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On nonlocal Dirichlet boundary value problem for sequential Caputo fractional Hahn integrodifference equations

Nichaphat Patanarapeelert[1], Tanapat Brikshavana[2] and Thanin Sitthiwirattham[2*]

*Correspondence: thanin_sit@dusit.ac.th
[2]Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, Thailand
Full list of author information is available at the end of the article

Abstract

The aim of this work is to study a nonlocal Dirichlet boundary value problem for sequential Caputo fractional Hahn integrodifference equation. The problem contains two fractional Hahn difference operators and a fractional Hahn integral with different numbers of order. We use the Banach fixed point theorem to prove the existence and uniqueness of the solution. In particular, the existence of at least one solution is presented by using the Schauder fixed point theorem.

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Keywords: fractional Hahn integrodifference equations; boundary value problems; existence; fixed point theorem

1 Introduction

In this paper, we study the quantum calculus that is the calculus without considering limits based on the sets of non-differentiable functions. Some examples of quantum difference operators are the Jackson q-difference operator, the forward (delta) difference operator, and the backward (nabla) difference operator. The quantum difference operators have been employed in many applications [1–9] of mathematical areas such as the theory of relativity, particle physics, and quantum mechanics.

In this work, we focus on the Hahn difference operator that is the generalization of the forward difference operator and the Jackson q-difference operator.

The Hahn difference operator $D_{q,\omega}$ is defined by [10]

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, \quad t \neq \omega_0 := \frac{\omega}{1-q}.$$

Note that

$$D_{q,\omega}f(t) = \Delta_\omega f(t) \quad \text{whenever } q = 1, \qquad D_{q,\omega}f(t) = D_q f(t) \quad \text{whenever } \omega = 0, \quad \text{and}$$
$$D_{q,\omega}f(t) = f'(t) \quad \text{whenever } q = 1, \omega \to 0.$$

This operator is used to construct families of orthogonal polynomials (see [11, 12] and the references therein).

In 2009, Aldwoah [13, 14] defined the right inverse of $D_{q,\omega}$ in terms of the Jackson q-integral containing the right inverse of D_q [15] and the Nörlund sum containing the right inverse of Δ_ω [15].

Next, the Hahn quantum variational calculus was introduced by Malinowska and Torres [16, 17] in 2010, and the generalized transversality conditions for the Hahn quantum variational calculus were described by Malinowska and Martins [18] in 2013.

The studies of existence and uniqueness results for the initial value problems for Hahn difference equations were presented in 2013 by Hamza *et al.* [19, 20] by using the method of successive approximations. In addition, they proved Gronwall's and Bernoulli's inequalities with respect to the Hahn difference operator and also established the mean value theorems for this calculus.

Recently, Sitthiwirattham [21] proposed a nonlinear Hahn difference equation with nonlocal boundary value conditions of the form

$$\begin{aligned} &D^2_{q,\omega}x(t) + f\bigl(t, x(t), D_{p,\theta}x(pt+\theta)\bigr) = 0, \quad t \in [\omega_0, T]_{q,\omega}, \\ &x(\omega_0) = \varphi(x), \\ &x(T) = \lambda x(\eta), \quad \eta \in (\omega_0, T)_{q,\omega}, \end{aligned} \tag{1.1}$$

where $0 < q < 1$, $0 < \omega < T$, $\omega_0 := \frac{\omega}{1-q}$, $1 \le \lambda < \frac{T-\omega_0}{\eta-\omega_0}$, $p = q^m$, $m \in \mathbb{N}$, $\theta = \omega(\frac{1-p}{1-q})$, $f : [\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and $\varphi : C([\omega_0, T]_{q,\omega}, \mathbb{R}) \to \mathbb{R}$ is a given functional.

In 2017, Sriphanomwan *et al.* [22] considered a nonlocal boundary value problem for second-order nonlinear Hahn integro-difference equation with integral boundary condition of the form

$$\begin{aligned} &D^2_{q,\omega}x(t) = f\bigl(t, x(t), D_{p,\theta}x(pt+\theta), \Psi_{p,\theta}x(pt+\theta)\bigr), \quad t \in [\omega_0, T]_{q,\omega}, \\ &x(\omega_0) = x(T), \qquad x(\eta) = \mu \int_{\omega_0}^{T} g(s)x(s)\, d_{q,\omega}s, \quad \eta \in (\omega_0, T)_{q,\omega}, \end{aligned} \tag{1.2}$$

where $0 < q < 1$, $0 < \omega < T$, $\omega_0 := \frac{\omega}{1-q}$, $\mu \int_{\omega_0}^{T} g(r)\, d_{q,\omega}r \neq 1$, $\mu \in \mathbb{R}$, $p = q^m$, $m \in \mathbb{N}$, $\theta = \omega(\frac{1-p}{1-q})$, $f \in C([\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $g \in C([\omega_0, T]_{q,\omega}, \mathbb{R}^+)$ are given functions, and for $\varphi \in C([\omega_0, T]_{q,\omega} \times [\omega_0, T]_{q,\omega}, [0, \infty))$,

$$\Psi_{p,\theta}x(t) := \int_{\omega_0}^{t} \varphi(t, ps+\theta)x(ps+\theta)\, d_{p,\theta}s.$$

For the fractional Hahn difference operators, Čermák and Nechvátal [23] introduced the fractional (q,h)-difference operator and the fractional (q,h)-integral for $q > 1$. Next, Čermák *et al.* [24] presented discrete Mittag-Leffler functions in linear fractional difference equations for $q > 1$. On discrete time scales for $q > 1$, Rahmat [25, 26] considered the (q,h)-Laplace transform and some (q,h)-analogues of integral inequalities. In 2016, Du *et al.* [27] studied the monotonicity and convexity for nabla fractional (q,h)-difference for $q > 0$, $q \neq 1$.

We observed from the above research that these operators are not fractional Hahn operators because the conditions are not satisfied with $0 < q < 1$. To fill this gap, Brikshavana and Sitthiwirattham [28] introduced the fractional Hahn operators.

In this paper, we study the boundary value problem for fractional Hahn difference equation containing a sequential Caputo fractional Hahn integrodifference equation with nonlocal Dirichlet boundary conditions. The governing problem is given by

$$\begin{aligned} &{}^{C}D^{\alpha}_{q,\omega}\,{}^{C}D^{\beta}_{q,\omega}\left[\frac{E_{\sigma_{q,\omega}}}{\rho_{q,\omega}(t)}+qD_{q,\omega}\right]u(t)=F\big(t,u(t),\Psi^{\gamma}_{q,\omega}u(t)\big),\quad t\in[\omega_0,T]_{q,\omega},\\ &u(\omega_0)=\phi(u),\\ &\rho_{q,\omega}(T)u(T)=\rho_{q,\omega}(\eta)u(\eta)=\psi(u),\quad \eta\in(\omega_0,T)_{q,\omega}, \end{aligned} \tag{1.3}$$

where $[\omega_0,T]_{q,\omega}=I^T_{q,\omega}:=\{q^kT+\omega[k]_q:k\in\mathbb{N}_0\}\cup\{\omega_0\}$; $\omega>0$, $q\in(0,1)$; $\alpha,\beta,\gamma\in(0,1]$; the shift operator $E_{\sigma_{q,\omega}}u(t):=u(\sigma_{q,\omega}(t))$; $F\in C(I^T_{q,\omega}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a given function; $\phi,\psi: C(I^T_{q,\omega},\mathbb{R})\to\mathbb{R}$ are given functionals; and for $\varphi\in C(I^T_{q,\omega}\times I^T_{q,\omega},[0,\infty))$, we define

$$\Psi^{\gamma}_{q,\omega}u(t):=\big(\mathcal{I}^{\gamma}_{q,\omega}\varphi u\big)(t)=\frac{1}{\Gamma_{q,\omega}(\gamma)}\int_{\omega_0}^{t}\big(t-\sigma_{q,\omega}(s)\big)^{\underline{\gamma-1}}_{q,\omega}\varphi(t,s)u(s)\,d_{q,\omega}s.$$

