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On four-point fractional q-integrodifference boundary value problems involving separate nonlinearity and arbitrary fractional order

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Abstract

In this paper, we study a sequential Caputo fractional q-integrodifference equation with fractional q-integral and Riemann–Liouville fractional q-derivative boundary value conditions. Our problem contains $2(M+N+1)$ different orders and six different numbers of q in derivatives and integrals. The problem contains separate nonlinear functions. To examine existence and uniqueness results of the problem, Banach's contraction principle and the Leray–Schauder nonlinear alternative are employed. An illustrative example is also provided.

MSC: 39A05; 39A13

Keywords: Existence; q-derivative; q-integral; q-integrodifference equation

1 Introduction

In the 20th century, q-difference calculus and fractional q-difference calculus play an important role in the areas of mathematics and applications [1–3] such as the applications to orthogonal polynomials and mathematical control theories. Essentially, for q-difference calculus, basic definitions and properties have been presented in Ref. [4]. For the fractional q-difference calculus proposed by Al-Salam [5] and Agarwal [6], see [7]. Recently, many researchers have extensively studied in q-difference equations and fractional q-difference equations (see [8–34]). However, there is a lack of research in boundary value problem of nonlinear q-difference equations. In what follows, we fill up this gap.

In 2014, Ahmad et al. [15] studied the existence of solutions for the Caputo fractional q-difference integral equation with nonlocal boundary conditions,

$$\begin{cases} {}^{C}D_q^{\beta}({}^{C}D_q^{\gamma}+\lambda)x(t)=pf(t,x(t))+kI_q^{\xi}g(t,x(t)), \quad t\in[0,1), \\ \alpha_1 x(0)-\beta_1(t^{(1-\gamma)}D_q x(0))_{t=0}=\sigma_1 x(\eta_1), \\ \alpha_2 x(1)-\beta_2 D_q(1)=\sigma_2 x(\eta_2), \end{cases} \tag{1.1}$$

where $\beta,\gamma,\xi\in(0,1)$, f,g are given continuous functions, λ,p,k are real constants and $\alpha_i,\beta_i,\sigma_i\in\mathbb{R}$, $\eta_i\in(0,1)$, $i=1,2$.

In 2016, Sitthiwirattham [31] examined the existence results of solutions to a fractional q-difference equation and a fractional q-integrodifference equation

$$D_q^\alpha x(t) = f\bigl(t, x(t), D_w^\nu x(t)\bigr), \tag{1.2}$$

$$D_q^\alpha x(t) = f\bigl(t, x(t), \Psi_w^\gamma x(t)\bigr), \quad t \in [0,T], \tag{1.3}$$

with nonlocal three-point fractional p-integral boundary conditions of the form

$$x(\eta) = \rho(x), \qquad I_p^\beta g(T)x(T) = 0,$$

where $p, q, w \in (0,1)$, $\alpha \in (1,2]$, $\nu \in (0,1]$, $\beta, \gamma > 0$ and $\eta \in (0,T)$ are given constants, $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C([0,T], \mathbb{R}^+)$ are given functions, and $\rho \in C([0,T], \mathbb{R}) \to \mathbb{R}$ is a given functional. For $\varphi \in C([0,T] \times [0,T], [0,\infty))$, define $\Psi_w^\gamma x(t) := (I_w^\gamma \varphi x)(t) = \frac{1}{\Gamma_w(\gamma)} \int_0^t (t - ws)^{(\gamma-1)} \varphi(t,s)x(s)\, d_w s$.

Recently, Patanarapeelert et al. [33] considered a sequential q-integrodifference boundary value problem involving two different orders and six different numbers of q in derivatives and integrals of the form

$$\begin{cases} D_q[\rho(t)D_p^\gamma(\kappa + D_o)]x(t) = f(t, x(t), D_w[e_o^{\kappa t} x(t)], \Psi_\nu x(t)), \\ x(0) = x(T), \\ (D_o[e_o^{\kappa t} x(t)])_{t=0} = D_o[e_o^{\kappa T} x(T)], \\ I_r^\theta \sigma(t)x(T) = 0, \end{cases} \tag{1.4}$$

where $t \in I_\alpha^T := \{\alpha^k T : k \in \mathbb{N}\} \cup \{0,T\}$, $\gamma, \theta \in (0,1]$, $p = \frac{p_1}{p_2}, q = \frac{q_1}{q_2}, o = \frac{o_1}{o_2}, r = \frac{r_1}{r_2}, w = \frac{w_1}{w_2}, \nu = \frac{\nu_1}{\nu_2}$, and $\alpha = \frac{1}{\mathrm{LCM}(p_2,q_2,o_2,r_2,w_2,\nu_2)}$ are proper fractions with $w \leq o$, LCM is the lest common multiple, $\kappa \leq \frac{1}{T}, \rho, \sigma \in C(I_\alpha^T, \mathbb{R}^+)$ and $f \in C(I_\alpha^T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are given functions.

In this paper, we aim to develop an understanding of nonlinear q-integrodifference equations. Particulary, our attention is to analyze existence and uniqueness for a four-point Riemann–Liouville fractional q-integrodifference boundary value problem for a sequential Caputo fractional q-integrodifference equation of the form

$$\begin{aligned} {}^C D_q^\alpha {}^C D_p^\beta u(t) = {} & \lambda_1 F\bigl(t, u(t), D_r^\gamma u(t), D_r^{\gamma-1} u(t), \ldots, D_r^{\gamma-M+1} u(t)\bigr) \\ & + \lambda_2 H\bigl(t, u(t), \Psi_w^\theta u(t), \Psi_w^{\theta-1} u(t), \ldots, \Psi_w^{\theta-N+1} u(t)\bigr), \quad t \in I_\chi^T, \end{aligned} \tag{1.5}$$

$$\begin{cases} D_p^k u(0) = D_p^{\beta+j} u(0) = 0, \quad k \in \mathbb{N}_{0,N-2}, j \in \mathbb{N}_{0,M-2}, \\ D_m^\nu u(0) = \mu D_m^\nu u(T), \\ u(\xi) = \tau I_n^\vartheta g(\eta)u(\eta), \end{cases} \tag{1.6}$$

where $I_\chi^T := \{\chi^k T : k \in \mathbb{N}\} \cup \{0,T\}$; $M, N \in \mathbb{N}_2 := \{2,3,\ldots\}$; $\alpha \in (M-1, M)$; $\beta \in (N-1, N)$; $\gamma \in (M-2, M-1)$; $\theta \in (N-2, N-1)$; $\nu, \vartheta \in (0,1)$; $\lambda_1, \lambda_2, \mu, \tau > 0$; $\xi, \eta \in I_\chi^T - \{0,T\}, \xi > \eta$; $p = \frac{p_1}{p_2}, q = \frac{q_1}{q_2}, r = \frac{r_1}{r_2}, w = \frac{w_1}{w_2}, m = \frac{m_1}{m_2}, n = \frac{n_1}{n_2}$ are simplest form of proper fractions and $\chi = \frac{1}{\mathrm{LCM}(p_2,q_2,r_2,w_2,m_2,n_2)}$, LCM is the least common multiple; $g \in C(I_\chi^T, \mathbb{R}^+)$, $F \in C(I_\chi^T \times \mathbb{R}^{M+2}, \mathbb{R})$, $H \in C(I_\chi^T \times \mathbb{R}^{N+2}, \mathbb{R})$ are given functions; and, for $\varphi \in C(I_\chi^T \times I_\chi^T, [0,\infty))$, define $\Psi_w^\theta u(t) := (I_w^\theta \varphi u)(t) = \frac{1}{\Gamma_w(\theta)} \int_0^t (t - ws)^{(\theta-1)} \varphi(t,s)u(s)\, d_w s$.

As is clear from our fractional q-integrodifference equation, there are $2(M + N + 1)$ different orders in derivatives and integral, and there are six different values of the q numbers consisting of q, p, r, m-derivatives and w, n-integrals. The rest of paper is organized as follows. Section 2 describes some basis definitions, some properties of the q-difference and fractional q-difference operators and lemma that are used to evaluate the results. In Sect. 3, we employ Banach's contraction mapping principle and the Leray–Schauder nonlinear alternative to prove an existence and uniqueness of solution of the problem (1.5)–(1.6). Finally, using our main results, we provide an example in Sect. 4.

2 Preliminaries

In this section, we provide some notations, definitions, and lemmas which are used in the main results. Let $q \in (0, 1)$ and define

$$[n]_q := \frac{1 - q^n}{1 - q} = q^{n-1} + \cdots + q + 1 \quad \text{and} \quad [n]_q! := \prod_{k=1}^{n} \frac{1 - q^k}{1 - q}, \quad n \in \mathbb{R}.$$

The q-analogue of the power function $(a - b)^{(n)}$ with $n \in \mathbb{N}_0 := [0, 1, \dots]$ is defined by

$$(a - b)^{(0)} := 1, \qquad (a - b)^{(n)} := \prod_{k=0}^{n-1} \bigl(a - bq^k\bigr), \quad a, b \in \mathbb{R}.$$

Generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} := a^\alpha \prod_{n=0}^{\infty} \frac{1 - (\frac{b}{a})q^n}{1 - (\frac{b}{a})q^{\alpha+n}}, \quad a \neq 0.$$

Specifically, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. In addition, $0^{(\alpha)} = 0$ for $\alpha > 0$. The q-gamma function is defined by

$$\Gamma_q(x) := \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \dots\},$$

and satisfies $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$.

For any $x, s > 0$, the q-beta function is defined by

$$\begin{aligned} B_q(x, s) &:= \int_0^1 t^{(x-1)}(1 - qt)^{(s-1)} \, d_q t \\ &= (1 - q) \sum_{n=0}^{\infty} q^n \bigl(1 - q^{n+1}\bigr)^{(\alpha-1)} \bigl(q^n\bigr)^{(x-1)} = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x + s)}. \end{aligned}$$

Definition 2.1 ([6]) For $q \in (0, 1)$, the q-derivative of a real function f is defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t} \quad \text{and} \quad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

The higher order q-derivatives of f is defined by

$$D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N} \quad \text{and} \quad D_q^0 f(t) = f(t).$$

For function f defined on the interval $[0,T]$, q-integral is defined as

$$I_q f(t) = \int_0^t f(s)\,d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f\bigl(tq^n\bigr),$$

where the infinite series is convergent.

