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On nonlocal Robin boundary value problems for Riemann–Liouville fractional Hahn integrodifference equation

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Abstract

In this paper, we study a nonlocal Robin boundary value problem for fractional Hahn integrodifference equation. Our problem contains three fractional Hahn difference operators and a fractional Hahn integral with different numbers of q, ω and order. The existence and uniqueness result is proved by using the Banach fixed point theorem. In addition, the existence of at least one solution is obtained by using Schauder's fixed point theorem.

MSC: 39A10; 39A13; 39A70

Keywords: Fractional Hahn integral; Riemann–Liouville fractional Hahn difference; Boundary value problems; Existence

1 Introduction

Recently, many researchers have extensively studied calculus without limit that deals with a set of non-differentiable functions, the so-called quantum calculus. Many types of quantum difference operators are employed in several applications of mathematical areas such as the calculus of variations, particle physics, quantum mechanics, and theory of relativity (see [1–12] and the references therein for some applications and new results of the quantum calculus).

In this paper, we study the Hahn quantum calculus that is one type of quantum calculus. Hahn [13] introduced the Hahn difference operator $D_{q,\omega}$ in 1949 as follows:

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, \quad t \neq \omega_0 := \frac{\omega}{1-q}.$$

The Hahn difference operator is a combination of two well-known difference operators: the forward difference operator and the Jackson q-difference operator. Notice that

$$D_{q,\omega}f(t) = \Delta_\omega f(t) \quad \text{whenever } q = 1, \qquad D_{q,\omega}f(t) = D_q f(t) \quad \text{whenever } \omega = 0 \quad \text{and}$$
$$D_{q,\omega}f(t) = f'(t) \quad \text{whenever } q = 1, \omega \to 0.$$

The Hahn difference operator has been employed to construct families of orthogonal polynomials and investigate some approximation problems (see [14–16] and the references therein).

In 2009, Aldwoah [17, 18] defined the right inverse of $D_{q,\omega}$ in terms of both the Jackson q-integral containing the right inverse of D_q [19] and the Nörlund sum containing the right inverse of Δ_ω [19].

In 2010, Malinowska and Torres [20, 21] introduced the Hahn quantum variational calculus. In 2013, Malinowska and Martins [22] studied the generalized transversality conditions for the Hahn quantum variational calculus. Later, Hamza and Ahmed [23, 24] studied the theory of linear Hahn difference equations, and investigated the existence and uniqueness results for the initial value problems for Hahn difference equations by using the method of successive approximations. Moreover, they proved Gronwall's and Bernoulli's inequalities with respect to the Hahn difference operator and established the mean value theorems for this calculus. In 2016, Hamza and Makharesh [25] investigated the Leibnitz's rule and Fubini's theorem associated with Hahn difference operator. In the same year, Sitthiwirattham [26] considered a nonlinear Hahn difference equation with nonlocal boundary value conditions of the form

$$\begin{aligned} &D^2_{q,\omega}x(t)+f\big(t,x(t),D_{p,\theta}x(pt+\theta)\big)=0, \quad t\in[\omega_0,T]_{q,\omega},\\ &x(\omega_0)=\varphi(x), \\ &x(T)=\lambda x(\eta), \quad \eta\in(\omega_0,T)_{q,\omega}, \end{aligned} \tag{1.1}$$

where $0<q<1$, $0<\omega<T$, $\omega_0:=\frac{\omega}{1-q}$, $1\le\lambda<\frac{T-\omega_0}{\eta-\omega_0}$, $p=q^m, m\in\mathbb{N}, \theta=\omega(\frac{1-p}{1-q})$, $f:[\omega_0,T]_{q,\omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given function, and $\varphi:C([\omega_0,T]_{q,\omega},\mathbb{R})\to\mathbb{R}$ is a given functional.

In 2017, Sriphanomwan *et al.* [27] considered a nonlocal boundary value problem for second-order nonlinear Hahn integrodifference equation with integral boundary condition of the form

$$\begin{aligned} &D^2_{q,\omega}x(t)=f\big(t,x(t),D_{p,\theta}x(pt+\theta),\Psi_{p,\theta}x(pt+\theta)\big), \quad t\in[\omega_0,T]_{q,\omega},\\ &x(\omega_0)=x(T), \qquad x(\eta)=\mu\int_{\omega_0}^{T}g(s)x(s)\,d_{q,\omega}s, \quad \eta\in(\omega_0,T)_{q,\omega}, \end{aligned} \tag{1.2}$$

where $0<q<1$, $0<\omega<T$, $\omega_0:=\frac{\omega}{1-q}$, $\mu\int_{\omega_0}^{T}g(r)\,d_{q,\omega}r\neq1$, $\mu\in\mathbb{R}$, $p=q^m, m\in\mathbb{N}, \theta=\omega(\frac{1-p}{1-q})$, $f\in C([\omega_0,T]_{q,\omega}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $g\in C([\omega_0,T]_{q,\omega},\mathbb{R}^+)$ are given functions, and for $\varphi\in C([\omega_0,T]_{q,\omega}\times[\omega_0,T]_{q,\omega},[0,\infty))$,

$$\Psi_{p,\theta}x(t):=\int_{\omega_0}^{t}\varphi(t,ps+\theta)x(ps+\theta)\,d_{p,\theta}s.$$

In 2010, Čermák and Nechvátal [28] proposed the fractional (q,h)-difference operator and the fractional (q,h)-integral for $q>1$. In 2011, Čermák *et al.* [29] studied discrete Mittag–Leffler functions in linear fractional difference equations for $q>1$, and Rahmat [30, 31] studied the (q,h)-Laplace transform and some (q,h)-analogues of integral inequalities on discrete time scales for $q>1$. In 2016, Du *et al.* [32] presented the monotonicity and convexity for nabla fractional (q,h)-difference for $q > 0, q \neq 1$. However, we realize that Hahn difference operator requires the condition $0 < q < 1$. Therefore, to fill the gap, Brikshavana and Sitthiwirattham [33] have introduced the fractional Hahn difference operators for $0 < q < 1$.

In 2017, Patanarapeelert and Sitthiwirattham [34] considered a Riemann–Liouville fractional Hahn difference boundary value problem for a fractional Hahn integrodifference equation of the form

$$\begin{aligned} &D^{\alpha}_{q,\omega}u(t) = F\bigl(t, u(t), \Psi^{\gamma}_{r,\phi}u(t)\bigr), \quad t \in [\omega_0, T]_{q,\omega}, \\ &u(\omega_0) = u(T), \\ &D^{\beta}_{p,\theta}u(\omega_0) = D^{\beta}_{p,\theta}u(pT+\theta), \end{aligned} \tag{1.3}$$

and a fractional Hahn integral boundary value problem for a Caputo fractional Hahn difference equation of the form

$$\begin{aligned} &{}^{C}D^{\alpha}_{q,\omega}u(t) = G\bigl(t, u(t), {}^{C}D^{\gamma}_{r,\phi}u(rt+\phi)\bigr), \quad t \in [\omega_0, T]_{q,\omega}, \\ &u(\omega_0) = \mathcal{A}(u), \\ &\mathcal{I}^{\beta}_{p,\theta}u(T) = \frac{1}{\Gamma_p(\beta)}\int_{\omega_0}^{T}\bigl(T - \sigma_{p,\omega}(s)\bigr)^{\underline{\beta-1}}_{p,\theta}u(s)\,d_{p,\theta}s = \mathcal{B}(u), \end{aligned} \tag{1.4}$$

where $[\omega_0, T]_{q,\omega} := \{q^kT + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\}$; $\alpha \in (1,2], \beta, \gamma \in (0,1]$, $\omega > 0, p, q, r \in (0,1), p = q^m, r = q^n, m, n \in \mathbb{N}$, $\theta = \omega(\frac{1-p}{1-q}), \phi = \omega(\frac{1-r}{1-q})$; $F, G \in C([\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function; $\mathcal{A}, \mathcal{B} : C([\omega_0, T]_{q,\omega}, \mathbb{R}) \to \mathbb{R}$ are given functionals; and for $\varphi \in C([\omega_0, T]_{q,\omega} \times [\omega_0, T]_{q,\omega}, [0,\infty))$, define

$$\Psi^{\gamma}_{r,\phi}u(t) := \bigl(\mathcal{I}^{\gamma}_{r,\phi}\varphi u\bigr)(t) = \frac{1}{\Gamma_r(\gamma)}\int_{\omega_0}^{t}\bigl(t - \sigma_{r,\phi}(s)\bigr)^{\underline{\gamma-1}}_{r,\phi}\varphi(t,s)u(s)\,d_{r,\phi}s.$$

