,k_\varphi s^{\beta-2}(1-s)^2\bigr)\,ds\right)<\varepsilon,
\end{aligned}$$

which implies that A is equicontinuous. Thus, the Ascoli–Arzela theorem guarantees A is a compact operator. It follows from Schauder's fixed point theorem that A has a fixed point w, i.e., $w=Aw$. Consequently, (3.8) has a solution.

Finally, we will show that BVP (1.1) has at least one positive solution. In fact, we only need to prove that $\varphi(t)\leq w(t)\leq\psi(t)$, $t\in[0,1]$. By (H_1), we have

$$f\bigl(t,\psi(t)\bigr)\leq g\bigl(t,w(t)\bigr)\leq f\bigl(t,\varphi(t)\bigr),\quad t\in[0,1].\tag{3.11}$$

It follows from (3.5) and (H_3) that

$$f\bigl(t,b(t)\bigr)\leq g\bigl(t,w(t)\bigr)\leq f\bigl(t,a(t)\bigr),\quad t\in[0,1].\tag{3.12}$$

Since $a(t)\in P$, by (3.3), we have

$$-D_{0^+}^{\alpha}\bigl(\varphi_p\bigl(D_{0^+}^{\beta}\psi(t)\bigr)\bigr)=-D_{0^+}^{\alpha}\bigl(\varphi_p\bigl(D_{0^+}^{\beta}(Ta)(t)\bigr)\bigr)=f\bigl(t,a(t)\bigr),\quad t\in(0,1).$$

By (3.4), (3.5), (3.11), and (3.12), we have

$$\begin{aligned}
&-D_{0^+}^{\alpha}\bigl(\varphi_p\bigl(D_{0^+}^{\beta}\psi(t)\bigr)\bigr)-\bigl[-D_{0^+}^{\alpha}\bigl(\varphi_p\bigl(D_{0^+}^{\beta}w(t)\bigr)\bigr)\bigr]=f\bigl(t,a(t)\bigr)-g\bigl(t,w(t)\bigr)\geq 0,\quad t\in[0,1],\\
&(\psi-w)(0)=(\psi-w)(1)=(\psi-w)'(0)=(\psi-w)'(1)=0,\\
&D_{0^+}^{\beta}(\psi-w)(0)=0,\qquad D_{0^+}^{\beta}(\psi-w)(1)=bD_{0^+}^{\beta}(\psi-w)(\eta).
\end{aligned}$$

Setting $z=\varphi_p(D_{0^+}^{\beta}\psi(t))-\varphi_p(D_{0^+}^{\beta}w(t))$, then

$$\begin{aligned}
&-D_{0^+}^{\alpha}z(t)=-D_{0^+}^{\alpha}\bigl(\varphi_p\bigl(D_{0^+}^{\beta}\psi(t)\bigr)\bigr)-\bigl[-D_{0^+}^{\alpha}\bigl(\varphi_p\bigl(D_{0^+}^{\beta}w(t)\bigr)\bigr)\bigr]\geq 0,\\
&z(0)=0,\quad z(1)=\varphi_p(b)z(\eta).
\end{aligned}$$

Hence, by Lemma 2.1, we get $z(t)\geq 0$, $t\in[0,1]$. Since φ_p is monotone increasing, we have $D_{0^+}^{\beta}\psi(t)\geq D_{0^+}^{\beta}w(t)$, that is, $D_{0^+}^{\beta}(\psi(t)-w(t))\geq 0$, $t\in[0,1]$. By Remark 2.1, we have $w(t)\leq$

$\psi(t)$ for $t \in [0,1]$. Similarly, $w(t) \geq \varphi(t)$ on $[0,1]$. Therefore, $w(t)$ is a positive solution of BVP (1.1). And $\varphi(t) \in P$ implies that there exists $m > 0$ such that $w(t) \geq \varphi(t) \geq mt^{\beta-2}(1-t)^2$, $t \in [0,1]$. This completes the proof. □

4 An example

Example 4.1 Consider the following fractional singular BVP:

$$\begin{cases} -D_{0^+}^{\frac{3}{2}}(\varphi_p(D_{0^+}^{\frac{7}{2}}u(t))) = (1-t)^{-\frac{1}{4}}u^{-\frac{1}{2}}, & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \qquad D_{0^+}^{\frac{7}{2}}u(0) = 0, & D_{0^+}^{\frac{7}{2}}u(1) = \frac{1}{4}D_{0^+}^{\frac{7}{2}}u(\frac{1}{2}), \end{cases} \tag{4.1}$$

where $\varphi_p(t) = |t|^{p-2}t$, $p > 1$. Then BVP (4.1) has a positive solution $w(t) \geq mt^{\frac{3}{2}}(1-t)^2$ for some $m > 0$.

In fact, let $\alpha = \frac{3}{2}$, $\beta = \frac{7}{2}$, $f(t,u) = (1-t)^{-\frac{1}{4}}u^{-\frac{1}{2}}$, $t \in (0,1)$. Obviously, $f(t,u)$ is singular at $t = 1$ and $u = 0$. It is easy to check that (H_1) in Theorem 3.1 is satisfied. For any constant $\lambda > 0$,

$$\begin{aligned} 0 &< \int_0^1 (1-s)^{\alpha-1} f\bigl(s, \lambda s^{\beta-2}(1-s)^2\bigr)\, ds \\ &= \int_0^1 (1-s)^{\frac{1}{2}}(1-s)^{-\frac{1}{4}}\bigl[\lambda s^{\frac{3}{2}}(1-s)^2\bigr]^{-\frac{1}{2}}\, ds \\ &= \lambda^{-\frac{1}{2}} \int_0^1 s^{-\frac{3}{4}}(1-s)^{-\frac{3}{4}}\, ds \\ &= \lambda^{-\frac{1}{2}} B\left(\frac{1}{4}, \frac{1}{4}\right) = \lambda^{-\frac{1}{2}} \frac{\Gamma^2(\frac{1}{4})}{\sqrt{\pi}} < +\infty, \end{aligned}$$

so (H_2) in Theorem 3.1 is satisfied.

Set $\mu = \frac{1}{2}$, then $f(t,u) \leq f(t,ru) \leq r^{-\mu}f(t,u)$ for any $r \in (0,1)$. Since $e(t) = t^{\frac{3}{2}}(1-t)^2 \in P$, by (3.2) we know $Te \in P$, $T^2e \in P$, then there exist positive numbers k and l such that $Te \geq ke$ and $T^2e \geq le$. Take $0 < r_0 < \min\{1, k, l^{\frac{1}{1-\mu^2}}\}$, then

$$T(r_0e) \geq Te \geq ke \geq r_0e, \qquad T^2(r_0e) \geq r_0^{\mu^2}T^2e \geq r_0^{\mu^2}le \geq r_0e.$$

If we take $a(t) = r_0t^{\frac{3}{2}}(1-t)^2$, then condition (H_3) of Theorem 3.1 is satisfied. Consequently, the above conclusion is guaranteed by Theorem 3.1.

5 Conclusion

In this paper, we consider the fractional singular three-point boundary value problem with p-Laplacian operator. It is worth pointing out that $f(t,u)$ may be singular at $t = 0, 1$ and $u = 0$. Some properties of the associated Green function are obtained. By using the upper and lower solutions method and a fixed point theorem, the existence result of positive solution is established.

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