Definition 2.2 ([6]) For $\alpha \geq 0$ and f defined on $[0,T]$, the fractional q-integral is defined by

$$\begin{aligned}\bigl(I_q^\alpha f\bigr)(x) &:= \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)} f(t)\,d_q t\\ &= \frac{x(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\bigl(x-xq^{n+1}\bigr)^{(\alpha-1)} f\bigl(xq^n\bigr)\\ &= \frac{x^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\bigl(1-q^{n+1}\bigr)^{(\alpha-1)} f\bigl(xq^n\bigr).\end{aligned}$$

We note that $(I_q^0 f)(x) = f(x)$.

Definition 2.3 ([8]) For $\alpha \geq 0$, m is the smallest integer such that $m \geq \alpha$ and f defined on $[0,T]$, the fractional q-derivative of the Riemann–Liouville type of order α is defined by

$$\bigl(D_q^\alpha f\bigr)(x) := \bigl(D_q^m I_q^{m-\alpha} f\bigr)(x) = \frac{1}{\Gamma_q(-\alpha)}\int_0^x (x-qt)^{(-\alpha-1)} f(t)\,d_q t, \quad \alpha > 0,$$

and

$$\bigl(D_q^0 f\bigr)(x) = f(x),$$

the fractional q-derivative of the Caputo type of order α is defined by

$$\bigl({}^C D_q^\alpha f\bigr)(x) := \bigl(I_q^{m-\alpha} D_q^m f\bigr)(x) = \frac{1}{\Gamma_q(m-\alpha)}\int_0^x (x-qt)^{(m-\alpha-1)} D_q^m f(t)\,d_q t, \quad \alpha > 0$$

and

$$\bigl({}^C D_q^0 f\bigr)(x) = f(x).$$

Lemma 2.1 ([6]) *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,T]$. Then the following properties hold*:

(i) $\bigl(I_q^\beta I_q^\alpha f\bigr)(x) = \bigl(I_q^{\alpha+\beta} f\bigr)(x)$,

(ii) $\bigl(D_q^\alpha I_q^\alpha f\bigr)(x) = f(x)$.

Lemma 2.2 ([8]) *Let $N-1 < \alpha \leq N$ and $N \in \mathbb{N}$. Then the following equality holds*:

$$\bigl(I_q^\alpha D_q^\alpha f\bigr)(x) = f(x) - \sum_{k=0}^{N-1} \frac{x^{\alpha-N+k}}{\Gamma_q(\alpha+k-N+1)}\bigl(D_q^{\alpha-N+k} f\bigr)(0).$$

Lemma 2.3 ([19]) *Let $N-1<\alpha\leq N$ and $N\in\mathbb{N}$. Then the following equality holds*:

$$\big(I_q^{\alpha\,C}D_q^{\alpha}f\big)(x)=f(x)-\sum_{k=0}^{N-1}\frac{x^k}{\Gamma_q(k+1)}\big(D_q^kf\big)(0),$$
$$\big({}^CD_q^{\alpha}I_q^{\alpha}f\big)(x)=f(x).$$

Lemma 2.4 ([17]) *Let $\alpha,\beta\geq 0$ and $0<p,q<1$. Then the following formulas hold*:

(i) $$\int_0^{\eta}(\eta-qt)^{(\alpha-1)}t^{(\beta)}\,d_qt=\eta^{\alpha+\beta}B_q(\alpha,\beta+1),$$

(ii) $$\int_0^{\eta}\int_0^{s}(\eta-ps)^{(\alpha-1)}(s-qt)^{(\beta-1)}\,d_qt\,d_ps=\frac{\eta^{\alpha+\beta}}{[\beta]_q}B_p(\alpha,\beta+1),$$

(iii) $$\int_0^{\eta}\int_0^{s}\int_0^{t}(\eta-ps)^{(\alpha-1)}(s-qt)^{(\beta-1)}(t-rv)^{(\gamma-1)}\,d_rv\,d_qt\,d_ps$$
$$=\frac{\eta^{\alpha+\beta+\gamma}}{[\gamma]_r}B_q(\beta,\gamma+1)B_p(\alpha,\beta+\gamma+1).$$

We next provided a lemma dealing with a linear variant of the boundary value problem. This lemma is used to define the solution of the boundary value problem (1.5)–(1.6).

Lemma 2.5 *Let $M,N\in\mathbb{N}_2$, $\alpha\in(M-1,M)$, $\beta\in(N-1,N)$, $\gamma\in(M-2,M-1)$, $\theta\in(N-2,N-1)$, $p=\frac{p_1}{p_2}$, $q=\frac{q_1}{q_2}$, $m=\frac{m_1}{m_2}$, $n=\frac{n_1}{n_2}$ be simplest proper fractions and $\phi=\frac{1}{\mathrm{LCM}(p_2,q_2,m_2,n_2)}$; $\lambda_1,\lambda_2,\mu,\tau>0$. For $f,h\in C(I_\phi^T,\mathbb{R})$ and $g\in C(I_\phi^T,\mathbb{R}^+)$, the solution for the boundary value problem*

$$D_q^{\alpha}D_p^{\beta}u(t)=\lambda_1f(t)+\lambda_2h(t),\quad t\in I_\phi^T \tag{2.1}$$

$$\begin{cases}D_p^ku(0)=D_p^{\beta+j}u(0)=0, & k\in\mathbb{N}_{0,N-2}, j\in\mathbb{N}_{0,M-2},\\ D_m^{\nu}u(0)=\mu D_m^{\nu}u(T),\\ u(\xi)=\tau I_n^{\vartheta}g(\eta)u(\eta),\end{cases} \tag{2.2}$$

is represented by

$$u(t)=\frac{\mathcal{P}[f+g]}{\Lambda}t^{N-1}-\frac{\mathcal{Q}[f+g]}{\Lambda}\int_0^t\frac{(t-ps)^{(\beta-1)}}{\Gamma_p(\beta)}s^{M-1}\,d_ps$$
$$+\int_0^t\int_0^s(t-ps)^{(\beta-1)}\frac{(s-qv)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)}\big[\lambda_1f(t)+\lambda_2h(t)\big]\,d_qv\,d_ps, \tag{2.3}$$

where the functionals $\mathcal{P}[f+g]$ and $\mathcal{Q}[f+g]$ are defined by

$$\mathcal{P}[f+g]$$
$$=\mu\int_0^T\int_0^s\frac{(T-ms)^{(-\nu-1)}(s-pv)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_m(-\nu)}v^{M-1}\,d_pv\,d_ms$$
$$\times\bigg[\tau\int_0^{\eta}\int_0^s\int_0^{v}\frac{(\eta-ns)^{(\vartheta-1)}(s-pv)^{(\beta-1)}(v-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_n(\vartheta)}$$

$$
\begin{aligned}
&\times g(\eta)\big[\lambda_1 f(x)+\lambda_2 h(x)\big]\,d_qx\,d_p\nu\,dns\\
&-\int_0^{\xi}\int_0^{s}\frac{(\xi-ps)^{(\beta-1)}(s-q\nu)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)}\big[\lambda_1 f(\nu)+\lambda_2 h(\nu)\big]\,d_q\nu\,d_ps\Bigg]\\
&+\Bigg[\int_0^{\xi}\frac{(\xi-ps)^{(\beta-1)}}{\Gamma_p(\beta)}s^{M-1}\,d_ps\\
&-\tau\int_0^{\eta}\int_0^{s}\frac{(\eta-ns)^{(\vartheta-1)}(s-p\nu)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_n(\vartheta)}g(\eta)\nu^{M-1}\,d_p\nu\,d_ns\Bigg]\\
&\times\mu\int_0^{T}\int_0^{s}\int_0^{\nu}\frac{(T-ms)^{(-\nu-1)}(s-p\nu)^{(\beta-1)}(\nu-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_m(-\nu)}\\
&\times\big[\lambda_1 f(x)+\lambda_2 h(x)\big]\,d_qx\,d_p\nu\,d_ms,
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
&\mathcal{Q}[f+g]\\
&=\mu\int_0^{T}\frac{(T-ms)^{(-\nu-1)}}{\Gamma_m(-\nu)}s^{M-1}\,d_ms\\
&\times\Bigg[\tau\int_0^{\eta}\int_0^{s}\int_0^{\nu}\frac{(\eta-ns)^{(\vartheta-1)}(s-p\nu)^{(\beta-1)}(\nu-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_n(\vartheta)}\\
&\times g(\eta)\big[\lambda_1 f(x)+\lambda_2 h(x)\big]\,d_qx\,d_p\nu\,d_ns\\
&-\int_0^{\xi}\int_0^{s}\frac{(\xi-ps)^{(\beta-1)}(s-q\nu)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)}\big[\lambda_1 f(\nu)+\lambda_2 h(\nu)\big]\,d_q\nu\,d_ps\Bigg]\\
&+\Bigg[\xi^{N-1}-\tau\int_0^{\eta}\frac{(\eta-ns)^{(\vartheta-1)}}{\Gamma_n(\vartheta)}g(\eta)s^{N-1}\,d_ns\Bigg]\\
&\times\mu\int_0^{T}\int_0^{s}\int_0^{\nu}\frac{(T-ms)^{(-\nu-1)}(s-p\nu)^{(\beta-1)}(\nu-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_m(-\nu)}\\
&\times\big[\lambda_1 f(x)+\lambda_2 h(x)\big]\,d_qx\,d_p\nu\,d_ms,
\end{aligned}
\tag{2.5}
$$

and the constant

$$
\begin{aligned}
\Lambda&=\mu\int_0^{T}\int_0^{s}\frac{(T-ms)^{(-\nu-1)}(s-px)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_m(-\nu)}\nu^{M-1}\,d_px\,d_ms\\
&\times\Bigg[\xi^{N-1}-\tau\int_0^{\eta}\frac{(\eta-ns)^{(\vartheta-1)}}{\Gamma_n(\vartheta)}g(\eta)s^{N-1}\,d_ns\Bigg]\\
&-\Bigg[\int_0^{\xi}\frac{(\xi-ps)^{(\beta-1)}}{\Gamma_p(\beta)}s^{M-1}\,d_ps\\
&-\tau\int_0^{\eta}\int_0^{s}\frac{(\eta-ns)^{(\vartheta-1)}(s-p\nu)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_n(\vartheta)}g(\eta)\nu^{M-1}\,d_p\nu\,d_ns\Bigg]\\
&\times\mu\int_0^{T}\frac{(T-ms)^{(-\nu-1)}}{\Gamma_m(-\nu)}s^{M-1}\,d_ms.
\end{aligned}
\tag{2.6}
$$