Presently, Patanarapeelert *et al.* [35] studied the boundary value problem for fractional Hahn difference equation containing a sequential Caputo fractional Hahn integrodifference equation with nonlocal Dirichlet boundary conditions

$$\begin{aligned} &{}^{C}D^{\alpha}_{q,\omega}\,{}^{C}D^{\beta}_{q,\omega}\biggl[\frac{E_{\sigma_{q,\omega}}}{\rho_{q,\omega}(t)} + qD_{q,\omega}\biggr]u(t) = F\bigl(t, u(t), \Psi^{\gamma}_{q,\omega}u(t)\bigr), \quad t \in [\omega_0, T]_{q,\omega}, \\ &u(\omega_0) = \phi(u), \\ &\rho_{q,\omega}(T)u(T) = \rho_{q,\omega}(\eta)u(\eta) = \psi(u), \quad \eta \in (\omega_0, T)_{q,\omega}, \end{aligned} \tag{1.5}$$

where $[\omega_0, T]_{q,\omega} = I^T_{q,\omega} := \{q^kT + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\}$; $\omega > 0, q \in (0,1)$; $\alpha, \beta, \gamma \in (0,1]$; the shift operator $E_{\sigma_{q,\omega}}u(t) := u(\sigma_{q,\omega}(t))$; $F \in C(I^T_{q,\omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function; $\phi, \psi : C(I^T_{q,\omega}, \mathbb{R}) \to \mathbb{R}$ are given functionals; and for $\varphi \in C(I^T_{q,\omega} \times I^T_{q,\omega}, [0,\infty))$, we define

$$\Psi^{\gamma}_{q,\omega}u(t) := \bigl(\mathcal{I}^{\gamma}_{q,\omega}\varphi u\bigr)(t) = \frac{1}{\Gamma_{q,\omega}(\gamma)}\int_{\omega_0}^{t}\bigl(t - \sigma_{q,\omega}(s)\bigr)^{\underline{\gamma-1}}_{q,\omega}\varphi(t,s)u(s)\,d_{q,\omega}s. \tag{1.6}$$

In quantum calculus, there are apparently few research works related to boundary value problems of fractional Hahn difference equations (see [34, 35]). Therefore, in this paper, we devote ourselves to studying a boundary value problem for fractional Hahn difference equation. Our problem is a nonlocal Robin boundary value problem for a fractional Hahn integrodifference equation of the form

$$\begin{aligned}
&D^{\alpha}_{q,\omega}u(t) = F\bigl(t,u(t),\Psi^{\gamma}_{r,\phi}u(t),D^{\nu}_{m,\chi}u(t)\bigr), \quad t\in I^T_{q,\omega},\\
&\lambda_1 u(\eta)+\lambda_2 D^{\beta}_{p,\theta}u(\eta) = \phi_1(u), \quad \eta\in I^T_{q,\omega}-\{\omega_0,T\},\\
&\mu_1 u(T)+\mu_2 D^{\beta}_{p,\theta}u(T) = \phi_2(u),
\end{aligned} \tag{1.7}$$

where $I^T_{q,\omega} := \{q^kT+\omega[k]_q : k\in\mathbb{N}_0\}\cup\{\omega_0\}$; $\alpha\in(1,2]$, $\beta,\gamma,\nu\in(0,1]$, $\omega>0$, $p,q,r\in(0,1)$, $p=q^a$, $r=q^b$, $m=q^c$, $a,b,c\in\mathbb{N}$, $\theta=\omega(\frac{1-p}{1-q})$, $\phi=\omega(\frac{1-r}{1-q})$, $\chi=\omega(\frac{1-m}{1-q})$; $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}^+$; $F\in C(I^T_{q,\omega}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a given function; $\phi_1,\phi_2 : C(I^T_{q,\omega},\mathbb{R})\to\mathbb{R}$ are given functionals, and $\Psi^{\gamma}_{r,\phi}u(t)$ is defined as (1.6).

In the next section, we briefly recall some definitions and lemmas used in this research work. In Sect. 3, we prove the existence and uniqueness of a solution to problem (1.7) by using the Banach fixed point theorem. In Sect. 4, we show the existence of at least one solution to problem (1.7) by using Schauder's fixed point theorem. Finally, an example is provided to illustrate our results in the last section.

2 Preliminaries

In this section, we present the notations, definitions, and lemmas used in the main results. Let $q\in(0,1)$, $\omega>0$ and define

$$[n]_q := \frac{1-q^n}{1-q} = q^{n-1}+\cdots+q+1 \quad\text{and}\quad [n]_q! := \prod_{k=1}^{n}\frac{1-q^k}{1-q}, \quad n\in\mathbb{R}.$$

The q-analogue of the power function $(a-b)^{\underline{n}}_q$ with $n\in\mathbb{N}_0 := [0,1,2,\ldots]$ is

$$(a-b)^{\underline{0}}_q := 1, \qquad (a-b)^{\underline{n}}_q := \prod_{k=0}^{n-1}\bigl(a-bq^k\bigr), \quad a,b\in\mathbb{R}.$$

The q,ω-analogue of the power function $(a-b)^{\underline{n}}_{q,\omega}$ with $n\in\mathbb{N}_0 := [0,1,2,\ldots]$ is

$$(a-b)^{\underline{0}}_{q,\omega} := 1, \qquad (a-b)^{\underline{n}}_{q,\omega} := \prod_{k=0}^{n-1}\bigl[a-\bigl(bq^k+\omega[k]_q\bigr)\bigr], \quad a,b\in\mathbb{R}.$$

More generally, if $\alpha\in\mathbb{R}$, we have

$$\begin{aligned}
&(a-b)^{\underline{\alpha}}_q = a^{\alpha}\prod_{n=0}^{\infty}\frac{1-(\frac{b}{a})q^n}{1-(\frac{b}{a})q^{\alpha+n}}, \quad a\neq 0,\\
&(a-b)^{\underline{\alpha}}_{q,\omega} = (a-\omega_0)^{\alpha}\prod_{n=0}^{\infty}\frac{1-(\frac{b-\omega_0}{a-\omega_0})q^n}{1-(\frac{b-\omega_0}{a-\omega_0})q^{\alpha+n}} = \bigl((a-\omega_0)-(b-\omega_0)\bigr)^{\underline{\alpha}}_q, \quad a\neq\omega_0.
\end{aligned}$$

Note that $a_q^{\underline{\alpha}} = a^\alpha$ and $(a-\omega_0)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha$. We also use the notation $(0)_q^{\underline{\alpha}} = (\omega_0)_{q,\omega}^{\underline{\alpha}} = 0$ for $\alpha > 0$. The q-gamma and q-beta functions are defined by

$$\Gamma_q(x) := \frac{(1-q)_q^{\underline{x-1}}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\},$$

$$B_q(x,s) := \int_0^1 t^{x-1}(1-qt)_q^{\underline{s-1}}\, d_qt = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)}.$$

Definition 2.1 For $q \in (0,1)$, $\omega > 0$ and f defined on an interval $I \subseteq \mathbb{R}$ which contains $\omega_0 := \frac{\omega}{1-q}$, the Hahn difference of f is defined by

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega} \quad \text{for } t \neq \omega_0,$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$. Provided that f is differentiable at ω_0, we call $D_{q,\omega}f$ the q,ω-derivative of f, and say that f is q,ω-differentiable on I.

Remarks

(1) $D_{q,\omega}[f(t)+g(t)] = D_{q,\omega}f(t) + D_{q,\omega}g(t)$.
(2) $D_{q,\omega}[\alpha f(t)] = \alpha D_{q,\omega}f(t)$.
(3) $D_{q,\omega}[f(t)g(t)] = f(t)D_{q,\omega}g(t) + g(qt+\omega)D_{q,\omega}f(t)$.
(4) $D_{q,\omega}[\frac{f(t)}{g(t)}] = \frac{g(t)D_{q,\omega}f(t) - f(t)D_{q,\omega}g(t)}{g(t)g(qt+\omega)}$.