We give some basic definitions and lemmas in the next section. In Section 3, we present the main results of the study. The existence and uniqueness of a solution to problem (1.3) are proved by using the Banach fixed point theorem. In addition, the existence of at least one solution to problem (1.3) is proved by using the Schauder fixed point theorem in Section 4. We end up with some example in the last section.

2 Preliminaries

In this section, we introduce notations, definitions, and lemmas used in the main results. For $q\in(0,1)$ and $\omega>0$, we define

$$[n]_q:=\frac{1-q^n}{1-q}=q^{n-1}+\cdots+q+1\quad\text{and}\quad[n]_q!:=\prod_{k=1}^{n}\frac{1-q^k}{1-q},\quad n\in\mathbb{R}.$$

The q-analogue of the power function $(a-b)^{\underline{n}}_q$ with $n\in\mathbb{N}_0:=[0,1,2,\ldots]$ is defined by

$$(a-b)^{\underline{n}}_q:=\prod_{k=0}^{n-1}\big(a-bq^k\big),\quad a,b\in\mathbb{R},$$

where $(a-b)^{\underline{0}}_q:=1$. The q,ω-analogue of the power function $(a-b)^{\underline{n}}_{q,\omega}$ with $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ is defined by

$$(a-b)^{\underline{n}}_{q,\omega}:=\prod_{k=0}^{n-1}\big[a-\big(bq^k+\omega[k]_q\big)\big],\quad a,b\in\mathbb{R},$$

where $(a-b)^{\underline{0}}_{q,\omega}:=1$. Generally, if $\alpha\in\mathbb{R}$, we have

$$\begin{aligned} &(a-b)^{\underline{\alpha}}_q=a^{\alpha}\prod_{n=0}^{\infty}\frac{1-(\frac{b}{a})q^n}{1-(\frac{b}{a})q^{\alpha+n}},\quad a\neq 0,\\ &(a-b)^{\underline{\alpha}}_{q,\omega}=(a-\omega_0)^{\alpha}\prod_{n=0}^{\infty}\frac{1-(\frac{b-\omega_0}{a-\omega_0})q^n}{1-(\frac{b-\omega_0}{a-\omega_0})q^{\alpha+n}}=\big((a-\omega_0)-(b-\omega_0)\big)^{\underline{\alpha}}_q,\quad a\neq\omega_0. \end{aligned}$$

Note that $a_q^{\underline{\alpha}} = a^\alpha$ and $(a-\omega_0)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha$. Here, we use the notation $(0)_q^{\underline{\alpha}} = (\omega_0)_{q,\omega}^{\underline{\alpha}} = 0$ for $\alpha > 0$. The following are the definitions of q-gamma and q-beta functions, respectively:

$$\Gamma_q(x) := \frac{(1-q)_q^{\underline{x-1}}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\},$$
$$B_q(x,s) := \int_0^1 t^{x-1}(1-qt)_q^{\underline{s-1}}\, d_q t = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)}.$$

Definition 2.1 ([10]) Assuming $q \in (0,1)$, $\omega > 0$ and letting f be the function defined on an interval $I \subseteq \mathbb{R}$ containing $\omega_0 := \frac{\omega}{1-q}$, the Hahn difference of f is defined by

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega} \quad \text{for } t \neq \omega_0.$$

In addition, $D_{q,\omega}f(\omega_0) = f'(\omega_0)$ when f is differentiable at ω_0. We call $D_{q,\omega}f$ the q,ω-derivative of f and f is q,ω-differentiable on I.

Remarks

(1) $D_{q,\omega}[f(t)+g(t)] = D_{q,\omega}f(t) + D_{q,\omega}g(t)$;
(2) $D_{q,\omega}[\alpha f(t)] = \alpha D_{q,\omega}f(t)$;
(3) $D_{q,\omega}[f(t)g(t)] = f(t)D_{q,\omega}g(t) + g(qt+\omega)D_{q,\omega}f(t)$;
(4) $D_{q,\omega}[\frac{f(t)}{g(t)}] = \frac{g(t)D_{q,\omega}f(t) - f(t)D_{q,\omega}g(t)}{g(t)g(qt+\omega)}$.

Letting $a, b \in I$, where $a < \omega_0 < b$ and $[k]_q = \frac{1-q^k}{1-q}$, $k \in \mathbb{N}_0$, we define the q,ω-interval by

$$\begin{aligned}[a,b]_{q,\omega} &:= \{q^k a + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{q^k b + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\} \\ &= [a,\omega_0]_{q,\omega} \cup [\omega_0, b]_{q,\omega} \\ &= (a,b)_{q,\omega} \cup \{a,b\} = [a,b)_{q,\omega} \cup \{b\} = (a,b]_{q,\omega} \cup \{a\}.\end{aligned}$$

For each $s \in [a,b]_{q,\omega}$, the sequence $\{\sigma_{q,\omega}^k(s)\}_{k=0}^\infty = \{q^k s + \omega[k]_q\}_{k=0}^\infty$ is uniformly convergent to ω_0.

We define the forward and backward jump operators as $\sigma_{q,\omega}^k(t) := q^k t + \omega[k]_q$ and $\rho_{q,\omega}^k(t) := \frac{t-\omega[k]_q}{q^k}$ for $k \in \mathbb{N}$, respectively.

Definition 2.2 ([13]) Let I be any closed interval of $\mathbb{R}$ containing a, b, and ω_0. Letting $f : I \to \mathbb{R}$ be a given function, we define q,ω-integral of f from a to b by

$$\int_a^b f(t)\, d_{q,\omega}t := \int_{\omega_0}^b f(t)\, d_{q,\omega}t - \int_{\omega_0}^a f(t)\, d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)\, d_{q,\omega}t := \big[x(1-q)-\omega\big]\sum_{k=0}^\infty q^k f\big(xq^k + \omega[k]_q\big), \quad x \in I.$$

Assuming that the series converges at $x = a$ and $x = b$, we say that f is q,ω-integrable on $[a,b]$, and the sum to the right-hand side is called the Jackson-Nörlund sum.

We note that the actual domain of function f is defined on $[a,b]_{q,\omega} \subset I$.

We next introduce the fundamental theorem of Hahn calculus in the following lemma.