Proof Using the q-integral of order α for (2.1), we obtain

$$
D_p^{\beta}u(t)=\sum_{i=0}^{M-1}C_it^i+\int_0^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big[\lambda_1 f(s)+\lambda_2 h(s)\big]\,d_qs.
\tag{2.7}
$$

Then we take the p-integral of order β for (2.7). We have

$$\begin{aligned} u(t) &= \sum_{i=0}^{N-1} C_{M+i} t^i + \sum_{i=0}^{M-1} C_i \int_0^t \frac{(t-ps)^{(\beta-1)}}{\Gamma_p(\beta)} s^i \, d_p s \\ &\quad + \int_0^t \int_0^s \frac{(t-ps)^{(\beta-1)}(s-qv)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \big[\lambda_1 f(v) + \lambda_2 h(v)\big] \, d_q v \, d_p s \\ &= \sum_{i=0}^{N-1} C_{M+i} t^i + \frac{1}{\Gamma_p(\beta)} \sum_{i=0}^{M-1} C_i B(i+1,\beta) t^{\beta+i} \\ &\quad + \int_0^t \int_0^s \frac{(t-ps)^{(\beta-1)}(s-qv)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \big[\lambda_1 f(v) + \lambda_2 h(v)\big] \, d_q v \, d_p s. \end{aligned} \tag{2.8}$$

Next, taking the p-derivative of order $k \in \mathbb{N}_{0,N-2}$ for (2.8) where $t \in I_\phi^T$, we get

$$\begin{aligned} D_p^k u(t) &= \sum_{i=0}^{N-1} C_{M+i} \int_0^t \frac{(t-ps)^{(-k-1)}}{\Gamma_p(-k)} s^i \, d_p s \\ &\quad + \frac{1}{\Gamma_p(\beta)} \sum_{i=0}^{M-1} C_i B(i+1,\beta) \int_0^t \frac{(t-ps)^{(-k-1)}}{\Gamma_p(-k)} s^{\beta+i} \, d_p s \\ &\quad + \int_0^t \int_0^s \int_0^x \frac{(t-ps)^{(-k-1)}(s-px)^{(\beta-1)}(x-qv)^{(\alpha-1)}}{\Gamma_p(-k)\Gamma_p(\beta)\Gamma_q(\alpha)} \\ &\quad \times \big[\lambda_1 f(v) + \lambda_2 h(v)\big] \, d_q v \, d_p x \, d_p s \\ &= \sum_{i=0}^{N-1} C_{M+i} \frac{\Gamma_p(i+1)}{\Gamma_p(i+1-k)} t^{i-k} + \sum_{i=0}^{M-1} C_i \frac{\Gamma_p(i+1)}{\Gamma_p(\beta+i+1-k)} t^{\beta+i-k} \\ &\quad + \int_0^t \int_0^s \int_0^x \frac{(t-ps)^{(-k-1)}(s-px)^{(\beta-1)}(x-qv)^{(\alpha-1)}}{\Gamma_p(-k)\Gamma_p(\beta)\Gamma_q(\alpha)} \\ &\quad \times \big[\lambda_1 f(v) + \lambda_2 h(v)\big] \, d_q v \, d_p x \, d_p s. \end{aligned} \tag{2.9}$$

Letting $t = 0$ in (2.9), and by the first conditions of (2.2) for $k \in \mathbb{N}_{0,N-2}$, we get

$$C_M = C_{M+1} = C_{M+2} = \cdots = C_{M+N-2} = 0 \quad \text{for } k \in \mathbb{N}_{0,N-2}, \text{ respectively.}$$

Substituting the constants $C_i, i \in \mathbb{N}_{M,M+N-2}$ into (2.8), we obtain

$$\begin{aligned} u(t) &= C_{M+N-1} t^{N-1} + \frac{1}{\Gamma_p(\beta)} \sum_{i=0}^{M-1} C_i B(i+1,\beta) t^{\beta+i} \\ &\quad + \int_0^t \int_0^s \frac{(t-ps)^{(\beta-1)}(s-qv)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \big[\lambda_1 f(v) + \lambda_2 h(v)\big] \, d_q v \, d_p s. \end{aligned} \tag{2.10}$$

Next, taking the p-derivative of order $\beta + j, j \in \mathbb{N}_{0,M-2}$ for (2.10), we get

$$D_p^{\beta+j} u(t) = C_{M+N-1} \int_0^t \frac{(t-ps)^{(-\beta-j-1)}}{\Gamma_p(-\beta-j)} s^{N-1} \, d_p s$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma_p(\beta)} \sum_{i=0}^{M-1} C_i B(i+1,\beta) \int_0^t \frac{(t-ps)^{(-\beta-j-1)}}{\Gamma_p(-\beta-j)} s^{\beta+i}\, d_p s \\
&+ \int_0^t \int_0^s \int_0^x \frac{(t-ps)^{(-\beta-j-1)}(s-px)^{(\beta-1)}(x-qv)^{(\alpha-1)}}{\Gamma_p(-\beta-j)\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\times \bigl[\lambda_1 f(v) + \lambda_2 h(v)\bigr]\, d_q v\, d_p x\, d_p s \\
= {} & C_{M+N-1} \frac{\Gamma_p(N)}{\Gamma_p(N-\beta-j)} t^{N-\beta-j-1} + \sum_{i=0}^{M-1} C_i \frac{\Gamma_p(i+1)}{\Gamma_p(i+1-j)} t^{i-j} \\
&+ \int_0^t \int_0^s \int_0^x \frac{(t-ps)^{(-\beta-j-1)}(s-px)^{(\beta-1)}(x-qv)^{(\alpha-1)}}{\Gamma_p(-\beta-j)\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\times \bigl[\lambda_1 f(v) + \lambda_2 h(v)\bigr]\, d_q v\, d_p x\, d_p s. \qquad (2.11)
\end{aligned}
$$

Letting $t = 0$ in (2.11), and by the first conditions of (2.2) for $j \in \mathbb{N}_{0,M-2}$, we get

$$
C_0 = C_1 = C_2 = \cdots = C_{M-2} = 0 \quad \text{for } j \in \mathbb{N}_{0,M-2}, \text{ respectively.}
$$

Substituting the constants $C_i, i \in \mathbb{N}_{0,M-2}$ into (2.10), we obtain

$$
\begin{aligned}
u(t) = {} & C_{M+N-1} t^{N-1} + C_{M-1} \int_0^t \frac{(t-ps)^{(\beta-1)}}{\Gamma_p(\beta)} s^{M-1}\, d_p s \\
&+ \int_0^t \int_0^s \frac{(t-ps)^{(\beta-1)}(s-qv)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \bigl[\lambda_1 f(v) + \lambda_2 h(v)\bigr]\, d_q v\, d_p s. \qquad (2.12)
\end{aligned}
$$

Next, taking the m-derivative of order ν for $u(t)$ where $t \in I_\phi^T$, we get

$$
\begin{aligned}
&D_m^\nu u(t) \\
&= C_{M+N-1} \int_0^t \frac{(t-ms)^{(-\nu-1)}}{\Gamma_m(-\nu)} s^{N-1}\, d_m s \\
&\quad + C_{M-1} \int_0^t \int_0^s \frac{(t-ms)^{(-\nu-1)}(s-pv)^{(\beta-1)}}{\Gamma_m(-\nu)\Gamma_p(\beta)} v^{M-1}\, d_p v\, d_m s \\
&\quad + \int_0^t \int_0^s \int_0^v \frac{(t-ms)^{(-\nu-1)}(s-pv)^{(\beta-1)}(v-qx)^{(\alpha-1)}}{\Gamma_m(-\nu)\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\quad \times \bigl[\lambda_1 f(x) + \lambda_2 h(x)\bigr]\, d_q x\, d_p v\, d_m s. \qquad (2.13)
\end{aligned}
$$

Letting $t = 0, T$ in (2.9), and by the second conditions of (2.2), we get

$$
\begin{aligned}
&C_{M+N-1}\mu \int_0^T \frac{(T-ms)^{(-\nu-1)}}{\Gamma_m(-\nu)} s^{N-1}\, d_m s \\
&\quad + C_{M-1}\mu \int_0^T \int_0^s \frac{(T-ms)^{(-\nu-1)}(s-pv)^{(\beta-1)}}{\Gamma_m(-\nu)\Gamma_p(\beta)} v^{M-1}\, d_p v\, d_m s \\
&= -\mu \int_0^T \int_0^s \int_0^v \frac{(T-ms)^{(-\nu-1)}(s-pv)^{(\beta-1)}(v-qx)^{(\alpha-1)}}{\Gamma_m(-\nu)\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\quad \times \bigl[\lambda_1 f(x) + \lambda_2 h(x)\bigr]\, d_q x\, d_p v\, d_m s. \qquad (2.14)
\end{aligned}
$$

Taking the n-integral of order ϑ for (2.12) where $t \in I_{\phi}^{T}$, we have

$$\begin{aligned} I_n^{\vartheta} u(t) &= C_{M+N-1} \int_0^t \frac{(t-ns)^{(\vartheta-1)}}{\Gamma_n(\vartheta)} s^{N-1} \, d_n s \\ &\quad + C_{M-1} \int_0^t \int_0^s \frac{(t-ns)^{(\vartheta-1)}(s-pv)^{(\beta-1)}}{\Gamma_n(\vartheta)\Gamma_p(\beta)} v^{M-1} \, d_p v \, d_n s \\ &\quad + \int_0^t \int_0^s \int_0^v \frac{(t-ns)^{(\vartheta-1)}(s-pv)^{(\beta-1)}(v-qx)^{(\alpha-1)}}{\Gamma_n(\vartheta)\Gamma_p(\beta)\Gamma_q(\alpha)} \\ &\quad \times \bigl[\lambda_1 f(x) + \lambda_2 h(x)\bigr] \, d_q x \, d_p v \, d_n s. \end{aligned} \tag{2.15}$$