Letting $a, b \in I \subseteq \mathbb{R}$ with $a < \omega_0 < b$ and $[k]_q = \frac{1-q^k}{1-q}, k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define the q,ω-interval by

$$\begin{aligned}
[a,b]_{q,\omega} &:= \{q^k a + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{q^k b + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\} \\
&= [a,\omega_0]_{q,\omega} \cup [\omega_0,b]_{q,\omega} \\
&= (a,b)_{q,\omega} \cup \{a,b\} = [a,b)_{q,\omega} \cup \{b\} = (a,b]_{q,\omega} \cup \{a\}.
\end{aligned}$$

We observe that for each $s \in [a,b]_{q,\omega}$, the sequence $\{\sigma_{q,\omega}^k(s)\}_{k=0}^\infty = \{q^k s + \omega[k]_q\}_{k=0}^\infty$ is uniformly convergent to ω_0.

In addition, we define the forward jump operator $\sigma_{q,\omega}^k(t) := q^k t + \omega[k]_q$ and the backward jump operator $\rho_{q,\omega}^k(t) := \frac{t-\omega[k]_q}{q^k}$ for $k \in \mathbb{N}$.

Definition 2.2 Let I be any closed interval of $\mathbb{R}$ that contains a, b, and ω_0. Letting $f : I \to \mathbb{R}$ be a given function, we define q,ω-integral of f from a to b by

$$\int_a^b f(t)\, d_{q,\omega}t := \int_{\omega_0}^b f(t)\, d_{q,\omega}t - \int_{\omega_0}^a f(t)\, d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)\, d_{q,\omega}t := [x(1-q)-\omega] \sum_{k=0}^\infty q^k f(xq^k + \omega[k]_q), \quad x \in I,$$

and the series converges at $x = a$ and $x = b$. We call f q,ω-integrable on $[a,b]$, and the sum to the right-hand side of the above equation is called the Jackson–Nörlund sum.

Note that the actual domain of function f is defined on $[a,b]_{q,\omega} \subset I$.

We next introduce the fundamental theorem of Hahn calculus.

Lemma 2.1 ([17]) *Let $f : I \to \mathbb{R}$ be continuous at ω_0 and define*

$$F(x) := \int_{\omega_0}^{x} f(t)\, d_{q,\omega}t, \quad x \in I.$$

Then F is continuous at ω_0. Furthermore, $D_{q,\omega_0}F(x)$ exists for every $x \in I$ and

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_{a}^{b} D_{q,\omega}F(t)\, d_{q,\omega}t = F(b) - F(a) \quad \text{for all } a, b \in I.$$

Lemma 2.2 ([26]) *Let $q \in (0,1)$, $\omega > 0$ and $f : I \to \mathbb{R}$ be continuous at ω_0. Then*

$$\int_{\omega_0}^{t}\int_{\omega_0}^{r} x(s)\, d_{q,\omega}s d_{q,\omega}r = \int_{\omega_0}^{t}\int_{qs+\omega}^{t} x(s)\, d_{q,\omega}r d_{q,\omega}s.$$

Lemma 2.3 ([26]) *Let $q \in (0,1)$ and $\omega > 0$. Then*

$$\int_{\omega_0}^{t} d_{q,\omega}s = t - \omega_0 \quad \text{and} \quad \int_{\omega_0}^{t} \big[t - \sigma_{q,\omega}(s)\big]\, d_{q,\omega}s = \frac{(t-\omega_0)^2}{1+q}.$$

Particulary, we introduce fractional Hahn integral and fractional Hahn difference of Riemann–Liouville type as follows.

Definition 2.3 For $\alpha, \omega > 0, q \in (0,1)$ and f defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn integral is defined by

$$\begin{aligned}\mathcal{I}^{\alpha}_{q,\omega}f(t) &:= \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}\big(t - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} f(s)\, d_{q,\omega}s \\ &= \frac{[t(1-q)-\omega]}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n \big(t - \sigma^{n+1}_{q,\omega}(t)\big)_{q,\omega}^{\underline{\alpha-1}} f\big(\sigma^{n}_{q,\omega}(t)\big),\end{aligned}$$

and $(\mathcal{I}^{0}_{q,\omega}f)(t) = f(t)$.

Definition 2.4 For $\alpha, \omega > 0, q \in (0,1)$, and f defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn difference of the Riemann–Liouville type of order α is defined by

$$\begin{aligned}D^{\alpha}_{q,\omega}f(t) &:= \big(D^{N}_{q,\omega}\mathcal{I}^{N-\alpha}_{q,\omega}f\big)(t) \\ &= \frac{1}{\Gamma_q(-\alpha)}\int_{\omega_0}^{t}\big(t - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{-\alpha-1}} f(s)\, d_{q,\omega}s,\end{aligned}$$

and $D^0_{q,\omega}f(t) = f(t)$, where N is the smallest integer that is greater than or equal to α.

Lemma 2.4 ([33]) *Letting* $\alpha > 0, q \in (0,1), \omega > 0$, *and* $f : I^T_{q,\omega} \to \mathbb{R}$, *we get*

$$\mathcal{I}^\alpha_{q,\omega} D^\alpha_{q,\omega} f(t) = f(t) + C_1(t-\omega_0)^{\alpha-1} + \cdots + C_N(t-\omega_0)^{\alpha-N}$$

for some $C_i \in \mathbb{R}, i = \mathbb{N}_{1,N}$ *and* $N-1 < \alpha \le N, N \in \mathbb{N}$.

Next, we give some auxiliary lemmas used for simplifying calculations.

Lemma 2.5 ([33]) *Letting* $\alpha, \beta > 0, p, q \in (0,1)$, *and* $\omega > 0$, *we have*

$$\begin{aligned}
&\int_{\omega_0}^t \bigl(t-\sigma_{q,\omega}(s)\bigr)^{\underline{\alpha-1}}_{q,\omega} (s-\omega_0)^{\underline{\beta}}_{q,\omega}\, d_{q,\omega}s = (t-\omega_0)^{\alpha+\beta} B_q(\beta+1,\alpha),\\
&\int_{\omega_0}^t \int_{\omega_0}^x \bigl(t-\sigma_{p,\omega}(x)\bigr)^{\underline{\alpha-1}}_{p,\omega} \bigl(x-\sigma_{q,\omega}(s)\bigr)^{\underline{\beta-1}}_{q,\omega}\, d_{q,\omega}s\, d_{p,\omega}x = \frac{(t-\omega_0)^{\alpha+\beta}}{[\beta]_q} B_p(\beta+1,\alpha).
\end{aligned}$$

The following lemma deals with the linear variant of problem (1.7) and gives a representation of the solution.

Lemma 2.6 *Let* $\alpha \in (1,2], \beta \in (0,1]$, $\omega > 0, p, q \in (0,1), p = q^m, m \in \mathbb{N}$, $\theta = \omega(\frac{1-p}{1-q})$; $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^+$; $h \in C(I^T_{q,\omega}, \mathbb{R})$ *is a given function*; $\phi_1, \phi_2 : C(I^T_{q,\omega}, \mathbb{R}) \to \mathbb{R}$ *are given functionals. Then the problem*

$$\begin{aligned}
&D^\alpha_{q,\omega} u(t) = h(t), \quad t \in I^T_{q,\omega},\\
&\lambda_1 u(\eta) + \lambda_2 D^\beta_{p,\theta} u(\eta) = \phi_1(u), \quad \eta \in I^T_{q,\omega} - \{\omega_0, T\},\\
&\mu_1 u(T) + \mu_2 D^\beta_{p,\theta} u(T) = \phi_2(u)
\end{aligned}\tag{2.1}$$

has the unique solution

$$\begin{aligned}
u(t) = {}& \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^t \bigl(t-\sigma_{q,\omega}(s)\bigr)^{\underline{\alpha-1}}_{q,\omega} h(s)\, d_{q,\omega}s\\
&- \frac{(t-\omega_0)^{\alpha-1}}{\Omega} \bigl\{\mathbf{B}_T \Phi_\eta[\phi_1,h] - \mathbf{B}_\eta \Phi_T[\phi_2,h]\bigr\}\\
&+ \frac{(t-\omega_0)^{\alpha-2}}{\Omega} \bigl\{\mathbf{A}_T \Phi_\eta[\phi_1,h] - \mathbf{A}_\eta \Phi_T[\phi_2,h]\bigr\},
\end{aligned}\tag{2.2}$$

where the functionals $\Phi_\eta[\phi_1,h], \Phi_T[\phi_2,h]$ *are defined by*

$$\begin{aligned}
\Phi_\eta[\phi_1,h] := {}& \phi_1(u) - \frac{\lambda_1}{\Gamma_q(\alpha)} \int_{\omega_0}^\eta \bigl(\eta-\sigma_{q,\omega}(s)\bigr)^{\underline{\alpha-1}}_{q,\omega} h(s)\, d_{q,\omega}s - \frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_p(-\beta)}\\
&\times \int_{\omega_0}^\eta \int_{\omega_0}^x \bigl(\eta-\sigma_{p,\theta}(s)\bigr)^{\underline{-\beta-1}}_{p,\theta} \bigl(x-\sigma_{q,\omega}(s)\bigr)^{\underline{\alpha-1}}_{q,\omega} h(s)\, d_{q,\omega}s\, d_{p,\theta}x,
\end{aligned}\tag{2.3}$$