Lemma 2.1 ([13]) *Let $f : I \to \mathbb{R}$ be continuous at ω_0 and define*

$$F(x) := \int_{\omega_0}^{x} f(t)\, d_{q,\omega}t, \quad x \in I.$$

Then F is continuous at ω_0. Furthermore, $D_{q,\omega_0}F(x)$ exists for every $x \in I$ and

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_{a}^{b} D_{q,\omega}F(t)\, d_{q,\omega}t = F(b) - F(a) \quad \text{for all } a, b \in I.$$

Lemma 2.2 ([21]) *Let $q \in (0,1)$, $\omega > 0$, and $f : I \to \mathbb{R}$ be continuous at ω_0. Then*

$$\int_{\omega_0}^{t}\int_{\omega_0}^{r} x(s)\, d_{q,\omega}s\, d_{q,\omega}r = \int_{\omega_0}^{t}\int_{qs+\omega}^{t} h(s)\, d_{q,\omega}r\, d_{q,\omega}s,$$

$$\int_{\omega_0}^{t} d_{q,\omega}s = t - \omega_0,$$

$$\int_{\omega_0}^{t} \bigl[t - \sigma_{q,\omega}(s)\bigr]\, d_{q,\omega}s = \frac{(t-\omega_0)^2}{1+q}.$$

In the sequel, we define fractional Hahn integral, fractional Hahn difference of Riemann-Liouville and Caputo types.

Definition 2.3 ([28]) For $\alpha, \omega > 0$, $q \in (0,1)$ and f defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn integral is defined by

$$\begin{aligned}
\mathcal{I}_{q,\omega}^{\alpha} f(t) &:= \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{t} \bigl(t - \sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{\alpha-1}} f(s)\, d_{q,\omega}s \\
&= \frac{[t(1-q)-\omega]}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n \bigl(t - \sigma_{q,\omega}^{n+1}(t)\bigr)_{q,\omega}^{\underline{\alpha-1}} f\bigl(\sigma_{q,\omega}^{n}(t)\bigr),
\end{aligned}$$

where $(\mathcal{I}_{q,\omega}^{0} f)(t) = f(t)$.

Definition 2.4 ([28]) For $\alpha, \omega > 0$, $q \in (0,1)$ and f defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn difference of the Caputo type of order α is defined by

$$\begin{aligned}
{}^{C}D_{q,\omega}^{\alpha} f(t) &:= \bigl(\mathcal{I}_{q,\omega}^{N-\alpha} D_{q,\omega}^{N} f\bigr)(t) \\
&= \frac{1}{\Gamma_q(N-\alpha)} \int_{\omega_0}^{t} \bigl(t - \sigma_{q,\omega}(s)\bigr)_{q,\omega}^{\underline{N-\alpha-1}} D_{q,\omega}^{N} f(s)\, d_{q,\omega}s,
\end{aligned}$$

and $D_{q,\omega}^{0} f(t) = {}^{C}D_{q,\omega}^{0} f(t) = f(t)$, where N is the smallest integer that is greater than α.

Lemma 2.3 ([28]) *Let* $\alpha > 0$, $q \in (0,1)$, $\omega > 0$, *and* $f : I_{q,\omega}^T \to \mathbb{R}$. *Then*

$$\mathcal{I}_{q,\omega}^{\alpha}\, {}^{C}D_{q,\omega}^{\alpha} f(t) = f(t) + C_0 + C_1(t-\omega_0) + \cdots + C_{N-1}(t-\omega_0)^{N-1}$$

for some $C_i \in \mathbb{R}$, $i = \mathbb{N}_{0,N-1}$ *and* $N-1 < \alpha \le N$, $N \in \mathbb{N}$.

The following lemma is used to simplify the calculations in this study.

Lemma 2.4 ([28]) *Letting* $\alpha, \beta > 0$, $p, q \in (0,1)$, *and* $\omega > 0$,

$$\begin{aligned}
&\int_{\omega_0}^{t} \big(t - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} (s-\omega_0)_{q,\omega}^{\underline{\beta}} \, d_{q,\omega}s = (t-\omega_0)^{\alpha+\beta} B_q(\beta+1,\alpha),\\
&\int_{\omega_0}^{t}\int_{\omega_0}^{x} \big(t - \sigma_{p,\omega}(x)\big)_{p,\omega}^{\underline{\alpha-1}} \big(x - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\beta-1}} \, d_{q,\omega}s \, d_{p,\omega}x = \frac{(t-\omega_0)^{\alpha+\beta}}{[\beta]_q} B_p(\beta+1,\alpha).
\end{aligned}$$

The next lemma presents the solution of the linear variant of problem (1.3).

Lemma 2.5 *Let* $\alpha, \beta \in (0,1)$, $\omega > 0$, $q \in (0,1)$, $h \in C(I_{q,\omega}^T, \mathbb{R})$ *be a given function, and* $\phi, \psi : C(I_{q,\omega}^T, \mathbb{R}) \to \mathbb{R}$ *be given functionals. Then the problem*

$$\begin{aligned}
&{}^{C}D_{q,\omega}^{\alpha}\, {}^{C}D_{q,\omega}^{\beta} \left[\frac{E_{\sigma_{q,\omega}}}{\rho_{q,\omega}(t)} + qD_{q,\omega} \right] u(t) = h(t), \quad t \in I_{q,\omega}^T,\\
&u(\omega_0) = \phi(u),\\
&\rho_{q,\omega}(T)u(T) = \rho_{q,\omega}(\eta)u(\eta) = \psi(u), \quad \eta \in I_{q,\omega}^T - \{\omega_0, T\}
\end{aligned} \tag{2.1}$$

has the unique solution

$$\begin{aligned}
u(t) = {}& \frac{1}{q^2 \rho_{p,\omega}(t)} \Bigg\{ q^2 \omega_0 \phi(u)\\
&+ \frac{1}{\Omega} \left[\mathcal{A}^*(h,u) \int_{\omega_0}^{t} (s-\omega) \, d_{q,\omega}s - \mathcal{B}^*(h,u) \int_{\omega_0}^{t} (s-\omega)(s-\omega_0) \, d_{q,\omega}s \right]\\
&+ \int_{\omega_0}^{t}\int_{\omega_0}^{s}\int_{\omega_0}^{x} \frac{(s-\omega)(s-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}} (x - \sigma_{q,\omega}(z))_{q,\omega}^{\underline{\alpha-1}}}{\Gamma_q(\alpha)\Gamma_q(\beta)} h(z) \, d_{q,\omega}z \, d_{q,\omega}x \, d_{q,\omega}s \Bigg\},
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
\Omega := {}& \int_{\omega_0}^{T} (s-\omega) \, d_{q,\omega}s \int_{\omega_0}^{\eta} (s-\omega)(s-\omega_0) \, d_{q,\omega}s - \int_{\omega_0}^{\eta} (s-\omega) \, d_{q,\omega}s \int_{\omega_0}^{T} (s-\omega)(s-\omega_0) \, d_{q,\omega}s\\
= {}& (\eta - T)(T-\omega_0)(\eta-\omega_0) \left[\frac{(T-\omega_0)(\eta-\omega_0)}{(q+1)(q^2+q+1)} + \frac{\omega_0 q (T+\eta-2\omega_0)}{q^2+q+1} + \frac{\omega_0^2 q^2}{q+1} \right],
\end{aligned} \tag{2.3}$$

the functionals $\mathcal{A}^*(h,u), \mathcal{B}^*(h,u)$ *are defined by*

$$\mathcal{A}^*(h,u) := \mathcal{A}(u) + \mathcal{A}_1(h) - \mathcal{A}_2(h), \tag{2.4}$$