Letting $t = \xi$ in (2.11), and by the third conditions of (2.2), we get

$$\begin{aligned} & C_{M+N-1}\biggl[\xi^{N-1} - \tau \int_0^{\eta} \int_0^s \frac{(\eta-ns)^{(\vartheta-1)}}{\Gamma_n(\vartheta)} g(\eta) s^{N-1} \, d_n s\biggr] \\ &\quad + C_{M-1}\biggl[\int_0^{\xi} \frac{(\xi-ps)^{(\beta-1)}}{\Gamma_p(\beta)} s^{M-1} \, d_p s \\ &\quad - \tau \int_0^{\eta} \int_0^s \frac{(\eta-ns)^{(\vartheta-1)}(s-pv)^{(\beta-1)}}{\Gamma_n(\vartheta)\Gamma_p(\beta)} g(\eta) v^{M-1} \, d_p v \, d_n s\biggr] \\ &= \biggl[\tau \int_0^{\eta} \int_0^s \int_0^v \frac{(\eta-ns)^{(\vartheta-1)}(s-pv)^{(\beta-1)}(v-qx)^{(\alpha-1)}}{\Gamma_n(\vartheta)\Gamma_p(\beta)\Gamma_q(\alpha)} \\ &\quad \times \bigl[\lambda_1 f(x) + \lambda_2 h(x)\bigr] g(\eta) \, d_q x \, d_p v \, d_n s \\ &\quad - \int_0^{\xi} \int_0^s \frac{(\xi-ps)^{(\beta-1)}(s-qv)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \bigl[\lambda_1 f(v) + \lambda_2 h(v)\bigr] \, d_q v \, d_p s\biggr]. \end{aligned} \tag{2.16}$$

Finally, solving the system of equations (2.14) and (2.12), we obtain

$$C_{M-1} = -\frac{\mathcal{Q}[f+h]}{\Lambda} \quad \text{and} \quad C_{M+N-1} = \frac{\mathcal{P}[f+h]}{\Lambda},$$

where $\mathcal{P}[f+g]$, $\mathcal{Q}[f+g]$ and Λ are defined by (2.4)–(2.6), respectively.

After substituting the constants C_{M-1}, C_{M+N-1} into (2.12), we obtain (2.3). The proof is complete. □

3 Main results

In order to obtain the main results, we first transform the boundary value problem (1.5)–(1.6) into a fixed point problem. Let $\mathcal{C} = C(I_{\chi}^{T}, \mathbb{R})$ be a Banach space of all continuous functions from I_{χ}^{T} to $\mathbb{R}$ such that $D_r^{\gamma-i}u(t)$ exists for $i \in \mathbb{N}_{0,M-1}$, $\gamma \in (M-2, M-1)$. Define a norm by

$$\|u\|_{\mathcal{C}} = \max_{i \in \mathbb{N}_{0,M-1}} \bigl\{\|u\|, \bigl\|D_r^{\gamma-i}u\bigr\|\bigr\},$$

where $\|u\| = \sup_{t \in I_{\chi}^{T}} |u(t)|$ and $\|D_r^{\gamma-i}u\| = \sup_{t \in I_{\chi}^{T}} |D_r^{\gamma-i}u(t)|$. Denote

$$F\bigl(t, u(t), D_r^{\gamma}u(t), D_r^{\gamma-1}u(t), \ldots, D_r^{\gamma-M+1}u(t)\bigr) := F\bigl[t, u(t), D_r^{\gamma-i}u(t)\bigr],$$

$$H\big(t,u(t),\Psi_w^{\theta}u(t),\Psi_w^{\theta-1}u(t),\dots,\Psi_w^{\theta-N+1}u(t)\big):=H\big[t,u(t),\Psi_w^{\theta-j}u(t)\big],$$

for $i\in\mathbb{N}_{0,M-1}$ and $j\in\mathbb{N}_{0,N-1}$. Define the operator $\mathcal{A}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned}
&(\mathcal{A}u)(t)\\
&\quad=\frac{\mathcal{P}[F(u)+G(u)]}{\Lambda}t^{N-1}-\frac{\mathcal{Q}[F(u)+G(u)]}{\Lambda}\int_0^t\frac{(t-ps)^{(\beta-1)}}{\Gamma_p(\beta)}s^{M-1}\,d_ps\\
&\qquad+\int_0^t\int_0^s(t-ps)^{(\beta-1)}\frac{(s-qv)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)}\big(\lambda_1F\big[t,u(t),D_r^{\gamma-i}u(t)\big]\\
&\qquad+\lambda_2H\big[t,u(t),\Psi_w^{\theta-j}u(t)\big]\big)\,d_qv\,d_ps,
\end{aligned}\tag{3.1}$$

where the functionals $\mathcal{P}[F(u)+G(u)]$ and $\mathcal{Q}[F(u)+G(u)]$ are defined by

$$\begin{aligned}
&\mathcal{P}\big[F(u)+H(u)\big]\\
&\quad=\mu\int_0^T\int_0^s\frac{(T-ms)^{(-\nu-1)}(s-px)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_m(-\nu)}x^{M-1}\,d_px\,d_ms\\
&\qquad\times\bigg[\tau\int_0^\eta\int_0^s\int_0^y\frac{(\eta-ns)^{(\vartheta-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_n(\vartheta)}g(\eta)\\
&\qquad\times\big(\lambda_1F\big[t,u(t),D_r^{\gamma-i}u(t)\big]+\lambda_2H\big[t,u(t),\Psi_w^{\theta-j}u(t)\big]\big)\,d_qx\,d_py\,dns\\
&\qquad-\int_0^\xi\int_0^s\frac{(\xi-ps)^{(\beta-1)}(s-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)}\big(\lambda_1F\big[t,u(t),D_r^{\gamma-i}u(t)\big]\\
&\qquad+\lambda_2H\big[t,u(t),\Psi_w^{\theta-j}u(t)\big]\big)\,d_qx\,d_ps\bigg]\\
&\qquad+\bigg[\int_0^\xi\frac{(\xi-ps)^{(\beta-1)}}{\Gamma_p(\beta)}s^{M-1}\,d_ps\\
&\qquad-\tau\int_0^\eta\int_0^s\frac{(\eta-ns)^{(\vartheta-1)}(s-px)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_n(\vartheta)}g(\eta)x^{M-1}\,d_px\,d_ns\bigg]\\
&\qquad\times\mu\int_0^T\int_0^s\int_0^y\frac{(T-ms)^{(-\nu-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_m(-\nu)}\\
&\qquad\times\big(\lambda_1F\big[t,u(t),D_r^{\gamma-i}u(t)\big]+\lambda_2H\big[t,u(t),\Psi_w^{\theta-j}u(t)\big]\big)\,d_qx\,d_py\,d_ms
\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}
&\mathcal{Q}\big[F(u)+H(u)\big]\\
&\quad=\mu\int_0^T\frac{(T-ms)^{(-\nu-1)}}{\Gamma_m(-\nu)}s^{M-1}\,d_ms\bigg[\tau\int_0^\eta\int_0^s\int_0^y\frac{(\eta-ns)^{(\vartheta-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_n(\vartheta)}\\
&\qquad\times g(\eta)\big(\lambda_1F\big[t,u(t),D_r^{\gamma-i}u(t)\big]+\lambda_2H\big[t,u(t),\Psi_w^{\theta-j}u(t)\big]\big)\,d_qx\,d_py\,d_ns\\
&\qquad-\int_0^\xi\int_0^s\frac{(\xi-ps)^{(\beta-1)}(s-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)}\big(\lambda_1F\big[t,u(t),D_r^{\gamma-i}u(t)\big]\\
&\qquad+\lambda_2H\big[t,u(t),\Psi_w^{\theta-j}u(t)\big]\big)\,d_qx\,d_ps\bigg]
\end{aligned}$$

$$+\left[\xi^{N-1}-\tau\int_0^{\eta}\frac{(\eta-ns)^{(\vartheta-1)}}{\Gamma_n(\vartheta)}g(\eta)s^{N-1}\,d_ns\right]$$
$$\times\mu\int_0^T\int_0^s\int_0^y\frac{(T-ms)^{(-\nu-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_m(-\nu)}$$
$$\times\left(\lambda_1F\left[t,u(t),D_r^{\gamma-i}u(t)\right]+\lambda_2H\left[t,u(t),\Psi_w^{\theta-j}u(t)\right]\right)d_qx\,d_py\,d_ms, \tag{3.3}$$

where Λ is defined by (2.6).

Clearly, the problem (1.5)–(1.6) has solutions if and only if the operator F has fixed points.