$$\begin{aligned}
\Phi_T[\phi_2,h] := {}& \phi_2(u) - \frac{\mu_1}{\Gamma_q(\alpha)} \int_{\omega_0}^T \bigl(T-\sigma_{q,\omega}(s)\bigr)^{\underline{\alpha-1}}_{q,\omega} h(s)\, d_{q,\omega}s - \frac{\mu_2}{\Gamma_q(\alpha)\Gamma_p(-\beta)}\\
&\times \int_{\omega_0}^T \int_{\omega_0}^x \bigl(T-\sigma_{p,\theta}(s)\bigr)^{\underline{-\beta-1}}_{p,\theta} \bigl(x-\sigma_{q,\omega}(s)\bigr)^{\underline{\alpha-1}}_{q,\omega} h(s)\, d_{q,\omega}s\, d_{p,\theta}x,
\end{aligned}\tag{2.4}$$

and the constants $\mathbf{A}_\eta, \mathbf{A}_T, \mathbf{B}_\eta, \mathbf{B}_T$, *and* Ω *are defined by*

$$\begin{aligned}
\mathbf{A}_\eta &:= \lambda_1(\eta-\omega_0)^{\alpha-1} + \frac{\lambda_2}{\Gamma_p(-\beta)}\int_{\omega_0}^{\eta}\big(\eta-\sigma_{p,\theta}(s)\big)_{p,\theta}^{\underline{-\beta-1}}(s-\omega_0)^{\alpha-1}\,d_{p,\theta}s \\
&= (\eta-\omega_0)^{\alpha-1}\left(\lambda_1 + \frac{\lambda_2(\eta-\omega_0)^{-\beta}\Gamma_p(\alpha)}{\Gamma_p(\alpha-\beta)}\right), \qquad (2.5)\\
\mathbf{A}_T &:= \mu_1(T-\omega_0)^{\alpha-1} + \frac{\mu_2}{\Gamma_p(-\beta)}\int_{\omega_0}^{T}\big(T-\sigma_{p,\theta}(s)\big)_{p,\theta}^{\underline{-\beta-1}}(s-\omega_0)^{\alpha-1}\,d_{p,\theta}s \\
&= (T-\omega_0)^{\alpha-1}\left(\mu_1 + \frac{\mu_2(T-\omega_0)^{-\beta}\Gamma_p(\alpha)}{\Gamma_p(\alpha-\beta)}\right), \qquad (2.6)\\
\mathbf{B}_\eta &:= \lambda_1(\eta-\omega_0)^{\alpha-2} + \frac{\lambda_2}{\Gamma_p(-\beta)}\int_{\omega_0}^{\eta}\big(\eta-\sigma_{p,\theta}(s)\big)_{p,\theta}^{\underline{-\beta-1}}(s-\omega_0)^{\alpha-2}\,d_{p,\theta}s \\
&= (\eta-\omega_0)^{\alpha-2}\left(\lambda_1 + \frac{\lambda_2(\eta-\omega_0)^{-\beta}\Gamma_p(\alpha-1)}{\Gamma_p(\alpha-\beta-1)}\right), \qquad (2.7)\\
\mathbf{B}_T &:= \mu_1(T-\omega_0)^{\alpha-2} + \frac{\mu_2}{\Gamma_p(-\beta)}\int_{\omega_0}^{T}\big(T-\sigma_{p,\theta}(s)\big)_{p,\theta}^{\underline{-\beta-1}}(s-\omega_0)^{\alpha-2}\,d_{p,\theta}s \\
&= (T-\omega_0)^{\alpha-2}\left(\mu_1 + \frac{\mu_2(T-\omega_0)^{-\beta}\Gamma_p(\alpha-1)}{\Gamma_p(\alpha-\beta-1)}\right), \qquad (2.8)\\
\Omega &:= \mathbf{A}_T\mathbf{B}_\eta - \mathbf{A}_\eta\mathbf{B}_T. \qquad (2.9)
\end{aligned}$$

Proof Taking fractional Hahn q,ω-integral of order α for (2.1), we obtain

$$\begin{aligned}
u(t) &= C_1(t-\omega_0)^{\alpha-1} + C_2(t-\omega_0)^{\alpha-2} + \mathcal{I}_{q,\omega}^{\alpha}h(t)\\
&= C_1(t-\omega_0)^{\alpha-1} + C_2(t-\omega_0)^{\alpha-2} + \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}\big(t-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}h(x)\,d_{q,\omega}s. \qquad (2.10)
\end{aligned}$$

Then we take fractional Hahn p,θ-difference of order β for (2.10) to get

$$\begin{aligned}
&D_{p,\theta}^{\beta}u(t)\\
&\quad= \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}\big(t-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}\big[C_1(s-\omega_0)^{\alpha-1} + C_2(s-\omega_0)^{\alpha-2}\big]\,d_{q,\omega}s\\
&\qquad+ \frac{1}{\Gamma_q(\alpha)\Gamma_p(-\beta)}\int_{\omega_0}^{t}\int_{\omega_0}^{x}\big(t-\sigma_{q,\omega}(x)\big)_{p,\theta}^{\underline{-\beta-1}}\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}h(s)\,d_{q,\omega}s\,d_{p,\theta}x. \qquad (2.11)
\end{aligned}$$

Substituting $t=\eta$ into (2.10) and (2.11) and employing the first condition of (2.1), we have

$$\mathbf{A}_\eta C_1 + \mathbf{B}_\eta C_2 = \Phi_\eta[\phi_1,h]. \qquad (2.12)$$

Taking $t=T$ into (2.10) and (2.11) and employing the second condition of (2.1), we have

$$\mathbf{A}_T C_1 + \mathbf{B}_T C_2 = \Phi_T[\phi_2,h]. \qquad (2.13)$$

The constants C_1 and C_2 are revealed from solving the system of equations (2.12)–(2.13) as

$$C_1 = \frac{\mathbf{B}_\eta \Phi_T - \mathbf{B}_T \Phi_\eta}{\Omega} \quad \text{and} \quad C_2 = \frac{\mathbf{A}_T \Phi_\eta - \mathbf{A}_\eta \Phi_T}{\Omega}.$$

Substituting the constants C_1, C_2 into (2.10), we obtain (2.2).

On the other hand, it is easy to show that (2.2) is the solution of problem (2.1). By taking fractional Hahn q,ω-difference of order α for (2.2), we obtain (2.1). This completes the proof. □

We next introduce Schauder's fixed point theorem used to prove the existence of a solution of problem (1.7).

Lemma 2.7 ([36] Arzelá–Ascoli theorem) *A set of functions in $C[a,b]$ with the sup norm is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 2.8 ([36]) *If a set is closed and relatively compact, then it is compact.*

Lemma 2.9 ([37] Schauder's fixed point theorem) *Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T : D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

3 Existence and uniqueness result

In this section, we consider the existence and uniqueness result for problem (1.7). Let $\mathcal{C} = C(I^T_{q,\omega}, \mathbb{R})$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \max_{t \in I^T_{q,\omega}} \bigl\{ \bigl|u(t)\bigr|, \bigl|D_{m,\chi} u(t)\bigr| \bigr\},$$

where $\alpha \in (1,2], \beta, \gamma \in (0,1]$, $\omega > 0, p, q, r \in (0,1), p = q^a, r = q^b, m = q^c, a, b, c \in \mathbb{N}$, $\theta = \omega(\frac{1-p}{1-q}), \phi = \omega(\frac{1-r}{1-q}), \chi = \omega(\frac{1-m}{1-q})$; $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^+$. Define an operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{F}u)(t) :=& \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^t \bigl(t - \sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}} F\bigl(s, u(s), \Psi^\gamma_{r,\phi} u(s), D^\nu_{m,\chi} u(s)\bigr)\, d_{q,\omega}s \\ &- \frac{(t-\omega_0)^{\alpha-1}}{\Omega} \bigl\{ \mathbf{B}_T \Phi^*_\eta[\phi_1, F_u] - \mathbf{B}_\eta \Phi^*_T[\phi_2, F_u] \bigr\} \\ &+ \frac{(t-\omega_0)^{\alpha-2}}{\Omega} \bigl\{ \mathbf{A}_T \Phi^*_\eta[\phi_1, F_u] - \mathbf{A}_\eta \Phi^*_T[\phi_2, F_u] \bigr\}, \end{aligned} \tag{3.1}$$