$$\mathcal{B}^*(h,u) := \mathcal{B}(u) + \mathcal{B}_1(h) - \mathcal{B}_2(h), \tag{2.5}$$

and

$$\mathcal{A}(u) := \big(\omega_0\phi(u) - \psi(u)\big)\int_{\eta}^{T}(s-\omega)(s-\omega_0)\,d_{q,\omega}s, \tag{2.6}$$

$$\mathcal{A}_1(h) := \frac{\int_{\omega_0}^{T}(s-\omega)(s-\omega_0)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \times \int_{\omega_0}^{\eta}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s, \tag{2.7}$$

$$\mathcal{A}_2(h) := \frac{\int_{\omega_0}^{\eta}(s-\omega)(s-\omega_0)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \times \int_{\omega_0}^{T}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s, \tag{2.8}$$

$$\mathcal{B}(u) := \big(\omega_0\phi(u) - \psi(u)\big)\int_{\eta}^{T}s\,d_{q,\omega}s, \tag{2.9}$$

$$\mathcal{B}_1(h) := \frac{\int_{\omega_0}^{T}(s-\omega)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \times \int_{\omega_0}^{\eta}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s, \tag{2.10}$$

$$\mathcal{B}_2(h) := \frac{\int_{\omega_0}^{\eta}(s-\omega)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \times \int_{\omega_0}^{T}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s. \tag{2.11}$$

Proof We first take fractional q,ω-integral of order α for (2.1) to obtain

$${}^{C}D_{q,\omega}^{\beta}\left[\frac{E_{\sigma_{q,\omega}}}{\rho_{q,\omega}(t)} + qD_{q,\omega}\right]u(t) = C_1 + \mathcal{I}_{q,\omega}^{\alpha}h(t). \tag{2.12}$$

Next, we take fractional q,ω-integral of order β for (2.12). Thus,

$$\begin{aligned}
&\frac{u(\sigma_{q,\omega}(t))}{\rho_{q,\omega}(t)} + qD_{q,\omega}u(t) = C_2 + C_1(t-\omega_0) + \mathcal{I}_{q,\omega}^{\beta}\mathcal{I}_{q,\omega}^{\alpha}h(t),\\
&\frac{u(\sigma_{q,\omega}(t))}{q} + \rho_{q,\omega}(t)D_{q,\omega}u(t) = C_2\frac{\rho_{q,\omega}(t)}{q} + C_1\frac{\rho_{q,\omega}(t)}{q}(t-\omega_0)\\
&\qquad + \frac{\rho_{q,\omega}(t)}{q\Gamma_q(\alpha)\Gamma_q(\beta)}\\
&\qquad \times \int_{\omega_0}^{t}\int_{\omega_0}^{s}\big(t-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}h(x)\,d_{q,\omega}x\,d_{q,\omega}s,
\end{aligned}$$

or

$$D_{q,\omega}\big[\rho_{q,\omega}(t)u(t)\big] = \frac{C_2}{q^2}(t-\omega) + \frac{C_1}{q^2}(t-\omega)(t-\omega_0) + \frac{1}{q^2\Gamma_q(\alpha)\Gamma_q(\beta)} \times \int_{\omega_0}^{t}\int_{\omega_0}^{s}(s-\omega)\big(t-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}}\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}h(x)\,d_{q,\omega}x\,d_{q,\omega}s. \tag{2.13}$$

Finally, we take q,ω-integral for (2.13) to obtain

$$\rho_{q,\omega}(t)u(t) = C_3 + C_2\frac{1}{q^2}\int_{\omega_0}^{t}(s-\omega)\,d_{q,\omega}s + \frac{C_1}{q^2}\int_{\omega_0}^{t}(s-\omega)(s-\omega_0)\,d_{q,\omega}s + \frac{1}{q^2\Gamma_q(\alpha)\Gamma_q(\beta)} \times \int_{\omega_0}^{t}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s. \tag{2.14}$$

By substituting $t=\omega_0$ into (2.14) and employing the first condition of (2.1), we have

$$C_3 = \omega_0\phi(u).$$

Therefore,

$$\rho_{q,\omega}(t)u(t) = \omega_0\phi(u) + C_2\frac{1}{q^2}\int_{\omega_0}^{t}(s-\omega)\,d_{q,\omega} + \frac{C_1}{q^2}\int_{\omega_0}^{t}(s-\omega)(s-\omega_0)\,d_{q,\omega}s + \frac{1}{q^2\Gamma_q(\alpha)\Gamma_q(\beta)} \times \int_{\omega_0}^{t}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s. \tag{2.15}$$

Further, by letting $t=\eta,T$ into (2.15) and employing the second and third conditions of (2.1), we have

$$C_1 = \frac{1}{\Omega}\bigg\{q^2\big(\psi(u)-\omega_0\phi(u)\big)\bigg[\int_{\omega_0}^{T}(s-\omega)\,d_{q,\omega} - \int_{\omega_0}^{\eta}(s-\omega)\,d_{q,\omega}s\bigg] - \frac{\int_{\omega_0}^{T}(s-\omega)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \times \int_{\omega_0}^{\eta}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s$$

$$+\frac{\int_{\omega_0}^{\eta}(s-\omega)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)}$$
$$\times\int_{\omega_0}^{T}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s\Bigg\},$$
$$C_2=-\frac{1}{\Omega}\Bigg\{q^2\big(\psi(u)-\omega_0\phi(u)\big)\Bigg[\int_{\omega_0}^{T}(s-\omega)(s-\omega_0)\,d_{q,\omega}s-\int_{\omega_0}^{\eta}(s-\omega)(s-\omega_0)\,d_{q,\omega}s\Bigg]$$
$$-\frac{\int_{\omega_0}^{T}(s-\omega)(s-\omega_0)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)}$$
$$\times\int_{\omega_0}^{\eta}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s$$
$$+\frac{\int_{\omega_0}^{\eta}(s-\omega)(s-\omega_0)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)}$$
$$\times\int_{\omega_0}^{T}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}h(z)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s\Bigg\},$$

where Ω is defined in (2.3).

To accomplish solution (2.2), we substitute the constants C_1, C_2 into (2.15). □