Theorem 3.1 *Assume* $F:I_\chi^T\times\mathbb{R}^{M+2}\to\mathbb{R}, H:I_\chi^T\times\mathbb{R}^{N+2}\to\mathbb{R}$, $g:I_\chi^T\to\mathbb{R}^+$ *and* $\varphi:I_\chi^T\times I_\chi^T\to[0,\infty)$ *are continuous, let* $\varphi_0:=\sup_{(t,s)\in I_\chi^T\times I_\chi^T}\{\varphi(t,s)\}$. *In addition,* F,H *and* g *satisfy the following conditions*:

(H_1) *there exist positive constants* $L_i, i\in\mathbb{N}_{0,M-1}$ *such that, for all* $t\in I_\chi^T$ *and* $u,v\in\mathbb{R}$

$$\left|F\left[t,u,D_r^{\gamma-i}u\right]-F\left[t,v,D_r^{\gamma-i}v\right]\right|\le L_M|u-v|+\sum_{i=0}^{M-1}L_i\left|D_r^{\gamma-i}u-D_r^{\gamma-i}v\right|,$$

(H_2) *there exist positive constants* $\ell_j, j\in\mathbb{N}_{0,N-1}$ *such that, for all* $t\in I_\chi^T$ *and* $u,v\in\mathbb{R}$

$$\left|H\left[t,u,\Psi_w^{\theta-j}u\right]-H\left[t,v,\Psi_w^{\theta-j}v\right]\right|\le \ell_N|u-v|+\sum_{j=0}^{N-1}\ell_j\left|\Psi_w^{\theta-j}u-\Psi_w^{\theta-j}v\right|,$$

(H_3) $0<g(t)<G$ *for all* $t\in I_\chi^T$,

(H_4) $\Theta:=[\lambda_1(L_M+L)+\lambda_2(\ell_N+\ell\frac{\varphi_0T^\theta}{\Gamma_w(\theta+1)})]\times\{\frac{\Omega_1}{|\Lambda|}T^{\beta-1}+\frac{\Omega_2}{|\Lambda|}\frac{T^{\alpha+\beta-1}\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)}+\frac{T^{\alpha+\beta}\Gamma_p(\alpha+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)}\}<1$.

Then the given boundary value problem (1.5)–(1.6) *has a unique solution, where*

$$L=\sum_{i=0}^{M-1}L_i,\qquad \ell=\sum_{j=0}^{N-1}\ell_j,$$
$$\Omega_1=\frac{\mu T^{M+\beta-\nu}\Gamma_p(M)\Gamma_m(M+\beta+1)\Gamma_p(\alpha+1)}{\Gamma_p(M+\beta)\Gamma_m(M+\beta-\nu)\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)}$$
$$\times\left\{\left|\xi^{\alpha+\beta}-\frac{\tau G\eta^{\alpha+\beta+\vartheta}\Gamma_n(\alpha+\beta+1)}{\Gamma_n(\alpha+\beta+\vartheta+1)}\right|\right.$$
$$\left.+\left|\xi^{M+\beta-1}-\frac{\tau G\eta^{M+\beta+\vartheta-1}\Gamma_n(M+\beta)}{\Gamma_n(M+\beta+\vartheta)}\right|\right\}, \tag{3.4}$$
$$\Omega_2=\mu T^{M-\nu-1}\left\{\left|\xi^{\alpha+\beta}-\frac{\tau G\eta^{\alpha+\beta+\vartheta}\Gamma_n(\alpha+\beta+1)}{\Gamma_n(\alpha+\beta+\vartheta+1)}\right|\frac{\Gamma_m(\alpha)}{\Gamma_m(\alpha-\nu)}\right.$$
$$+\left|\xi^{N-1}-\frac{\tau G\eta^{N+\vartheta-1}\Gamma_n(N)}{\Gamma_n(N+\vartheta)}\right|$$
$$\left.\times\frac{T^{\beta+1}\Gamma_p(\alpha)\Gamma_m(\alpha+\beta+1)\Gamma_p(\alpha+1)}{\Gamma_p(\alpha+\beta)\Gamma_m(\alpha+\beta-\nu+1)\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)}\right\}.$$

Proof We transform the boundary value problem (1.5)–(1.6) into a fixed point problem $u = \mathcal{A}u$, where $\mathcal{A}:\mathcal{C} \to \mathcal{C}$ is defined by (3.1). For $t \in I_\chi^T$, letting

$$\bigl|\mathcal{F}\bigl[t,u,v,D_r^{\gamma-i}\bigr]\bigr| := \bigl|F\bigl[t,u(t),D_r^{\gamma-i}u(t)\bigr] - F\bigl[t,v(t),D_r^{\gamma-i}v(t)\bigr]\bigr|, \quad i \in \mathbb{N}_{0,M-1}$$

and

$$\bigl|\mathcal{H}\bigl[t,u,v,\Psi_w^{\theta-j}\bigr]\bigr| := \bigl|H\bigl[t,u(t),\Psi_w^{\theta-j}u(t)\bigr] - H\bigl[t,v(t),\Psi_w^{\theta-j}v(t)\bigr]\bigr|, \quad j \in \mathbb{N}_{0,N-1},$$

we find that

$$\begin{aligned}
&\bigl|\mathcal{P}\bigl[F(u)+H(u)\bigr] - \mathcal{P}\bigl[F(v)+H(v)\bigr]\bigr| \\
&\quad\le \mu \int_0^T \int_0^s \frac{(T-ms)^{(-\nu-1)}(s-px)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_m(-\nu)} x^{M-1}\, d_px\, d_ms \\
&\qquad\times \biggl|\tau \int_0^\eta \int_0^s \int_0^\nu \frac{(\eta-ns)^{(\vartheta-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_n(\vartheta)} g(\eta) \\
&\qquad\times \bigl[\lambda_1\bigl|\mathcal{F}\bigl[x,u,v,D_r^{\gamma-i}\bigr]\bigr| + \lambda_2\bigl|\mathcal{H}\bigl[x,u,v,\Psi_w^{\theta-j}\bigr]\bigr|\bigr]\, d_qx\, d_py\, d_ns \\
&\qquad- \int_0^\xi \int_0^s \frac{(\xi-ps)^{(\beta-1)}(s-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \bigl[\lambda_1\bigl|\mathcal{F}\bigl[x,u,v,D_r^{\gamma-i}\bigr]\bigr| \\
&\qquad+ \lambda_2\bigl|\mathcal{H}\bigl[x,u,v,\Psi_w^{\theta-j}\bigr]\bigr|\bigr]\, d_qx\, d_ps\biggr| \\
&\qquad+ \biggl|\int_0^\xi \frac{(\xi-ps)^{(\beta-1)}}{\Gamma_p(\beta)} s^{M-1}\, d_ps \\
&\qquad- \tau \int_0^\eta \int_0^s \frac{(\eta-ns)^{(\vartheta-1)}(s-px)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_n(\vartheta)} x^{\alpha-1} g(\eta)\, d_px\, d_nMs\biggr| \\
&\qquad\times \mu \int_0^T \int_0^s \int_0^y \frac{(T-ms)^{(-\nu-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_m(-\nu)} \\
&\qquad\times \bigl[\lambda_1\bigl|\mathcal{F}\bigl[x,u,v,D_r^{\gamma}\bigr]\bigr| + \lambda_2\bigl|\mathcal{H}\bigl[x,u,v,\Psi_w^{\theta}\bigr]\bigr|\bigr]\, d_qx\, d_py\, d_ms \\
&\quad\le \Biggl[\lambda_1\Biggl(L_M|u-v| + \sum_{i=0}^{M-1} L_i\bigl|D_r^{\gamma-i}u - D_r^{\gamma-i}v\bigr|\Biggr) \\
&\qquad+ \lambda_2\Biggl(\ell_N|u-v| + \sum_{j=0}^{N-1} \ell_j\bigl|\Psi_w^{\theta-j}u - \Psi_w^{\theta-j}v\bigr|\Biggr)\Biggr] \\
&\qquad\times \biggl|\frac{\xi^{\alpha+\beta}\Gamma_p(\alpha+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)} - \frac{\tau G\eta^{\alpha+\beta+\vartheta}\Gamma_p(\alpha+1)\Gamma_n(\alpha+\beta+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_n(\alpha+\beta+\vartheta+1)\Gamma_q(\beta+1)}\biggr| \\
&\qquad\times \frac{\mu T^{M+\beta-\nu-1}\Gamma_p(M)\Gamma_m(M+\beta)}{\Gamma_p(M+\beta)\Gamma_m(M+\beta-\nu)} \\
&\qquad+ \Biggl[\lambda_1\Biggl(L_M|u-v| + \sum_{i=0}^{M-1} L_i\bigl|D_r^{\gamma-i}u - D_r^{\gamma-i}v\bigr|\Biggr) \\
&\qquad+ \lambda_2\Biggl(\ell_N|u-v| + \sum_{j=0}^{N-1} \ell_j\bigl|\Psi_w^{\theta-j}u - \Psi_w^{\theta-j}v\bigr|\Biggr)\Biggr]
\end{aligned}$$

$$
\begin{aligned}
&\times \left|\frac{\xi^{M+\beta-1}\Gamma_p(M)}{\Gamma_p(M+\beta)} - \frac{\tau G\eta^{M+\beta+\vartheta-1}\Gamma_p(M)\Gamma_n(M+\beta)}{\Gamma_p(M+\beta)\Gamma_n(M+\beta+\vartheta)}\right| \frac{\mu T^{\alpha+\beta-\nu-1}\Gamma_p(\alpha)\Gamma_m(\alpha+\beta)}{\Gamma_p(\alpha+\beta)\Gamma_m(\alpha+\beta-\nu)} \\
\leq{}& \left[\lambda_1(L_M+L)\|u-v\|_{\mathcal{C}} + \lambda_2\left(\ell_N + \ell \max_{j\in\mathbb{N}_{0,N-1}}\left\{\frac{\varphi_0 T^{\theta-j}}{\Gamma_w(\theta-j+1)}\right\}\right)\|u-v\|\right] \\
&\times \left|\xi^{\alpha+\beta} - \frac{\tau G\eta^{\alpha+\beta+\vartheta}\Gamma_n(\alpha+\beta+1)}{\Gamma_n(\alpha+\beta+\vartheta+1)}\right| \\
&\times \frac{\mu T^{M+\beta-\nu-1}\Gamma_p(M)\Gamma_m(M+\beta)\Gamma_p(\alpha+1)}{\Gamma_p(M+\beta)\Gamma_m(M+\beta-\nu)\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)} \\
&+ \left[\lambda_1(L_M+L)\|u-v\|_{\mathcal{C}} + \lambda_2\left(\ell_N + \ell \max_{j\in\mathbb{N}_{0,N-1}}\left\{\frac{\varphi_0 T^{\theta-j}}{\Gamma_w(\theta-j+1)}\right\}\right)\|u-v\|\right] \\
&\times \left|\xi^{M+\beta-1} - \frac{\tau G\eta^{M+\beta+\vartheta-1}\Gamma_n(M+\beta)}{\Gamma_n(M+\beta+\vartheta)}\right| \\
&\times \frac{\mu T^{\alpha+\beta-\nu}\Gamma_p(M)\Gamma_m(\alpha+\beta+1)\Gamma_p(\alpha+1)}{\Gamma_p(M+\beta)\Gamma_m(\alpha+\beta-\nu+1)\Gamma_q(\beta+1)\Gamma_p(\alpha+\beta+1)} \\
<{}& \|u-v\|_{\mathcal{C}}\left[\lambda_1(L_M+L) + \lambda_2\left(\ell_N + \ell\frac{\varphi_0 T^{\theta}}{\Gamma_w(\theta+1)}\right)\right] \\
&\times \frac{\mu T^{M+\beta-\nu}\Gamma_p(M)\Gamma_m(M+\beta+1)\Gamma_p(\alpha+1)}{\Gamma_p(M+\beta)\Gamma_m(M+\beta-\nu)\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)} \\
&\times \left\{\left|\xi^{\alpha+\beta} - \frac{\tau G\eta^{\alpha+\beta+\vartheta}\Gamma_n(\alpha+\beta+1)}{\Gamma_n(\alpha+\beta+\vartheta+1)}\right| + \left|\xi^{M+\beta-1} - \frac{\tau G\eta^{M+\beta+\vartheta-1}\Gamma_n(M+\beta)}{\Gamma_n(M+\beta+\vartheta)}\right|\right\} \\
={}& \|u-v\|_{\mathcal{C}}\left[\lambda_1(L_M+L) + \lambda_2\left(\ell_N + \ell\frac{\varphi_0 T^{\theta}}{\Gamma_w(\theta+1)}\right)\right]\Omega_1. \qquad (3.5)
\end{aligned}
$$