where the functionals $\Phi^*_\eta[\phi_1, F_u], \Phi^*_T[\phi_2, F_u]$ are defined by

$$\begin{aligned}\Phi^*_\eta[\phi_1, F_u] :=& \phi_1(u) - \frac{\lambda_1}{\Gamma_q(\alpha)} \int_{\omega_0}^\eta \bigl(\eta - \sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}} F\bigl(s, u(s), \Psi^\gamma_{r,\phi} u(s), D^\nu_{m,\chi} u(s)\bigr)\, d_{q,\omega}s \\ &- \frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_p(-\beta)} \int_{\omega_0}^\eta \int_{\omega_0}^x \bigl(\eta - \sigma_{p,\theta}(s)\bigr)_{p,\theta}^{\underline{-\beta-1}} \bigl(x - \sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}} \\ &\times F\bigl(s, u(s), \Psi^\gamma_{r,\phi} u(s), D^\nu_{m,\chi} u(s)\bigr)\, d_{q,\omega}s\, d_{p,\theta}x, \end{aligned} \tag{3.2}$$

$$\begin{aligned}\Phi_T^*[\phi_2, F_u] &:= \phi_2(u) - \frac{\mu_1}{\Gamma_q(\alpha)} \int_{\omega_0}^{T} \bigl(T - \sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}} F\bigl(s, u(s), \Psi_{r,\phi}^{\gamma} u(s), D_{m,\chi}^{\nu} u(s)\bigr)\, d_{q,\omega}s \\ &\quad - \frac{\mu_2}{\Gamma_q(\alpha)\Gamma_p(-\beta)} \int_{\omega_0}^{T} \int_{\omega_0}^{x} \bigl(T - \sigma_{p,\theta}(s)\bigr)_{p,\theta}^{\underline{-\beta-1}} \bigl(x - \sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}} \\ &\quad \times F\bigl(s, u(s), \Psi_{r,\phi}^{\gamma} u(s), D_{m,\chi}^{\nu} u(s)\bigr)\, d_{q,\omega}s\, d_{p,\theta}x,\end{aligned} \tag{3.3}$$

and the constants $\mathbf{A}_\eta, \mathbf{A}_T, \mathbf{B}_\eta, \mathbf{B}_T, \Omega$ are defined by (2.5)–(2.9), respectively.

We find that problem (1.7) has a solution if and only if the operator $\mathcal{F}$ has a fixed point.

Theorem 3.1 *Assume that* $F : I_{q,\omega}^T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous*, $\varphi : I_{q,\omega}^T \times I_{q,\omega}^T \to [0,\infty)$ *is continuous with* $\varphi_0 = \max\{\varphi(t,s) : (t,s) \in I_{q,\omega}^T \times I_{q,\omega}^T\}$. *In addition, suppose that the following conditions hold*:

(H_1) *There exist constants* $\ell_1, \ell_2, \ell_3 > 0$ *such that, for each* $t \in I_{q,\omega}^T$ *and* $u, v \in \mathbb{R}$,

$$\begin{aligned}&\bigl|F\bigl(t, u, \Psi_{r,\phi}^{\gamma} u, D_{m,\chi}^{\nu} u\bigr) - F\bigl(t, v, \Psi_{r,\phi}^{\gamma} v, D_{m,\chi}^{\nu} v\bigr)\bigr| \\ &\quad \le \ell_1 |u - v| + \ell_2 \bigl|\Psi_{r,\phi}^{\gamma} u - \Psi_{r,\phi}^{\gamma}\bigr| + \ell_3 \bigl|D_{m,\chi}^{\nu} u - D_{m,\chi}^{\nu} v\bigr|.\end{aligned}$$

(H_2) *There exist constants* $\vartheta_1, \vartheta_2 > 0$ *such that, for each* $u, v \in \mathcal{C}$,

$$\bigl|\phi_1(u) - \phi_1(v)\bigr| \le \vartheta_1 \|u - v\|_{\mathcal{C}} \quad \text{and} \quad \bigl|\phi_2(u) - \phi_2(v)\bigr| \le \vartheta_2 \|u - v\|_{\mathcal{C}}.$$

(H_3) $\mathcal{O} < 1$,

where

$$\mathcal{L} := \ell_1 + \ell_2 \varphi_0 \frac{(T - \omega_0)^{\gamma}}{\Gamma_r(\gamma + 1)}, \tag{3.4}$$

$$\Theta_1 := \left\{ \frac{\lambda_1 (\eta - \omega_0)^{\alpha}}{\Gamma_q(\alpha + 1)} + \frac{\lambda_2 (\eta - \omega_0)^{\alpha - \beta} \Gamma_q(-\beta)}{\Gamma_q(\alpha - \beta + 1)\Gamma_p(-\beta)} \right\}, \tag{3.5}$$

$$\Theta_2 := \left\{ \frac{\mu_1 (T - \omega_0)^{\alpha}}{\Gamma_q(\alpha + 1)} + \frac{\mu_2 (T - \omega_0)^{\alpha - \beta} \Gamma_q(-\beta)}{\Gamma_q(\alpha - \beta + 1)\Gamma_p(-\beta)} \right\}, \tag{3.6}$$

$$\begin{aligned}\mathcal{O} &:= \frac{(\mathcal{L} + \ell_3)(T - \omega_0)^{\alpha}}{\Gamma_q(\alpha + 1)} + \frac{[\vartheta_1 + (\mathcal{L} + \ell_3)\Theta_1]}{|\Omega|} \bigl\{|\mathbf{B}_T|(T - \omega_0)^{\alpha-1} + |\mathbf{A}_T|(T - \omega_0)^{\alpha-2}\bigr\} \\ &\quad + \frac{[\vartheta_2 + (\mathcal{L} + \ell_3)\Theta_2]}{|\Omega|} \bigl\{|\mathbf{B}_\eta|(T - \omega_0)^{\alpha-1} + |\mathbf{A}_\eta|(T - \omega_0)^{\alpha-2}\bigr\}.\end{aligned} \tag{3.7}$$

Then problem (1.7) *has a unique solution in* $I_{q,\omega}^T$.

Proof To show that F is a contraction, we denote that

$$\mathcal{H}|u - v|(t) := \bigl|F\bigl(t, u(t), \Psi_{r,\phi}^{\gamma} u(t), D_{m,\chi}^{\nu} u(t)\bigr) - F\bigl(t, v(t), \Psi_{r,\phi}^{\gamma} v(t), D_{m,\chi}^{\nu} v(t)\bigr)\bigr|$$

for each $t \in I_{q,\omega}^T$ and $u, v \in \mathcal{C}$. We find that

$$\begin{aligned}&\bigl|\Phi_\eta^*[\phi_1, F_u] - \Phi_\eta^*[\phi_1, F_v]\bigr| \\ &\quad \le \bigl|\phi_1(u) - \phi_1(v)\bigr| + \frac{\lambda_1}{\Gamma_q(\alpha)} \int_{\omega_0}^{\eta} \bigl(\eta - \sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}} \mathcal{H}|u - v|(s)\, d_{q,\omega}s\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_p(-\beta)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{x}\bigl(\eta-\sigma_{p,\theta}(s)\bigr)_{p,\theta}^{\underline{-\beta-1}}\bigl(x-\sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\,d_{q,\omega}s\,d_{p,\theta}x\\
&\leq \vartheta_1\|u-v\|_{\mathcal{C}}+\bigl(\ell_1|u-v|+\ell_2\bigl|\Psi_{r,\phi}^{\gamma}u-\Psi_{r,\phi}^{\gamma}\bigr|+\ell_3\bigl|D_{m,\chi}^{\nu}u-D_{m,\chi}^{\nu}v\bigr|\bigr)\\
&\quad\times\left|\frac{\lambda_1(\eta-\omega_0)^{\alpha}}{\Gamma_q(\alpha+1)}+\frac{\lambda_2(\eta-\omega_0)^{\alpha-\beta}\Gamma_q(-\beta)}{\Gamma_q(\alpha-\beta+1)\Gamma_p(-\beta)}\right|\\
&\leq \vartheta_1\|u-v\|_{\mathcal{C}}+\left(\left[\ell_1+\ell_2\varphi_0\frac{(T-\omega_0)^{\gamma}}{\Gamma_r(\gamma+1)}\right]|u-v|+\ell_3\bigl|D_{m,\chi}^{\nu}u-D_{m,\chi}^{\nu}v\bigr|\right)\Theta_1\\
&\leq\bigl[\vartheta_1+(\mathcal{L}+\ell_3)\Theta_1\bigr]\|u-v\|_{\mathcal{C}}.
\end{aligned}
$$