3 Existence and uniqueness of solution

In this section, we present the existence and uniqueness of solution for problem (1.3). Let $\mathcal{C}=C(I_{q,\omega}^{T},\mathbb{R})$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}}=\max_{t\in I_{q,\omega}^{T}}\big\{|u|\big\},$$

where $\omega>0$, $q\in(0,1)$. Define an operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned}(\mathcal{F}u)(t):={}&\frac{1}{q^2\rho_{q,\omega}(t)}\Bigg\{q^2\omega_0\phi(u)+\frac{1}{\Omega}\Bigg[\mathcal{A}^{**}(F,u)\int_{\omega_0}^{t}(s-\omega)\,d_{q,\omega}s\\&-\mathcal{B}^{**}(F,u)\int_{\omega_0}^{t}(s-\omega)(s-\omega_0)\,d_{q,\omega}s\Bigg]\\&+\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\\&\times\int_{\omega_0}^{t}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}\\&\times F\big(z,u(z),\Psi_{q,\omega}^{\gamma}u(z)\big)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s\Bigg\},\end{aligned}\tag{3.1}$$

where $\Omega\neq 0$ is defined in (2.3), the functionals $\mathcal{A}^{**}(F,u)$, $\mathcal{B}^{**}(F,u)$ are defined by

$$\mathcal{A}^{**}(F,u):=\mathcal{A}(u)+\mathcal{A}_1^{*}(F)-\mathcal{A}_2^{*}(F),\tag{3.2}$$

$$\mathcal{B}^{**}(F,u):=\mathcal{B}(u)+\mathcal{B}_1^{*}(F)-\mathcal{B}_2^{*}(F),\tag{3.3}$$

and

$$\mathcal{A}_1^*(F) := \frac{\int_{\omega_0}^T (s-\omega)(s-\omega_0)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{\eta}\int_{\omega_0}^{s}\int_{\omega_0}^{x} (s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}} \times F\big(z,u(z),\Psi_{q,\omega}^{\gamma}u(z)\big)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s, \tag{3.4}$$

$$\mathcal{A}_2^*(F) := \frac{\int_{\omega_0}^{\eta} (s-\omega)(s-\omega_0)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{T}\int_{\omega_0}^{s}\int_{\omega_0}^{x} (s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}} \times F\big(z,u(z),\Psi_{q,\omega}^{\gamma}u(z)\big)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s, \tag{3.5}$$

$$\mathcal{B}_1^*(F) := \frac{\int_{\omega_0}^T (s-\omega)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{\eta}\int_{\omega_0}^{s}\int_{\omega_0}^{x} (s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}} \times F\big(z,u(z),\Psi_{q,\omega}^{\gamma}u(z)\big)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s, \tag{3.6}$$

$$\mathcal{B}_2^*(F) := \frac{\int_{\omega_0}^{\eta} (s-\omega)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{T}\int_{\omega_0}^{s}\int_{\omega_0}^{x} (s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}} \times F\big(z,u(z),\Psi_{q,\omega}^{\gamma}u(z)\big)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s, \tag{3.7}$$

with functionals $\mathcal{A}(u)$, $\mathcal{B}(u)$ defined in (2.6), (2.9), respectively.

We find that problem (1.3) has solutions if and only if the operator $\mathcal{F}$ has fixed points.

Theorem 3.1 *Assume that the following conditions hold*:

(H_1) *There exist constants* $\lambda_1, \lambda_2 > 0$ *such that, for each* $t \in I_{q,\omega}^T$ *and* $u, v \in \mathbb{R}$,

$$\big|F\big(t,u,\Psi_{r,\omega}^{\gamma}u\big) - F\big(t,v,\Psi_{q,\omega}^{\gamma}v\big)\big| \le \lambda_1|u-v| + \lambda_2\big|\Psi_{q,\omega}^{\gamma}u - \Psi_{q,\omega}^{\gamma}v\big|.$$

(H_2) *There exist constants* $\ell_1, \ell_2 > 0$ *such that, for each* $u, v \in \mathcal{C}$,

$$\big|\phi(u)-\phi(v)\big| \le \ell_1\|u-v\|_{\mathcal{C}} \quad \textit{and} \quad \big|\psi(u)-\psi(v)\big| \le \ell_2\|u-v\|_{\mathcal{C}}.$$

(H_3) $\mathcal{L}[\Theta_1+\Theta_2] < q^2\omega_0$, *where*

$$\mathcal{L} := \max\left\{\ell, \lambda_1 + \lambda_2\varphi_0\frac{(T-\omega_0)_{q,\omega}^{\underline{\gamma}}}{\Gamma_r(\gamma+1)}\right\}, \quad \ell = |\omega_0\ell_1 - \ell_2| \tag{3.8}$$

$$\Theta_1 := \omega_0 + \frac{(T-\omega_0)^2}{|\Omega|}\big(\Lambda_{11} + (T-\omega_0)\Lambda_{12}\big), \tag{3.9}$$

$$\Theta_2 := \frac{(T-\omega_0)^2}{|\Omega|}\big(\Lambda_{21} + (T-\omega_0)[\Lambda_{22}+\mathcal{O}]\big) + (T-\omega_0)\mathcal{O}, \tag{3.10}$$

and

$$\Lambda_{11} := (T-\eta)\left|\frac{T^2+T\eta+\eta^2-3\omega_0(T+\eta-\omega_0)}{q^2+q+1} + \omega_0 q\frac{(T+\eta-2\omega_0)}{1+q}\right|, \tag{3.11}$$

$$\Lambda_{12} := \frac{(\eta-\omega_0)^3(T-\omega_0)^3}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\beta+1,\alpha)\big|(T-\omega_0)^{\alpha+\beta-1} - (\eta-\omega_0)^{\alpha+\beta-1}\big|, \tag{3.12}$$

$$\Lambda_{21} := \frac{T-\eta}{1+q}\big|T+\eta-\omega_0\big(2-q-q^2\big)\big|, \tag{3.13}$$

$$\Lambda_{22} := \frac{(\eta-\omega_0)^2(T-\omega_0)^2}{\Gamma_q(\alpha)\Gamma_q(\beta+1)} B_q(\beta+1,\alpha)\left|(T-\omega_0)^{\alpha+\beta}-(\eta-\omega_0)^{\alpha+\beta}\right|, \tag{3.14}$$

$$\mathcal{O} := \frac{(T-\omega_0)^{\alpha+\beta+2}}{\Gamma_q(\alpha)\Gamma_q(\beta+1)} B_q(\beta+1,\alpha). \tag{3.15}$$

Then problem (1.3) *has a unique solution.*

Proof In order to prove that our problem has a unique solution, we show that F is a contraction. Since

$$\mathcal{H}|u-v|(t) := \left|F\big(t,u(t),\Psi^{\gamma}_{q,\omega}u(t)\big)-F\big(t,v(t),\Psi^{\gamma}_{q,\omega}v(t)\big)\right|,$$

for each $t \in I^T_{q,\omega}$ and $u,v \in \mathcal{C}$, we obtain

$$\begin{aligned}
&\left|\mathcal{A}(u)-\mathcal{A}(v)\right| \\
&\quad = \big(\omega_0\left|\phi(u)-\phi(v)\right| - \left|\psi(u)-\psi(v)\right|\big)\left|\int_\eta^T (s-\omega)(s-\omega_0)\,d_{q,\omega}s\right| \\
&\quad \le \ell\|u-v\|_{\mathcal{C}}\left|\frac{(T-\omega_0)^3-(\eta-\omega_0)^3}{q^2+q+1}+\omega_0 q\frac{(T-\omega_0)^2-(\eta-\omega_0)^2}{1+p}\right| \\
&\quad = \ell\|u-v\|_{\mathcal{C}}(T-\eta)\left|\frac{T^2+T\eta+\eta^2-3\omega_0(T+\eta-\omega_0)}{q^2+q+1}+\omega_0 q\frac{(T+\eta-2\omega_0)}{1+q}\right|.
\end{aligned} \tag{3.16}$$

Similarly, we have

$$\left|\mathcal{B}(u)-\mathcal{B}(v)\right| \le \ell\|u-v\|_{\mathcal{C}}\frac{T-\eta}{1+q}\left|T+\eta-\omega_0\big(2-q-q^2\big)\right|. \tag{3.17}$$