Similarly to above, we have

$$
\begin{aligned}
&\bigl|\mathcal{Q}\bigl[F(u)+H(u)\bigr] - \mathcal{Q}\bigl[F(v)+H(v)\bigr]\bigr| \\
\leq{}& \left|\mu\int_0^T \frac{(T-ms)^{(-\nu-1)}}{\Gamma_m(-\nu)} s^{M-1}\,d_m s\left[\tau\int_0^\eta\int_0^s\int_0^y \frac{(\eta-ns)^{(\vartheta-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_n(\vartheta)}\right.\right. \\
&\times g(\eta)\bigl[\lambda_1\bigl|\mathcal{F}\bigl[x,u,v,D_r^{\gamma-i}\bigr]\bigr| + \lambda_2\bigl|\mathcal{H}\bigl[x,u,v,\Psi_w^{\theta-j}\bigr]\bigr|\bigr]\,d_q x\,d_p y\,d_n s \\
&- \int_0^\xi\int_0^s \frac{(\xi-ps)^{(\beta-1)}(s-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)}\bigl[\lambda_1\bigl|\mathcal{F}\bigl[x,u,v,D_r^{\gamma-i}\bigr]\bigr| \\
&\left. + \lambda_2\bigl|\mathcal{H}\bigl[x,u,v,\Psi_w^{\theta-j}\bigr]\bigr|\bigr]\,d_q x\,d_p s\right] \\
&+ \left[\xi^{N-1} - \tau\int_0^\eta \frac{(\eta-ns)^{(\vartheta-1)}}{\Gamma_n(\vartheta)} s^{N-1} g(\eta)\,d_n s\right] \\
&\times \mu\int_0^T\int_0^s\int_0^y \frac{(T-ms)^{(-\nu-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_m(-\nu)} \\
&\left.\times \bigl[\lambda_1\bigl|\mathcal{F}\bigl[x,u,v,D_r^{\gamma-i}\bigr]\bigr| + \lambda_2\bigl|\mathcal{H}\bigl[x,u,v,\Psi_w^{\theta-j}\bigr]\bigr|\bigr]\,d_q x\,d_p y\,d_m s\right| \\
\leq{}& \|u-v\|_{\mathcal{C}}\left[\lambda_1(L_M+L) + \lambda_2\left(\ell_N + \ell \max_{j\in\mathbb{N}_{0,N-1}}\left\{\frac{\varphi_0 T^{\theta-j}}{\Gamma_w(\theta-j+1)}\right\}\right)\right]\mu T^{M-\nu-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{ \left| \xi^{\alpha+\beta} - \frac{\tau G \eta^{\alpha+\beta+\vartheta} \Gamma_n(\alpha+\beta+1)}{\Gamma_n(\alpha+\beta+\vartheta+1)} \right| \frac{\Gamma_m(M)}{\Gamma_m(M-\nu)} \right. \\
&\left. + \left| \xi^{N-1} - \frac{\tau G \eta^{N+\vartheta-1} \Gamma_n(N)}{\Gamma_n(N+\vartheta)} \right| \frac{T^{\beta+1}\Gamma_p(\alpha)\Gamma_m(\alpha+\beta+1)\Gamma_p(\alpha+1)}{\Gamma_p(\alpha+\beta)\Gamma_m(\alpha+\beta-\nu+1)\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)} \right\} \\
&< \|u-v\|_{\mathcal{C}} \left[\lambda_1(L_M+L) + \lambda_2 \left(\ell_N + \ell \frac{\varphi_0 T^\theta}{\Gamma_w(\theta+1)} \right) \right] \Omega_2.
\end{aligned} \tag{3.6}
$$

Therefore, we find that

$$
\begin{aligned}
&\left| (\mathcal{A}u)(t) - (\mathcal{A}v)(t) \right| \\
&\leq \left| \frac{t^{N-1}}{\Lambda} \left[\mathcal{P}\left[F(u)+H(u)\right] - \mathcal{P}\left[F(v)+H(v)\right] \right] \right. \\
&\quad - \frac{1}{\Lambda} \left[\mathcal{Q}\left[F(u)+H(u)\right] - \mathcal{Q}\left[F(v)+H(v)\right] \right] \\
&\quad \times \int_0^t \frac{(t-ps)^{(\beta-1)}}{\Gamma_p(\beta)} s^{M-1} \, d_p s + \int_0^t \int_0^s (t-ps)^{(\beta-1)} \frac{(s-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\quad \left. \times \left(\lambda_1 F\left[x,u(x),D_r^{\gamma-i}u(x)\right] + \lambda_2 H\left[x,u(x),\Psi_w^{\theta-j}u(x)\right] \right) d_q x \, d_p s \right| \\
&< \|u-v\|_{\mathcal{C}} \left[\lambda_1(L_M+L) + \lambda_2 \left(\ell_N + \ell \frac{\varphi_0 T^\theta}{\Gamma_w(\theta+1)} \right) \right] \\
&\quad \times \left\{ \frac{\Omega_1}{|\Lambda|} T^{\beta-1} + \frac{\Omega_2}{|\Lambda|} \frac{T^{\alpha+\beta-1}\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)} + \frac{T^{\alpha+\beta}\Gamma_p(\alpha+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)} \right\} \\
&= \|u-v\|_{\mathcal{C}} \Theta.
\end{aligned} \tag{3.7}
$$

Next, taking r-derivative of order γ for (3.1), we have

$$
\begin{aligned}
\left(D_r^{\gamma-i}\mathcal{A}u\right)(t) &= \frac{\mathcal{P}[F(u)+H(u)]}{\Lambda} \int_0^t \frac{(t-rs)^{(i-\gamma-1)}}{\Gamma_r(i-\gamma)} s^{N-1} \, d_r s \\
&\quad - \frac{\mathcal{Q}[F(u)+H(u)]}{\Lambda} \int_0^t \int_0^s \frac{(t-rs)^{(i-\gamma-1)}}{\Gamma_r(i-\gamma)} \frac{(s-px)^{(\beta-1)}}{\Gamma_p(\beta)} x^{M-1} \, d_p x \, d_r s \\
&\quad + \int_0^t \int_0^s \int_0^y \frac{(t-rs)^{(i-\gamma-1)}}{\Gamma_r(i-\gamma)} \frac{(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\quad \times \left(\lambda_1 F\left[x,u(x),D_r^{\gamma-i}u(x)\right] + \lambda_2 H\left[x,u(x),\Psi_w^{\theta-j}u(x)\right] \right) d_q x \, d_p y \, d_r s.
\end{aligned} \tag{3.8}
$$

Further, for any $u, v \in \mathcal{C}$ and $t \in I_\chi^T$, we obtain

$$
\begin{aligned}
&\left| \left(D_r^{\gamma-i}\mathcal{A}u\right)(t) - \left(D_r^{\gamma-i}\mathcal{A}v\right)(t) \right| \\
&\leq \left| \frac{1}{\Lambda} \left[\mathcal{P}\left[F(u)+H(u)\right] - \mathcal{P}\left[F(v)+H(v)\right] \right] \int_0^t \frac{(t-rs)^{(i-\gamma-1)}}{\Gamma_r(i-\gamma)} s^{N-1} \, d_r s \right. \\
&\quad - \frac{1}{\Lambda} \left[\mathcal{Q}\left[F(u)+G(u)\right] - \mathcal{Q}\left[F(v)+G(v)\right] \right] \\
&\quad \times \int_0^t \int_0^s \frac{(t-rs)^{(i-\gamma-1)}}{\Gamma_r(i-\gamma)} \frac{(s-px)^{(\beta-1)}}{\Gamma_p(\beta)} x^{M-1} \, d_p x \, d_r s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_0^s\int_0^y \frac{(t-rs)^{(i-\gamma-1)}}{\Gamma_r(i-\gamma)}\frac{(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\times \bigl(\lambda_1 F\bigl[x,u(x),D_r^{\gamma-i}u(x)\bigr]+\lambda_2 H\bigl[x,u(x),\Psi_w^{\theta-j}u(x)\bigr]\bigr)\,d_qx\,d_py\,d_rs\biggr| \\
&< \|u-v\|_{\mathcal{C}}\left[\lambda_1(L_M+L)+\lambda_2\left(\ell_N+\ell\frac{\varphi_0 T^{\theta}}{\Gamma_w(\theta+1)}\right)\right]\left\{\frac{\Omega_1}{|\Lambda|}\frac{T^{N-\gamma-1}\Gamma_r(N)}{\Gamma_r(N-\gamma)}\right. \\
&\left.+\frac{\Omega_2}{|\Lambda|}\frac{T^{M+\beta-\gamma-1}\Gamma_p(M)\Gamma_r(M+\beta)}{\Gamma_p(M+\beta)\Gamma_r(M+\beta-\gamma)}+\frac{T^{\alpha+\beta-\gamma}\Gamma_p(\alpha+1)\Gamma_r(\alpha+\beta+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)\Gamma_r(\alpha+\beta-\gamma+1)}\right\} \\
&< \|u-v\|_{\mathcal{C}}\Theta. \qquad (3.9)
\end{aligned}
$$

Hence, we obtain

$$\|\mathcal{A}u-\mathcal{A}v\|_{\mathcal{C}} \le \|u-v\|_{\mathcal{C}}\Theta. \qquad (3.10)$$

We can conclude from (H_4) that $\mathcal{A}$ is a contraction. The proof is completed by using Banach's contraction mapping principle. □

The following theorems show the existence of at least one solution to the boundary value problem (1.6) by employing the Leray–Schauder nonlinear alternative.