Similarly,

$$
\bigl|\Phi_T^*[\phi_2,F_u]-\Phi_T^*[\phi_2,F_v]\bigr|\leq\bigl[\vartheta_2+(\mathcal{L}+\ell_3)\Theta_2\bigr]\|u-v\|_{\mathcal{C}}.
$$

Next, we have

$$
\begin{aligned}
&\bigl|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\bigr|\\
&\quad\leq\frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}\bigl(T-\sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\,d_{q,\omega}s\\
&\qquad+\frac{(T-\omega_0)^{\alpha-1}}{|\Omega|}\bigl\{|\mathbf{B}_T|\bigl|\Phi_\eta^*[\phi_1,F_u]-\Phi_\eta^*[\phi_1,F_v]\bigr|+|\mathbf{B}_\eta|\bigl|\Phi_T^*[\phi_2,F_u]-\Phi_T^*[\phi_2,F_v]\bigr|\bigr\}\\
&\qquad+\frac{(T-\omega_0)^{\alpha-2}}{|\Omega|}\bigl\{|\mathbf{A}_T|\bigl|\Phi_\eta^*[\phi_1,F_u]-\Phi_\eta^*[\phi_1,F_v]\bigr|+|\mathbf{A}_\eta|\bigl|\Phi_T^*[\phi_2,F_u]-\Phi_T^*[\phi_2,F_v]\bigr|\bigr\}\\
&\quad\leq\biggl[\frac{(\mathcal{L}+\ell_3)(T-\omega_0)^{\alpha}}{\Gamma_q(\alpha+1)}+\frac{[\vartheta_1+(\mathcal{L}+\ell_3)\Theta_1]}{\Omega}\bigl\{|\mathbf{B}_T|(T-\omega_0)^{\alpha-1}+|\mathbf{A}_T|(T-\omega_0)^{\alpha-2}\bigr\}\\
&\qquad+\frac{[\vartheta_2+(\mathcal{L}+\ell_3)\Theta_2]}{|\Omega|}\bigl\{|\mathbf{B}_\eta|(T-\omega_0)^{\alpha-1}+|\mathbf{A}_\eta|(T-\omega_0)^{\alpha-2}\bigr\}\biggr]\|u-v\|_{\mathcal{C}}\\
&\quad=\mathcal{O}\|u-v\|_{\mathcal{C}}.
\end{aligned}\tag{3.8}
$$

We take fractional Hahn m,χ-difference of order γ for (3.1) to obtain

$$
\begin{aligned}
&\bigl(D_{m,\chi}^{\gamma}\mathcal{F}u\bigr)(t)\\
&\quad=\frac{1}{\Gamma_m(-\gamma)\Gamma_q(\alpha)}\int_{\omega_0}^{t}\int_{\omega_0}^{x}\bigl(t-\sigma_{m,\chi}(x)\bigr)_{m,\chi}^{\underline{-\gamma-1}}\bigl(x-\sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}}\\
&\qquad\times F\bigl(s,u(s),\Psi_{r,\phi}^{\gamma}u(s),D_{m,\chi}^{\nu}u(s)\bigr)\,d_{q,\omega}s\,d_{m,\chi}x\\
&\qquad-\frac{1}{\Omega\Gamma_m(-\gamma)}\bigl\{\mathbf{B}_T\Phi_\eta^*[\phi_1,F]-\mathbf{B}_\eta\Phi_T^*[\phi_2,F]\bigr\}\int_{\omega_0}^{t}\bigl(t-\sigma_{m,\chi}(s)\bigr)_{m,\chi}^{\underline{-\gamma-1}}(s-\omega_0)^{\alpha-1}\,d_{m,\chi}s\\
&\qquad+\frac{1}{\Omega\Gamma_m(-\gamma)}\bigl\{\mathbf{A}_T\Phi_\eta^*[\phi_1,F]-\mathbf{A}_\eta\Phi_T^*[\phi_2,F]\bigr\}\\
&\qquad\times\int_{\omega_0}^{t}\bigl(t-\sigma_{m,\chi}(s)\bigr)_{m,\chi}^{\underline{-\gamma-1}}(s-\omega_0)^{\alpha-2}\,d_{m,\chi}s.
\end{aligned}\tag{3.9}
$$

Using the same argument as above, we have

$$\left|\left(D^{\gamma}_{m,\chi}\mathcal{F}u\right)(t)-\left(D^{\gamma}_{m,\chi}\mathcal{F}v\right)(t)\right|<\mathcal{O}\|u-v\|_{\mathcal{C}}. \tag{3.10}$$

(3.8) and (3.10) imply that

$$\|\mathcal{F}u-\mathcal{F}v\|_{\mathcal{C}}\leq\mathcal{O}\|u-v\|_{\mathcal{C}}.$$

By (H_3) we can conclude that $\mathcal{F}$ is a contraction. Therefore, by using the Banach fixed point theorem, $\mathcal{F}$ has a fixed point which is a unique solution of problem (1.7) on $I^T_{q,\omega}$. □

4 Existence of at least one solution

In this section, we present the existence of a solution to (1.7) by using Schauder's fixed point theorem.

Theorem 4.1 *Suppose that* (H_1) *and* (H_3) *hold. Then problem* (1.7) *has at least one solution on* $I^T_{q,\omega}$.

Proof We divide the proof into three steps as follows.

Step I. Verify that $\mathcal{F}$ maps bounded sets into bounded sets in $B_R=\{u\in\mathcal{C}:\|u\|_{\mathcal{C}}\leq R\}$. We consider $B_R=\{u\in C(I^T_{q,\omega}):\|u\|_{\mathcal{C}}\leq R\}$. Set $\max_{t\in I^T_{q,\omega}}|F(t,0,0,0)|=K$, $\sup_{u\in\mathcal{C}}|\phi_1(u)|=M_1$, $\sup_{u\in\mathcal{C}}|\phi_2(u)|=M_2$ and choose a constant

$$R\geq\frac{M_1+M_2+\frac{K}{|\Omega|}[(T-\omega_0)^{\alpha-1}(|\mathbf{B}_T|\Theta_1+|\mathbf{B}_\eta|\Theta_2)+(T-\omega_0)^{\alpha-2}(|\mathbf{A}_T|\Theta_1+|\mathbf{A}_\eta|\Theta_2)]}{1-\mathcal{O}}. \tag{4.1}$$

Denote that

$$\left|\mathcal{S}(t,u,0)\right|=\left|F\left(t,u(t),\Psi^{\gamma}_{r,\phi}u(t),D^{\gamma}_{m,\chi}u(t)\right)-F(t,0,0,0)\right|+\left|F(t,0,0,0)\right|.$$

For each $t\in I^T_{q,\omega}$ and $u\in B_R$, we obtain

$$\begin{aligned}
&\left|\Phi^*_\eta[\phi_1,F_u]\right|\\
&\quad\leq M_1+\frac{\lambda_1}{\Gamma_q(\alpha)}\int_{\omega_0}^{\eta}\left(\eta-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}\left|\mathcal{S}(s,u,0)\right|d_{q,\omega}s\\
&\qquad+\frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_p(-\beta)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{x}\left(\eta-\sigma_{p,\theta}(s)\right)_{p,\theta}^{\underline{-\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}\left|\mathcal{S}(s,u,0)\right|d_{q,\omega}s\,d_{p,\theta}x\\
&\quad\leq M_1+\left(\left[\ell_1+\ell_2\varphi_0\frac{(T-\omega_0)^{\gamma}}{\Gamma_r(\gamma+1)}\right]|u|+\ell_3\left|D^{\nu}_{m,\chi}u\right|+K\right)\Theta_1\\
&\quad\leq\vartheta_1+K\Theta_1+(\mathcal{L}+\ell_3)\Theta_1\|u\|_{\mathcal{C}}\\
&\quad\leq\vartheta_1+K\Theta_1+(\mathcal{L}+\ell_3)\Theta_1R.
\end{aligned} \tag{4.2}$$

Similarly,

$$\left|\Phi^*_T[\phi_2,F_u]\right|\leq M_2+K\Theta_2+(\mathcal{L}+\ell_3)\Theta_2R. \tag{4.3}$$

From (4.2)–(4.3), we find that

$$\begin{aligned}\bigl|(\mathcal{F}u)(t)\bigr| &\leq R + M_1 + M_2 + \frac{K}{|\Omega|} \\ &\quad \times \bigl[(T-\omega_0)^{\alpha-1}\bigl(|\mathbf{B}_T|\Theta_1 + |\mathbf{B}_\eta|\Theta_2\bigr) + (T-\omega_0)^{\alpha-2}\bigl(|\mathbf{A}_T|\Theta_1 + |\mathbf{A}_\eta|\Theta_2\bigr)\bigr] \\ &\leq R. \end{aligned} \tag{4.4}$$

In addition, we obtain

$$\bigl|\bigl(D^{\gamma}_{m,\chi}\mathcal{F}u\bigr)(t)\bigr| < R. \tag{4.5}$$

Therefore, $\|\mathcal{F}u\|_{\mathcal{C}} \leq R$, which implies that $\mathcal{F}$ is uniformly bounded.