In addition, we have

$$\begin{aligned}
&\left|\mathcal{A}_1^*\big(F(u)\big)-\mathcal{A}_1^*\big(F(v)\big)\right| \\
&\quad = \frac{\int_{\omega_0}^T (s-\omega)(s-\omega_0)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^\eta\int_{\omega_0}^s\int_{\omega_0}^x (s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}} \\
&\qquad \times \mathcal{H}|u-v|(s)\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s \\
&\quad \le \frac{\frac{(T-\omega_0)^2}{1+q+q^2}\left|T-\omega_0\left(\frac{1-q^2-q^3}{1+q}\right)\right|}{\Gamma_q(\alpha)\Gamma_q(\beta)}\big(\lambda_1|u-v|+\lambda_2\left|\Psi^{\gamma}_{q,\omega}u-\Psi^{\gamma}_{q,\omega}v\right|\big) \\
&\qquad \times \int_{\omega_0}^\eta\int_{\omega_0}^s\int_{\omega_0}^x (s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s \\
&\quad \le \big(\lambda_1|u-v|+\lambda_2\left|\Psi^{\gamma}_{q,\omega}u-\Psi^{\gamma}_{q,\omega}v\right|\big)\left\{\frac{(\eta-\omega_0)(T-\omega_0)^3}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\beta+1,\alpha)\int_{\omega_0}^\eta s^{\alpha+\beta}\,d_{q,\omega}s\right\} \\
&\quad \le \left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u-v\|_{\mathcal{C}}\left\{\frac{(\eta-\omega_0)^{\alpha+\beta+2}(T-\omega_0)^3}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\beta+1,\alpha)\right\},
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&\left|\mathcal{A}_2^*\big(F(u)\big)-\mathcal{A}_2^*\big(F(v)\big)\right|\\&\quad\le\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u-v\|_{\mathcal{C}}\left\{\frac{(\eta-\omega_0)^3(T-\omega_0)^{\alpha+\beta+2}}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\beta+1,\alpha)\right\},\end{aligned}\tag{3.19}$$

$$\begin{aligned}&\left|\mathcal{B}_1^*\big(F(u)\big)-\mathcal{B}_1^*\big(F(v)\big)\right|\\&\quad\le\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u-v\|_{\mathcal{C}}\left\{\frac{(\eta-\omega_0)^{\alpha+\beta+2}(T-\omega_0)^2}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\beta+1,\alpha)\right\},\end{aligned}\tag{3.20}$$

$$\begin{aligned}&\left|\mathcal{B}_2^*\big(F(u)\big)-\mathcal{B}_2^*\big(F(v)\big)\right|\\&\quad\le\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u-v\|_{\mathcal{C}}\left\{\frac{(\eta-\omega_0)^2(T-\omega_0)^{\alpha+\beta+2}}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\beta+1,\alpha)\right\}.\end{aligned}\tag{3.21}$$

Based on (3.16)-(3.21), we have

$$\begin{aligned}&\left|\mathcal{A}^{**}(F,u)-\mathcal{A}^{**}(F,v)\right|\\&\quad\le\ell\|u-v\|_{\mathcal{C}}\Lambda_{11}+\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u-v\|_{\mathcal{C}}\Lambda_{12},\end{aligned}\tag{3.22}$$

$$\begin{aligned}&\left|\mathcal{B}^{**}(F,u)-\mathcal{B}^{**}(F,v)\right|\\&\quad\le\ell\|u-v\|_{\mathcal{C}}\Lambda_{21}+\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u-v\|_{\mathcal{C}}[\Lambda_{22}+\mathcal{O}],\end{aligned}\tag{3.23}$$

where Λ_{11}, Λ_{12}, Λ_{21}, Λ_{22}, and $\mathcal{O}$ are defined on (3.11)-(3.15), respectively.

Consider

$$\begin{aligned}&\left|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\right|\\&\quad\le\frac{1}{q^2\rho_{q,\omega}(\omega_0)}\Bigg\{\omega_0\ell\|u-v\|_{\mathcal{C}}\\&\qquad+\frac{1}{|\Omega|}\Big[(T-\omega_0)^2\left|\mathcal{A}^{**}(F,u)-\mathcal{A}^{**}(F,v)\right|\\&\qquad+(T-\omega_0)^3\left|\mathcal{B}^{**}(F,u)-\mathcal{B}^{**}(F,v)\right|\Big]\\&\qquad+\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u-v\|_{\mathcal{C}}(T-\omega_0)\mathcal{O}\Bigg\}\\&\quad\le\ell\|u-v\|_{\mathcal{C}}\frac{\Theta_1}{q^2\omega_0}+\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u-v\|_{\mathcal{C}}\frac{\Theta_2}{q\omega_0}\\&\quad\le\frac{\mathcal{L}}{q^2\omega_0}\|u-v\|_{\mathcal{C}}[\Theta_1+\Theta_2],\end{aligned}\tag{3.24}$$

where $\mathcal{L}$, Θ_1, and Θ_2 are defined on (3.8)-(3.10), respectively.

From (H_3), this implies that $\mathcal{F}$ is a contraction. Therefore, by using the Banach fixed point theorem, we can conclude that $\mathcal{F}$ has a fixed point which is a unique solution of problem (1.3) on $t \in I^T_{q,\omega}$. □

4 Existence of at least one solution

In this section, we also deduce the existence of a solution to (1.3) by using the following Schauder fixed point theorem.

Lemma 4.1 ([29] (Arzelá-Ascoli theorem)) *A set of functions in $C[a,b]$ with the sup norm is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 4.2 ([29]) *If a set is closed and relatively compact, then it is compact.*

Lemma 4.3 ([30] (Schauder's fixed point theorem)) *Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T: D \to D$ be the map such that the set $Tu: u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

Theorem 4.1 *Suppose that (H_1)-(H_2) hold. Then* (1.3) *has at least one solution.*

Proof We divide the proof into three steps as follows.

Step I. Verify that $\mathcal{F}$ maps bounded sets to bounded sets in $B_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \le R\}$. Let $B_R = \{u \in C(I^T_{q,\omega}) : \|u\|_{\mathcal{C}} \le R\}$, $\max_{t\in I^T_{q,\omega}} |F(t,0,0)| = K$, $\sup_{u\in\mathcal{C}} |\phi(u)| = M$, $\sup_{u\in\mathcal{C}} |\psi(u)| = N$, and choose a constant

$$R \ge \frac{\Theta_1|\omega_0 M - N| + \Theta_2 K}{q^2\omega_0 - [\Theta_1\ell + \Theta_2(\lambda_1 + \frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)^{\underline{\gamma+1}}_{q,\omega})]}. \tag{4.1}$$

We note that $|\mathcal{S}(t,u,0)| = |F(t,u(t),\Psi^\gamma_{q,\omega}u(t)) - F(t,0,0)| + |F(t,0,0)|$.