Theorem 3.2 (Nonlinear alternative for single valued maps [35]) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact [that is, $F(\overline{U})$ is a relatively compact subset of C] map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\sigma \in (0,1)$ with $u = \sigma F(u)$.*

Theorem 3.3 *Assume that (H_3) holds and the functions F, H satisfy the following conditions:*

(H_5) *for $i = 1,2$ there exist continuous nondecreasing functions $\psi, \phi_i : [0,\infty) \to (0,\infty)$ and functions $z_i \in L^1(I_\chi^T, \mathbb{R}^+)$, such that for all $(t,u) \in I_\chi^T \times \mathbb{R}$*

$$
\begin{aligned}
&\bigl|F\bigl[t,u(t),D_r^{\gamma-i}u(t)\bigr]\bigr| \le z_1(t)\psi_1\bigl(\|u\|\bigr), \quad i \in \mathbb{N}_{0,M-1} \quad \text{and} \\
&\bigl|H\bigl[t,u(t),\Psi_w^{\theta-j}u(t)\bigr]\bigr| \le z_2(t)\psi_2\bigl(\|u\|\bigr), \quad j \in \mathbb{N}_{0,N-1},
\end{aligned}
$$

(H_6) *there exists a constant $K > 0$ such that*

$$\frac{K}{(\lambda_1\psi_1(K)\|z_1\|_{L^1}+\lambda_2\psi_2(K)\|z_2\|_{L^1})\Theta} > 1.$$

Then the boundary value problem (1.5)–(1.6) has at least one solution on I_χ^T.

Proof To show that $\mathcal{A}$ *maps bounded sets (balls) into bounded sets in* $\mathcal{C}$, the constructive proof is as follows. For a positive number ρ, let $B_\rho = \{u \in C(I_\chi^T,\mathbb{R}) : \|u\|_{\mathcal{C}} \le \rho\}$ be a bounded ball in $C(I_\chi^T,\mathbb{R})$. Then for $t \in I_\chi^T$ we have

$$
\begin{aligned}
&\bigl|\mathcal{P}\bigl[F(u)+H(u)\bigr]\bigr| \\
&\le \mu\int_0^T\int_0^s \frac{(T-ms)^{(-\nu-1)}(s-py)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_m(-\nu)}y^{\alpha-1}\,d_py\,d_ms
\end{aligned}
$$

$$
\begin{aligned}
&\times \left|\tau \int_0^{\eta}\int_0^{s}\int_0^{y} \frac{(\eta - ns)^{(\vartheta-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_n(\vartheta)} g(\eta)\right.\\
&\times \left(\lambda_1\psi_1\left(\|u\|\right)\left|z_1(x)\right| + \lambda_2\psi_2\left(\|u\|\right)\left|z_2(x)\right|\right) d_qx\, d_py\, d_ns\\
&\left.- \int_0^{\xi}\int_0^{s} \frac{(\xi - ps)^{(\beta-1)}(s-qy)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)} \left(\lambda_1\psi_1\left(\|u\|\right)\left|z_1(y)\right| + \lambda_2\psi_2\left(\|u\|\right)\left|z_2(y)\right|\right) d_qy\, d_ps\right|\\
&+ \left|\int_0^{\xi} \frac{(\xi - ps)^{(\beta-1)}}{\Gamma_p(\beta)} s^{\alpha-1}\, d_ps - \tau \int_0^{\eta}\int_0^{s} \frac{(\eta - ns)^{(\vartheta-1)}(s-py)^{(\beta-1)}}{\Gamma_p(\beta)\Gamma_n(\vartheta)} g(\eta) y^{\alpha-1}\, d_py\, d_ns\right|\\
&\times \mu \int_0^{T}\int_0^{s}\int_0^{y} \frac{(T - ms)^{(-\nu-1)}(s-py)^{(\beta-1)}(y-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_m(-\nu)}\\
&\times \left(\lambda_1\psi_1\left(\|u\|\right)\left|z_1(x)\right| + \lambda_2\psi_2\left(\|u\|\right)\left|z_2(x)\right|\right) d_qx\, d_py\, d_ms\\
\le{}& \left(\lambda_1\psi_1\left(\|u\|\right)\|z_1\|_{L^1} + \lambda_2\psi_2\left(\|u\|\right)\|z_2\|_{L^1}\right)\\
&\times \frac{\mu T^{M+\beta-\nu}\Gamma_p(M)\Gamma_m(M+\beta)\Gamma_p(\alpha+1)}{\Gamma_p(M+\beta)\Gamma_m(M+\beta-\nu)\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)}\\
&\times \left\{\left|\xi^{\alpha+\beta} - \frac{\tau G \eta^{\alpha+\beta+\vartheta}\Gamma_n(\alpha+\beta+1)}{\Gamma_n(\alpha+\beta+\vartheta+1)}\right|\right.\\
&\left.+ \left|\xi^{M+\beta-1} - \frac{\tau G \eta^{M+\beta+\vartheta-1}\Gamma_n(M+\beta)}{\Gamma_n(M+\beta+\vartheta)}\right|\right\}\\
={}& \left(\lambda_1\psi_1\left(\|u\|\right)\|z_1\|_{L^1} + \lambda_2\psi_2\left(\|u\|\right)\|z_2\|_{L^1}\right)\Omega_1. \qquad (3.11)
\end{aligned}
$$

Using the same argument as above, we have

$$
\begin{aligned}
&\left|\mathcal{Q}\left[F(u)+H(u)\right] - \mathcal{Q}\left[F(v)+H(v)\right]\right|\\
&\le \left(\lambda_1\psi_1\left(\|u\|\right)\|z_1\|_{L^1} + \lambda_2\psi_2\left(\|u\|\right)\|z_2\|_{L^1}\right)\mu T^{M-\nu-1}\\
&\quad\times \left\{\left|\xi^{\alpha+\beta} - \frac{\tau G \eta^{\alpha+\beta+\vartheta}\Gamma_n(\alpha+\beta+1)}{\Gamma_n(\alpha+\beta+\vartheta+1)}\right| \frac{\Gamma_m(M)}{\Gamma_m(M-\nu)} + \left|\xi^{N-1} - \frac{\tau G \eta^{N+\vartheta-1}\Gamma_n(N)}{\Gamma_n(N+\vartheta)}\right|\right.\\
&\quad\left.\times \frac{T^{\beta+1}\Gamma_p(\alpha)\Gamma_m(\alpha+\beta+1)\Gamma_p(\alpha+1)}{\Gamma_p(\alpha+\beta)\Gamma_m(\alpha+\beta-\nu+1)\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)}\right\}\\
&= \left(\lambda_1\psi_1\left(\|u\|\right)\|z_1\|_{L^1} + \lambda_2\psi_2\left(\|u\|\right)\|z_2\|_{L^1}\right)\Omega_2. \qquad (3.12)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\left|(\mathcal{A}u)(t)\right| &\le \left(\lambda_1\psi_1\left(\|u\|\right)\|z_1\|_{L^1} + \lambda_2\psi_2\left(\|u\|\right)\|z_2\|_{L^1}\right)\\
&\quad\times \left\{\frac{\Omega_1}{|\Lambda|} T^{\beta-1} + \frac{\Omega_2}{|\Lambda|} \frac{T^{\alpha+\beta-1}\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)} + \frac{T^{\alpha+\beta}\Gamma_p(\alpha+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)}\right\}\\
&:= \left(\lambda_1\psi_1\left(\|u\|\right)\|z_1\|_{L^1} + \lambda_2\psi_2\left(\|u\|\right)\|z_2\|_{L^1}\right)\Theta, \qquad (3.13)
\end{aligned}
$$

and, for $i \in \mathbb{N}_{0,M-1}$,

$$
\begin{aligned}
&\left|\left(D_r^{\gamma-i}\mathcal{A}u\right)(t)\right|\\
&\le \left(\lambda_1\psi_1\left(\|u\|\right)\|z_1\|_{L^1} + \lambda_2\psi_2\left(\|u\|\right)\|z_2\|_{L^1}\right)\left\{\frac{\Omega_1}{|\Lambda|} \frac{T^{N-\gamma-1}\Gamma_r(N)}{\Gamma_r(N-\gamma)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\Omega_2}{|\Lambda|} \frac{T^{M+\beta-\gamma-1}\Gamma_p(M)\Gamma_r(M+\beta)}{\Gamma_p(M+\beta)\Gamma_r(M+\beta-\gamma)} + \frac{T^{\alpha+\beta-\gamma}\Gamma_p(\alpha+1)\Gamma_r(\alpha+\beta+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)\Gamma_r(\alpha+\beta-\gamma+1)} \Bigg\} \\
&\leq \big(\lambda_1\psi_1\big(\|u\|\big)\|z_1\|_{L^1} + \lambda_2\psi_2\big(\|u\|\big)\|z_2\|_{L^1}\big)\Theta. \qquad (3.14)
\end{aligned}
$$

Consequently, $\|\mathcal{A}u\|_{\mathcal{C}} \leq (\lambda_1\psi_1(\|u\|)\|z_1\|_{L^1} + \lambda_2\psi_2(\|u\|)\|z_2\|_{L^1})\Theta$.