Step II. We can conclude that the operator $\mathcal{F}$ is continuous on B_R by the continuity of F.

Step III. In this step, we examine that $\mathcal{F}$ is equicontinuous on B_R. For any $t_1, t_2 \in I^T_{q,\omega}$ with $t_1 < t_2$, we have

$$\begin{aligned}&\bigl|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\bigr| \\ &\quad\leq \frac{\|F\|}{\Gamma_q(\alpha+1)}\bigl|(t_2-\omega_0)^{\alpha} - (t_1-\omega_0)^{\alpha}\bigr| \\ &\qquad + \frac{|(t_2-\omega_0)^{\alpha-1} - (t_1-\omega_0)^{\alpha-1}|}{|\Omega|}\bigl\{|\mathbf{B}_T|\Phi^*_\eta[\phi_1,F] + |\mathbf{B}_\eta|\Phi^*_T[\phi_2,F]\bigr\} \\ &\qquad + \frac{|(t_2-\omega_0)^{\alpha-2} - (t_1-\omega_0)^{\alpha-2}|}{|\Omega|}\bigl\{|\mathbf{A}_T|\Phi^*_\eta[\phi_1,F] + |\mathbf{A}_\eta|\Phi^*_T[\phi_2,F]\bigr\} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}&\bigl|\bigl(D^{\gamma}_{m,\chi}\mathcal{F}u\bigr)(t_1) - \bigl(D^{\gamma}_{m,\chi}\mathcal{F}u\bigr)(t_2)\bigr| \\ &\quad\leq \frac{\|F\|\Gamma_q(-\gamma)}{\Gamma_m(-\gamma)\Gamma_q(\alpha-\gamma+1)}\bigl|(t_2-\omega_0)^{\alpha-\gamma} - (t_1-\omega_0)^{\alpha-\gamma}\bigr| \\ &\qquad + \frac{\Gamma_q(\alpha)\Gamma_q(-\gamma)}{|\Omega|\Gamma_m(-\gamma)\Gamma_q(\alpha-\gamma)}\bigl\{|\mathbf{B}_T|\Phi^*_\eta[\phi_1,F] + |\mathbf{B}_\eta|\Phi^*_T[\phi_2,F]\bigr\} \\ &\qquad \times \bigl|(t_2-\omega_0)^{\alpha-\gamma-1} - (t_1-\omega_0)^{\alpha-\gamma-1}\bigr| \\ &\qquad + \frac{\Gamma_q(\alpha-1)\Gamma_q(-\gamma)}{|\Omega|\Gamma_m(-\gamma)\Gamma_q(\alpha-\gamma-1)}\bigl\{|\mathbf{A}_T|\Phi^*_\eta[\phi_1,F] + |\mathbf{A}_\eta|\Phi^*_T[\phi_2,F]\bigr\} \\ &\qquad \times \bigl|(t_2-\omega_0)^{\alpha-\gamma-2} - (t_1-\omega_0)^{\alpha-\gamma-2}\bigr|. \end{aligned} \tag{4.7}$$

We observe that the right-hand side of (4.7) tends to be zero when $|t_2 - t_1| \to 0$. So $\mathcal{F}$ is relatively compact on B_R.

This implies that the set $\mathcal{F}(B_R)$ is an equicontinuous set. As a consequence of Steps I to III together with the Arzelá–Ascoli theorem, we get that $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is completely continuous. By Schauder's fixed point theorem, we can conclude that problem (1.7) has at least one solution. □

5 Example

Consider the following boundary value problem for fractional Hahn difference equation:

$$\begin{aligned}
D^{\frac{4}{3}}_{\frac{1}{2},\frac{2}{3}}u(t) &= \frac{1}{(1000e^3+t^2)(1+|u(t)|)}\Big[e^{-3t}\big(u^2+2|u|\big)+e^{-(\pi+\sin^2\pi t)}\big|\Psi^{\frac{1}{2}}_{\frac{1}{8},\frac{7}{6}}u(t)\big| \\
&\quad + e^{-(1+\cos^2\pi t)}\big|D^{\frac{2}{5}}_{\frac{1}{2},\frac{2}{3}}u(t)\big|\Big], \quad t\in\left[\frac{4}{3},10\right]_{\frac{1}{2},\frac{2}{3}}, \qquad (5.1)\\
\frac{1}{10e}u\left(\frac{15}{8}\right)&+100eD^{\frac{3}{4}}_{\frac{1}{4},1}u\left(\frac{15}{8}\right)=\sum_{i=0}^{\infty}\frac{C_i|u(t_i)|}{1+|u(t_i)|}, \quad t_i\in 10\left(\frac{1}{2}\right)^i+\frac{2}{3}[i]_{\frac{1}{2}},\\
200\pi u(10)&+\frac{1}{10\pi}D^{\frac{3}{4}}_{\frac{1}{4},1}u(10)=\sum_{i=0}^{\infty}\frac{D_i|u(t_i)|}{1+|u(t_i)|}, \quad t_i\in 10\left(\frac{1}{2}\right)^i+\frac{2}{3}[i]_{\frac{1}{2}},
\end{aligned}$$

where $\varphi(t,s)=\frac{e^{-2|s-t|}}{(t+10)^3}$ and C_i, D_i are given constants with $\frac{1}{500t^3}\le\sum_{i=0}^{\infty}C_i\le\frac{\pi}{500t^3}$ and $\frac{1}{1000t^2}\le\sum_{i=0}^{\infty}D_i\le\frac{\pi}{1000t^2}$.

We provide $\alpha=\frac{4}{3},\beta=\frac{3}{4},\gamma=\frac{1}{2},\nu=\frac{2}{5},q=\frac{1}{2},p=\frac{1}{4},r=\frac{1}{8},m=\frac{1}{2},\ \omega=\frac{2}{3},\theta=1,\phi=\frac{7}{6},\chi=\frac{2}{3},\omega_0=\frac{\omega}{1-q}=\frac{4}{3},T=10,\eta=10(\frac{1}{2})^4+\frac{2}{3}[4]_{\frac{1}{2}}=\frac{15}{8},\ \lambda_1=\frac{1}{10e},\lambda_2=100e,\mu_1=200\pi,\mu_2=\frac{1}{10\pi}$, $\phi_1(u)=\sum_{i=0}^{\infty}\frac{C_i|u(t_i)|}{1+|u(t_i)|},\phi_2=\sum_{i=0}^{\infty}\frac{D_i|u(t_i)|}{1+|u(t_i)|}$, and $F(t,u(t),\Psi^{\gamma}_{r,\phi}u(t),D^{\nu}_{m,\chi}u(t))=\frac{1}{(1000e^3+t^2)(1+|u(t)|)}\times[e^{-3t}(u^2+2|u|)+e^{-(\pi+\sin^2\pi t)}|\Psi^{\frac{1}{2}}_{\frac{1}{8},\frac{7}{6}}u(t)|+e^{-(1+\cos^2\pi t)}|D^{\frac{2}{5}}_{\frac{1}{2},\frac{2}{3}}u(t)|]$.