For each $u \in B_R$, we obtain

$$\begin{aligned} \bigl|\mathcal{A}(u)\bigr| &= \bigl|\omega_0\bigl(\bigl|\phi(u)-\phi(0)\bigr| + \bigl|\phi(0)\bigr|\bigr) - \bigl(\bigl|\psi(u)-\psi(0)\bigr| + \bigl|\psi(0)\bigr|\bigr)\bigr| \\ &\quad\times \biggl|\int_\eta^T (s-\omega)(s-\omega_0)\,d_{q,\omega}s\biggr| \\ &\le \bigl(\ell\|u\|_{\mathcal{C}} + |\omega_0 M - N|\bigr)(T-\eta) \\ &\quad\times \biggl|\frac{T^2 + T\eta + \eta^2 - 3\omega_0(T+\eta-\omega_0)}{q^2+q+1} + \omega_0 q\frac{(T+\eta-2\omega_0)}{1+q}\biggr|. \end{aligned} \tag{4.2}$$

Similarly,

$$\bigl|\mathcal{B}(u)\bigr| \le \bigl(\ell\|u\|_{\mathcal{C}} + |\omega_0 M - N|\bigr)\frac{T-\eta}{1+q}\bigl|T + \eta - \omega_0\bigl(2 - q - q^2\bigr)\bigr|. \tag{4.3}$$

In addition,

$$\begin{aligned}
&\left|\mathcal{A}_1^*(F(u))\right| \\
&\quad= \frac{\int_{\omega_0}^{T}(s-\omega)(s-\omega_0)\,d_{q,\omega}s}{\Gamma_q(\alpha)\Gamma_q(\beta)} \\
&\qquad\times \int_{\omega_0}^{\eta}\int_{\omega_0}^{s}\int_{\omega_0}^{x}(s-\omega)\big(s-\sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{\beta-1}}\big(x-\sigma_{q,\omega}(z)\big)_{q,\omega}^{\underline{\alpha-1}}\big|\mathcal{S}(t,u,0)\big|\,d_{q,\omega}z\,d_{q,\omega}x\,d_{q,\omega}s \\
&\quad\leq \left[\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u\|_{\mathcal{C}}+K\right] \\
&\qquad\times\left\{\frac{(\eta-\omega_0)^{\alpha+\beta+2}(T-\omega_0)^3}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\alpha,\beta+1)\right\}. \qquad (4.4)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left|\mathcal{A}_2^*(F(u))\right| &\leq \left[\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u\|_{\mathcal{C}}+K\right] \\
&\quad\times\left\{\frac{(\eta-\omega_0)^3(T-\omega_0)^{\alpha+\beta+2}}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\alpha,\beta+1)\right\}, \qquad (4.5)\\
\left|\mathcal{B}_1^*(F(u))\right| &\leq \left[\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u\|_{\mathcal{C}}+K\right] \\
&\quad\times\left\{\frac{(\eta-\omega_0)^{\alpha+\beta+2}(T-\omega_0)^2}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\alpha,\beta+1)\right\}, \qquad (4.6)\\
\left|\mathcal{B}_2^*(F(u))\right| &\leq \left[\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u\|_{\mathcal{C}}+K\right] \\
&\quad\times\left\{\frac{(\eta-\omega_0)^2(T-\omega_0)^{\alpha+\beta+2}}{\Gamma_q(\alpha)\Gamma_q(\beta+1)}B_q(\alpha,\beta+1)\right\}. \qquad (4.7)
\end{aligned}$$

From (4.2)-(4.7), we obtain

$$\begin{aligned}
\left|\mathcal{A}^{**}(F,u)\right| &\leq \big(\ell\|u\|_{\mathcal{C}}+|\omega_0M-N|\big)\Lambda_{11} \\
&\quad+\left[\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u\|_{\mathcal{C}}+K\right]\Lambda_{12}, \qquad (4.8)\\
\left|\mathcal{B}^{**}(F,u)\right| &\leq \big(\ell\|u\|_{\mathcal{C}}+|\omega_0M-N|\big)\Lambda_{21} \\
&\quad+\left[\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\|u\|_{\mathcal{C}}+K\right][\Lambda_{22}+\mathcal{O}], \qquad (4.9)
\end{aligned}$$

where Λ_{11}, Λ_{12}, Λ_{21}, Λ_{22}, and $\mathcal{O}$ are defined in (3.11)-(3.15), respectively.

From (4.8) and (4.9), we have

$$\begin{aligned}
\big|(\mathcal{F}u)(t)\big| &\leq \frac{R}{q^2\omega_0}\left[\Theta_1\ell+\Theta_2\left(\lambda_1+\frac{\lambda_2\varphi_0}{\Gamma_q(\gamma+1)}(T-\omega_0)_{q,\omega}^{\underline{\gamma+1}}\right)\right] \\
&\quad+\frac{1}{q^2\omega_0}\big[|\omega_0M-N|\Theta_1+K\Theta_2\big], \qquad (4.10)
\end{aligned}$$

where Θ_1 and Θ_2 are defined in (3.8)-(3.10), respectively.

Therefore, $\|\mathcal{F}u\|_{\mathcal{C}}\leq R$. This implies that $\mathcal{F}$ is uniformly bounded.

Step II. Show that $\mathcal{F}$ is continuous on B_R. From the continuity of F, we can conclude that the operator $\mathcal{F}$ is continuous on B_R.

Step III. Examine that $\mathcal{F}$ is equicontinuous with B_R. For any $t_1, t_2 \in I_{q,\omega}^T$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\bigl|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\bigr| \\
&\leq \omega_0 M \left| \frac{1}{t_2 - \omega} - \frac{1}{t_1 - \omega} \right| + \left| \frac{\mathcal{A}^{**}}{\Omega} \right| \\
&\quad \times \left| \frac{1}{(t_2 - \omega)} \int_{\omega_0}^{t_2} (s-\omega)\, d_{q,\omega}s - \frac{1}{(t_1 - \omega)} \int_{\omega_0}^{t_1} (s-\omega)\, d_{q,\omega}s \right| \\
&\quad + \left| \frac{\mathcal{B}^{**}}{\Omega} \right| \left| \frac{1}{(t_2 - \omega)} \int_{\omega_0}^{t_2} (s-\omega)(s-\omega_0)\, d_{q,\omega}s - \frac{1}{(t_1 - \omega)} \int_{\omega_0}^{t_1} (s-\omega)(s-\omega_0)\, d_{q,\omega}s \right| \\
&\quad + \|F\| \left| \frac{1}{t_2 - \omega} \int_{\omega_0}^{t_2} \int_{\omega_0}^{s} \int_{\omega_0}^{x} \frac{(s-\omega)(s-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}} (x - \sigma_{q,\omega}(z))_{q,\omega}^{\underline{\alpha-1}}}{\Gamma_q(\alpha)\Gamma_q(\beta)}\, d_{q,\omega}z\, d_{q,\omega}x\, d_{q,\omega}s \right. \\
&\quad \left. - \frac{1}{t_1 - \omega} \int_{\omega_0}^{t_1} \int_{\omega_0}^{s} \int_{\omega_0}^{x} \frac{(s-\omega)(s-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}} (x - \sigma_{q,\omega}(z))_{q,\omega}^{\underline{\alpha-1}}}{\Gamma_q(\alpha)\Gamma_q(\beta)}\, d_{q,\omega}z\, d_{q,\omega}x\, d_{q,\omega}s \right| \\
&< \frac{\omega_0 M}{t_1 - \omega} |t_1 - t_2| + \left| \frac{\mathcal{A}^{**}}{\Omega(t_1 - \omega)} \right| \left| \int_{t_1}^{t_2} (s-\omega)\, d_{q,\omega}s \right. \\
&\quad + \left| \frac{\mathcal{B}^{**}}{\Omega(t_1 - \omega)} \right| \left| \int_{t_1}^{t_2} (s-\omega)(s-\omega_0)\, d_{q,\omega}s \right. \\
&\quad + \frac{\|F\|}{t_1 - \omega} \int_{t_1}^{t_2} \int_{\omega_0}^{s} \int_{\omega_0}^{x} \frac{(s-\omega)(s-\sigma_{p,\omega}(x))_{q,\omega}^{\underline{\beta-1}} (x - \sigma_{q,\omega}(z))_{q,\omega}^{\underline{\alpha-1}}}{\Gamma_q(\alpha)\Gamma_p(\beta)}\, d_{q,\omega}z\, d_{p,\omega}x\, d_{q,\omega}s.
\end{aligned}$$

When $|t_2 - t_1| \to 0$, the right-hand side of the above inequality tends to be zero. Therefore, $\mathcal{F}$ is relatively compact on B_R.