Further, we will show that $\mathcal{A}$ *maps bounded sets into equicontinuous sets of* $C(I_\chi^T,\mathbb{R})$. Letting $t_1, t_2 \in I_\chi^T$ with $t_1 \leq t_2$ and $u \in B_\rho$, we have

$$
\begin{aligned}
&\big|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)\big| \\
&\leq \frac{1}{|\Lambda|}\mathcal{P}\big[F(u)+G(u)\big]\big|t_2^{N-1} - t_1^{N-1}\big| \\
&+ \frac{1}{|\Lambda|}\mathcal{Q}\big[F(u)+G(u)\big]\bigg|\int_0^{t_2}\frac{(t_2-ps)^{(\beta-1)}}{\Gamma_p(\beta)}s^{M-1}\,d_ps - \int_0^{t_1}\frac{(t_1-ps)^{(\beta-1)}}{\Gamma_p(\beta)}s^{M-1}\,d_ps\bigg| \\
&+ \big(\lambda_1\psi_1\big(\|u\|\big)\|z_1\|_{L^1} + \lambda_2\psi_2\big(\|u\|\big)\|z_2\|_{L^1}\big) \\
&\times \bigg|\int_0^{t_2}\int_0^s (t_2-ps)^{(\beta-1)}\frac{(s-qy)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)}\,d_qy\,d_ps \\
&- \int_0^{t_1}\int_0^s (t_1-ps)^{(\beta-1)}\frac{(s-qx)^{(\alpha-1)}}{\Gamma_p(\beta)\Gamma_q(\alpha)}\,d_qx\,d_ps\bigg| \\
&\leq \big(\lambda_1\psi_1\big(\|u\|\big)\|z_1\|_{L^1} + \lambda_2\psi_2\big(\|u\|\big)\|z_2\|_{L^1}\big)\bigg\{\frac{\Omega_1}{|\Lambda|}\big|t_2^{N-1} - t_1^{N-1}\big| \\
&+ \frac{\Omega_2}{|\Lambda|}\frac{\Gamma_p(M)}{\Gamma_p(M+\beta)}\big|t_1^{M+\beta-1} - t_2^{M+\beta-1}\big| \\
&+ \frac{\Gamma_p(\alpha+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)}\big|t_2^{\alpha+\beta} - t_1^{\alpha+\beta}\big|\bigg\}, \qquad (3.15)
\end{aligned}
$$

and, for $i \in \mathbb{N}_{0,M-1}$,

$$
\begin{aligned}
&\big|\big(D_r^{\gamma-i}\mathcal{A}u\big)(t_2) - \big(D_r^{\gamma-i}\mathcal{A}u\big)(t_1)\big| \\
&\leq \big(\lambda_1\psi_1\big(\|u\|\big)\|z_1\|_{L^1} + \lambda_2\psi_2\big(\|u\|\big)\|z_2\|_{L^1}\big) \\
&\times \bigg\{\frac{\Omega_1}{|\Lambda|}\frac{\Gamma_r(N)}{\Gamma_r(N-\gamma)}\big|t_2^{N-\gamma-1} - t_1^{N-\gamma-1}\big| \\
&+ \frac{\Omega_2}{|\Lambda|}\frac{\Gamma_p(M)\Gamma_r(M+\beta)}{\Gamma_p(M+\beta)\Gamma_r(M+\beta-\gamma)}\big|t_1^{M+\beta-\gamma-1} - t_2^{M+\beta-\gamma-1}\big| \\
&+ \frac{\Gamma_p(\alpha+1)\Gamma_r(\alpha+\beta+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)\Gamma_r(\alpha+\beta-\gamma+1)}\big|t_2^{\alpha+\beta-\gamma} - t_1^{\alpha+\beta-\gamma}\big|\bigg\}. \qquad (3.16)
\end{aligned}
$$

As $t_2 - t_1 \to 0$, the right hand side of (3.16) tends to zero independently of $u \in B_\rho$. As $\mathcal{A}$ satisfies the above assumptions, it follows by the Arzelá–Ascoli theorem that $\mathcal{A}: C(I_\chi^T,\mathbb{R}) \to C(I_\chi^T,\mathbb{R})$ is completely continuous.

Let u be a solution. Proceeding by similar computations to the first step for $t \in I_\chi^T$, we have

$$
\big|u(t)\big| \leq \big(\lambda_1\psi_1\big(\|u\|\big)\|z_1\|_{L^1} + \lambda_2\psi_2\big(\|u\|\big)\|z_2\|_{L^1}\big)\Theta.
$$

Hence,

$$\frac{\|u\|_{\mathcal{C}}}{(\lambda_1\psi_1(\|u\|)\|z_1\|_{L^1}+\lambda_2\psi_2(\|u\|)\|z_2\|_{L^1})\Theta}\le 1.$$

Under (H_6), there exists K such that $\|u\|_{\mathcal{C}}\neq K$. We set

$$U=\{u\in C(I_\chi^T,\mathbb{R}):\|u\|_{\mathcal{C}}<K\}.$$

Note that the operator $\mathcal{A}:\overline{U}\to C(I_\chi^T,\mathbb{R})$ is continuous and completely continuous. We find that there is no $u\in\partial U$ such that $u=\sigma\mathcal{A}u$ for some $\sigma\in(0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Theorem 3.2), we can conclude that $\mathcal{A}$ has a fixed point $u\in\overline{U}$ which is a solution of the problem (1.5)–(1.6). This completes the proof. □

4 Example

The following boundary value problem is an example illustrating our main result. Consider the second-order q-difference equation with q-integral boundary conditions

$$\begin{aligned}
{}^{C}D_{\frac{1}{2}}^{\frac{5}{2}}\,{}^{C}D_{\frac{1}{3}}^{\frac{4}{3}}u(t)&=\frac{e^{-\sin^2(2\pi t+5)}}{200+e^{\cos^2(2\pi t)}}\cdot\frac{|u(t)|+|D_{\frac{3}{4}}^{\frac{4}{3}}u(t)|+|D_{\frac{3}{4}}^{\frac{1}{3}}u(t)|}{[1+|u(t)|]}\\
&\quad+\frac{e^{-\cos^2(2\pi t)}|u(t)|+|\Psi_{\frac{2}{5}}^{\frac{1}{4}}u(t)|}{(t+10)^2[1+|u(t)|]},\\
u(0)&=D_{\frac{1}{3}}u(0)=D_{\frac{1}{3}}^{\frac{4}{3}}u(0)=D_{\frac{1}{3}}^{\frac{7}{3}}u(0)=D_{\frac{1}{3}}^{\frac{10}{3}}u(0)=0,\\
D_{\frac{2}{3}}^{\frac{1}{5}}u(0)&=10D_{\frac{2}{3}}^{\frac{1}{5}}u(6),\qquad u\left(\frac{1}{10}\right)=2I_{\frac{1}{4}}^{\frac{3}{4}}e^{\cos(\frac{\pi}{36{,}000})}u\left(\frac{\pi}{36{,}000}\right),
\end{aligned}\tag{4.1}$$

where $t\in I_{\frac{1}{60}}^{6}=\{6(\frac{1}{60})^n:n\in\mathbb{N}\}\cup\{0,3\}$ and $\Psi_{\frac{2}{5}}^{\frac{1}{4}}u(t)=\frac{1}{\Gamma_{\frac{2}{3}}(\frac{1}{4})}\int_0^t\frac{e^{-s}}{(t+20)^2}\cdot u(s)\,d_{\frac{2}{3}}s$.

We apply Theorem 3.1 when $q=\frac{1}{2}$, $p=\frac{1}{3}$, $r=\frac{3}{4}$, $w=\frac{2}{5}$, $m=\frac{2}{3}$, $n=\frac{1}{4}$, $M=3, N=2$, $\alpha=\frac{5}{2}, \beta=\gamma=\frac{4}{3}, \theta=\frac{1}{4}, \nu=\frac{1}{5}, \vartheta=\frac{3}{4}, T=6$, $\lambda_1=e^{-5}, \lambda_2=1, \mu=10, \tau=2$, $\chi=\frac{1}{60}, \xi=\frac{1}{10}, \eta=\frac{1}{36{,}000}$ and $\varphi_0=\sup\{\varphi(t,s)\}=\frac{1}{400}$.

Since

$$\begin{aligned}
\big|F(t,u,D_r^\gamma u)-F(t,v,D_r^\gamma v)\big|&\le\frac{1}{201e}\big[|u-v|+\big|D_r^\gamma u-D_r^\gamma v\big|+\big|D_r^{\gamma-1}u-D_r^{\gamma-1}v\big|\big],\\
\big|H(t,u,\Psi_w^\theta u)-H(t,v,\Psi_w^\theta v)\big|&\le\frac{1}{100e}|u-v|+\frac{1}{100}\big|\Psi_w^\theta u-\Psi_w^\theta v\big|,
\end{aligned}$$

so (H_1)–(H_2) are satisfied with $L_1=L_2=L_3=\frac{1}{201e}, L=\frac{2}{201e}$ and $\ell_2=\frac{1}{100e}, \ell_1=\frac{1}{100}=\ell$.

In addition, since $\frac{1}{e}\le g(t)\le e$, then (H_3) is satisfied with $G=e$. Moreover, we can show that

$$|\Lambda|=22.0923,\qquad\Omega_1=10.3252\quad\text{and}\quad\Omega_2=28.9938.$$

We obtain

$$\Theta \approx 0.0329 < 1.$$

It implies that (H_4) holds. From Theorem 3.1, we can conclude that the assigned problem (4.1) has a unique solution on $I^6_{\frac{1}{60}}$.

5 Conclusion

We have proved the existence results of the four-point fractional q-integral and Riemann–Liouville fractional q-derivative boundary value problem for a sequential Caputo fractional q-integrodifference equation involving separate nonlinearity (1.5)–(1.6), by using the Banach contraction mapping principle as regards the existence and uniqueness of a solution, and the Leray–Schauder nonlinear alternative for the existence of at least a solution. Our problem contains $2(M + N + 1)$ different orders and six different numbers of q in derivatives and integral, which is a new idea.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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