We can find that

$$|\mathbf{A}_\eta|=173.1815,\qquad |\mathbf{A}_T|=1290.6198,\qquad |\mathbf{B}_\eta|=1312.8836,$$
$$|\mathbf{B}_T|=148.9158\quad\text{and}\quad |\Omega|=1.7202\times10^6.$$

For all $t\in[\frac{4}{3},10]_{\frac{1}{2},\frac{2}{3}}$ and $u,v\in\mathbb{R}$, we have

$$\begin{aligned}
&\big|F\big(t,u,\Psi^{\gamma}_{r,\phi}u,D^{\nu}_{m,\chi}u\big)-F\big(t,v,\Psi^{\gamma}_{r,\phi}v,D^{\nu}_{m,\chi}v\big)\big|\\
&\quad\le\frac{1}{e^4(1000e^3+\frac{16}{9})}|u-v|+\frac{1}{e^{\pi}(1000e^3+\frac{16}{9})}\big|\Psi^{\gamma}_{r,\phi}u-\Psi^{\gamma}_{r,\phi}v\big|\\
&\qquad+\frac{1}{e(1000e^3+\frac{16}{9})}\big|D^{\nu}_{m,\chi}u-D^{\nu}_{m,\chi}v\big|.
\end{aligned}$$

Thus, (H_1) holds with $\ell_1=9.118\times10^{-7},\ell_2=2.1513\times10^{-6}$, and $\ell_3=0.0000183$.

For all $u,v\in\mathcal{C}$,

$$\big|\phi_1(u)-\phi_1(v)\big|=\frac{\pi}{500t^3}\|u-v\|_{\mathcal{C}},$$
$$\big|\phi_2(u)-\phi_2(v)\big|=\frac{e}{1000t^2}\|u-v\|_{\mathcal{C}}.$$

So, (H_2) holds with $\vartheta_1=0.00265$ and $\vartheta_2=0.00153$.

Also, we find that

$$\mathcal{L}=9.163\times10^{-7},\qquad \Theta_1=440.682,\qquad \Theta_2=248.882.$$

Therefore, (H_3) holds with

$$\mathcal{O} \approx 0.000307 < 1.$$

Hence, by Theorem 3.1, problem (5.1) has a unique solution.

6 Conclusion

We have proved the existence and uniqueness result of the nonlocal Robin boundary problem for a fractional Hahn integrodifference equation (1.7) by using the Banach fixed point theorem, and the existence of at least one solution by Schauder's fixed point theorem. Our problem contains three fractional Hahn difference operators and a fractional Hahn integral with different numbers of q, ω and order, which is a new idea.

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References

1. Annaby, M.H., Mansour, Z.S.: *q*-Fractional Calculus and Equations. Springer, Berlin (2012)
2. Kac, V., Cheung, P.: Quantum Calculus. Springer, New York (2002)
3. Jagerman, D.L.: Difference Equations with Applications to Queues. Dekker, New York (2000)
4. Aldowah, K.A., Malinowska, A.B., Torres, D.F.M.: The power quantum calculus and variational problems. Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms **19**, 93–116 (2012)
5. Birto da Cruz, A.M.C., Martins, N., Torres, D.F.M.: Symmetric differentiation on time scales. Appl. Math. Lett. **26**(2), 264–269 (2013)
6. Cruz, B., Artur, M.C.: Symmetric Quantum Calculus. Ph.D. thesis, Aveiro University (2012)
7. Wu, G.C., Baleanu, D.: New applications of the variational iteration method from differential equations to q-fractional difference equations. Adv. Differ. Equ. **2013**, 21 (2013)
8. Tariboon, J., Ntouyas, S.K.: Quantum calculus on finite intervals and applications to impulsive difference equations. Adv. Differ. Equ. **2013**, 282 (2013)
9. Álvarez-Nodarse, R.: On characterization of classical polynomials. J. Comput. Appl. Math. **196**, 320–337 (2006)
10. Agarwal, R.P., Baleanu, D., Hedayati, V., Rezapour, S.: Two fractional derivative inclusion problems via integral boundary condition. Appl. Math. Comput. **257**, 205–212 (2015)
11. Agarwal, R.P., Baleanu, D., Rezapour, S., Salehi, S.: The existence of solution for some fractional finite difference equations via sum boundary conditions. Adv. Differ. Equ. **2014**, 282 (2014)

12. Nyamoradi, N., Baleanu, D., Agarwal, R.P.: Existence and uniqueness of positive solutions to fractional boundary value problems with nonlinear boundary conditions. Adv. Differ. Equ. **2013**, 266 (2013)
13. Hahn, W.: Über Orthogonalpolynome, die q-Differenzenlgleichungen genügen. Math. Nachr. **2**, 4–34 (1949)
14. Costas-Santos, R.S., Marcellán, F.: Second structure relation for q-semiclassical polynomials of the Hahn tableau. J. Math. Anal. Appl. **329**, 206–228 (2007)
15. Kwon, K.H., Lee, D.W., Park, S.B., Yoo, B.H.: Hahn class orthogonal polynomials. Kyungpook Math. J. **38**, 259–281 (1998)
16. Foupouagnigni, M.: Laguerre–Hahn orthogonal polynomials with respect to the Hahn operator: fourth-order difference equation for the rth associated and the Laguerre–Freud equations recurrence coefficients. Ph.D. thesis, Université Nationale du Bénin, Bénin (1998)
17. Aldwoah, K.A.: Generalized time scales and associated difference equations. Ph.D. thesis, Cairo University (2009)
18. Annaby, M.H., Hamza, A.E., Aldwoah, K.A.: Hahn difference operator and associated Jackson–Nörlund integrals. J. Optim. Theory Appl. **154**, 133–153 (2012)
19. Jackson, F.H.: Basic integration. Q. J. Math. **2**, 1–16 (1951)
20. Malinowska, A.B., Torres, D.F.M.: The Hahn quantum variational calculus. J. Optim. Theory Appl. **147**, 419–442 (2010)
21. Malinowska, A.B., Torres, D.F.M.: Quantum Variational Calculus. Springer Briefs in Electrical and Computer Engineering-Control, Automation and Robotics. Springer, Berlin (2014)
22. Malinowska, A.B., Martins, N.: Generalized transversality conditions for the Hahn quantum variational calculus. Optimization **62**(3), 323–344 (2013)
23. Hamza, A.E., Ahmed, S.M.: Theory of linear Hahn difference equations. J. Adv. Math. **4**(2), 441–461 (2013)
24. Hamza, A.E., Ahmed, S.M.: Existence and uniqueness of solutions of Hahn difference equations. Adv. Differ. Equ. **2013**, 316 (2013)
25. Hamza, A.E., Makharesh, S.D.: Leibniz' rule and Fubinis theorem associated with Hahn difference operator. J. Adv. Math. **12**(6), 6335–6345 (2016)
26. Sitthiwirattham, T.: On a nonlocal boundary value problem for nonlinear second-order Hahn difference equation with two different q,ω-derivatives. Adv. Differ. Equ. **2016**, 116 (2016)
27. Sriphanomwan, U., Tariboon, J., Patanarapeelert, N., Ntouyas, S.K., Sitthiwirattham, T.: Nonlocal boundary value problems for second-order nonlinear Hahn integro-difference equations with integral boundary conditions. Adv. Differ. Equ. **2017**, 170 (2017)
28. Čermák, J., Nechvátal, L.: On (q,h)-analogue of fractional calculus. J. Nonlinear Math. Phys. **17**(1), 51–68 (2010)
29. Čermák, J., Kisela, T., Nechvátal, L.: Discrete Mittag–Leffler functions in linear fractional difference equations. Abstr. Appl. Anal. **2011**, Article ID 565067 (2011)
30. Rahmat, M.R.S.: The (q,h)-Laplace transform on discrete time scales. Comput. Math. Appl. **62**, 272–281 (2011)
31. Rahmat, M.R.S.: On some (q,h)-analogues of integral inequalities on discrete time scales. Comput. Math. Appl. **62**, 1790–1797 (2011)
32. Du, F., Jai, B., Erbe, L., Peterson, A.: Monotonicity and convexity for nabla fractional (q,h)-difference. J. Differ. Equ. Appl. **22**(9), 1224–1243 (2016)
33. Brikshavana, T., Sitthiwirattham, T.: On fractional Hahn calculus with the delta operators. Adv. Differ. Equ. **2017**, 354 (2017)
34. Patanarapeelert, N., Sitthiwirattham, T.: Existence results for fractional Hahn difference and fractional Hahn integral boundary value problems. Discrete Dyn. Nat. Soc. **2017**, Article ID 7895186 (2017)
35. Patanarapeelert, N., Brikshavana, T., Sitthiwirattham, T.: On nonlocal Dirichlet boundary value problem for sequential Caputo fractional Hahn integrodifference equations. Bound. Value Probl. **2018**, 6 (2018)
36. Griffel, D.H.: Applied Functional Analysis. Ellis Horwood, Chichester (1981)
37. Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cone. Academic Press, Orlando (1988)