Thus, the set $\mathcal{F}(B_R)$ is an equicontinuous set. As a consequence of Steps I to III together with the Arzelá-Ascoli theorem, $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is completely continuous. By the Schauder fixed point theorem mentioned above, we can conclude that problem (1.3) has at least one solution. The proof is completed. □

5 Example

In this section, we present a boundary value problem for fractional Hahn difference equation to illustrate our results as follows:

$$\begin{aligned}
&{}^{C}D_{\frac{1}{2},2}^{\frac{1}{2}}\, {}^{C}D_{\frac{1}{2},2}^{\frac{2}{5}} \left[\frac{E_{\sigma_{\frac{1}{2},2}}}{\rho_{\frac{1}{2},2}(t)} + \frac{1}{2} D_{\frac{1}{2},2} \right] u(t) \\
&\quad = \frac{e^{-10t}}{\pi t^2} \cdot \frac{|u| + 1}{1 + \cos^2 u} \\
&\qquad + \frac{\arctan(\cos^2 \pi t)}{(t + 100)^4} \mathcal{I}_{\frac{1}{2},2}^{\frac{2}{3}} (\varphi u)(t), \quad t \in \left[\frac{67}{16}, 10 \right]_{\frac{1}{2},2}, \qquad (5.1) \\
&u\left(\frac{67}{16}\right) = \frac{|u| \sin^2 |\pi u|}{(200e)^6}, \\
&\rho_{\frac{1}{2},2}(10) u(10) = \rho_{\frac{1}{2},2}\left(\frac{19}{4}\right) u\left(\frac{19}{4}\right) = \sum_{i=0}^{\infty} \frac{C_i |u(t_i)|}{1 + |u(t_i)|}, \quad t_i = 10\left(\frac{1}{30}\right)^i + 2[i]_{\frac{1}{30}},
\end{aligned}$$

where $\varphi(t,s) = \frac{e^{-2|s-t|}}{2000\pi^3}$ and C_i are given constants with $\frac{1}{(100\pi)^4} \leq \sum_{i=0}^{\infty} C_i \leq \frac{e}{(100\pi)^4}$.

We let $\alpha = \frac{1}{2}$, $\beta = \frac{2}{5}$, $\gamma = \frac{2}{3}$, $q = \frac{1}{2}$, $\omega = 2$, $\omega_0 = \frac{\omega}{1-\chi} = \frac{60}{29}$, $T = 10$, $\eta = 10(\frac{1}{30})^3 + 2[3]_{\frac{1}{30}} = \frac{5587}{2700}$, $\phi(u) = \frac{|u|\sin^2|\pi u|}{(200e)^6}$, $\psi(u) = \sum_{i=0}^{\infty} \frac{C_i|u(t_i)|}{1+|u(t_i)|}$, and $F(t,u(t),\Psi_{r,\omega}^{\gamma}u(t)) = \frac{e^{-10t}}{\pi t^2} \cdot \frac{|u|+1}{1+\cos^2 u} + \frac{\arctan(\cos^2 \pi t)}{(t+100)^4} \mathcal{I}_{\frac{4}{5},2}^{\frac{2}{3}}(\varphi u)(t)$.

We find that

$$|\Omega| = 0.644, \qquad \Lambda_{11} = 15{,}130.184, \qquad \Lambda_{12} = 198{,}828.485, \qquad \Lambda_{21} = 14{,}966.095,$$
$$\Lambda_{22} = 78{,}845.778, \quad \text{and} \quad \mathcal{O} = 60.577.$$

For all $t \in [\frac{60}{29}, 10]_{\frac{1}{30},2}$ and $u, v \in \mathbb{R}$, we have

$$\bigl|F\bigl(t,u,\Psi_{r,\omega}^{\gamma}u\bigr) - F\bigl(t,v,\Psi_{r,\omega}^{\gamma}v\bigr)\bigr| \leq \frac{1}{e^{\frac{600}{29}}\pi(\frac{60}{29})^2}|u-v| + \frac{\pi}{4(\frac{60}{29}+100)^4}\bigl|\Psi_{r,\omega}^{\gamma}u - \Psi_{r,\omega}^{\gamma}v\bigr|.$$

Thus, (H_1) holds with $\gamma_1 = 7.69 \times 10^{-11}$, $\gamma_2 = 7.236 \times 10^{-9}$, and $\varphi_0 = [\frac{1}{2000\pi^3}]$.

For all $u, v \in \mathcal{C}$,

$$\bigl|\phi(u) - \phi(v)\bigr| = \frac{1}{(200e)^6}\|u-v\|_{\mathcal{C}},$$
$$\bigl|\psi(u) - \psi(v)\bigr| = C_i \sum_{i=0}^{\infty}\bigl|u(t_i) - v(t_i)\bigr| \leq \frac{e}{(100\pi)^4}\|u-v\|_{\mathcal{C}}.$$

So, (H_2) holds with $\ell_1 = 3.873 \times 10^{-17}$ and $\ell_2 = 2.791 \times 10^{-10}$.

Hence, from Theorem 4.1, problem (5.1) has at least one solution. □

In addition, we find that

$$\mathcal{L} = \max\left\{|\omega_0\ell_1 - \ell_2|, \lambda_1 + \lambda_2\varphi_0\frac{(T-\omega_0)_{r,\omega}^{\underline{\gamma}}}{\Gamma_r(\gamma+1)}\right\} = 2.791 \times 10^{-10},$$
$$\Theta_1 = \omega_0 + \frac{(T-\omega_0)^2}{|\Omega|}\bigl(\Lambda_{11} + (T-\omega_0)\Lambda_{12}\bigr) = 1.961 \times 10^8,$$
$$\Theta_2 = \frac{(T-\omega_0)^2}{|\Omega|}\bigl(\Lambda_{21} + (T-\omega_0)[\Lambda_{22} + \mathcal{O}]\bigr) + T\mathcal{O} = 7.891.$$

Therefore, (H_3) holds with

$$0.055 = \mathcal{L}[\Theta_1 + \Theta_2] < q^2\omega_0 = 1.034.$$

Hence, problem (5.1) has a unique solution by Theorem 3.1. □

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Authors' contributions
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Author details
[1]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand. [2]Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, Thailand.

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